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Analyticity Properties and Asymptotic Expansions of Conformal Covariant Green's Functions.

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Summary. — Conformal covariant Green's functions and operator-product expansions for more than two operators are discussed. Analyticity properties are investigated and asymptotic expressions derived. Extension in D-dimensional space-time is studied and the peculiar role of $D=2$, both group-theoretically and as connected to infinite-momentum frames, is pointed out. The role of shadow singularities in relation to Euclidean metrics is discussed.

1. – Introduction.

The possible role of conformal covariance in the description of asymptotic properties of renormalizable field theories has recently attracted much attention. Recent work has used conformal covariance in connection with skeleton graph expansions in renormalizable field theories (1). Such an approach has revealed

(1) G. Mack and K. Symanzik: *Comm. Math. Phys.*, **27**, 247 (1972); G. Mack: in *Scale and Conformal Symmetry in Hadron Physics* (New York, 1973).

itself of great success as it in principle allows for calculations of scale dimensions and coupling constants ([2]). A different, but perhaps related, approach uses conformal covariance in conjunction with the hypothesis of operator-product expansion at short distance ([3]). Such an approach has a direct relationship to measurable quantities which are known to be connected to matrix elements of operators occurring in Wilson expansions, or in its extension over the entire light-cone ([4]). Conformal covariance, applied to operator-product expansions, appears as a powerful constraint, fixing the relative strengths of infinite chains of operators ([5]). Conformal covariant Wilson expansions allow for the construction of conformal covariant Green's functions. Particular examples are the two- and three-point functions, directly connected to the operator-product expansion for two operators (disconnected and connected part respectively), and the four-point function. For the latter it is possible to derive an expansion into a sequence of irreducible graphs, where each graph corresponds to the exchange of infinite towers of local operators, *i.e.* to exchange of infinite-dimensional representations of conformal algebra ([6]). Such sequences are labelled by two conformal quantum numbers, the spin and the dimension of the exchanged local operator of lowest dimensionality. In relativistic field theory, the natural frame of reference for an understanding of conformal invariance is obviously offered by the Ward identities for the conformal currents. This approach leads to the Callan-Symanzik equations ([7]). To be specific, let us consider a definite model, namely the $g\phi^4$ self-interacting theory. The Ward identities ([7,8]) read

$$\sum_{i=1}^{n}[x_i^\mu \partial_\mu^i + l(g)]\langle 0|T(\varphi(x_1)\,\dots\,\varphi(x_n))|0\rangle = \int \mathrm{d}^4z \langle 0|T(\theta(z)\varphi(x_1)\,\dots\,\varphi(x_n))|0\rangle$$

and

$$\sum_{i=1}^{n}[2x_\mu x^i\partial_i - x_i^2\partial_\mu^i + 2l(g)]\langle 0|T(\varphi(x_1)\dots\varphi(x_n))|0\rangle = \int \mathrm{d}^4z z_\mu \langle 0|T(\theta(z)\varphi(x_1)\dots\varphi(x_n))|0\rangle,$$

where $\theta(x) = -\eta(g)\varphi^2(x) - \beta(g)\varphi^4(x)$. Here $l(g)$, $\eta(g)$ and $\beta(g)$ are functions of the coupling constant g. These functions summarize the breaking of the naive

([2]) G. MACK: *Lectures Notes in Physics*, Vol. **17** (Berlin, 1972); G. PARISI and L. PELITI: *Lett. Nuovo Cimento*, **2**, 627 (1971).
([3]) K. WILSON: *Phys. Rev.*, **179**, 1499 (1969).
([4]) R. A. BRANDT and G. PREPARATA: *Nucl. Phys.*, **27** B, 541 (1971); Y. FRISHMAN: *Phys. Rev. Lett.*, **25**, 960 (1970).
([5]) S. FERRARA, R. GATTO and A. F. GRILLO: *Nucl. Phys.*, **34** B, 349 (1971).
([6]) S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Nucl. Phys.*, **49** B, 77 (1972).
([7]) C. CALLAN: *Phys. Rev. D*, **2**, 1541 (1970); K. SYMANZIK: *Comm. Math. Phys.*, **18**, 227 (1970).
([8]) B. SCHROER: *Lett. Nuovo Cimento*, **2**, 867 (1971); G. PARISI: *Phys. Lett.*, **39** B, 643 (1972).

classical symmetry from mass and renormalization effects. Such Ward identities, as is well known, do not generally admit of powerlike solutions. However, asymptotically the « soft » $\eta\phi^2$ term in $\theta(x)$ may be neglected; if $\beta(g)$ then happens to vanish (identically as in the Thirring model or due to the fact that g is such as to make it zero), one sees that the only difference from « naive » application of scale and conformal symmetry would lie in the possible deviation of $l(g)$, the coupling-dependent scale dimension, from its canonical value. However, it is also known [7,9] that, under general assumptions, the mere existence of a zero of $\beta(g)$ (besides the trivial perturbation theory zero) can ensure asymptotic scale and conformal symmetry for the solutions of the homogeneous Ward identities (masses neglected). In this paper we shall directly consider a situation for which $\beta(g)=0$ at least for the physical value of g. In the light of the general results we have just recalled, this might be too a strong condition and in fact most of the results are presumably extendible, under some restrictions, to the more general case. However, under such an assumption we can deal with conformal symmetry algebraically and derive, in a sense, a maximal set of implications, whose extension to more general situations may then be rediscussed with the help of Callan-Symanzik techniques. It will generally be useful to deal with a D-dimensional space-time. In fact most of the results exhibit simple analyticity properties in D. The value $D=2$ turns out to be of particular interest because of the simpler group-theoretical structure and also, from a physical viewpoint, because of its connection to asymptotic limits.

In Sect. **2** we present a general discussion of operator expansion. In Sect. **3** we turn to the four-point function and establish its conformal properties. Section **4** is devoted to the study of the analyticity properties. In Sect. **5** we examine the various light-cone limits and introduce the collinear conformal group. Section **6** summarizes some constraints on normalizations. In Appendix A we report some rarely used properties of double hypergeometric functions. In Appendix B we give the momentum-space representation of conformal covariant vertices. In Appendix C we examine the inclusion of a larger class of representations of possible interest.

2. – Operator expansion in a D-dimensional space-time.

Let us start with the formulation of a conformal covariant Wilson expansion in a D-dimensional space-time; the corresponding Lorentz group is isomorphic to $O_{D-1,1}$. One has $g_\mu{}^\mu = D$. The conformal group is isomorphic to the pseudorthogonal group $O_{D,2}$ in $2+D$ dimensions.

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(9) S. COLEMAN: *Rendiconti S.I.F.*, Course LIV, edited by R. GATTO (New York, 1973), p. 280.

Let us consider three local operators $A(x)$, $B(x)$, $O(x)$ which for simplicity we assume to be conformal scalars (*i.e.* they are Lorentz scalars and moreover they satisfy $[A(0), K_\mu] = 0$ and $[A(0), D] = il_A A(0)$, where K_μ is the vector generator of special conformal transformations) (*). The only nonvanishing Casimir operator of the conformal group is for these representations C_I, $C_I = l(D-l)$. We call $l^* = D - l$ the « shadow dimension » of the dimension l ([10]). Note that $C_I = ll^*$ and $l + l^* = D$. The coupling of local operators of the type $\{\partial_\alpha\}^m \cdot \Box^n O(x)$, where $\{\partial_\alpha\}^m = \partial_{\alpha_1} \dots \partial_{\alpha_m}$, to the operator product $A(x)B(0)$ turns out to be of the following form ([11]):

$$(2.1)\qquad A(x)B(0) = \left(\frac{1}{x^2}\right)^{(\Sigma_{AB}-l)/2} \frac{\Gamma(l)}{\Gamma\big((l+\Delta_{AB})/2\big)\Gamma\big((l-\Delta_{AB})/2\big)} C^0_{AB} \cdot$$
$$\cdot \int_0^1 d\lambda\, \lambda^{(\Delta_{AB}+l)/2-1}(1-\lambda)^{-(\Delta_{AB}+l)/2-1}\, {}_0F_1\left(l+1-\frac{D}{2}; \frac{-x^2}{4}\lambda(1-\lambda)\Box_x\right) O(\lambda x) + \dots,$$

where $\Sigma_{AB} = l_A + l_B$ and $\Delta_{AB} = l_A - l_B$ respectively.

The light-cone restriction turns out to be

$$(2.2)\qquad A(x)B(0) \underset{x^2\to 0}{\simeq} \left(\frac{1}{x^2}\right)^{(\Sigma_{AB}+l)/2} C^0_{AB}\, {}_1F_1\left(\frac{l+\Delta_{AB}}{2}; l; x\cdot\partial\right) O(0) + \dots;$$

it is interesting to observe that, apart from the dynamical coefficient C^0_{AB}, (2.2) is D-independent.

Generalization to conformal group representations with nonzero spin can be given.

The general way to obtain the operator-product expansion (OPE) is to start with the vertex-graph identity for T^*-ordered products

$$(2.3)\qquad \langle 0|T^*(A(x)B(0)O(z))|0\rangle = \frac{\Gamma\big((l^*_C+\Delta_{AB})/2\big)\Gamma\big((l^*-\Delta_{AB})/2\big)\Gamma(l)}{\Gamma\big((l+\Delta_{AB})/2\big)\Gamma\big((l-\Delta_{AB})/2\big)\Gamma(D/2-l)} \cdot$$
$$\cdot \left(\frac{1}{x^2}\right)^{(\Sigma_{AB}-l^*)/2} \int d^D t \left(\frac{1}{t^2}\right)^{(l^*-\Delta_{AB})/2} \left[\frac{1}{(t-x)^2}\right]^{(l^*+\Delta_{AB})/2} \left[\frac{1}{(t-z)^2}\right]^{l},$$

which can be rewritten as

$$\langle 0|T^*(A(x)B(0)O(z))|0\rangle \propto \int d^D t \langle 0|T^*(A(x)B(0)O^*(t))|0\rangle\, \langle 0|T^*(O(t)O(z))|0\rangle\,.$$

(*) D is the dilatation operator; note that the same letter is used also for the number of dimensions, but there should be no confusion.

([10]) S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: *Lett. Nuovo Cimento*, **4**, 115 (1972).

([11]) Notations are the same as in ref. ([5]).

If we perform a Wick rotation $x_0 \to ix_0$, the T^* product is just equivalent to the Schwinger's function and so one can write

$$A(x)B(0) \propto \int \mathrm{d}^D t \, \langle 0|A(x)B(0)O^*(z)|0\rangle \, O(t) + \dots . \tag{2.4}$$

For the correct prescription in pseudo-Euclidean space see ref. ([12]).

