IL NUOVO CIMENTO Vol. XLVIII A, N. 2 21 Marzo 1967

Transverse-Momentum Distribution of Secondaries from High-Energy Nuclear Interactions and Interpretation by Means of a Statistical Model.

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(ricevuto il 16 Settembre 1966)

Summary. — An analysis of the experimental transverse momentum p_t of secondary particles produced in high-energy nuclear interactions in the accelerator energy region is carried out. The average value of p_t for each kind of particle is almost constant for primary energy from 6 GeV up to cosmic-ray energy. Several theoretical p_t distributions are fitted to the experimental p_t distribution. The experimental p_t distributions of π-mesons, K-mesons, nucleons, Λ and Σ were well fitted by the theoretical p_t distributions derived from the momentum distributions of Planck for mesons and of Fermi for baryons. The characteristic parameter kT involved in the theoretical p_t distribution is nearly constant for all kinds of particles and is ~ 0.125 GeV. It is shown that the p_t distribution of protons from large-angle p-p elastic scattering and that of secondaries from inelastic interactions are similar in shape and are governed by the same statistical law. The relations between the above two p_t distributions are discussed. On the basis of statistical theory, a derivation of the theoretical p_t distribution is given.

1. – Introduction.

Though extensive studies have been performed on the distribution of the transverse momentum, p_t, of protons of elastic p-p scattering in the accelerator energy region, relatively few works have been done on the p_t distribution of multiply produced particles from inelastic collisions. The experimental data are accumulated for both comic-ray and accelerator energy regions and the attention of research workers has been directed not only to the constancy of the mean value of p_t but also to the shape of the p_t distribution, because the latter reflects the nature and the type of the interaction more sensitively. As the mechanism of multiple particle production in inelastic collisions is much more complicated than that of inelastic p-p scattering, relatively fewer analyses have been performed.

In a recent paper (1), the experimental p_t distribution of the secondary π-mesons produced in cosmic-ray jets of $p_t = (0 \div 1.0)$ GeV/c has been studied. It is found that the experimental p_t distribution fits well the distributions: I) $f(p_t)\mathrm{d}p_t = y^2 \sum_{n=1}^{\infty} K_1(ny)\,\mathrm{d}y$, $y = (M^2 + p_t^2)^{\frac{1}{2}}/kT$ with $kT = 0.125$ GeV, which is derived from Planck's momentum distribution, and II) $f(p_t)\mathrm{d}p_t = p_t \exp[-p_t/p_0]\mathrm{d}p_t$, with $p_0 = 0.154$ GeV/c, but does not fit the distribution III) $f(p_t)\mathrm{d}p_t = p_t \exp[-p_t^2/\alpha^2]\mathrm{d}p_t$.

On the other hand, in ref. (2) it has been pointed out that the p_t distribution of protons from large-angle elastic p-p scattering (3) can be represented in the form of IV) $p_t \exp[-p_t/0.151]\mathrm{d}p_t$.

It is worth-while to mention that, though the one is the p_t distribution of the multiply produced π-mesons in cosmic-ray jets (1,4) and the other is that of protons of large-angle elastic p-p scattering, both the p_t distributions II) and IV) have a common shape of p_t distribution functions.

It is not unreasonable to suppose that the production of protons from large-angle p-p elastic scattering and that of secondaries from inelastic interactions are governed by the same statistical law, so that the p_t distributions of both the protons of large-angle elastic scattering and that of secondary π-mesons in cosmic-ray jets have same p_t distribution function.

COCCONI (5) conjectured that all the secondary particles emerging from p-p interactions, elastic as well as inelastic, have a transverse momentum distribution of the form $p_t \exp[-p_t/p_0]\mathrm{d}p_t$, $p_0 = 0.16$ GeV/c.

It is interesting to study the p_t distribution of all the secondaries produced by interactions in the accelerator energy region by extending the previous analysis of the p_t distribution in cosmic-ray jets. This approach seems to be quite crude to study experimental results in the accelerator energy region compared with the general methods of analysis adopted in this field. However, even in the accelerator energy region, inelastic events are too complicated to be treated by rigorous theory and it is almost impossible to apply rigorous theory to superposed events of various types of interactions in which still unknown processes are taking part. The approach described in this paper, therefore, is of importance as it visualizes gross features of multiple particle production as a whole

(1) K. IMAEDA and J. AVIDAN: *Nuovo Cimento*, **32**, 1497 (1964).
(2) J. OREAR: *Phys. Rev. Lett.*, **12**, 112 (1964).
(3) W. F. BAKER, R. L. COOL, E. W. JENKINS, T. F. KYCIA, S. J. LINDENBAUM, W. A. LOVE, D. LÜERS, J. A. NIEDERER, S. OZAKI, A. L. READ, J. J. RUSSELL and L. C. L. YUAN: *Phys. Rev. Lett.*, **7**, 101 (1961).
(4) G. COCCONI, L. J. KOESTER and D. H. PERKINS: UCID-10022, p. 167 (1961).
(5) G. COCCONI: *Nuovo Cimento*, **33**, 643 (1964).

by treating superposed events of different types of interactions to find out the empirical law of the p_t distribution.

In this paper, the dependence of the average value of p_t of secondary particles on the primary energy and the values of $\bar{p}_t$ (average value of p_t) for different kind of particles are studied (Sect. **2**). The best-fit theoretical p_t distributions are determined for baryons and mesons by fitting theoretical p_t distribution functions to experimental p_t distributions (Sect. **3**). The theoretical p_t distributions are derived from the momentum distribution of secondaries of Planck for mesons and of Fermi for baryons with plausible assumptions (Sect. **4**). The momentum distribution of the secondary π-mesons in the c.m.s. and the relation between the p_t distribution of the secondaries of inelastic events and that of protons of elastic p-p scattering are discussed (Sect. **5** and **6**).

This analysis leads to a statistical model which enables us to describe the p_t distribution of all kinds of secondary particles in a unified manner.

2. – Analysis of experimental data.

2·1. *Average value of* p_t. – The average values of p_t of mesons and baryons produced in inelastic interactions for various primary energies collected from published papers ([6-23]) are shown in Fig. 1*a*)–*e*).

([6]) M. S. SOLOVJEW: *Proc. of the 1960 High-Energy Phys. Conf. at Rochester*, p. 338.

([7]) M. H. BLUE, J. J. LORD, J. G. PARKS and C. H. TSAO: *Nuovo Cimento*, **20**, 274 (1961).

([8]) S. J. GOLDSACK, L. RIDDIFORD, B. TALLINI, B. R. FRENCH, W. W. NEALE, J. R. NORBURY, I. O. SKILLICORN, W. T. DAVIES, M. DERRICK, J. H. MULVEY and D. RADOJIČIĆ: *Nuovo Cimento*: **23**, 941 (1962).

([9]) J. BARTKE, R. BUDDE, W. A. COOPER, H. FILTHUTH, Y. GOLDSCHMIDT-CLERMONT, G. R. MACLEOD, A. DE MARCO, A. MINGUZZI-RANZI, L. MONTANET, D. R. D. MORRISON, S. NILSSON, C. PEYROU, R. SOSNOWSKI, A. BIGI, R. CARRARA, C. FRANZINETTI, I. MANNELLI, G. BRAUTTI, M. CESCHIA and L. CHERSOVANI: *Nuovo Cimento*, **24**, 876 (1962).

([10]) F. F. ABRAHAM and R. M. KALBACH: *Nuovo Cimento*, **26**, 717 (1962).

([11]) R. J. PISERCHIO and R. M. KALBACH: *Nuovo Cimento*, **26**, 729 (1962).

([12]) S. CIURLO, E. PICASSO, G. TOMASINI, A. GAINOTTI, C. LAMBORIZIO and S. MORA: *Nuovo Cimento*, **27**, 791 (1963).

([13]) G. BELLINI, E. FIORINI, A. J. HERZ, P. NEGRI, S. RATTI, C. BAGLIN, H. BINGHAM, M. BLOCH, D. DRIJARD, A. LAGARRIGUE, P. MITTNER, A. ORKIN-LECOURTOIS, P. RANÇON, A. ROUSSET, B. DE RAAD, R. SALMERON and R. VOSS: *Nuovo Cimento*, **27**, 816 (1963).

([14]) M. I. FERRERO, C. M. GARELLI, A. MARZARI-CHIESA and M. VIGONE: *Nuovo Cimento*, **27**, 1066 (1963).

([15]) J. BARTKE: *Nuovo Cimento*, **28**, 712 (1963).

([16]) T. FARBEL and H. TAFT: *Nuovo Cimento*, **28**, 1214 (1963).

([17]) J. BARTKE, W. A. COOPER, D. R. O. MORRISON, S. NILSSON, CH. PEYROU, It is seen in Fig. 1*a*) that $\bar{p}_t$ of π-mesons increases with increasing primary energy in a region of primary energy $E_p < 6$ GeV for π-p collisions whereas $\bar{p}_t$ is almost constant for E_p from 6 GeV to 25 GeV. When we compare the value of $\bar{p}_t$ of π-mesons of cosmic-ray jets ([1,4]) to that in the accelerator energy region, it can be said that the value of $\bar{p}_t$ is practically constant for all energy regions above 6 GeV.

