Quaternionic Multiplication Rule as a Local Q-Metric.

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It is known that all associative classical field theories can be formulated in terms of quaternions (see *e.g.* the review paper [1]). Recently, a revival of interest in the quaternionic formulation has been exhibited in papers, where quaternions were used mainly for purposes of obtaining desirable (spinor) forms of equations, for providing easier proofs or for providing more general interpretations [2].

It is herein suggested that the quaternionic approach has much more profound meaning in physics than being just a convenient mathematical tool. In this connection consider an interesting and convincing example in which a quaternion-based metric generates the Schrödinger-Pauli spin equation.

Take the special representation of constant «imaginary» quaternionic units: $q_k = -i\sigma_k$, where σ_k are the Pauli matrices; Latin indices run through 1, 2, 3. Then it is easily verified that the following multiplication rule holds: $q_i q_j = -\delta_{ij} + \varepsilon_{ijk} q_k$ (δ_{ij} is the Kronecker symbol, ε_{ijk} is the 3-dimensional Levi-Civita tensor, the summation convention is valid). As each of the three q_i can be regarded as determining a direction independent and orthogonal to the other two, or as a unit vector of a frame in certain 3-space, it is possible to associate the quaternionic multiplication rule with a local metric of this particular space:

$$g_{ij} = q_i \bar{q}_j = \delta_{ij} - \varepsilon_{ijk} q_k \,. \tag{1}$$

The standard multiplication rule has been slightly modified, its sign has been changed by quaternionic conjugation (or «space reflection») $\bar{q}_i = -q_i$. Obviously the metric (1), (from now on we call it Q-metric) consists of two parts: a symmetric part, which is the usual flat-space Cartesian metric multiplied by the (2×2) unit matrix, and an antisymmetric containing (2×2)-matrices q_i with vanishing trace. There is a case where the antisymmetric term aquires great significance.

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(1) P. RASTALL: *Rev. Mod. Phys.*, **36**, 820 (1964).

(2) F. A. DORIA: *Lett. Nuovo Cimento*, **14**, 480 (1975); V. MAGERNIC and M. NAGY: *Lett. Nuovo Cimento*, **16**, 265 (1976); A. SINGH: *Lett. Nuovo Cimento*, **33**, 457 (1982).

With aid of the Q-metric, it is possible to form a Hamiltonian function for a particle having mass m and generalized momentum $\tilde{P}_i$:

$$H = \frac{1}{2m} \tilde{P}_i \tilde{P}_j q_{ij} = \frac{1}{2m} (\tilde{P}_i \tilde{P}_i - \tilde{P}_i \tilde{P}_j \varepsilon_{ijk} q_k) . \tag{2}$$

In classical theory the second term in (2) must vanish, since it is a contraction of symmetric and antisymmetric tensors, but in quantum mechanics for a particle with charge e, in the Maxwell field A_i, $\tilde{P}_i = p_i - (e/c) A_i$, which when substituted in the second term of (2) gives

$$-\frac{e}{c} (p_i A_j - A_j p_i) \varepsilon_{ijk} q_k . \tag{3}$$

The sum in brackets is not symmetric under the operational substitution $p_k = -i\hbar\partial_k$. After performing the commutation $p_i A_j = -i\hbar A_{j,i} + A_j p_i$, (3) becomes

$$\frac{i\hbar e}{2c} F_{ij} \varepsilon_{ijk} q_k = \frac{e\hbar}{c} B_k \sigma_k , \tag{4}$$

and (2) when rewritten in vector notation aquires the familiar form

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 - \frac{e\hbar}{2mc} \boldsymbol{B}\cdot\boldsymbol{\sigma} . \tag{5}$$

Equation (5) represents the energy operator for a charged particle with spin $\frac{1}{2}$ moving in a magnetic field, *i.e.* the Hamiltonian of the Schrödinger-Pauli spin quantum-mechanical equation, which originally had been derived from heuristic considerations. Here the coefficient $e\hbar/2mc$ (Bohr magneton) appears naturally with the correct sign, hence there is no need to evaluate an arbitrary multiplier β, usually attached to the Pauli spin term, from the relativistic Dirac equations (see *e.g.* (3)). The use of the quaternionic metric produces the correct coupling constant directly.

This result renders strong support to the idea of considering the multiplication rule of quaternionic basis as a local metric in a 3-space. This space possesses a richer geometrical content, since the q-metric displays under certain physical conditions (in quantum mechanics for instance) the spin structure, normally hidden in the Euclidian geometry of classical physics.

The possible construction of a variable Q-metric and its 4-dimensional extension will be given elsewhere.

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(3) L. D. LANDAU and E. M. LIFSHITZ: *Quantum Mechanics* (London, 1958), p. 473.