To obtain (2.1) from (2.4) we insert a Riemann-Liouville fractional transform into (2.4)

$$A(x)B(0) = \frac{\Gamma(l)\Gamma(l^*)}{\Gamma((l+\Delta_{AB})/2)\Gamma((l-\Delta_{AB})/2)\Gamma(D/2-l)} \left(\frac{1}{x^2}\right)^{(\Sigma_{AB}-l)/2} \cdot \\ \cdot \int_0^1 \mathrm{d}\lambda \, \lambda^{(l+\Delta_{AB})/2-1}(1-\lambda)^{(l-\Delta_{AB})/2-1} \int \mathrm{d}^D t [t^2 + x^2\lambda(1-\lambda)]^{-l^*} \cdot \\ \cdot [x^2\lambda(1-\lambda)]^{D/2-l} \exp[t\cdot\partial] O(x) , \tag{2.5}$$

and use the relation

$$\int \mathrm{d}^D t \exp[t\cdot\partial][t^2 + x^2\lambda(1-\lambda)]^{-l^*} = \\ = \frac{1}{\Gamma(l^*)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{l^*-1} \int \mathrm{d}^D t \exp[-\alpha(t^2 + \lambda(1-\lambda)x^2) + t\cdot\partial] = \\ = \frac{1}{\Gamma(l^*)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{l^*-1} \exp[-\alpha\lambda(1-\lambda)x^2] \int \mathrm{d}^D t \exp[-\alpha t^2 + t\cdot\partial] = \\ = \frac{1}{\Gamma(l^*)} \int_0^\infty \mathrm{d}\alpha \, \alpha^{D/2-l-1} \exp[-\alpha\lambda(1-\lambda)x^2 + \Box/4\alpha] = \\ = \frac{1}{\Gamma(l^*)} (\lambda(1-\lambda)x^2)^{\frac{1}{2}(l-D/2)} \Box^{\frac{1}{2}(D/2-l)} K_{l-D/2}\left((-\lambda(1-\lambda)x^2\Box)^{\frac{1}{2}}\right) . \tag{2.6}$$

The prescription in Minkowski space is obtained according to the substitution

$$K_{l-D/2}\left((-\lambda(1-\lambda)x^2\Box)^{\frac{1}{2}}\right) \to \frac{1}{\sin\pi(l-D/2)} J_{l-D/2}\left((\lambda(1-\lambda)x^2\Box)^{\frac{1}{2}}\right);$$

we obtain eq. (2.1) by recalling the relation

$$\exp[-(i/2)\nu\pi] J_\nu\left(z\exp\left[\frac{1}{2}i\pi\right]\right) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} {}_0F_1\left(\nu+1; \frac{1}{4}z^2\right). \tag{2.7}$$

([12]) S. Ferrara, A. F. Grillo and G. Parisi: *Lett. Nuovo Cimento*, **5**, 147 (1972).

We note that the K_ν-function is even in its index. Therefore it would include in the Wilson expansion the « shadow » singularity

$$(2.8)\qquad A(x)B(0)=\left[\left(\frac{1}{x^2}\right)^{(\Sigma_{AB}-l)/2}\frac{\Gamma(l)}{\Gamma((l+\Delta_{AB})/2)\Gamma((l-\Delta_{AB})/2)}\cdot\right.$$
$$\left.\cdot\int_0^1 d\lambda\,\lambda^{(l+\Delta_{AB})/2-1}(1-\lambda)^{(l-\Delta_{AB})/2-1}\,{}_0F_1\left(l+1-\frac{D}{2};\frac{-x^2}{4}\lambda(1-\lambda)\Box_x\right)+l\leftrightarrow l^*\right]O(\lambda x)\,.$$

This clearly would violate the Wilson dimensional rule for the coefficient singularity whenever $l > D/2$.

Generalization to higher-spin representations is straightforward. It is sufficient to consider the vertex-graph identity in the general form

$$(2.9)\qquad \langle 0|T^*(A_n(x)B_m(0)O_J(z))|0\rangle =$$
$$=\int d^Dt\,\langle 0|T^*(A_n(x)B_m(0)O^*_{J'}(t))|0\rangle\,\langle 0|T^*(O_{J'}(t)O_J(z))|0\rangle$$

and the Wick-rotated version in O_D

$$(2.10)\qquad \langle 0|A_n(x)B_m(0)O_J(z)|0\rangle=\int d^Dt\langle 0|A_n(x)B_m(0)O^*_{J'}(t)|0\rangle\,\langle 0|O_{J'}(t)O_J(z)|0\rangle\,.$$

The OPE in O_D becomes

$$(2.11)\qquad A_n(x)B_m(0)=\int d^Dt\,\langle 0|A_n(x)B_m(0)O^*_{J'}(t)|0\rangle\,O_{J'}(t)+\ldots\,.$$

Clearly expression (2.11) is symmetric under $l\leftrightarrow l^*$, but such a symmetry in the pseudo-Euclidean space can be removed as in the scalar case.

For A, B scalars, the kernel in (2.11) becomes

$$(2.12)\qquad \langle 0|A(x)B(0)O^*_J(t)|0\rangle=(x^2)^{-\frac{1}{2}(\Sigma_{AB}-\tau^*_J)}(t^2)^{-\frac{1}{2}(\tau^*_J-\Delta_{AB})}\cdot$$
$$\cdot(x-t)^{2-\frac{1}{2}(\tau^*_J+\Delta_{AB})}\left[\left(\frac{1}{x-t}\right)_{\alpha_1}+\left(\frac{1}{t}\right)_{\alpha_1}\right]\ldots\left[\left(\frac{1}{x-t}\right)_{\alpha_J}+\left(\frac{1}{t}\right)_{\alpha_J}\right],$$

where $\tau^*_J=D-l_J-J$, and it is understood that the traces are subtracted.

3. – Covariant expansion of the 4-point function.

As is well known, conformal symmetry puts strong constraints on the various Green's functions of the theory. In general, for an n-point Green's function one has an arbitrary function of $n(n-3)/2$ parameters if

$$\frac{n(n-3)}{2}\leqslant nD-\frac{(D+2)(D+1)}{2}$$

and

$$nD - \frac{(D+2)(D+1)}{2}$$

otherwise. It follows that the functional form of two- and three-point functions is completely fixed, while the four-point function depends on an arbitrary function of two conformal invariant parameters ($D > 1$). In particular the most general form of the four-point function for arbitrary conformal scalars has the following form:

$$(3.1)\qquad \langle 0|A(x)B(y)C(z)D(t)|0\rangle = [(x-y)^2]^{-\frac{1}{2}(\Sigma_{AB}-\Delta_{AB})}\cdot \\ \cdot [(x-z)^2]^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}[(x-t)^2]^{-\frac{1}{2}(\Delta_{AB}-\Delta_{CD})}[(z-t)^2]^{-\frac{1}{2}(\Sigma_{CD}-\Delta_{AB})} f(\varrho,\eta)\,,$$

where

$$\varrho = \frac{(x-t)^2(z-y)^2}{(x-y)^2(z-t)^2}\,,\qquad \eta = \frac{(x-z)^2(y-t)^2}{(x-y)^2(z-t)^2}\,.$$

Insertion of the Wilson expansion in the operator product $A(x)B(y)$ allows for a generalized conformal covariant partial-wave analysis of the scattering amplitude. Each term of this expansion corresponds to the exchange of a conformal tensor operator $O_{\alpha_1\ldots\alpha_n}(x)$ together with the infinite set of local tensors $\{\partial_\alpha\}^m \Box^n O_{\alpha_1\ldots\alpha_n}(x)$ of an infinite-dimensional representation of conformal algebra.

Let us first consider the exchange of a local scalar operator $O(x)$ of dimension l. One has simply to insert the ansatz (2.1) into the left-hand side of eq. (3.1). Using the same procedure as in ref. [6] one finds

$$(3.2)\qquad f_0(\varrho,\eta) = \eta^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\Gamma(l)}{\Gamma((l+\Delta_{AB})/2)\Gamma((l-\Delta_{AB})/2)}\cdot \\ \cdot\int\limits_0^1 \mathrm{d}\sigma\,\sigma^{\frac{1}{2}(\Delta_{AB}-\Delta_{CD})-1}(1-\sigma)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})-1}\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-\frac{1}{2}(\Delta_{CD}+l)}\cdot \\ \cdot {}_2F_1\left(\frac{1}{2}(l-\Delta_{CD}),\frac{1}{2}(l+\Delta_{CD});\, l+1-\frac{D}{2};\,\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right) = \\ = \Gamma(l)\eta^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\varrho^{-\frac{1}{2}(l+\Delta_{CD})}\left\{\left[\frac{\Gamma(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma((l-\Delta_{AB})/2)\Gamma((l+\Delta_{AB})/2)}\cdot\right.\right. \\ \left.\cdot F_4\left(\frac{1}{2}(l+\Delta_{CD}),\frac{1}{2}(l+\Delta_{AB});\, l+1-\frac{D}{2};\, 1+\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\,\frac{1}{\varrho};\,\frac{\eta}{\varrho}\right)\right] + \\ \left. + \left(\frac{\varrho}{\eta}\right)^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}[\Delta_{AB}\to-\Delta_{AB},\,\Delta_{CD}\to-\Delta_{CD}]\right\}\,,$$

where $F_4(\alpha,\beta;\gamma;\gamma';x;y)$ is a double hypergeometric function. For details see Appendix A. The previous formula simplifies in the special case of $D=2$. In fact for this case $O_{2,2}=O_{2,1}\otimes O_{2,1}$ and the two factor groups $O_{2,1}$ act as projective transformations on the light-cone variables $x^{\pm}=x^0\pm x^3$ ($x_T=0$). Using a general formula for the double hypergeometric function, valid for

$\alpha+\beta=\gamma+\gamma'+1,$

$$F_4(\alpha,\beta;\gamma;\gamma';x(1-y);y(1-x)) = {}_2F_1(\alpha,\beta;\gamma;x)\,{}_2F_1(\alpha,\beta;\gamma';y)\,, \tag{3.3}$$

one obtains that the term in the curly bracket on the right-hand side of eq. (3.2) simplifies into

$$\begin{aligned}&\left\{\left[\frac{\Gamma(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma((l-\Delta_{AB})/2)\Gamma((l-\Delta_{CD})/2)}\cdot\right.\right.\\ &\cdot{}_2F_1\left(\frac{1}{2}(l+\Delta_{AB}),\frac{1}{2}(l+\Delta_{CD});l;\frac{(x^--y^-)(t^--z^-)}{(x^--t^-)(y^--z^-)}\right)\cdot\\ &\left.\cdot{}_2F_1\left(\frac{1}{2}(l+\Delta_{AB}),\frac{1}{2}(l+\Delta_{CD});1+\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\frac{(x^+-z^+)(y^+-t^+)}{(x^+-t^+)(y^+-z^+)}\right)\right]+\\ &\left.+\left(\frac{(x^+-t^+)(y^+-z^+)(x^--t^-)(y^--z^-)}{(x^+-z^+)(y^+-t^+)(x^--z^-)(y^--t^-)}\right)^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}[\Delta_{AB}\to-\Delta_{AB},\,\Delta_{CD}\to-\Delta_{CD}]\right\}.\end{aligned} \tag{3.4}$$

The factorization of (3.4) obviously reflects the factorized form of the conformal group for $D=2$. We will return to the two-dimensional conformal group, in connection to light-cone limits, in the next Sections. In order to generalize formula (3.2) to the case of an exchanged tensor, $O_{\alpha_1\ldots\alpha_n}(x)$, of arbitrary order n it is useful to work with the Wick-rotated version of eq. (3.2). This is obtained with the OPE (2.4) instead of (2.1) inserted into eq. (3.1). Using the results of ref. ([12]) one obtains

$$\begin{aligned}f^{\mathrm{WR}}(\varrho,\eta)&=\frac{\Gamma(l)\Gamma((l^*-\Delta_{AB})/2)\Gamma((l^*+\Delta_{AB})/2)}{\Gamma((l+\Delta_{CD})/2)\Gamma((l-\Delta_{CD})/2)\Gamma(D/2-l)\Gamma(D/2)}\cdot\\ &\cdot\int_0^1 d\sigma\,\sigma^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})-1}(1-\sigma)^{\frac{1}{2}(\Delta_{AB}-\Delta_{CD})-1}\cdot\\ &\qquad\cdot{}_2F_1\left(\frac{1}{2}(l^*-\Delta_{AB}),\frac{1}{2}(l-\Delta_{AB});\frac{D}{2};1-\frac{\eta}{\sigma}-\frac{\varrho}{1-\sigma}\right).\end{aligned} \tag{3.5}$$