For other particles, the experimental data illustrated in Fig. 1*b*) to 1*e*) assure the same conclusion that the $\bar{p}_t$ is constant for $E_p > 6$ GeV, although the constant differs for different particle masses. The average value and the statistical error of $\bar{p}_t$ are given in Table I for various kinds of particles.

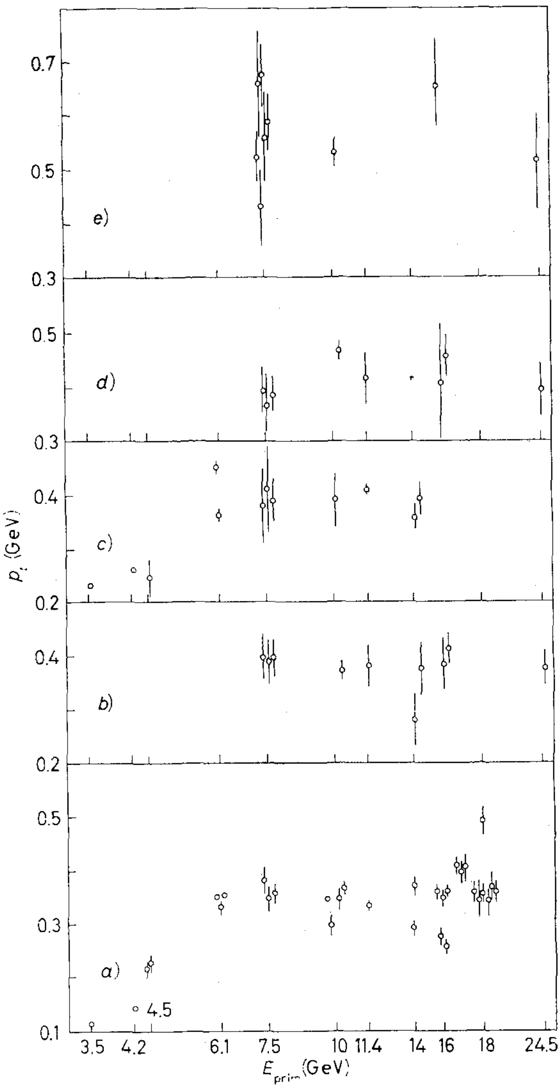

Fig. 1. - Average values of p_t for *a*) π-mesons, *b*) K-mesons, *c*) nucleons, *d*) Λ, *e*) Σ.

R. Sosnowski, A. Bigi, R. Carrara, C. Franzinetti and I. Mannelli: *Nuovo Cimento*, **29**, 8 (1963).

(18) S. Feminò, S. Jannelli and F. Mezzanares: *Nuovo Cimento*, **31**, 273 (1964).

(19) M. Csejthey-Barth: *Nuovo Cimento*, **32**, 545 (1964).

(20) F. R. Huson and W. B. Fretter: *Nuovo Cimento*, **33**, 1 (1964).

(21) A. Bigi, S. Brandt, A. de Marco-Trabucco, Ch. Peyrou, R. Sosnowski and A. Wróblewski: *Nuovo Cimento*, **33**, 1249 (1964).

(22) A. Bigi, S. Brandt, A. de Marco-Trabucco, Ch. Peyrou, R. Sosnowski and A. Wróblewski: *Nuovo Cimento*, **33**, 1265 (1964).

(23) G. Bellini, M. di Corato, F. Duimio and E. Fiorini: *Nuovo Cimento*, **40 A**, 948 (1965).

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TABLE I. – *Average value* $\bar{p}_t$, *statistical error* $\Delta\bar{p}_t$ *and dispersion* $0.67\sigma_{\bar{p}_t}$ *of experimental values of* p_t.

	π-meson	K-meson	Nucleon	Λ	Σ	Ξ
$\bar{p}_t$ (MeV/c)	347.9	377.6	404.9	420.3	528.9	580
$\Delta\bar{p}_t$ (MeV/c)	2.1	10.6	4.8	8.8	23.5	6
$0.67\sigma_{\bar{p}_t}$ (MeV/c)	16.3	9.2 (*)	19.5	17.3	12.3 (*)	4.0

(*) Some experimental $\bar{p}_t$ values which lie outside of $4\sigma_{\bar{p}_t}$ are excluded.

As was already pointed out in ref. [21,22], the value of $\bar{p}_t$ increases with the mass of the particles. This dependence can be understood on the basis of statistical theory of multiple particle production which will be described in Sect. 4. On the basis of this statistical theory, the experimental data give evidence that the value of the temperature T of the final-state interaction involved in the theory is the same for all kind of particles, *i.e.* the experimental value of kT is 0.125 GeV for all kinds of particles. In Fig. 2, the theoretical values of $\bar{p}_t$ for $kT = 0.125$ GeV for mesons and baryons are plotted.

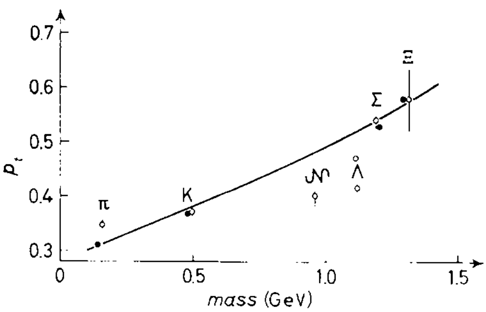

Fig. 2. – Average value of p_t *vs.* mass of particles. The solid line shows the p_t calculated by PD for mesons and FD for baryons of $kT = 0.125$ GeV.

It is seen that the theoretical values of $\bar{p}_t$ for various kinds of particles agree with those of experiments within experimental errors.

2·2. *Dependence of* $\bar{p}_t$ *on various characteristic features of the events.* – Although the average value of p_t remains constant for interactions in a wide range of primary energies, the average value seems to fluctuate with the characteristic features of the events. Thus, it is necessary to study whether the values of $\bar{p}_t$ obtained in various experiments are consistent with a constant value within a statistical error or the value fluctuates due to the variation of the parameter of the interaction which characterizes the types of events.

The weighted average $\bar{p}_t$, the statistical error $\Delta\bar{p}_t$ and the dispersion $\sigma_{\bar{p}_t}$, which are defined in the following, are calculated for the sample of experimental values of $(\bar{p}_{ti})_{\text{exp}}$ given in various papers.

When $(\bar{p}_{ti}) \pm \Delta\bar{p}_{ti}$ is the average value and the statistical error in $\bar{p}_{ti}$ of the i-th experiment, one can define the weighted average $\bar{p}_t$, the statistical error $\Delta\overline{p}_t$ and the dispersion $\sigma_{\overline{p}_t}$ for the sample obtained by experiments $i = 1, 2, ..., n$ as follows:

$$\overline{p}_t = C \sum_i \left((\overline{p}_{ti})/(\Delta\overline{p}_{ti})^2\right), \qquad \sigma^2_{\overline{p}_t} = C \sum_i \left((\overline{p}_{ti}) - \overline{p}_t)^2/(\Delta\overline{p}_{ti})^2\right),$$

$$1/C = \sum_i \left(1/(\Delta\overline{p}_{ti})^2\right) \qquad \text{and} \qquad \Delta\overline{p}_t = 1/C^{\frac{1}{2}}.$$

The values of $\overline{p}_t$, $\Delta\overline{p}_t$ and $\sigma_{\overline{p}_t}$ calculated by the above equations using the experimental data (6-23) are given in Table I. Except for K-mesons and $\Sigma^{\pm}$, $\sigma_{\overline{p}_t}$ is much larger than $\Delta\overline{p}_t$. This result suggests that the fluctuation can be ascribed not only to the systematic errors existing in various experiments but also to a real fluctuation of the $\overline{p}_t$ value due to the difference in type of interaction and in primary energy.

As examples, several authors have given the result that the value of $\overline{p}_t$ is different for different multiplicities n_s of events. In Fig. 3, $\overline{p}_t$ vs. n_s collected from refs. (8,12,17,19,23) is plotted. The result indicates that, for π-mesons, $\overline{p}_t$ decresases as the multiplicity increases.

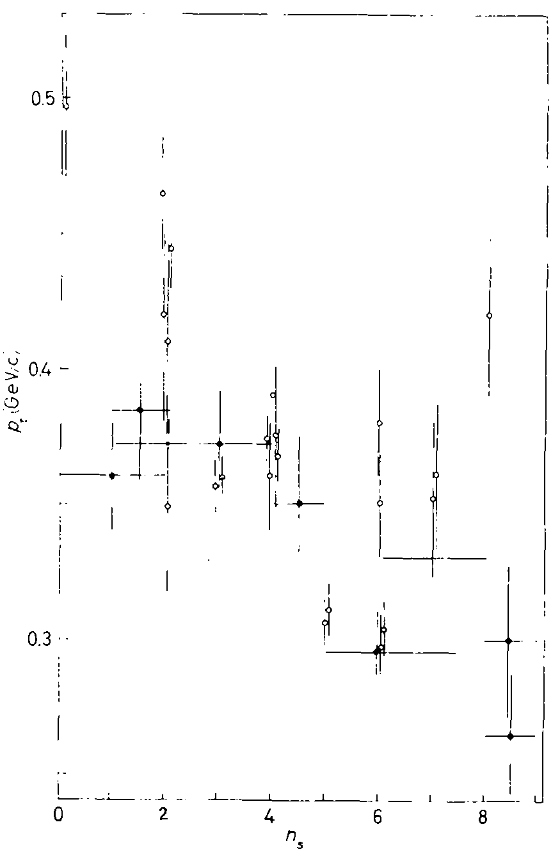

Fig. 3. – Average value of p_t of π-mesons *vs.* n_s, the multiplicity of events.