Formula (3.5) can be rewritten using the decomposition

$$\begin{aligned}&{}_2F_1\left(\frac{1}{2}(l^*-\Delta_{AB}),\frac{1}{2}(l-\Delta_{AB});\frac{D}{2};1-\frac{\eta}{\sigma}-\frac{\varrho}{1-\sigma}\right)=\\ &=\frac{\Gamma(D/2)\Gamma(D/2-l)}{\Gamma((l^*-\Delta_{AB})/2)\Gamma((l^*+\Delta_{AB})/2)}\left(\frac{\eta}{\sigma}+\frac{\varrho}{1-\sigma}\right)^{\frac{1}{2}(\Delta_{AB}-l)}\cdot\\ &\cdot{}_2F_1\left(\frac{l-\Delta_{AB}}{2},\frac{l+\Delta_{AB}}{2};l+1-\frac{D}{2};\left(\frac{\eta}{\sigma}+\frac{\varrho}{1-\sigma}\right)^{-1}\right)+\\ &+\frac{\Gamma(D/2)\Gamma(l-D/2)}{\Gamma((l-\Delta_{AB})/2)\Gamma((l+\Delta_{AB})/2)}\left(\frac{\eta}{\sigma}+\frac{\varrho}{1-\sigma}\right)^{\frac{1}{2}(\Delta_{AB}-l^*)}\cdot\\ &\qquad\cdot{}_2F_1\left(\frac{1}{2}(l^*-\Delta_{AB}),\frac{1}{2}(l^*+\Delta_{AB});l+1-\frac{D}{2};\left(\frac{\eta}{\sigma}+\frac{\varrho}{1-\sigma}\right)^{-1}\right).\end{aligned} \tag{3.6}$$

The relation between the Wick-rotated and the pseudo-Euclidean version is simply obtained by considering the first term in the decomposition (3.6) and observing the overall symmetry in the substitution $\sigma \leftrightarrow 1-\sigma$, $\Delta_{AB} \leftrightarrow -\Delta_{CD}$, $x \leftrightarrow t$, $z \leftrightarrow y$ ($\varrho \leftrightarrow \varrho$, $\eta \leftrightarrow \eta$). The important point is that, for the Wick-rotated form, generalization to arbitrary tensors is trivial.

In fact in this case one simply considers the ansatz (2.11) with $n = m = 0$ and arbitrary J, for which the kernel of the integral is given by eq. (2.12).

Using the vertex-graph identity one gets

$$(3.7)\qquad \langle 0|A(x)B(y)C(z)D(t)|0\rangle_n = \int \mathrm{d}^D\xi \langle 0|A(x)B(y)O_{\alpha_1\ldots\alpha_n}(\xi)|0\rangle\langle 0|O^{*\alpha_1\ldots\alpha_n}(\xi)C(z)D(t)|0\rangle =$$

$$= \left[\frac{1}{(x-y)^2}\right]^{(\Sigma_{AB}-\tau_n)/2}\left[\frac{1}{(z-t)^2}\right]^{(\Sigma_{CD}-\tau_n^*)/2}\int \mathrm{d}^D\xi\left\{\left[\left(\frac{1}{x-\xi}\right)_{\alpha_1}+\left(\frac{1}{\xi-y}\right)_{\alpha_1}\right]\ldots\right.$$

$$\left.\ldots\left[\left(\frac{1}{x-\xi}\right)_{\alpha_n}+\left(\frac{1}{\xi-y}\right)_{\alpha_n}\right]\left[\left(\frac{1}{\xi-z}\right)^{\alpha_1}+\left(\frac{1}{t-\xi}\right)^{\alpha_1}\right]\ldots\left[\left(\frac{1}{\xi-z}\right)^{\alpha_n}+\left(\frac{1}{t-\xi}\right)^{\alpha_n}\right]\right\} - \text{traces.}$$

Collecting together the various terms one can write (3.7) in the following form:

$$(3.8)\qquad \langle 0|A(x)B(y)C(z)D(t)|0\rangle_n = \left[\frac{1}{(x-y)^2}\right]^{\Sigma_{AB}/2}\left[\frac{1}{(z-t)^2}\right]^{\Sigma_{CD}/2}\cdot$$

$$\cdot\int \mathrm{d}^D\mu(\xi z t)\left[\frac{(\xi-y)^2}{(\xi-x)^2}\right]^{\Delta_{AB}/2}\left[\frac{(\xi-t)^2}{(\xi-z)^2}\right]^{\Delta_{CD}/2}\Lambda_\xi^{ln}(xy, zt)\,C_n^{D/2-1}\big(-\Omega_\xi(xy, zt)\big)\,,$$

where

$$(3.9)\qquad -\Omega_\xi(xy, zt) = \cos\widehat{XY} = \frac{1}{2[(x-y)^2(z-t)^2]^{\frac{1}{2}}}\cdot$$

$$\cdot\{X_\xi^{yt,xz}(x-z)^2 + X_\xi^{xz,yt}(y-t)^2 - X_\xi^{xt,yz}(y-z)^2 - X_\xi^{yz,xt}(x-t)^2\}$$

and

$$(3.10)\qquad \begin{cases} X_\xi^{x_1x_2x_3x_4} = \left[\dfrac{(\xi-x_1)^2(\xi-x_2)^2}{(\xi-x_3)^2(\xi-x_4)^2}\right]^{\frac{1}{2}} \\ \Lambda_\xi(xy, zt) = \left[\dfrac{(\xi-z)^2(\xi-t)^2(x-y)^2}{(\xi-x)^2(\xi-y)^2(z-t)^2}\right]^{\frac{1}{2}}, \\ \mathrm{d}^D\mu(\xi z t) = \left[\dfrac{(z-t)^2}{(\xi-z)^2(\xi-t)^2}\right]^{D/2}\mathrm{d}^D\xi\,, \end{cases}\qquad \begin{cases} X_\mu = \left(\dfrac{1}{\xi-x}+\dfrac{1}{y-\xi}\right)_\mu, \\ Y_\mu = \left(\dfrac{1}{z-\xi}+\dfrac{1}{\xi-t}\right)_\mu, \end{cases}$$

and $C_n^{D/2-1}$ is the spherical harmonic of the homogeneous Lorentz group O_D in D dimensions.

Expression (3.8) is manifestly conformal covariant. In fact it is manifestly inversion covariant: under $x_i \to x_i/x^2$, $A_i(x_i)$ transforms into $(1/x_i^2)^{l_i} A_i(1/x_i)$. On the right-hand side of eq. (3.8) Λ_ξ, Ω_ξ and $\mathrm{d}^D\mu$ are inversion invariant and covariant respectively and the other factors transform as the left-hand side. Moreover from the property

$$C_n^{D/2-1}(x) = (-1)^n C_n^{D/2-1}(-x) \tag{3.11}$$

one obtains the reflection properties

$$\begin{aligned} W_n(xyzt|\Delta_{AB}\Delta_{CD}) &= (-1)^n W_n(yxzt|-\Delta_{AB}\Delta_{CD}) = \\ &= (-1)^n W_n(xytz|\Delta_{AB}-\Delta_{CD}) = W_n(yxtz|-\Delta_{AB}-\Delta_{CD}) \,. \end{aligned} \tag{3.12}$$

This in particular implies that, if A and B and (or) C and D are equal, only even tensors are present in the operator-product expansions.

The general form (3.8) for the conformal covariant expansion of the four-point function in a Euclidean D-dimensional space allows us to perform a generalized expansion of the scattering amplitude. Each term corresponds to a graph in which an irreducible representation of the conformal group is exchanged. This representation is labelled by a Lorentz quantum number n and by the dilatation quantum number l_n. So one can write in general, as a consequence of the Wilson expansion,

$$\begin{aligned} \langle 0|A(x)B(y)C(z)D(t)|0\rangle &= \left[\frac{1}{(x-y)^2}\right]^{\Sigma_{AB}/2}\left[\frac{1}{(z-t)^2}\right]^{\Sigma_{CD}/2} \cdot \\ &\cdot \sum_{n,l_n}\int \mathrm{d}^D\mu(\xi zt)\left[\frac{(\xi-y)^2}{(\xi-x)^2}\right]^{\Delta_{AB}/2}\left[\frac{(\xi-t)^2}{(\xi-z)^2}\right]^{\Delta_{CD}/2} \Lambda_\xi^{l_n}(xy,zt)\, C_n^{D/2-1}(-\Omega_\xi(xy,zt))\, c_n^{ABCD} \,. \end{aligned}$$

Summation over n, l_n suggests an integral representation for the four-point function

$$\begin{aligned} \langle 0|A(x)B(y)C(z)D(t)|0\rangle &= \left[\frac{1}{(x-y)^2}\right]^{\Sigma_{AB}/2}\left[\frac{1}{(z-t)^2}\right]^{\Sigma_{CD}/2} \cdot \\ &\cdot \int \mathrm{d}^D\mu(\xi zt)\left[\frac{(\xi-y)^2}{(\xi-x)^2}\right]^{\Delta_{AB}/2}\left[\frac{(\xi-t)^2}{(\xi-z)^2}\right]^{\Delta_{CD}/2} g(\Lambda_\xi(xy,zt),\,\Omega_\xi(xy,zt)) \,, \end{aligned} \tag{3.13}$$

where g is an arbitrary function of the two variables Λ_ξ, Ω_ξ.

In order to give a meaning to eq. (3.8) in the physical pseudo-Euclidean space $O_{D-1,1}$ it is sufficient to recall that, according to ref. [6], eq. (3.8) is equivalent to a finite sum of terms having the same structure of the scalar contribution. One introduces the shift operators $\Delta_{AB}^\pm$, $\Delta_{CD}^\pm$ with the property of shifting Δ_{AB}, Δ_{CD} of ± 1 respectively and satisfying

$$[\Delta_{AB}^\pm, \Delta_{CD}^\pm] = 0 \,, \qquad \Delta_{AB}^+\Delta_{AB}^- = 1 = \Delta_{CD}^+\Delta_{CD}^- \,. \tag{3.14}$$

One has

$$\langle 0|A(x)B(y)C(z)D(t)|0\rangle_n = C_n^{D/2-1}(T)\langle 0|A(x)B(y)C(z)D(t)|0\rangle_0 , \tag{3.15}$$

where $C_n^{D/2-1}(T)$ is an operator-valued function of the operator

$$T = \frac{1}{2[(x-y)^2(z-t)^2]^{\frac{1}{2}}}[(y-z)^2\Delta_{AB}^-\Delta_{CD}^+ + (x-t)^2\Delta_{AB}^+\Delta_{CD}^- - \\ -(x-z)^2\Delta_{AB}^+\Delta_{CD}^+ - (y-t)^2\Delta_{AB}^-\Delta_{CD}^-] . \tag{3.16}$$

4. – Analyticity properties.

In this Section we discuss the analyticity properties of the contribution to the four-point function from a single conformal covariant partial wave in momentum space. Properties in configuration space will be derived as consequences.

We start from formula (4.1) for the Wightman function which is valid in Minkowski space; we perform a Wick rotation and we consider the Wightman function for imaginary time arguments, which coincides with the so-called Schwinger function in Euclidean space [13]. It is well known that the Fourier transform of the Schwinger function in Euclidean space coincides at imaginary time arguments with the Fourier transform of the standard time-ordered products in Minkowski space [14]. We are then able to write the explicit expression for the time-ordered product which corresponds to the Wightman function.

The final formula for the time-ordered product is an analytic function of the external momenta with cuts only in the forward and backward tubes. We will use a theorem [15] which [16] states that, if the time-ordered product has the correct analyticity properties, there always exists a unique Wightman function, which corresponds to the time-ordered product and has the right analyticity properties.

The contribution of a scalar can be written as

$$W_0(xytz) = \int_0^\infty dm^2 \int_0^1 du \int_0^1 dv \left(\frac{m}{\Omega}\right)^{D/2-1} K_{D/2-1}(m\Omega) \cdot \\ \cdot J_{l-D/2}(mA) J_{l-D/2}(mB) f_{AB0}(u) f_{CD0}(v) , \tag{4.1}$$

([13]) J. SCHWINGER: *Proc. Nat. Acad. Sci.*, **44**, 956 (1958); T. NAKANO: *Progr. Theor. Phys.* (*Kyoto*), **21**, 241 (1959).

([14]) K. SYMANZIK: *Journ. Math. Phys.*, **7**, 510 (1966).

([15]) O. STEINMAN: *Helv. Phys. Acta*, **33**, 257 (1960).

([16]) D. RUELLE: *Nuovo Cimento*, **19**, 356 (1961).

where

$$\begin{aligned} A^2 &= u(1-u)(x-y)^2\,,\\ B^2 &= v(1-v)(z-t)^2\,,\\ \Omega^2 &= [ux+(1-u)y-v(z+(1-v)t)]^2\,,\\ f_{AB0}(u) &= u^{\frac{1}{2}(l_A-l_B+l)-1}(1-u)^{\frac{1}{2}(l_B-l_A+l)-1}\,. \end{aligned}$$

The function in (4.1) depends only on the invariant scalar products and its expression is valid when all distances are spacelike. As discussed above it coincides with the corresponding Schwinger function in Euclidean space, provided we use the metric $p^2>0$ for spacelike vectors.

It can be shown that the Green's function eq. (4.1) has singularities for spacelike distances. This follows from the representation in terms of the double hypergeometric function (eq. (3.2)) and its singularity structure ([17]). In fact the function $F_4(\alpha,\beta;\gamma;\gamma';x,y)$ has singularities on the line $x^{\frac{1}{2}}+y^{\frac{1}{2}}=1$, which correspond to spacelike points in the expression of the four-point function.

A particular case in which this phenomenon is clear can be explicitly worked out by fixing $l_A=l_B$, $l_C=l_D$, $l=2$ and $D=4$ ([18]).

Consider new the Fourier transform of eq. (4.1):

$$\begin{aligned} (4.2)\qquad &\delta^D(p+p'+q'+q)G(p^2,q^2,p'^2,q'^2,s^2,k^2)=\\ &=\left(\frac{1}{2\pi}\right)^{2D}\int \mathrm{d}^Dx\,\mathrm{d}^Dy\,\mathrm{d}^Dz\,\mathrm{d}^Dt\exp[i(px+qy+p'z+q't)]W_0(x,y,z,t)\,, \end{aligned}$$

where

$$k^2=(p+q)^2=(p'+q')^2\,,\qquad s^2=(p-p')^2=(t-t')^2\,.$$

With the aid of the convolution theorem such transform can be written as

$$(4.3)\qquad \int_0^\infty \frac{\mathrm{d}m^2}{m^2-k^2}\,\bar\Gamma_{ABl}(p^2,q^2,k^2,-m^2)\,\bar\Gamma_{CDl}(p'^2,q'^2,k^2,-m^2)(m^2)^{D/2-l}\,,$$

where

$$\begin{aligned} (4.4)\qquad \bar\Gamma_{ABl}(p^2,q^2,r^2,-m^2) &= (m^2)^{\frac{1}{2}(l-D/2)}\int \mathrm{d}^Dx\exp[ipx]\left(\frac{1}{x^2}\right)^{\frac{1}{2}(l_A+l_B-l)-1}\cdot\\ &\cdot\int_0^1 \mathrm{d}u\exp[-iur\cdot x]f_{ABl}(u)J_{l-D/2}\{(u(1-u)m^2x^2)^{\frac{1}{2}}\}\qquad [r=p+q]\,. \end{aligned}$$

([17]) *Bateman Manuscript Project*, Vol. **1**, Chap. V (New York), p. 227.

([18]) S. FERRARA, R. GATTO, A. F. GRILLO and G. PARISI: in *Scale and Conformal Symmetry in Hadrons Physics* (New York, 1973).

An explicit computation shows that the function $\bar{\Gamma}$ is described by two different analytic functions in the regions

$$[(p^2)^{\frac{1}{2}}+(q^2)^{\frac{1}{2}}]^2-(p-q)^2 \gtrless m^2 .$$

As a consequence the amplitude eq. (4.3) is not an analytic function of k^2 and this momentum-space structure reflects the presence of spacelike singularities in the co-ordinate space.

However a new amplitude can be defined as the dispersive integral of the imaginary part of eq. (4.3). In fact the $\bar{\Gamma}$ is a real function of k^2 so that the only contribution to the imaginary part comes from the denominator.

We obtain therefore the new amplitude

$$\tilde{G}(p^2, q^2, p'^2, q'^2, s^2, k^2) = \int_0^\infty \frac{dm^2}{m^2+k^2} \Gamma_{ABl}(p^2, q^2, -m^2)\Gamma_{CDl}(p'^2, q'^2, -m^2)(m^2)^{D/2-1}, \tag{4.5}$$

where we have used the relation

$$\bar{\Gamma}_{ABl}(p^2, q^2, -m^2, -m^2) = \Gamma_{ABl}(p^2, q^2, -m^2)$$

and Γ is defined as [12]

$$\Gamma_{ABl}(p^2, q^2, m^2) = V(p^2, q^2, m^2+i\varepsilon) - V(p^2, q^2, m^2-i\varepsilon), \tag{4.6}$$

where V is the Fourier transform of the vertex. The properties of the function Γ are discussed in Appendix B. The Fourier transform of the Schwinger function or of the time-ordered product does not factorize in momentum space. Only its discontinuity in k^2 factorizes. Another interesting propery of eq. (4.5) is that it is finite at $p^2=p'^2\neq 0$, $q^2=q'^2\neq 0$, $k^2=0$ if $l>D/2$.

It may be interesting to compare this result with that obtained by using a Wilson expansion where the shadow singularities are not subtracted. The time-ordered function would factorize at all momenta in the product of vertices and propagators and the function would never be finite in the forward direction.

We now quote a theorem [6,15], which allows us to derive the properties of the Wightman function from those of the time-ordered product $G(p)$ in momentum space:

1) if $G(p)$ is invariant under the inhomogeneous Lorentz group;

2) if the only singularities of $G(p)$ are located at

$$\mathrm{Im}\,[\textstyle\sum' p_i^0] = 0\,, \qquad (\textstyle\sum' p_i)^2 > \mu^2\,,$$

where the sum $\sum'$ is restricted to any subset of momenta;

3) if the boundary values of $G(p)$ at real p are tempered distributions;

then the function $G(p)$ uniquely determines a Wightman function $W(z)$ such that the Fourier transform of the time-ordered product, constructed from $W(z)$, is $G(p)$.

The function W is invariant under the inhomogeneous Lorentz group, the boundary values of W are tempered distributions, and W is singular only if two of its arguments $x_1 = (z_1, \underline{x}_1)$, $x_2 = (z_2, \underline{x}_2)$, are such that $\mathrm{Re}(z_1 - z_2) = 0$ and $x_1 - x_2$ is not spacelike.

This theorem is valid only in a theory where there exists a mass gap μ. However this difficulty can be easily solved, following MACK and TODOROV ([19]), by introducing a regulating mass μ, proving all the properties in the presence of this regulating mass, and afterwards sending the mass to zero. As long as no infra-red divergences appear in this limit the zero-mass limit of the Wightman function will satisfy the analyticity properties and coincide with the zero-mass Wightman function.

Provided the dispersive integral in (4.5) converges it is easy to show that the properties of the vertex proven in Appendix B imply that the function in (4.5) satisfies the condition of the above-mentioned theorem.

If we denote by $\bar{l}_A$ the min (l_A, l_A^*), the asymptotic behaviour of the vertex in the region of large momenta implies that the integral in (4.5) converges for large m if $\bar{l}_A + \bar{l}_B + \bar{l}_C + \bar{l}_D > D$ irrespectively of the internal dimension.

Convergence for small m is present only if $l > D/2 - 1$, which is the bound on the dimensions coming from positivity of the Hilbert space. For values of the dimensions in this range no problem arises; however for general values of the dimensions an analytic continuation in the dimensions can be performed, without loosing any desired property, at least for $\bar{l}_A + \bar{l}_B + \bar{l}_C + \bar{l}_D \neq D - N$, where N is a nonnegative integer.

These results concern the scalar contribution; the functions coming from the exchange of a higher-spin tensor can be written as sums of scalar contributions multiplied by distances taken at some integer power.

5. – Light-cone limits and the collinear conformal group $O_{2,1}$.

The light-cone (LC) limit of the covariant expansion can now be obtained. Let us consider for example the scalar contribution given by eq. (3.2). In the light-cone limit $(x-y)^2 \to 0$, $1/\varrho \to 0$. Using the reduction formula of the double hypergeometric function

$$F_4(\alpha_1 \beta; \gamma; \gamma'; 0; y) = {}_2F_1(\alpha_1 \beta; \gamma'; y) , \tag{5.1}$$

([19]) G. MACK and I. T. TODOROV: *Phys. Rev.* (to be published).

one obtains in the limit $\varrho\to\infty$, η/ϱ fixed,

$$(5.2)\qquad f_0(\varrho,\eta)\simeq\Gamma(l)\eta^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\varrho^{-\frac{1}{2}(l+\Delta_{CD})}\frac{\Gamma(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma((l-\Delta_{AB})/2)\Gamma((l-\Delta_{CD})/2)}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}(l+\Delta_{AB}),\frac{1}{2}(l+\Delta_{CD});\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1;\frac{\eta}{\varrho}\right)+$$
$$+\left(\frac{\varrho}{\eta}\right)^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\Gamma(\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma((l+\Delta_{AB})/2)\Gamma((l+\Delta_{CD})/2)}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}(l-\Delta_{AB}),\frac{1}{2}(l-\Delta_{CD});1-\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\frac{\eta}{\varrho}\right)\simeq$$
$$\simeq\eta^{\frac{1}{2}(\Delta_{AB}-l)}{}_2F_1\left(\frac{1}{2}(l+\Delta_{CD}),\frac{1}{2}(l-\Delta_{AB});l;1-\frac{\varrho}{\eta}\right).$$

So the LC limit of the scalar contribution is

$$(5.3)\qquad \langle 0|A(x)B(y)C(z)D(t)|0\rangle\underset{(x-y)^2\to 0}{\sim}\left[\frac{1}{(x-y)^2}\right]^{(\Sigma_{AB}-l)/2}\left[\frac{1}{(z-t)^2}\right]^{(\Sigma_{CD}-l)/2}\cdot$$
$$\cdot\left[\frac{1}{(x-z)^2}\right]^{\frac{1}{2}(\Delta_{CD}+l)}\left[\frac{1}{(x-t)^2}\right]^{\frac{1}{2}(\Delta_{AB}-\Delta_{CD})}\left[\frac{1}{(y-t)^2}\right]^{\frac{1}{2}(l-\Delta_{AB})}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}(l-\Delta_{AB}),\frac{1}{2}(l+\Delta_{CD});l;1-\frac{(x-t)^2(y-z)^2}{(x-z)^2(y-t)^2}\right).$$

Note that the functional form of (3.3) is D-independent. In the same limit the contribution of an exchanged tensor of Lorentz spin n is

$$(5.4)\qquad \langle 0|A(x)B(y)C(z)D(t)|0\rangle=\left[\frac{1}{(x-y)^2}\right]^{(\Sigma_{AB}-\tau_n)/2}\cdot$$
$$\cdot\left[\frac{1}{(z-t)^2}\right]^{(\Sigma_{CD}-\tau_n)/2}\left[\frac{1}{(x-z)^2}\right]^{\frac{1}{2}(\Delta_{CD}+d_n)}\left[\frac{1}{(x-t)^2}\right]^{\frac{1}{2}(\Delta_{AB}-\Delta_{CD})}\left[\frac{1}{(y-t)^2}\right]^{\frac{1}{2}(d_n-\Delta_{AB})}\cdot$$
$$\cdot{}_2F_1\left(\frac{1}{2}(d_n-\Delta_{AB}),\frac{1}{2}(d_n+\Delta_{CD});d_n;1-\frac{(x-t)^2(y-z)^2}{(x-z)^2(y-t)^2}\right),$$

where $\tau_n(d_n)=l_n\mp n$ respectively.