For the other particles, there is an indication of increase of $\overline{p}_t$ with multiplicity. However, for the latter, we need more statistics before we can give any definite conclusion.

Reference (12) shows that in the interaction of 16 GeV π-mesons, the $\overline{p}_t$ of π-meson secondaries with emission angles smaller than 5° in the laboratory system is definitely smaller than those with greater angles and ref. (18,20,22) showed that $\overline{p}_t$ depends on p_l the longitudinal momentum.

Reference [23] gives a detailed study of $\bar{p}_t$ values for different types of events and shows that $\bar{p}_t$ differs for different types of interactions. Thus, though the $\bar{p}_t$ value fluctuates according to the characteristic features of interactions, the fluctuation is not very large and the $\bar{p}_t$ value stays constant in a wide range of primary energies greater than about 6 GeV.

3. – The shape of the p_t distribution.

The experimental p_t distribution given in ref. [22] was compared with the following theoretical distribution functions.

I) PD and FD.

The PD and FD are the p_t distributions defined by the following eq. (1) which are derived from the momentum distribution of Planck for mesons and of Fermi for baryons of the secondaries in the c.m.s. of colliding particles with some assumptions as described in Sect. **4**:

$$f_{\pm}(p_t)\,\mathrm{d}p_t = \frac{y^2}{F_{\pm}(a)}\sum_{n=1}^{\infty}(\pm 1)^{n+1}K_1(ny)\,\mathrm{d}y\,, \tag{1}$$

where $+$ and $-$ refer to PD and FD, respectively,

$$F_{\pm}(a) = a^2\sum_{n=1}^{\infty}(\pm 1)^{n+1}\frac{K_2(na)}{n}\,, \quad y \equiv (M^2+p_t^2)^{\frac{1}{2}}/kT\,,$$

$K_n(x)$ is the modified Bessel function of the second kind, $a = M/kT$, M is the mass of particle, k and T are Boltzmann constant and temperature and the velocity of light c is put equal to one.

II) KD.

The p_t distribution which is referred to as KD is given by

$$f_K(p_t)\,\mathrm{d}p_t = \frac{y^2K_1(y)}{a^2K_2(a)}\,\mathrm{d}y\,. \tag{2}$$

III) LD.

The p_t distribution which is referred to as LD is defined by [1,4]

$$f_L(p_t)\,\mathrm{d}p_t = \frac{p_t}{p_0^2}\exp\left[-p_t/p_0\right]\mathrm{d}p_t\,. \tag{3}$$

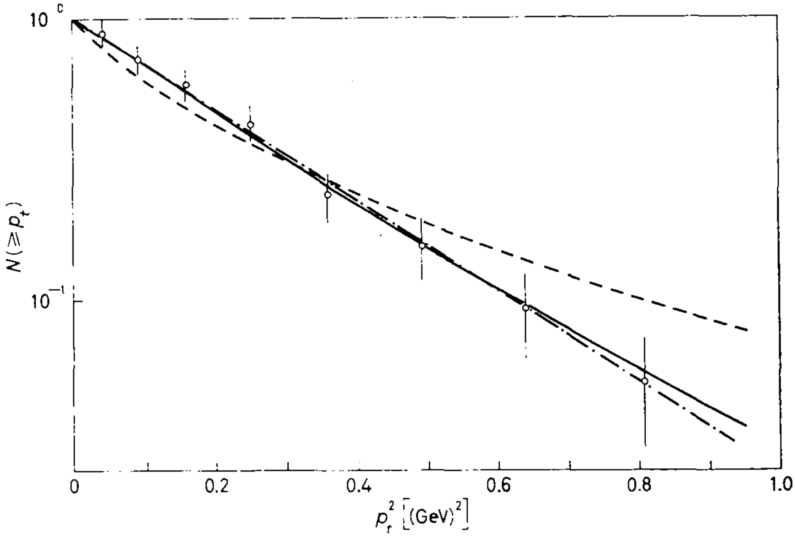

Fig. 4. – Experimental integral p_t distribution of Λ. ——: FD, – – –: LD and —·—·—: BD.

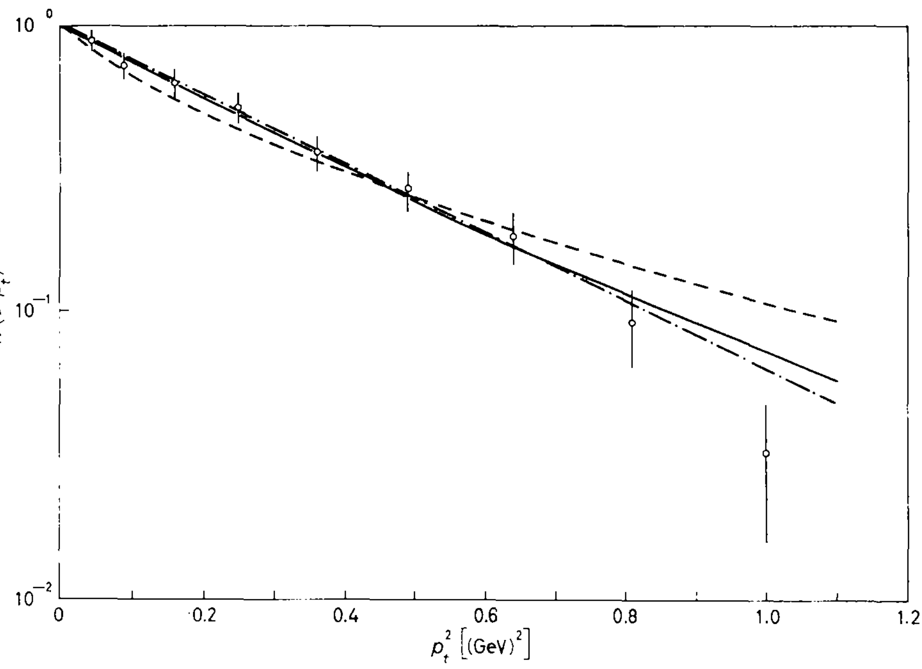

Fig. 5. – Experimental integral p_t distribution of Σ. – FD, – – – LD, —·—·— BD.

IV) BD.

The distribution which is referred to as BD is given by ([1,24,25])

$$f_B(p_t)\,\mathrm{d}p_t=\frac{2p_t}{\alpha^2}\exp\left[-p_t^2/\alpha^2\right]\mathrm{d}p_t\,. \tag{4}$$

In the above kT, P_0 and α are constants. The derivation of the above eq. (1)–(4) is based on the consideration of the statistical theory of multiple particle production together with assumptions and approximations which are given in Sect. **4**.

In Fig. 4, 5, 6 and 7, the experimental integral p_t distributions of $\pi^\pm+\pi^0$ mesons, K^0-mesons, Λ^0, $\Sigma^\pm$ given in ref. ([22]) are plotted and compared with those calculated from the various theoretical p_t distributions (1)–(4). The result is as follows: i) For Λ^0 and $\Sigma^\pm$, as shown in Fig. 4 and 5; BD, FD (and KD) fit well to the experimental distributions but LD does not. ii) For K-mesons,

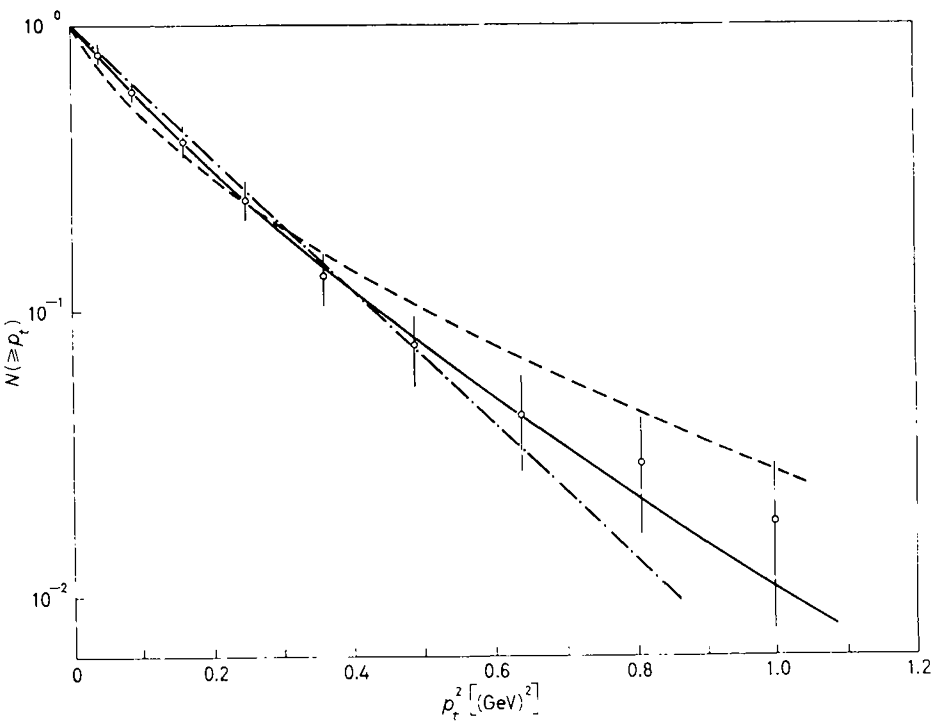

Fig. 6. – Experimental integral p_t distribution of K-mesons. ——— KD; - - - - LD, —·—·— BD.