The fact that the LC contribution in (5.3), (5.4) is D-independent has a simple geometrical meaning.

In fact the regular part of the four-point function (as well as the Wilson expansion) is restricted in the limit by covariance under that subgroup of $O_{D,2}$ which corresponds to collinear transformations on the light-cone. This subgroup is $O_{2,1}\subset O_{D,2}$ independently of D ($D\geqslant 2$). In fact only the number of transverse co-ordinates x_T, which is $D-2$, is D-dependent, but it does not affect the light-cone structure. A general D-dimensional vector can be written in terms of (x_+, x_-, x_T). Performing a Lorentz transformation we can set $x_T \to 0$, so in the LC limit $x_- \to 0$, and only the x_+-dependence is relevant for the finite part of the Green's functions.

Conformal transformations become projective ($SL_{2,R}$) transformations on the straight line $x_- = 0$. We consider in more detail the properties of the collinear algebra $O_{2,1}$. We will see that such an algebra is sufficient to determine the behaviour (finite part) of the various Green's functions of the theory when all points lie on the same straight line on the light-cone. Conformal invariance will correspond to projective invariance on the co-ordinate x_+.

We consider a conformal irreducible tensor $(n/2, n/2)\, O_{\alpha_1 \dots \alpha_n}(x)$, *i.e.*

$$(5.5)\qquad [O_{\alpha_1 \dots \alpha_n}(0), K_\lambda] = 0\,, \qquad [O_{\alpha_1 \dots \alpha_n}(0), D] = i l_n O_{\alpha_1 \dots \alpha_n}(0)$$

(see Appendix C for a more general situation).

Consider now the subgroup $O_{2,2} \subset O_{D,2}$ generated by

$$(5.6)\qquad P_\pm = \tfrac{1}{2}(P_0 \pm P_z)\,, \qquad K_\pm = \tfrac{1}{2}(K^0 \pm K^z)\,, \qquad D_\pm = \tfrac{1}{2}(D \pm M_{0z})\,.$$

By making the identification $L_+^\pm = P_\pm$, $L_-^\pm = K_\pm$, $L_0^\pm = D_\pm$, one obtains two commuting $O_{2,1}$ subgroups. It is convenient to consider components of the type $O_{++\dots--\dots TT\dots}(0)$. Then under $L_i^\pm$ one has

$$(5.7)\qquad \begin{cases} [O_{++\dots--\dots TT\dots}(0), L_-^\pm] = 0\,, \\[2mm] [O_{++\dots--\dots TT\dots}(0), L_+^\pm] = i\left(\dfrac{\mathrm{d}}{\mathrm{d}x_+} O_{++\dots--\dots TT\dots}\right)(0)\,, \\[2mm] [O_{++\dots--\dots TT\dots}(0), L_0^\pm] = i\,\dfrac{l_n \pm \Delta}{2}\, O_{++\dots--\dots TT\dots}(0)\,, \end{cases}$$

where Δ is the difference between the number of $+$ and $-$ components. In particular

$$(5.8)\qquad \begin{cases} [O_{++\dots+}(0), L_0^\pm] = i\left(\dfrac{l_n \pm n}{2}\right) O_{++\dots+}(0)\,, \\[2mm] [O_{--\dots-}(0), L_0^\pm] = i\left(\dfrac{l_n \mp n}{2}\right) O_{--\dots-}(0)\,. \end{cases}$$

Consider now the LC expansion of two operators: $\lim_{x^2 \to 0} \{O^{\{\alpha\}}(x)\, O^{\{\beta\}}(0)\}$. If one fixes $x_T = 0$, then $x^2 = x_+ x_-$ ($x_\pm = t \pm z$) and the LC limit is $x_- \to 0$, x_+ fixed, *i.e.* one obtains a short-distance expansion in x_-. If one calls τ, d the two quantum numbers associated to the two $O_{2,1}$, then for a given representation one gets

$$(5.9)\qquad O^{\tau_A d_A}(x_+ x_-)\, O^{\tau_B d_B}(0) \underset{x \to 0}{\sim} (x_-)^{-\frac{1}{2}(\tau_A + \tau_B - \tau_n)} (x_+)^{-\frac{1}{2}(d_A + d_B - d_n)}\, C^n_{AB} \cdot$$
$$\cdot \int_0^1 \mathrm{d}\lambda\, \lambda^{(d_A - d_B + d_n)/2 - 1} (1 - \lambda)^{(d_B - d_A + d_n)/2 - 1}\, O^{\tau_n d_n}(\lambda x_+, 0)\,.$$

For A, B scalars one has $\tau_A = d_A = l_A$, $\tau_B = d_B = l_B$, $\tau_n = l_n - n$, $d_n = l_n + n$

$$(5.10)\qquad O^A(x_+x_-)O^B(0) \underset{x_-\to 0}{\sim} (x_-)^{-\frac{1}{2}(l_A+l_B-\tau_n)}(x_+)^{-\frac{1}{2}(l_A+l_B-d_n)}\, C^n_{AB}\cdot \cdot\int_0^1 \mathrm{d}\lambda\, \lambda^{(l_A-l_B+d_n)/2-1}(1-\lambda)^{(l_B-l_A+d_n)/2-1}O_{++\ldots+}(\lambda x_+, 0)$$

and one recovers the usual conformal covariant LC expansion. We note that the algebraic meaning of the twist τ_n is therefore that of the quantum number associated to projective transformations along x_-, while d_n refers to projective transformations along x_+. In particular the expansion (5.10) is universal, *i.e.* it is independent (apart from C^n_{AB}) of the dimension D of the space-time. This implies that also the behaviour of the generalized partial waves of the various Green's functions is D-independent (up to normalization) when all points lie on the some straight line along the light-cone. For example one simply recovers (5.4) by using projective covariance. By making use of (5.10) one gets

$$(5.11)\qquad A(x_+x_-)B(0) \underset{x_-\to 0}{\sim} (x_-)^{-\frac{1}{2}(\Sigma_{AB}-\tau_n)}(x_+)^{-\frac{1}{2}(\Sigma_{AB}-d_n)}\cdot \cdot\int_0^1 \mathrm{d}\lambda\, \lambda^{(\Delta_{AB}+d_n)/2-1}(1-\lambda)^{-(\Delta_{AB}+d_n)/2-1}O_n(\lambda x_+, 0)$$

(where O_n stands for $O_{++\ldots+}$).

The contribution to the 4-point function is

$$(5.12)\qquad \langle 0|A(x_+x_-)B(0)C(z_+z_-)D(t_+t_-)|0\rangle \simeq (x_+)^{-\frac{1}{2}(\Sigma_{AB}-d_n)}(x_-)^{-\frac{1}{2}(\Sigma_{AB}-\tau_n)}\cdot \cdot\int_0^1 \mathrm{d}\lambda\, \lambda^{(\Delta_{AB}+d_n)/2-1}(1-\lambda)^{(-\Delta_{AB}+d_n)/2-1}\langle 0|O_n(\lambda x_+0)C(z_+z_-)D(t_+t_-)|0\rangle$$

and

$$(5.13)\qquad \langle 0|O_n(\lambda x_+0)\, C(z_+z_-)\, D(t_+t_-)|0\rangle \simeq \simeq (z_-)^{-\frac{1}{2}(\Delta_{CD}+\tau_n)}(t_-)^{-\frac{1}{2}(\tau_n-\Delta_{CD})}(z_- - t_-)^{-\frac{1}{2}(\Sigma_{CD}-\tau_n)}\cdot \cdot(\lambda x_+ - z_+)^{-\frac{1}{2}(\Delta_{CD}+d_n)}(\lambda x_+ - t_+)^{-\frac{1}{2}(d_n-\Delta_{CD})}(z_+ - t_+)^{-\frac{1}{2}(\Sigma_{CD}-d_n)}.$$

We have performed the sequence of limits

$$\frac{x_i^- - x_{i+1}^-}{x_{i+1}^- - x_{i-2}^-}\to 0\,, \qquad i.e.\ \frac{x_-}{z_-}\to 0\,, \quad \frac{z_-}{t_- - z_-}\to 0\,,$$

such as to prescribe an order.

Inserting (5.13) into (5.12) and using the fundamental integral representation for hypergeometric functions one has

$$(5.14)\qquad \langle 0|A(x_+x_-)B(0)C(z_+z_-)D(t_+t_-)|0\rangle_n \simeq$$
$$\simeq (x_+)^{-\frac{1}{2}(\Sigma_{AB}-d_n)}(x_-)^{-\frac{1}{2}(\Sigma_{CD}-\tau_n)}(z_-)^{-\frac{1}{2}(\Delta_{CD}+\tau_n)}(t_-)^{-\frac{1}{2}(\tau_n-\Delta_{CD})}(z_- - t_-)^{-\frac{1}{2}(\Sigma_{CD}-\tau_n)}\cdot$$
$$\cdot(z_+ - t_+)^{-\frac{1}{2}(\Sigma_{CD}-d_n)}(t_+)^{-\frac{1}{2}(d_n-\Delta_{CD})}z_+^{-\frac{1}{2}(\Delta_{CD}+d_n)}\int_0^1 \mathrm{d}\lambda\,\lambda^{(\Delta_{AB}+d_n)/2-1}\cdot$$
$$\cdot(1-\lambda)^{-(\Delta_{AB}+d_n)/2-1}\left(1-\lambda\frac{x_+}{z_+}\right)^{-\frac{1}{2}(\Delta_{CD}+d_n)}\left(1-\lambda\frac{x_+}{t_+}\right)^{-\frac{1}{2}(d_n-\Delta_{CD})} =$$
$$= (x_-)^{-\frac{1}{2}(\Sigma_{AB}-\tau_n)}(z_-)^{-\frac{1}{2}(\Delta_{CD}+\tau_n)}(t_-)^{-\frac{1}{2}(\tau_n-\Delta_{CD})}(z_- - t_-)^{-\frac{1}{2}(\Sigma_{CD}-\tau_n)}\cdot$$
$$\cdot(z_+ - t_+)^{-\frac{1}{2}(\Sigma_{CD}-\tau_n)}(t_+)^{-\frac{1}{2}(d_n-\Delta_{AB})}(t_+ - x_+)^{-\frac{1}{2}(\Delta_{AB}-\Delta_{CD})}(z_+ - t_+)^{-\frac{1}{2}(d_n+\Delta_{CD})}\cdot$$
$$\cdot(x_+)^{-\frac{1}{2}(\Sigma_{AB}-d_n)}\,{}_2F_1\left(\frac{1}{2}(d_n-\Delta_{AB}),\frac{1}{2}(d_n+\Delta_{CD});\,d_n;\,1-\frac{z_+(t_+-x_+)}{t_+(z_+-x_+)}\right).$$

One just recovers (5.4) by observing that in the previous sequences of limits one has $z_-(t_- - x_-)/t_-(z_- - x_-) \to 1$.

Multiple OPE can be obtained at short and light line distances. The contribution of a single operator in the OPE in the sequence of limits $(x_i^- - x_{i+1}^-)/(x_{i+1}^- - x_{i+2}^-) \to 0$ is

$$(5.15)\qquad A_1(x_1)\dots A_n(x_n) \underset{x_i\to 0}{\sim}$$
$$\underset{x_i\to 0}{\sim}\left(\frac{1}{|x_1-x_2|}\right)^{l_1+l_2-l_{12}}\left(\frac{1}{|x_2-x_3|}\right)^{l_{12}+l_3-l_{13}}\dots\left(\frac{1}{|x_{n-1}-x_n|}\right)^{l_{1n-1}+l_n-l} O_l(x_n),$$

where $l_{12}\dots l_{1n-1}$ is a sequence of $n-2$ operators which add up to a conformal partial wave in the $(n+1)$-point function $\langle 0|A_1(x_1)\dots A_n(x_n)\,O_l(0)|0\rangle$ in the short-distance limit. (5.15) is a simple consequence of dilatation invariance at short distances.

The same analysis can be carried out when all points lie along a straight line on the light-cone. If we put $x_i = (x_{i+}, x_{i-}, x_{Ti}=0)$, then in the limit

$$x_{i-}\to 0 \qquad \frac{x_{i-}-x_{i+1-}}{x_{i+1-}-x_{i+2-}}\to 0$$

and fixed x_{i+}, one obtains

$$(5.16)\qquad A_1(x_{1+}x_{1-})\dots A_n(x_{n+}x_{n-}) \underset{x_{i-}\to 0,\; x_{n+}\geqslant x_{n-1+}\dots\geqslant x_{1+}\geqslant 0}{\simeq}$$
$$\simeq (x_{1+}-x_{2+})^{\frac{1}{2}(d_{12}-d_1-d_2)}(x_{1-}-x_{2-})^{\frac{1}{2}(\tau_{12}-\tau_1-\tau_2)}\cdot$$
$$\cdot\prod_{J=2}^{n-1}(x_J^-)^{d_J}\int_0^1\prod_{J=1}^{n-1}\mathrm{d}\lambda_J\,\lambda_J^{a_J-1}(1-\lambda_J)^{b_J-1}\prod_{k=1}^{n-2}X_k^{c_k}\,O_{1n}(\omega_n),$$

where we have put

$$(5.17)\qquad \begin{cases} a_J = \frac{1}{2}(d_{1J} - d_{J+1} + d_{1J+1}) , \\ b_J = \frac{1}{2}(d_{J+1} - d_{1J} + d_{1J+1}) , \\ c_J = \frac{1}{2}(d_{1J+1} - d_{1J} - d_{J+1}) , \\ d_J = \frac{1}{2}(\tau_{1J+1} - \tau_{1J} - \tau_{J+1}) , \\ \omega_n = \lambda_{n-1}(\omega_{n-1} - x_{n+}) + x_{n+} = \sum_{i=1}^{n-1}\left(\prod_{J=1}^{n-1}\lambda_J\right)(x_{i+} - x_{i+1+}) + x_{n+} , \\ X_n = \omega_{n-1} - x_{n+} . \end{cases}$$

Here (d, τ) are the quantum numbers associated to $O_{2,2} \subset O_{D,2}$ of the several operators involved in the expansion. (5.16) can be rewritten in more compact form which turns out to be manifestly causal:

$$(5.18)\qquad A_1(x_{1+}x_{1-}) \ldots A_n(x_{n+}x_{n-}) = \\ = F_n(x_{1+} - x_{2+}, x_{1-} - x_{2-}, \{x_-\})\int_{x_{1+}}^{x_{n+}} d\omega\, L(\{\omega, \{x_+\}) \, O_{1n}(\omega) ,$$

where

$$F_n(x_{1+} - x_{2+}, x_{1-} - x_{2-}, \{x_-\}) = \\ = (x_{1+} - x_{2+})^{-\frac{1}{2}(d_1 + d_2 - d_{12})}(x_{1-} - x_{2-})^{-\frac{1}{2}(\tau_1 + \tau_2 - \tau_{12})}\prod_{J=2}^{n-1} x_{J-}^{-\frac{1}{2}(\tau_{1J} + \tau_{J+1} - \tau_{J+1})} ,$$

and

$$L(\omega, \{x_+\}) = \sum_{i=1}^{n-1}\theta(\omega - x_{1+})\theta(\omega_{i+1} - \omega)\prod_{J=1}^{i-1}\int_0^1 d\lambda_J \prod_{k=1}^{n-1}\int_{(\omega - x_{k+})/(\omega_{k-1} - x_{k+})}^{1} d\lambda_k f_n(\omega, \{\lambda\}) f_n(\omega, \{\lambda\}) = \\ = \int_0^1 d\lambda_{n-1}\prod_{J=1}^{n-1}\lambda_J^{a_J - 1}(1 - \lambda_J)^{b_J - 1}\prod_{k=1}^{n-2}\left[\sum_{i=1}^{k-1}(x_{i+} - x_{i+1+})\prod_{J=1}^{k-1}\lambda_J\right]\delta(\omega - \omega_n) .$$

Note that the function $L(\omega\{x\})$ is a sort of ordered product on the light line $x_- = 0$ and its appearance ensures the causal support properties of the right-hand side of eq. (5.18).

6. – Constraints on normalizations.

A useful consequence of conformal symmetry is that it relates the overall normalizations of Green's functions and the coefficients of OPEs. For example, using the OPE one gets the relation

$$(6.1)\qquad C_{ABO} = C^O_{AB}\, C_{OO} ,$$

where C_{ABO}, C_{OO} are the normalizations of the three- and two-point functions $\langle ABO\rangle$, $\langle OO\rangle$ respectively and C^O_{AB} is the coefficient which couples O to the OP $A(x)B(0)$.

In the case of the four-point function one obtains, in exactly the same way, that the overall normalization of the partial wave $\langle O|ABCD|O\rangle_n$ is given by

$$C^n_{ABCD} = C^n_{AB}\, C_{CDn} = C^n_{AB}\, C^n_{CD}\, C_{nn}\,, \tag{6.2}$$

where C^n_{AB}, C^n_{CD} are the coefficients of O_n in the OPEs $A(x)B(0)$, $C(x)D(0)$, and $\langle O_n O_n\rangle$ its propagator normalization. Of course similar relations can be obtained from higher-order Green's functions.

7. – Conclusions.

The conformally irreducible contributions to the generalized partial-wave expansion have been shown to separately exhibit the correct analyticity properties. The expansion, which obtains on the explicit assumption of Wilson operator-product expansion, is different, though in principle related, to the field-theoretic skeleton expansions, which more directly must exhibit explicit analyticity properties.

We have worked in a general D-dimensional space-time and found analytic properties in D and in the Lorentz spin n (lowest spin of local operators exchanged in a conformally irreducible graph). The value $D = 2$ has physical importance and exhibits mathematical peculiarities. On the light-cone the projective group $O_{2,1}$ summarizes the relevant algebraic restrictions, independently of the dimensions D of the space-time from which one has started (irrelevance of the number of transverse dimensions). In D-dimensional space-time the « shadow » dimension of l is $l^* = D - l$ corresponding to the symmetry of the Casimir operator. We have discussed the mechanism which eliminates in pseudo-Euclidean spaces the « shadow » singularities in the Wilson expansion.

The four-point function has been constructed by explicit assumption of operator-product expansions. The central role is summarized in the appearance of the double hypergeometric function.

The peculiar role of $D = 2$ is here exhibited through a factorization property of F_4, in that case, into a product of two ${}_2F_1$-functions. Group-theoretically this property corresponds to the factorization of $O_{2,2}$ into $O_{2,1} \otimes O_{2,1}$. Each of the two factor groups acts projectively on the light-cone variables x_+, x_-. General integral representations can be given for the four-point function and again the different role of shadow symmetry in Euclidean and pseudo-Euclidean spaces is exhibited. The formal discussion of covariance and the deduced partial-wave expansion allows for a direct verification of the analyticity properties. This is obtained by introducing a suitable amplitude. In discussing light-cone limits we have taken advantage of the singular structure of the $O_{2,1}$ algebra. In particular this has allowed us to construct operator-product expansions for more than two operators and to verify explicitly their causality properties. We have thus extended the known result on the causal properties of the conformal covariant light-cone expansion for the product of two operators, which appears to be an important consequence of approximate conformal covariance. Useful consequences of conformal covariance are also the relations among normalizations of Green's functions and coefficients in operator expansions, which in some cases are directly connected to physical processes. As a by-product of our investigation we have also derived momentum-space representations of conformal covariant vertices and discussed their analyticity and support properties.

Appendix A

Representation in terms of double hypergeometric functions.

In this Appendix we briefly derive an expression for the scalar contribution of dimension l to the Wightman function $\langle 0|A(x)B(y)C(z)D(t)|0\rangle$ in terms of double hypergeometric functions.

We start from the integral representation obtained by insertion of the conformal covariant Wilson expansion (see eq. (3.3))

$$\text{(A.1)}\qquad \frac{\Gamma(l)}{\Gamma\big((l+\Delta_{AB})/2\big)\Gamma\big((l-\Delta_{AB})/2\big)}\int\limits_0^1 \mathrm{d}\sigma\,\sigma^{\frac{1}{2}(\Delta_{AB}-\Delta_{CD})-1}(1-\sigma)^{\frac{1}{2}(\Delta_{AB}+\Delta_{CD})-1}\cdot$$

$$\cdot\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-\frac{1}{2}(\Delta_{CD}+l)}{}_2F_1\left(\frac{1}{2}(l-\Delta_{CD}),\frac{1}{2}(l+\Delta_{CD});\,l-\frac{D}{2}+1;\,\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right).$$

Expanding in power series the hypergeometric function

$$\text{(A.2)}\qquad {}_2F_1\left(\frac{1}{2}(l-\Delta_{CD}),\frac{1}{2}(l+\Delta_{CD});\,l-\frac{D}{2}+1;\,\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-1}\right)=$$

$$=\frac{\Gamma(l+D/2+1)}{\Gamma(\frac{1}{2}(l-\Delta_{CD}))\Gamma(\frac{1}{2}(l+\Delta_{CD}))}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\varrho}{\sigma}+\frac{\eta}{1-\sigma}\right)^{-n}\cdot$$

$$\cdot\Gamma\left(\frac{1}{2}(l-\Delta_{CD})+n\right)\Gamma\left(\frac{1}{2}(l+\Delta_{CD})+n\right)\Big/\Gamma\left(l-\frac{D}{2}+1+n\right),$$

we get from (A.1)

$$\text{(A.3)}\qquad \int_0^1 \mathrm{d}\sigma\,\sigma^{\frac{1}{2}(\Delta_{AB}+l)+n-1}(1-\sigma)^{\frac{1}{2}(-\Delta_{AB}+l)+n-1}(1-\sigma z)^{-n-\frac{1}{2}(l+\Delta_{CD})}=$$

$$=\frac{\Gamma\left(\frac{1}{2}(\Delta_{AB}+l)+n\right)\Gamma\left(\frac{1}{2}(-\Delta_{AB}+l)+n\right)}{\Gamma(l+2n)}\cdot$$

$$\cdot{}_2F_1\left(n+\frac{1}{2}(l+\Delta_{CD}),\frac{1}{2}(l+\Delta_{AB})+n;\,l+2n;\,z\right)=$$

$$=\frac{\Gamma\left(\frac{1}{2}(l+\Delta_{AB})+n\right)\Gamma\left(\frac{1}{2}(-\Delta_{AB}+l)+n\right)}{\Gamma(l+2n)}\cdot$$

$$\cdot\left[\frac{\Gamma(l+2n)\Gamma\left(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)}{\Gamma\left(\frac{1}{2}(l-\Delta_{CD})+n\right)\Gamma\left(\frac{1}{2}(l-\Delta_{AB})+n\right)}\cdot\right.$$