([24]) H. H. Aly, M. F. Kaplon and M. L. Shen: *Nuovo Cimento*, **32**, 905 (1964).
([25]) E. M. Friedländer: *Nuovo Cimento*, **41** A, 417 (1966).

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as shown in Fig. 6, KD (and PD) fit well but LD and BD do not. iii) For π-mesons, as shown in Fig. 7, LD, KD and PD fit but BD does not.

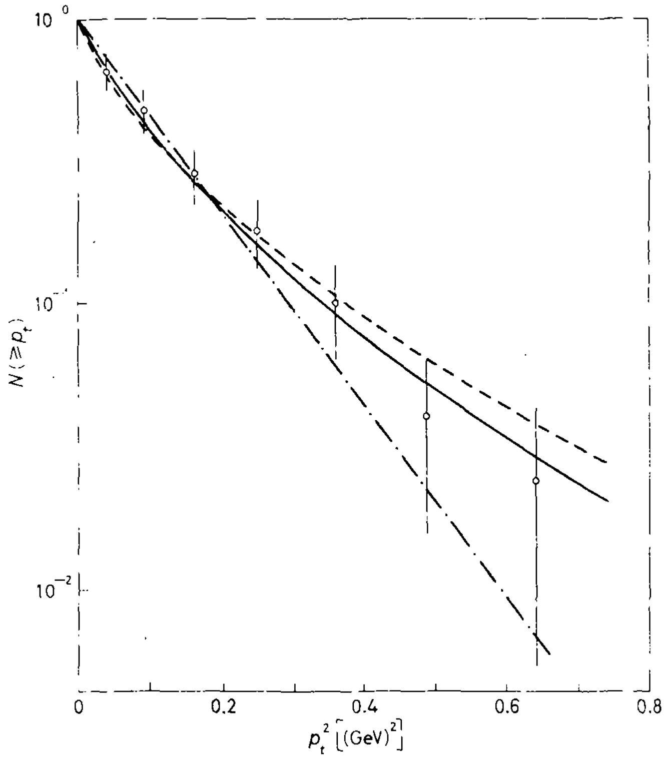

Fig. 7. – Experimental integral p_t distribution of π-mesons. ——— PD, – – – LD — · — · — BD.

For a quantitative study, the experimental values of dispersion σ of the p_t distribution defined by

$$\sigma = \left(\langle \overline{p_t^2}\rangle - (\overline{p}_t)^2\right)^{\frac{1}{2}}$$

have been estimated and the value of $S \equiv \sigma/\overline{p}_t$ is compared with those calculated using analytical expressions of $\overline{p}_t$ and $\overline{p_t^2}$ given in Table IV for the theoretical p_t distributions (1)–(4) (see Sect. **4**).

Since S is constant for LD (0.707) and BD (0.526) and is sensitive to the shape of distribution in the range of $p_t = (0 \div 1.0)$ GeV/c, we use the relation of experimental values of S *vs.* p_t/M as the criterion for the fitness of the theoretical p_t distributions. In Fig. 8, the experimental values of S *vs.* $\overline{p}_t/M$ are plotted together with the theoretical curves.

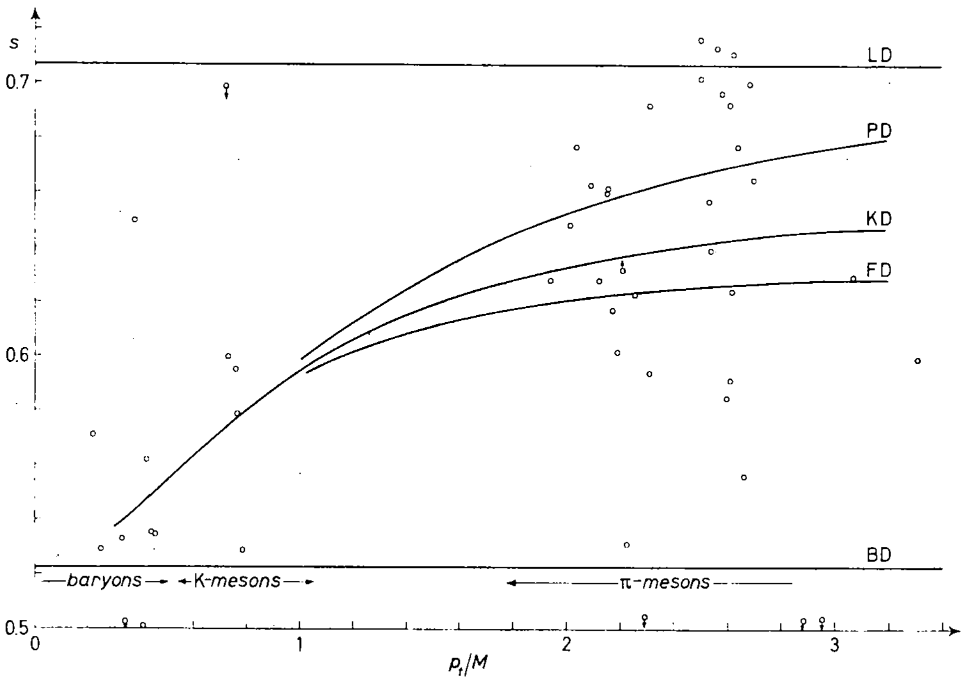

Fig. 8. – Plot of the experimental values of S vs. p_t/M.

Allowing for experimental errors both in $\bar{p}_t$ and S, the result shows that BD, FD and KD fit the experimental points of baryons; KD and PD fit those of K-mesons and LD, PD and KD fit those of π-mesons.

Although the experimental data are not sufficient to reject any one of KD, BD and FD for baryons, any one of PD and KD for K-mesons and any one of KD, LD and PD for π-mesons, when we discuss all the experimental points of S vs. $\bar{p}_t/M$ in Fig. 8 for all kinds of particles as a whole, BD and LD cannot be accepted, as these distributions fail to give correct values of S vs. $\bar{p}_t/M$ for some particles. The KD fits both mesons and baryons and is a good approximation for both PD and FD when $\exp[E/kT]\gg 1$ and is applicable to both mesons and baryons. The values of $\bar{p}_t/M$, S and $(\bar{p}_t^2)^{\frac{1}{2}}/M$ for KD are plotted as a function of $a = M/Tk$ and is shown in Fig. 9.

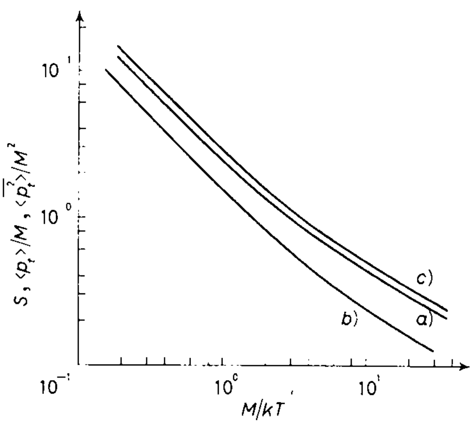

Fig. 9. – Relation of p_t/M, S and $(\overline{p_t^2}/M^2)^{\frac{1}{2}}$ vs. $a = M/kT$: curve a) p_t/M, curve b) S and curve c) $(\overline{p_t^2}/M^2)^{\frac{1}{2}}$.

The experimental values of S for π-mesons are plotted in Fig. 10. The expectation values of S for LD, KD and PD for $\bar{p}_t/M = 2.48$ are 0.707, 0.667 and 0.641 respectively and are also shown in Fig. 10.

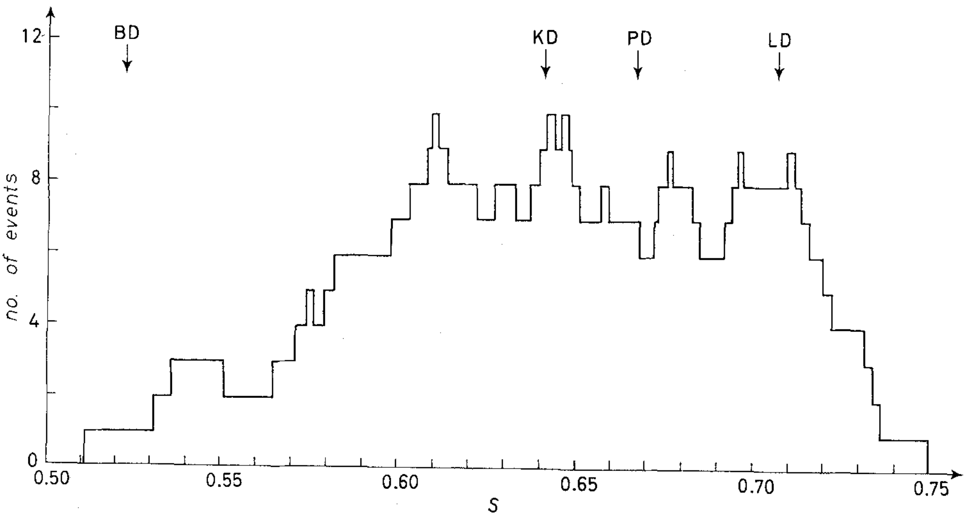

Fig. 10. – Distribution of S of π-mesons.

Although we cannot determine uniquely the best-fit distribution among LD, PD and KD for π-mesons, the value of S of LD is slightly larger than most of the experimental values and those of KD and PD are quite consistent with the experimental values.

Thus, in conclusion, the experimental p_t distributions are well represented by FD for baryons with $kT = (0.110 \div 0.125)$ GeV and by PD for mesons with $kT \sim 0.125$ GeV, whereas BD with $\alpha^2 = (0.2 \div 0.4)\,(\text{GeV}/c)^2$ for baryons and LD with $p_0 = 0.16$ GeV/c for π-mesons also fit the experimental data.

For all kind of particles, KD with $kT = 0.125$ GeV expresses well the experimental p_t distributions. The experimental values of kT for various kinds of particles are listed in Table II.

From Table II and Fig. 2, we can see that the value of kT is nearly constant for all kind of particles and is ~ 0.125 GeV.

TABLE II. – *Temperature of emitting system estimated from the experiment of* BIGI *et al.* [21] *for various kind of particles.*

	π^0	$\pi^\pm$	K	Λ^0	$\Sigma^\pm$	Ξ^-
$\bar{p}_t$ (GeV/c)	0.30	0.32	0.37	0.47	0.53	0.58
kT (GeV)	0.120	0.128	0.118	0.111	0.123	0.125

The $\bar{p}_t$ value calculated from PD for mesons and FD for baryons using $kT = 0.125$ GeV is shown in Fig. 2. The agreement with the experimental values is fairly good.

4. – Derivation of various p_t distribution functions from statistical theory.

In Sect. **3**, it is concluded that the gross features of the p_t distribution and its dispersion σ are well accounted for by PD for mesons and FD for baryons by adjusting a parameter $a = M/kT$ in which kT is practically constant for all kinds of particles and is ~ 0.125 GeV. Also, KD alone fits well for all kinds of particles: baryons and mesons.

This conclusion suggests the fact that the $\bar{p}_t$ of secondary particles are determined by a statistical law in the final-state interaction in multiple particle production processes.

Let us now consider the p_t distribution of different kinds of secondary particles produced in various types of events of given primary energy. For the sample composed of various types of events, the parameters which characterize a specified process or interaction, such as the decay process of resonance states of mesons and baryons, become insignificant because the p_t values of secondaries of a specified event are integrated over samples which are a superposition of various types of events. Therefore, the p_t distribution which reflects the characteristics of multiple particle production process as a whole is governed by statistical factors which are common to all the events, *i.e.* the phase-space volume available for the secondary particles in the final-state interaction of the production process.

Regarding the final-state interaction, it is not possible to specify a model or a process exactly by the p_t distribution alone, as it requires a good knowledge of the differential cross-section for the production of various kinds of particles. However, Fermi's and Landau's multiple particle production theories successfully derived the characteristic features of multiple particle production by making use of plausible assumptions as to the nature of this final-state interaction.

Assuming the secondary particles to be produced through states of thermal equilibrium of final-state interaction, whatever the processes through which the particles are produced may be and not necessarily those assumed by FERMI and LANDAU, the momentum distribution of the produced particles in the production system will be given approximately by Planck's distribution for mesons and by Fermi's distribution for baryons when the number of produced particles is large. As we are dealing with inelastic collisions as a whole, it is almost impossible to calculate the partition function through which all the details of production of the secondaries can be drawn, because we cannot know all the processes involved in the particle production process, as the characteristic quantities such as temperature, energy of the secondaries, inelasticity of the interaction, cannot be estimated by the model, but are to be determined by experiments and need not be equal to those given by the theory (*).

One may derive the p_t distribution function in the production system using Planck's or Fermi's momentum distribution

$$\varrho_{\pm}(p, \cos\theta, \varphi)\,\mathrm{d}p\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi = \frac{1}{N_{\pm}}\,\frac{p^2}{\exp[E/kT]-(\pm 1)}\,\mathrm{d}p\,\mathrm{d}\cos\theta\,\mathrm{d}\varphi\,, \tag{5}$$

where $N_{\pm}$ is the normalization constant and is equal to

$$N_{\pm} = 4\pi a^2 \sum_{n=1}^{\infty} (\pm 1)^{n+1} \frac{K_2(na)}{n} = 4\pi F_{\pm}(a)\,,$$

E, p, M, θ and φ are the energy, momentum, mass, space angle and azimuthal angle of the emitted particle, T is the temperature of the emitting system, k is the Boltzmann constant, and $+$ refers to Planck's distribution and $-$ to Fermi's distribution.

As p_t is invariant under Lorentz transformation of the emitting system moving in the direction of the primary, the p_t distribution function is unaffected by the velocity of the emitting system whatever the longitudinal velocity of the emitting system may be. In a jet, the emitting system might be moving with a velocity having a component perpendicular to the direction of motion of the primary. However, the lack of positive experimental evidence of azimuthal anisotropy indicates that the velocity of the emitting system perpendicular to the direction of the primary is not large enough compared to the longitudinal velocity of the system to give rise to azimuthal anisotropy. We may neglect the transversal velocity of the emitting system. Thus, we assume here that the emitting system is moving in the direction of the primary.

As the p_t distribution in the emitting system and in the l.s. (laboratory system) are the same, we can calculate the p_t distribution in the emitting system using eq. (5).

Changing the variables from $(p, \cos\theta)$ to $(p_t, p_l \equiv p\cos\theta)$ and integrating eq. (5) with respect to φ and p_l, we find eq. (1).

In the case of $E/kT \gg 1$, which holds both for baryons and K-mesons, we

(*) *Note added in proof.* - On a basis of a statistical thermodynamics of strong interaction, R. HAGEDORN, starting from the partition function, derived the p_t distribution of KD (*Suppl. Nuovo Cimento*, **3**, 147 (1965)).

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may neglect ± 1 in eq. (5) and thus get

$$(6)\qquad \varrho(p, \cos\theta)\,\mathrm{d}p\,\mathrm{d}\cos\theta = \frac{p^2}{N_K}\exp[-E/kT]\,\mathrm{d}p\,\mathrm{d}\cos\theta\,, \qquad N_K \equiv a^2 K_2(a)\,.$$

A) KD is obtained by integration of (6) with respect to p_l, we get eq. (2).

B) BD is obtained as follows. By the further approximation $p^2/M^2 \ll 1$ and using $E \approx M + p^2/2M$, we integrate eq. (6) with respect to p_l and obtain the expression of BD given by eq. (4), where $\alpha^2 = 2MkT$.

C) LD is obtained as follows. By another approximation $p_t > (M, p_l)$, putting $E \approx p_t + M + p_l$, we integrate eq. (6) over p_l and get eq. (3).

One can see that, from the above derivation, LD will not fit baryons and K-mesons because the approximation $p_t/M \gg 1$ does not hold. Also, BD cannot be applied to π-mesons and K-mesons, as the approximation $p/M \ll 1$ does not hold.

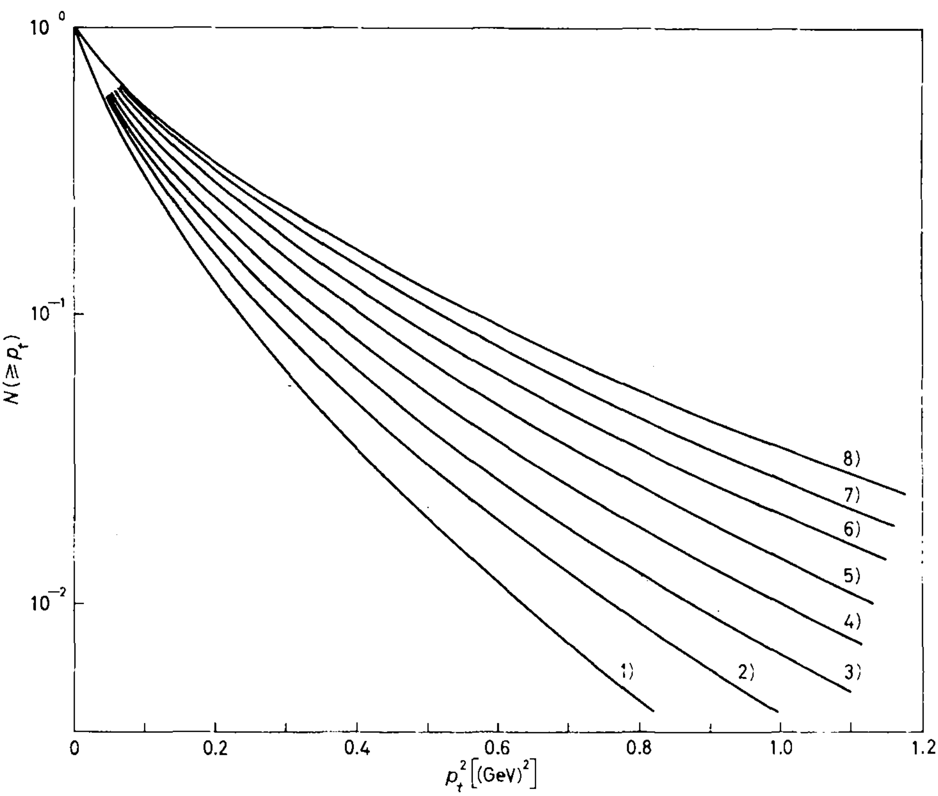

Fig. 11. – Integral p_t distribution of π-mesons calculated by PD. Curves 1)-8) represent those with $\bar{p}_t = 260, 280, 300, 320, 340, 360, 380, 400$ MeV/c, respectively.