$$\cdot{}_2F_1\left(n+\frac{1}{2}(l+\Delta_{CD}),\frac{1}{2}(l+\Delta_{AB})+n;\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\,1-z\right)+$$

$$+(1-z)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\Gamma(l+2n)\Gamma\left(\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)}{\Gamma\left(n+\frac{1}{2}(l+\Delta_{CD})\right)\Gamma\left(n+\frac{1}{2}(l+\Delta_{AB})\right)}\cdot$$

$$\left.\cdot{}_2F_1\left(\frac{1}{2}(l-\Delta_{CD})+n,\frac{1}{2}(l-\Delta_{AB})+n;\,-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1;\,1-z\right)\right]=$$

$$=\frac{\Gamma\left(\frac{1}{2}(l+\Delta_{AB})+n\right)\Gamma\left(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)}{\Gamma\left(\frac{1}{2}(l-\Delta_{CD})+n\right)}\cdot$$

$$\cdot{}_2F_1\left(n+\frac{1}{2}(l+\Delta_{CD}),\,n+\frac{1}{2}(l+\Delta_{AB});\,1+\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\,1-z\right)+$$

$$+(1-z)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\Gamma\left(\frac{1}{2}(l-\Delta_{AB})+n\right)\Gamma\left(\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)}{\Gamma\left(n+\frac{1}{2}(l+\Delta_{CD})\right)}\cdot$$

$$\cdot{}_2F_1\left(\frac{1}{2}(l-\Delta_{CD})+n,\frac{1}{2}(l_{AB}-\Delta)+n;\,1-\frac{1}{2}(\Delta_{AB}+\Delta_{CD});\,1-z\right),$$

where

$$z=1-\eta/\varrho\,.$$

If we expand the two hypergeometric functions in power series (A.1) becomes

$$\text{(A.4)}\qquad \frac{\Gamma(l-D/2+1)\Gamma(l)}{\Gamma((l-\Delta_{AB})/2)\Gamma((l+\Delta_{AB})/2)\Gamma((l-\Delta_{CD})/2)\Gamma((l+\Delta_{CD})/2)}\sum_{nm}\frac{1}{n!}\frac{1}{m!}\cdot$$

$$\cdot\frac{\Gamma\left(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)\Gamma\left(\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1\right)\Gamma\left(n+m+\frac{1}{2}(l+\Delta_{CD})\right)}{\Gamma(l-D/2+1+n)\Gamma\left(\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1+m\right)}\cdot$$

$$\cdot\Gamma\left(n+m+\frac{1}{2}(l+\Delta_{AB})\right)(1-z)^m\varrho^{-n+\frac{1}{2}(-l-\Delta_{CD})}+\sum_{nm}\frac{1}{n!}\frac{1}{m!}\cdot$$

$$\cdot\frac{\Gamma\left(\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)\Gamma\left(1-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})\right)\Gamma\left(n+m+\frac{1}{2}(l-\Delta_{AB})\right)}{\Gamma(l-D/2+1+n)\Gamma\left(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1+m\right)}\cdot$$

$$\cdot\Gamma\left(n+m+\frac{1}{2}(l-\Delta_{CD})\right)(1-z)^m(1-z)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\varrho^{-n-\frac{1}{2}(l+\Delta_{CD})}\,.$$

It is easy to see that the two double series are nothing but the power series which define the fourth double hypergeometric functions. One obtains the final result

$$\text{(A.5)}\qquad \Gamma(l)\varrho^{-\frac{1}{2}(l+\Delta_{CD})}\left[\frac{\Gamma(-\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma((l-\Delta_{AB})/2)\Gamma((l-\Delta_{CD})/2)}\cdot\right.$$

$$\cdot F_4\left(\frac{1}{2}(l+\Delta_{CD}),\frac{1}{2}(l+\Delta_{AB});\frac{1}{2}(\Delta_{AB}+\Delta_{CD})+1;l+1-\frac{D}{2};1-z;\frac{1}{\varrho}\right)+$$

$$+(1-z)^{-\frac{1}{2}(\Delta_{AB}+\Delta_{CD})}\frac{\Gamma(\frac{1}{2}(\Delta_{AB}+\Delta_{CD}))}{\Gamma((l+\Delta_{AB})/2)\Gamma((l+\Delta_{CD})/2)}\cdot$$

$$\cdot F_4\left(\frac{1}{2}(l-\Delta_{CD}),\frac{1}{2}(l-\Delta_{AB});1-\frac{1}{2}(\Delta_{AB}+\Delta_{CD});l+1-\frac{D}{2};1-z,\frac{1}{\varrho}\right),$$

where

$$\frac{1}{\varrho}=\frac{(x-y)^2(z-t)^2}{(x-t)^2(z-y)^2},\qquad 1-z=\frac{\eta}{\varrho}=\frac{(x-z)^2(y-t)^2}{(x-t)^2(y-z)^2}$$

are conformal invariant quantities.

Appendix B

Momentum-space representation of a conformal covariant vertex.

In this Appendix we study the properties of the conformal covariant vertex of three scalars in momentum space.

We consider the vertex function for three operators $A(x)$, $B(y)$, $C(z)$ of dimensions l_A, l_B, l_C respectively in a D-dimensional space-time.

Conformal invariance fixes the vertex to be

$$\text{(B.1)}\qquad \langle 0|T(A(x)B(y)C(z))|0\rangle =$$

$$=\frac{1}{\Gamma(D-\Sigma)}\frac{1}{\Gamma(D/2+l_A-\Sigma)}\frac{1}{\Gamma(D/2+l_B-\Sigma)}\frac{1}{\Gamma(D/2+l_C-\Sigma)}\cdot$$

$$\cdot\frac{1}{[(x-y)^2-i\varepsilon]^{\Sigma-l_C}}\frac{1}{[(x-z)^2-i\varepsilon]^{\Sigma-l_B}}\frac{1}{[(y-z)^2-i\varepsilon]^{\Sigma-l_A}},\qquad \Sigma=\frac{1}{2}(l_A+l_B+l_C)\,.$$

Its Fourier transform is

$$\text{(B.2)}\qquad \frac{1}{(2\pi)^{\frac{3}{2}D}}\int d^Dx\,d^Dy\,d^Dz\,\langle 0|T(A(x)B(y)C(z))|0\rangle\exp[ipx+iqy+itz]=$$

$$=\frac{1}{(2\pi)^{D/2}}\delta^D(p+q+t)\,V_{ABC}(p^2,q^2,t^2)\,.$$

The function V_{ABC} can be computed directly in Euclidean space by performing a Wick rotation. The following integral rappresentation is used:

$$\text{(B.3)}\qquad \frac{1}{(p^2)^\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^\infty \frac{d\alpha}{\alpha}\,\alpha^a \exp[-\alpha p^2]\,.$$

The final result is

$$\text{(B.4)}\qquad V_{ABC}(p^2, q^2, t^2) = C\int_0^1 \frac{d\alpha}{\alpha}\int_0^1 \frac{d\beta}{\beta}\int_0^1 \frac{d\gamma}{\gamma}\,\delta(\alpha+\beta+\gamma-1)\cdot$$
$$\cdot\,\alpha^{D/2+l_A-\Sigma}\beta^{D/2+l_B-\Sigma}\gamma^{D/2+l_C-\Sigma}(\alpha\beta t^2+\gamma\alpha q^2+\gamma\beta p^2)^{\Sigma-D}\,,$$

where

$$\text{(B.5)}\qquad C = \pi^{D/2}2^{2\Sigma-\frac{3}{2}D}\Gamma^{-1}(\Sigma-l_A)\Gamma^{-1}(\Sigma-l_B)\Gamma^{-1}(\Sigma-l_C)\cdot$$
$$\cdot\,\Gamma^{-1}\left(\frac{D}{2}-\Sigma+l_A\right)\Gamma^{-1}\left(\frac{D}{2}-\Sigma+l_B\right)\Gamma^{-1}\left(\frac{D}{2}-\Sigma+l_C\right).$$

An elegant formula for V_{ABC} can be obtained by noting that (B.4) can be rewritten as

$$\text{(B.6)}\qquad V_{ABC}(p^2, q^2, t^2) = \frac{C}{\Gamma(D-\Sigma)\Gamma(D/2-\Sigma)}\int_0^\infty \frac{d\alpha}{\alpha}\int_0^\infty \frac{d\beta}{\beta}\int_0^\infty \frac{d\gamma}{\gamma}\int_0^\infty \frac{d\lambda}{\lambda}\cdot$$
$$\cdot\,\alpha^{l_A-D/2}\beta^{l_B-D/2}\gamma^{l_C-D/2}\,\lambda^{\Sigma-D/2}\exp\left[-\frac{p^2}{\alpha}+\frac{q^2}{\beta}+\frac{t^2}{\gamma}+\lambda\alpha+\lambda\beta+\lambda\gamma\right].$$

The above integral factorizes into the product of three functions of α, β, γ. Using the integral representation

$$\text{(B.7)}\qquad \int_0^\infty \frac{d\alpha}{\alpha}\,\alpha^A \exp\left[-\frac{p^2}{\alpha}-\lambda\alpha\right] = 2\left(\frac{p^2}{\lambda}\right)^{A/2} K_A[2(p^2\lambda)^{\frac{1}{2}}]\,,$$

we obtain the nearly factorized form

$$\text{(B.8)}\qquad V_{ABC}(p^2, q^2, t^2) = \frac{8C}{\Gamma(D-\Sigma)\Gamma(D/2-\Sigma)}\cdot$$
$$\cdot\int_0^\infty \frac{d\lambda}{\lambda}\,\lambda^{D/4}F(l_A, p^2, \lambda)F(l_B, q^2, \lambda)F(l_C, t^2, \lambda)\,,$$

where

$$F(l_A, p^2, \lambda) = (p^2)^{l_A/2-D/4} K_{l_A-D/2}[2(\lambda p^2)^{\frac{1}{2}}]\,. \tag{B.9}$$

The integration in (B.8) can be explicitly done by recalling the relation among the K and J Bessel functions

$$K_\nu(z) = \frac{\pi}{2\sin\nu\pi}\left[(i)^\nu J_{-\nu}(iz) - (i)^{-\nu} J_\nu(iz)\right] \tag{B.10}$$

and the integral representation for the double hypergeometric function F_4

$$\begin{aligned}\int_0^\infty & x^{\lambda-1} J_\mu(\alpha x) J_\nu(\beta x) K_\varrho(\gamma x)\,\mathrm{d}x = \\ &= 2^{\lambda-2}\frac{\alpha^\mu\beta^\nu\gamma^{-\lambda-\mu-\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\Gamma\left(\frac{\lambda+\mu+\nu-\varrho}{2}\right)\Gamma\left(\frac{\lambda+\mu+\nu+\varrho}{2}\right)\cdot \\ &\quad\cdot F_4\left(\frac{\lambda+\mu+\nu-\varrho}{2}, \frac{\lambda+\mu+\nu+\varrho}{2}; \mu+1; \nu+1; \frac{-\alpha^2}{\gamma^2}; \frac{-\beta^2}{\gamma^2}\right).\end{aligned} \tag{B.11}$$

The final answer is

$$\begin{aligned}V_{ABC}(p^2, q^2, t^2) &= \frac{2C}{\Gamma(D-\Sigma)\Gamma(D/2-\Sigma)}\sum_{\substack{-1\\h\neq 0}}^{1}\sum_{\substack{-1\\k\neq 0}}^{1} hk(p^2)^{((k+1)/2)(l_A-D/2)}\cdot \\ &\cdot(q^2)^{((k+1)/2)(l_B-D/2)}(t^2)^{-l_C/2+D/4-k(l_A/2-D/4)-h(l_B/2-D/4)}\cdot \\ &\cdot\frac{\pi^2}{\sin\pi k(l_A-D/2)\sin\pi h(l_B-D/2)}\cdot \\ &\cdot\frac{\Gamma[l_C/2-D/4+h(l_B/2-D/4)+k(l_A/2-D/4)-D/4]}{\Gamma[k(l_A-D/2)+1]\Gamma[h(l_B-D/2)+1]}(i)^{-h(l_A-D/2)-k(l_B-D//2)}\cdot \\ &\cdot\Gamma\left(\frac{l_C}{2}-\frac{D}{4}+h\left(\frac{l_B}{2}-\frac{D}{4}\right)+k\left(\frac{l_A}{2}-\frac{D}{4}\right)+\frac{D}{4}\right)\cdot \\ &\cdot F_4\left[\frac{l_C}{2}-\frac{D}{2}+h\left(\frac{l_B}{2}-\frac{D}{4}\right)+k\left(l_A-\frac{D}{4}\right), \frac{l_C}{2}+h\left(\frac{l_B}{2}-\frac{D}{4}\right)+\right. \\ &\quad\left.+k\left(\frac{l_A}{2}-\frac{D}{4}\right), k\left(l_A-\frac{D}{2}\right), h\left(l_B-\frac{D}{2}\right); \frac{p^2}{t^2}, \frac{q^2}{t^2}\right].\end{aligned} \tag{B.12}$$

From the above integral representation one can derive the following properties of V_{ABC}.