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Table III shows the validity of the assumptions for various kind of particles in the course of the derivation of various theoretical distribution function.

TABLE III. – *Validity of various theoretical p_t distribution functions for various kinds of particles in a region where* $p_t = (0 \div 1.0)$ GeV/c. (O: applicable, X: not applicable).

Distribution function	π-meson	K-meson	Baryon
PD	O	O	X
FD	X	X	O
KD	O	O	O
BD	X	X	O
LD	O	X	X

However, the above validity rests on statistical theory and from the point of view of fitting the p_t distribution functions to the experimental one, we have no *a priori* reasons to believe that any one of the above distribution functions is more fundamental than the others.

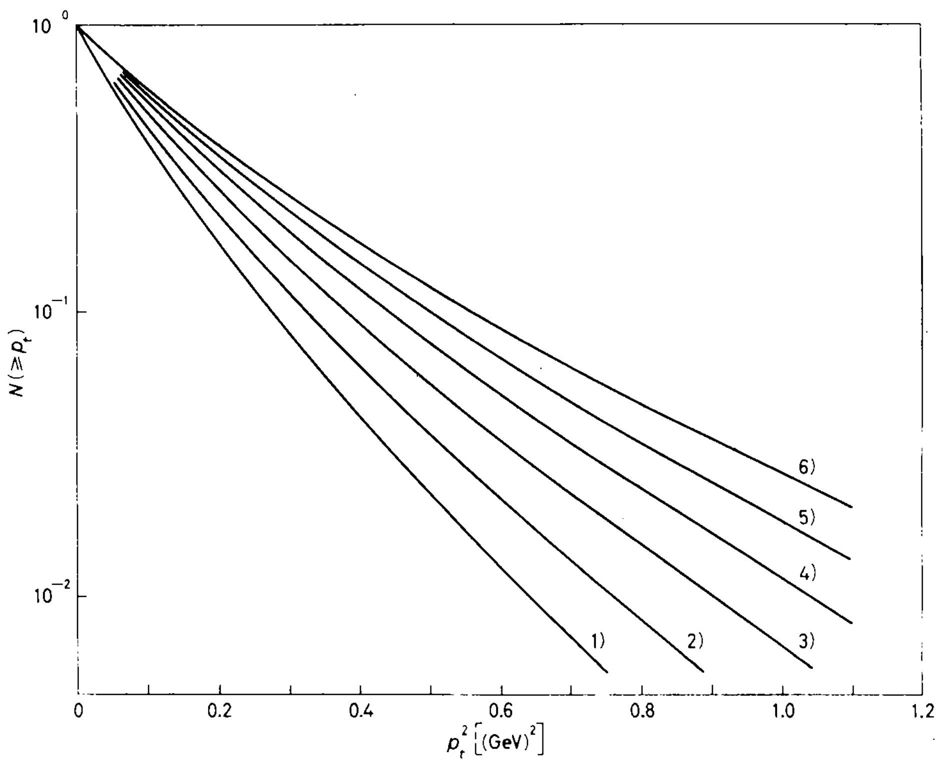

Fig. 12. – Integral p_t distribution of K-meson calculated by KD. Curves 1)-6) represent those with $\bar{p}_t = 300, 325, 350, 375, 400, 425$ MeV/c, respectively.

Regarding the analytical expression of the p_t distribution, ref. [25] insists that only the BD is compatible with the assumption of axial symmetry (A.A.S.) of jets and thus rejects all other possible expressions of the p_t distribution functions because, on the basis of the deficiency of evidence of azimuthal asymmetry and of the result of the deduction of BD in ref. [24], the assumption of A.A.S. leads only to the BD for the p_t distribution of secondary particles.

However, we adopt all the above expressions (1)–(4), because the p_t distribution functions given by eqs. (1)–(4) are not incompatible with A.A.S. of jets and it seems that the inference given in ref. [25] cannot be justified for actual jets. The details of the derivation of BD based on A.A.S. are shown in the Appendix.

The analytical expressions for the integral p_t distribution $N(\geqslant p_t)$, $\bar{p}_t/M$ and $\overline{p_t^2}/M^2$ for the theoretical p_t distributions given by eqs. (1)–(4) are listed in Table IV.

The curve of the integral p_t distribution for π-mesons by PD, that for K-mesons by KD ($\approx$ PD) and those for nucleons by KD ($\approx$ FD) for various values of $\bar{p}_t$ are given in Fig. 11-13.

TABLE IV. – *Analytical expressions of integral distribution functions, average values, $\bar{p}_t$ and $\overline{p_t^2}$ for various theoretical distributions.*

	$f(p_t)\,\mathrm{d}p_t$	$N(\geqslant p_t)$	$\bar{p}_t/M$	$\overline{p_t^2}/M^2$
FD (+) FD (—)	$\dfrac{G_\pm(y)}{F_\pm(a)}\,\mathrm{d}y$	$\dfrac{F_\pm(y)}{F_\pm(a)}$	$\dfrac{H_\pm(a)}{F_\pm(a)}$	$\dfrac{I_\pm(a)}{F_\pm(a)}$
KD	$\dfrac{y^2K_1(y)}{a^2K_2(a)}\,\mathrm{d}y$	$\dfrac{y^2K_2(y)}{a^2K_2(y)}$	$\dfrac{(\pi/2)^{\frac12}a^{\frac32}K_{\frac52}(a)}{a^2K_2(a)}$	$\dfrac{2aK_3(a)}{a^2K_2(a)}$
BD	$\dfrac{2p_t}{\alpha^2}\exp[-p_t^2/\alpha^2]\,\mathrm{d}p_t$	$\exp[-p_t^2/\alpha^2]$	$(\pi^{\frac12}\alpha)/2M$	$4M^2/\alpha^2$
LD	$\dfrac{p_t}{p_0^2}\exp[-p_t/p_0]\,\mathrm{d}p_t$	$\left(1+\dfrac{p_t}{p_0}\right)\exp[-p_t/p_0]$	$\dfrac{2p_0}{M}$	$\dfrac{6p_0^2}{M^2}$

In the above $y\equiv(M^2+p_t^2)^{\frac12}/kT$, $a\equiv M/kT$, $\alpha^2=2MkT$ (*), $p_0=kT$ (*),

$$G_\pm(y)=y^2\sum_{n=1}^{\infty}(\pm 1)^{n+1}K_1(ny),\qquad F_\pm(y)=y^2\sum_{n=1}^{\infty}(\pm 1)^{n+1}K_2(ny)/n,$$

$$H_\pm(a)=(\pi/2)^{\frac12}a^{\frac32}\sum_{n=1}^{\infty}(\pm 1)^{n+1}K_{\frac52}(na)/n^{\frac32},\quad I_\pm(a)=2a\sum_{n=1}^{\infty}(\pm 1)^{n+1}K_3(na)/n^2.$$

(*) Dependence of α and p_0 on kT are derived in Sect. 4.

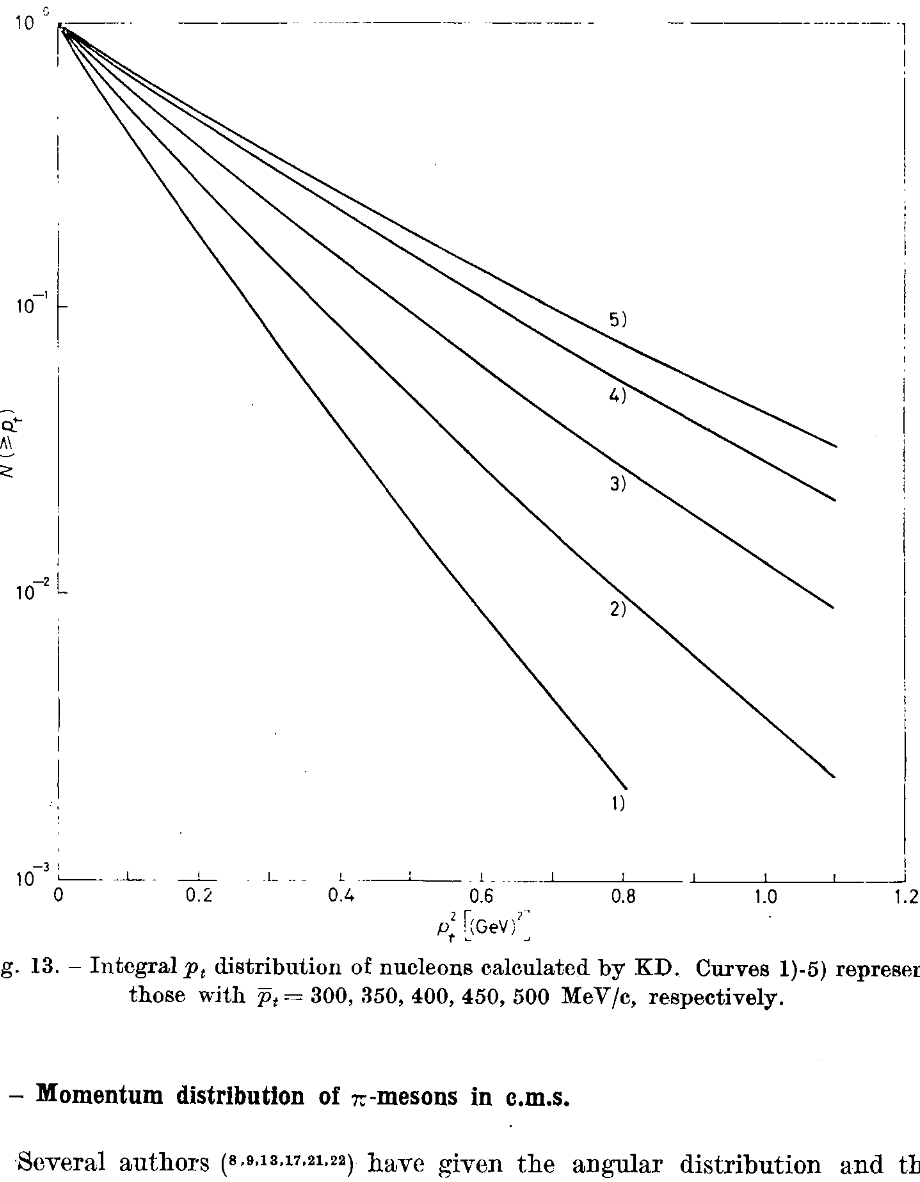

Fig. 13. – Integral p_t distribution of nucleons calculated by KD. Curves 1)-5) represent those with $\bar{p}_t = 300, 350, 400, 450, 500$ MeV/c, respectively.

5. – Momentum distribution of π-mesons in c.m.s.

Several authors (8,9,13,17,21,22) have given the angular distribution and the p_t *vs.* p_l plot (in c.m.s.) of the secondary particles from nuclear interactions. One can see that, in general, $\bar{p}_t < \bar{p}_l$ holds for the secondaries from π-p and p-p interactions but one can see also that the angular distribution of π-mesons from π-p collisions of multiplicity $n_s \geq 6$ and those of π-mesons from p-p collisions are almost isotropic in c.m.s. except for a small fraction of forward or backward collimated ones, and thus the system emitting π-mesons is almost at rest in c.m.s. Thus, if the assumption made in the last Section concerning the momentum distribution of π-mesons in the emitting system holds good, the momentum distribution of π-mesons in c.m.s. would be that of Planck. Thus, the momentum distribution of Planck directly fits the experimental momentum distribution in c.m.s. of π-mesons from p-p interactions. This was confirmed, for the experimental data of 24 GeV p-p interaction given in ref. ([26]), by ref. ([27]) which shows good agreement between the experimental momentum distribution of π-mesons and the theoretical distribution of Planck.

This supports the previous assumption and the p_t distribution derived from eq. (5) fits the experimental p_t distribution. However, for particles other than π-mesons in p-p collisions and all the particles of high-multiplicity events from π-p interactions except for mesons of charge opposite to the primary meson, the emitting system of particles is moving in c.m.s. in the direction of the primary and the velocity distribution of the emitting system depends on the type of collision. Thus, unless we assume the distribution of the longitudinal velocity of the emitting system of particles, it is not possible to compare the experimental momentum distribution with the theoretical one.

6. – Relation between the multiple particle production of inelastic interactions and elastic scattering.

6'1. *Elastic* p-p *scattering*. – For the experimental p_t distributions of small-angle elastic scattering given in ref. ([3]), those of protons from inelastic collisions which are given in ref ([25]) and those of baryons in this paper, the distribution functions are expressed by the same distribution function: BD. We will now compare the above three cases.

We have already shown that the experimental p_t distribution for baryon secondaries from inelastic nuclear interactions are well expressed by FD, and and that FD is approximated by KD for $y \gg 1$:

$$(6') \qquad f_K(p_t)\,\mathrm{d}p_t = \frac{1}{N_k}\, y^2 K_1(y)\,\mathrm{d}y \sim y^{\frac{3}{2}} \exp[-y]\,\mathrm{d}y\,, \qquad y = (M^2 + p_t^2)^{\frac{1}{2}}/kT\,.$$

Equation (6) reduces to the following expression of BD when $p_t < Mc$ and $M/kT > 1$:

$$(7) \qquad p_t \exp[-p_t^2/\alpha^2]\,\mathrm{d}p_t\,, \qquad \alpha^2 = 2kTM$$

([26]) P. Dodd, M. Jobes, J. Kinson, B. Tallini, B. R. French, H. J. Sherman, I. O. Skillicorn, W. T. Davies, M. Derrick and D. Radojičić: *Proc. Aix-en-Provence Conf.*, vol. **1** (1961), p. 435.

([27]) K. Imaeda and T. P. Shah: *Nuovo Cimento*, **41**, 405 (1966).

On the other hand, for elastic p-p scattering, NARAYAN and SHARMA ([28]) adopted the following expression for $p_t = (0-5)$ GeV/c:

$$(8) \qquad (\exp[-ap_t^2] + p_t^2 \exp[-bp_t])2p_t \mathrm{d}p_t\,,$$

where $a = 9$ (GeV/c)$^{-2}$, $b = 8.4$ (GeV/c)$^{-1}$. For small p_t, only the first term is appreciable and thus, eq. (8) is $2p_t \exp[-ap_t^2]\mathrm{d}p_t$ and, for large p_t, only the second term is appreciable: $2p_t^3 \exp[-bp_t]\mathrm{d}p_t$.

OREAR ([2]) gave the expression of BD for protons from diffraction scattering:

$$(9) \qquad \exp[-At|\mathrm{d}t \simeq \exp - \left[\frac{R_\pi^2 p_t^2}{4\hbar^2}\right] \mathrm{d}p_t^2,$$

where R_π is the r.m.s. radius of the π-meson and is $1.2 \cdot 10^{-13}$ cm, t is the momentum transfer and is related to p_t by $p_t^2 = -t - t^2/4p^2 \sim -t$ for large p of the incident protons in c.m.s. and in eq. (9) this approximation is used.

CHAVDA and NARAYAN ([29]) introduced an expression of the p_t distribution for protons from elastic p-p scattering:

$$(10) \qquad \exp\left[-c\sqrt{d + p_t^2}\right] \mathrm{d}p_t^2/(d + p_t^2)^{\frac{1}{2}}\,,$$

where $c = 7.38$ (GeV/c)$^{-1}$, $d = 0.1225$ (GeV)^2 to explain both the large-angle scattering of type similar to LD and the diffraction scattering of type BD.

Expression (6) is similar to eq. (10) when $c = 1/kT$ and $d = M^2$. By comparison of eqs. (3) through (7) for the p_t distribution of particles from inelastic collision and eqs. (8) through (10) for that of elastic p-p scattering one might expect that there should be close correlation between those two distributions, namely, the p_t distribution of the particles from multiple production and that of elastic p-p scattering are governed by the same law.

Now, compare the characteristic constants T, M, p_0, α, c, d, a and b involved in the expressions (3) to (10).

As one can see that $1/kT = 8$ (GeV)$^{-1}$ agrees with the corresponding constants $b = 8.6$ (GeV/c)$^{-1}$ and $c = 7.38$ (GeV/c)$^{-1}$. These constants characterize large-angle elastic p-p scattering and multiple particle production and the agreement of these constants means that all the distributions tend to LD for large p_t as will be discussed in the next Section.

On the other hand the constant $1/\alpha^2 = (2.5 \div 5.0)$ (GeV/c)$^{-2}$ is much smaller than the corresponding constants $A = R_\pi^2/4\hbar^2 = a = 9$ (GeV/c)$^{-2}$ and $c/2d^{\frac{1}{2}} = 10.5$ (GeV/c)$^{-2}$. This means that, though the distributions can be expres-

([28]) D. S NARAYAN and K. V. L. SARMA: *Phys. Lett.*, **5**, 365 (1963).
([29]) L. K. CHAVDA and D. S. NARAYAN: *Nuovo Cimento*, **43 A**, 382 (1964).

sed by BD for both cases, the experimental slope $a = c/2d^{\frac{1}{2}}$ of the p_t distribution of diffraction scattering is about three times $(= a/(1/\alpha^2))$ steeper than that of the inelastic one. If one attempts to explain diffraction scattering by the same mechanism as multiple particle production, the interpretation of kT or the mass M in eq. (2) should be altered so that $A = 1/\alpha^2 = 1/2MkT$.

However, neither of them can be modified because kT is a constant for all kinds of particles and the mass in also assigned to each kind of particle and thus cannot be altered. This shows that the p_t distributions of the two processes are governed by a different law.