If the integral in (B.8) is convergent, V_{ABC} is a real analytic function of q^2, p^2, t^2.

Dispersion relations of the type

$$V_{ABC}(p^2, q^2, t^2) = \int_0^\infty \frac{\mathrm{d}m^2}{m^2+t^2}\Gamma_{ABC}(p^2 q^2, -m^2) \tag{B.13}$$

are valid, where Γ_{ABC} itself is a real analytic function of p^2 and q^2 and satisfies dispersion relations in p^2, q^2. There exists the relation

$$\Gamma_{ABC}(p^2, q^2, t^2) \propto \int_0^\infty \frac{d\lambda}{\lambda} \lambda^{D/4} F(l_A, p^2, \lambda)\, F(l_B, q^2, \lambda)\, G(l_C, t^2, \lambda) \theta(-t^2)\,, \tag{B.14}$$

where

$$G(l_C, t^2, \lambda) \propto t^{2(l_C/2)-D/4} J_{l_C-D/2}[2(-\lambda t^2)^{\frac{1}{2}}] \propto \propto \lambda^{D/4-l_C/2}\, t^{2(l_C-D/2)}\, {}_0F_1\left(\frac{D}{2}-l_C+1\,;\; \lambda t^2\right). \tag{B.15}$$

The integrals (B.8) and (B.14) are always convergent in the large-λ region whenever some momentum is different from zero, due to the exponential damping of K Bessel functions at infinity. The only troubles may arise from the small-λ region. (B.2) and (B.8) are indeed divergent if $l_A + l_B + l_C > 2D$. However V_{ABC} can be defined for $l_A + l_B + l_C > 2D$ by analytic continuation from the other region. The process of analytic continuations in the dimensions cannot ruin the « good » analyticity properties in momentum space: eqs. (B.13)-(B.15) are still valid apart from possible subtraction in the dispersion integral.

If we send the momentum t^2 to infinity the arguments of the F_4-function in (B.12) goes to zero: the behaviour at large t^2 is ruled by the explicit powers of t^2 $\left(F_4(\alpha, \beta;\, \gamma;\, \delta;\, 0, 0) = 1\right)$; the leading term of V_{ABC} is proportional to $(t^2)^{-\sigma}$, where

$$\begin{cases} \sigma = \dfrac{l_C}{2} - \dfrac{D}{4} - \left|\dfrac{l_A}{2} - \dfrac{D}{4}\right| - \left|\dfrac{l_B}{2} - \dfrac{D}{4}\right| = \dfrac{l_C}{2} + \tilde{l}_A + \tilde{l}_B - \dfrac{3}{4} D\,, \\ \tilde{l}_A = \min(l_A, D - l_A) = \min(l_A, l_A^*)\,. \end{cases} \tag{B.16}$$

From (B.12) the function V_{ABC} in the small-t^2 region is the sum of two contributions, one regular, the second one proportional to $(t^2)^{l_C - D/2}$.

Appendix C

Indecomposable representations of the conformal algebra.

Throughout the previous Sections we have considered only the simplest representations of the conformal group, namely singlets of local operators with fixed dimensions and annihilated by K_λ.

Both the above assumptions can be relaxed: for what concerns dilatations we obtain the so-called « dilatation multiplets » which have been studied to some extent.

A second case corresponds to *indecomposable* representations of the conformal algebra and will be studied here in some detail using the $O_{2,1}$ formalism.

For definiteness, let us consider the $\pm$ projection of a set of N tensor operators $O_{\alpha_1\ldots\alpha_n}(x)$

$$O_n(x_+, x_-) = O_{+\ldots+}(x_+, x_-) \tag{C.1}$$

with transformation properties under $O_{2,1+}$

$$[L_1^+, O_n(x_+, x_-)] = \partial_+ O_n(x_+, x_-) , \tag{C.2}$$

$$[L_0^+, O_n(x_+, x_-)] = [\tfrac{1}{2}(d_n + n) + x_+\partial_+]O_n(x_+, x_-) , \tag{C.3}$$

$$[L_{-1}^+, O_n(x_+, x_-)] = [x_+^2\partial_+ + x_+(d_n + n)]O_n(x_+, x_-) + k_{n-1}O_{n-1}(x_+, x_-) . \tag{C.4}$$

In writing eq. (C.4) we have made a definite choice, namely that L_{-1}^+ lowers the spin n as well as the dimension, this choice being the only one physically interesting since it leads to correlated dimensions

$$l_n = l + n ,$$

as can be easily verified by inspection. The other possible choice would have implied increasing « twist » within the multiplet giving nonleading terms on the light-cone.

Obviously we must have at least $k_{-1} = 0$ so that in such a case $O_0(x_+, x_-)$ is the lowest element of the multiplet, *i.e.* has dimension l and spin zero, however in general the multiplet can begin also at a higher spin and in that case we would have $k_n = 0$ for n less than some fixed n'.

Note again that every subset of the O's, starting with the lowest one (say O_0) is by itself a representation of the same kind (*i.e.* transforms as in eqs. (C.2)-(C.4)): this is the origin of the name of these representation since they are reducible (there exists a hierarchy of invariant subspaces each spanned by a subset of O's) but not completely reducible.

The light-cone expansion can easily be derived for such multiplets.

Firstly, one notes that for every O_n new operators can be defined that are annihilated by L_{-1}^+

$$\tilde{O}_n(x_+x_-) = \sum_{i=1}^{n} c_{n-i}^n (\partial_+)^{n-i} O_i(x_+x_-) , \tag{C.5}$$

$$[L_{-1}^+, \tilde{O}_n(0)] = 0 , \tag{C.6}$$

The constants c_{n-i}^n are recursively fixed by eq. (C.6):

$$c_{n-i}^n = \frac{(-1)^{n-i}}{(n-i)!} \frac{\Gamma(2d_n)}{\Gamma(2d_n + n - i)} K(n, i), \tag{C.7}$$

$$K(n, i) = \prod_{J=0}^{n-i} k_J . \tag{C.8}$$

Equation (C.7) follows from the defining commutators with the help of the formula

$$[L_{-1}^+, (\partial_+)^h O_n(0)] = h(2d_n + h - 1)(\partial_+)^{h-1} O_n(0) + k_{n-1}(\partial_+)^h O_n(0) , \tag{C.9}$$

which can be proved by induction.

$\tilde{O}_n$ is then an operator of the same kind as that for which the expansion has been derived, so that, if A and B are «normal» (irreducible), one has

$$\text{(C.10)}\quad \begin{cases} A(x_+x_-)\,B(0) \underset{x_-\to 0}{\sim} (x_+)^{-\frac{1}{2}(d_A+d_B-d_n)}(x_-)^{-\frac{1}{2}(\tau_A+\tau_B-\tau_n)}\int\limits_0^1 \mathrm{d}\lambda f(\lambda)\tilde{O}_n(\lambda x_+, 0)\,, \\ f(\lambda) = \lambda^{\frac{1}{2}(d_A-d_B+d_n)-1}(1-\lambda)^{\frac{1}{2}(d_B-d_A+d_n)-1}\,. \end{cases}$$

By partial integration the derivatives in O_n can be transferred to act on $f(\lambda)$; the «surface» terms do not contribute since

$$f(0) = f(1) = f'(0) = f'(1) = \dots = 0\,.$$

The explicit calculation is lengthy but straightforward and gives

$$\text{(C.11)}\quad A(x_+x_-)B(0) \underset{x_-\to 0}{\simeq} (x_+)^{-\frac{1}{2}(d_A+d_B-d_n)}(x_-)^{-\frac{1}{2}(\tau_A+\tau_B-\tau_n)}\cdot$$
$$\cdot\sum_{i=1}^n c^n_{n-i}x_+^{i-n}\,\frac{\Gamma\big((d_A-d_B+d_n)/2\big)}{\Gamma\big((d_A-d_B+d_n)/2+i-n\big)}\int\limits_0^1 \mathrm{d}\lambda\,\psi_i(\lambda)O_i(\lambda x_+, 0)\,,$$

where

$$\text{(C.12)}\quad \psi_i(\lambda) = \lambda^{\frac{1}{2}(d_A-d_B+d_n)-1}(1-\lambda)^{\frac{1}{2}(d_B-d_A+d_n)+i-n-1}\cdot$$
$$\cdot\,{}_2F_1\left(i-n, 1-\frac{1}{2}(d_A-d_B+d_n); \frac{1}{2}(d_B-d_A+d_n)+i-n; \frac{\lambda-1}{\lambda}\right) =$$
$$= (1-\lambda)^{\frac{1}{2}(d_B-d_A+d_n)+i-n-1}\cdot$$
$$\cdot\,{}_2F_1\left(\frac{1}{2}(d_B-d_A+d_n), 1-\frac{1}{2}(d_A-d_B+d_n); \frac{1}{2}(d_B-d_A+d_n)+i-n; 1-\lambda\right).$$

The first form in eq. (C.12) exhibits the fact that the ${}_2F_1$-function is really a polynomial in $(\lambda-1)/\lambda$, while the second form can possibly be used to continue in n (infinite-dimensional representations).

Note added in proofs.

A similar amplitude as that discussed in Sect. 4 has been introduced by A. M. POLYAKOV in a recent preprint (A. M. POLYAKOV, Chernogolovka, 1973).

● RIASSUNTO

Si discutono le funzioni di Green covarianti conformi e gli sviluppi operatoriali per più di due operatori. Si studiano le proprietà di analiticità e si danno espressioni asintotiche. Si costruisce la generalizzazione per spazio-tempo a D dimensioni, mettendo così in luce il ruolo peculiare di $D=2$, sia da un punto di vista gruppale sia in relazione a sistemi di riferimento di impulso infinito. Si discute il ruolo delle singolarità ombra in relazione alla metrica euclidea.

Аналитические свойства и асимптотические разложения конформно-ковариантных. функций Грина.

Резюме (*). — Обсуждаются конформно-ковариантные функции Грина и разложения операторных произведений более чем двух операторов. Исследуются аналитические свойства и выводятся асимптотические выражения. Исследуется расширение в D-мерном пространстве-времени и отмечается особая роль $D=2$, как с точки зрения теории групп, так и в системах бесконечного импульса. Обсуждается роль теневых сингулярностей в связи с эвклидовой метрикой.

(*) *Переведено редакцией.*