In the case of inelastic events, the exponential slope $1/\alpha^2$ of the p_t distribution of baryons has mass dependence as can be seen in eqs. (4) and (6) and the smaller the mass of the particles the steeper the exponential slope, whereas in the case of diffraction scattering the slope $R_\pi^2/4\hbar^2$ is almost constant and does not depend so much on the mass of the particle.

6'2. *Cocconi's conjecture.* – The p_t distribution of the π-mesons from nuclear interactions both of accelerator energy and of cosmic-ray energy ([1,4]) and that of protons from elastic p-p scattering at large angles ([2]) of $p_t = (0.5 \div \text{few})$ GeV/c are given by a common expression: $p_t \exp[-p_t/p_0] dp_t$, $p_0 = (0.15 \div 0.16)$ GeV/c.

Several authors noticed the similarity between the p_t distribution of protons from large-angle elastic p-p scattering and that of π-mesons from inelastic scattering.

COCCONI ([5]) conjectured that all kinds of secondary particles from nuclear interactions have a common p_t distribution expressed by LD given above.

Now we compare our result on the p_t distributions to those of protons of large-angle elastic p-p scattering.

FD and PD reduce to KD for $E/kT \gg 1$ which, when $p_t/M \ll 1$, does not agree with LD.

Thus, in a region of p_t from 0 to 1 GeV/c, the p_t distributions of K-mesons and baryons do not agree with Cocconi's conjecture and LD should be replaced by KD.

However, for $p_t \gg Mc$ and p_l, if the statistical theory still holds good in this region and kT is independent of the masses of the particles, the p_t distribution becomes independent of mass and the expression becomes LD as Cocconi conjectured. The condition $p_t \gg Mc$ puts the limit $p_t \gg 0.14$ GeV/c for π-mesons and $p_t \gg 1$ GeV/c for baryons and this is the reason why the p_t distribution of π-mesons of cosmic-ray jets and that of protons from large-angle elastic p-p scattering of $p_t > 1$ GeV/c have a common expression of LD. Thus, the result on whether the shape of the p_t distribution in a region of $p_t > 1$ GeV/c for all kinds of particles is LD or not and whether the shape depends on mass or not offers a crucial test for the applicability of statistical theory of multiple particle production and Cocconi's conjecture.

However, since the data of $p_t > 1$ GeV/c for inelastic events are at present deficient, we cannot give any firm conclusion on the validity of LD for $p_t > 1$ GeV/c.

7. – Conclusions.

1) $\bar{p}_t$ increases with increasing primary energy in the region of $E_p < 6$ GeV but stays constant in the region of E_p from 6 GeV up to cosmic-ray energy.

2) $\bar{p}_t$ increases with the mass of secondary particles.

3) $\bar{p}_t$ depends slightly on the multiplicity, emission angle and production process of the secondaries.

4) $\bar{p}_t$ of π-mesons decreases with increasing multiplicity.

5) The experimental p_t distributions of mesons (π, K) and baryons ($\mathcal{N}$, Λ, Σ, Ξ) of $p_t = (0 \div 1.0)$ GeV/c are well represented by the p_t distribution derived from the momentum distribution of Planck and that of Fermi with $kT = 0.125$ GeV respectively.

6) KD represents well the experimental p_t distribution of all kinds of particles of $kT = 0.125$ GeV. It is an approximation to both PD and FD.

7) LD and BD represent well the experimental p_t distribution of π-mesons and baryons, respectively, with $p_0 \simeq 0.16$ GeV/c and $\alpha^2 = (0.2 \div 0.4)$ (GeV/c)^2.

8) The shape of the p_t distribution of nucleons from large-angle elastic p-p scattering is similar to that of baryons from inelastic events. The p_t distributions for all these particles are explained on the basis of the statistical theory of multiple particle production.

9) The shape of the p_t distribution for nucleons from small-angle p-p elastic scattering is different from those of inelastic scattering.

10) It is conjectured that, if the production mechanisms are the same for particles in the region of p_t greater than 1 GeV/c as in that of p_t smaller than 1 GeV/c, the p_t distribution in the region where $p_t > 1$ GeV/c becomes LD with $p_0 = 0.16$ GeV/c and is independent of the mass of the secondaries as COCCONI has conjectured. Conversely, if the Cocconi's conjecture holds good in the region where $p_t > 1$ GeV/c, we can conclude that the multiple particle production process is governed by a statistical law. However, at present, very few experimental data are available. We cannot give any firm conclusion on this conjecture.

Appendix

The assumption of axial symmetry alone is not sufficient to deduce the BD, because the axial symmetry restricts the momentum distribution function $F(p, \cos\theta, \varphi)$ of the secondary particle such that $F(p, \cos\theta, \varphi)$ is independent of φ, the azimuthal angle.

To derive BD, the assumption of statistical independence, *i.e.* assumption 2) in the following, is necessary, but assumption 2) does not necessarily hold for secondary particles ([30]).

The BD function is derived on the basis of the following three assumptions:

1) Assumption of independence of the p_t distribution on the angle θ:

$$F(p, \cos\theta, \varphi) = F'(p_t, \varphi) F''(\cos\theta) .$$

2) Assumption of statistical independence of $p_x = p_t \cos\varphi$, $p_y = p_t \sin\varphi$:

$$F'(p_t, \varphi) = f_1(p_x) f_2(p_y) .$$

This assumption of statistical independence is necessary to derive BD, because without this assumption, the A.A.S. alone leads to the p_t distribution function $F'(p_t, \varphi)$ independent of φ and the function $F'(p_t)$ is an arbitrary function of p_t and not necessarily BD. But this assumption is not generally assumed in cosmic-ray jets.

3) Assumption of axial symmetry. The distribution function does not change under rotation of angle φ around the z-axis:

$$p'_x = p_x \cos\varphi - p_y \sin\varphi \qquad \text{and} \qquad p'_y = p_x \sin\varphi + p_y \cos\varphi .$$

Using $dp'_x dp'_y = dp_x dp_y$ and putting $\varphi = -90°$, we obtain, from $f_1(p'_x) \cdot f_2(p'_y) \cdot dp'_x dp'_y = f_1(p_x) \cdot f_2(p_y) dp_x dp_y$ $f_1(p_x) = f_2(p_x)$ and f_1 and f_2 are even functions of the argument, *i.e.*

$$f_1(p_x) f_2(p_y) = F(p_x^2) F(p_y^2) . \tag{A.1}$$

From eq. (A.1), we can derive $F(p_t^2) = C \exp[-p_t^2/a]$, by the same procedure as in ref. ([24]).

([30]) J. R. Wayland and T. Bowen: preprint, July 21 (1966). This paper has been received after completion of this work. The authors pointed out that BD is a too strong condition imposed on jet and reached the same conclusion for BD.

RIASSUNTO (*)

Si esegue un'analisi del momento trasversale sperimentale p_t delle particelle secondarie prodotte in interazioni nucleari di alta energia nella zona di energia degli acceleratori. Il valore medio di p_t per ciascuna specie di particelle è quasi costante per energie primarie da 6 GeV sino all'energia dei raggi cosmici. Si adattano molte distribuzioni teoriche di p_t alla distribuzione sperimentale di p_t. Le distribuzioni sperimentali di pt dei mesoni π, mesoni K, nucleoni, Λ e Σ sono bene approssimate dalle distribuzioni teoriche di p_t dedotte dalle distribuzioni di Planck per i mesoni e di Fermi per i barioni. Il parametro caratteristico kT usato nella distribuzione teorica di p_t è quasi costante per tutte le specie di particelle ed è ~ 0.125 GeV. Si dimostra che la distribuzione di p_t dei protoni da scattering elastico p-p di grande angolo e quella dei secondari da interazioni anelastiche sono di forma simile e sono governate dalla stessa legge statistica. Si discute la relazione fra le due suddette distribuzioni di p_t. Sulla base della teoria statistica si riporta una deduzione della distribuzione teorica di p_t.

(*) *Traduzione a cura della Redazione.*

Распределение поперечных импульсов вторичных частиц при ядерных взаимодействиях при высоких энергиях и интерпретация посредством статистической модели.

Резюме (*). — Производится экспериментальный анализ поперечных импульсов p_t вторичных частиц, рожденных в ядерных взаимодействиях при высоких энергиях, получаемых на ускорителях. Средняя величина p_t для отдельного сорта частиц является почти постоянной для начальной энергии от 6 ГэВ вплоть до энергии космических лучей. Некоторые теоретические p_t распределения соответствуют экспериментальному p_t распределению. Экспериментальные распределения π-мезонов, K-мезонов, нуклонов, Λ и Σ хорошо описываются посредством теоретических p_t распределений, полученных из импульсного распределения Планка для мезонов и Ферми для барионов. Характеристический параметр kT, входящий в теоретическое p_t распределение, приблизительно равен константе для всех сортов частиц и равен ~ 0.125 ГэВ. Показывается, что p_t распределение протонов для p-p упругого рассеяния на большие углы и p_t распределение вторичных частиц при неупругих взаимодействиях являются одинаковыми по форме и описываются одним и тем же статистическим законом. Обсуждается связь между этими двумя p_t распределениями. На основе статистической теории приводится вывод теоретического p_t распределения.

(*) *Переведено редакцией.*

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