ON LIFTS OF EXTREMAL QUASICONFORMAL MAPPINGS*

By

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Introduction

1. Let R and S be compact Riemann surfaces and let f be an extremal quasiconformal mapping of R onto S. Let $\hat{f}$ be a quasiconformal mapping of the universal covering surface $\hat{R}$ of R onto the universal covering surface $\hat{S}$ of S which is a lift of f. If the genus g of R is one, which means that $\hat{R} = \hat{S} = \mathbf{C}$, an extremal quasiconformal mapping of $\hat{R}$ onto $\hat{S}$ is conformal. Therefore the lift $\hat{f}$ of f is extremal if and only if f is conformal. It has been an open problem what the situation is for $g \geqq 2$. Here, the universal covering surface is the disk, and a lift $\hat{f}$ of f induces a boundary homeomorphism which is invariant under simultaneous cover transformations. Is $\hat{f}$ extremal for its boundary values (without the group relation in the interior of the disk)? The question was asked by I. Kra (in a letter to E. Reich, Nov. 1973) and others. O. Lehto proved[†]: If the extremal problem on the disk, for the induced boundary homeomorphism, has a unique solution, then it actually is the lifted mapping; but there is no proof of uniqueness. In fact, the problem already came up in E. Blums thesis[‡]: He was able to prove tht the lift of a horizontal stretching of a doubly connected domain to its universal covering surface is extremal for its boundary values. His proof failed for triply connected domains.

I believe that, for compact surfaces, the lift is never extremal except for the trivial case of a conformal mapping. In this paper, an extremal mapping of a compact surface of genus two is constructed and it is shown that its lift is not extremal for its boundary values. We work with a special two sheeted model of the surface R, with certain identifications, and with an extremal mapping f which is a horizontal stretching. This mapping is lifted to a planar covering surface $\tilde{R}$ of infinite

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† Group isomorphisms induced by quasiconformal mappings, in *Contributions to Analysis* (1974), 241–244.

‡ Die Extremalität gewisser Teichmüllerscher Abbildungen des Einheitskreises. Comment. Math. Helv. 44 (1969), 319–340.

connectivity. The lift is called $\tilde{f}$. A piecewise linear variation $\tilde{f}_1$ of $\tilde{f}$ is now constructed with a maximal dilatation $\tilde{K}_1 < \tilde{K} = K$. Lifting both mappings, $\tilde{f}$ and $\tilde{f}_1$, to the universal covering surface of $\tilde{R}$, which is the same as $\hat{R}$, gives the desired counterexample. A similar construction is possible for any genus $g \geqq 2$.

2. Somehow the simplest example of a lift which is not extremal for its boundary values is provided by the once punctured torus $\dot{R}$. One can work with the universal covering surface of the torus R, which is the plane. The puncture of R is lifted into a lattice consisting of the vertices of a fundamental polygon and their translates. The universal covering surface of $\dot{R}$ is represented by the universal covering surface of the plane punctured at the points of the lattice. Here, the construction of a piecewise linear variation $\tilde{f}$ of $\hat{f}$ becomes particularly simple and transparent. We therefore start with this example of a non compact surface.

First example: once punctured torus

3. Let R and S be tori. To simplify the construction we assume that R can be represented by the unit square $R_0 : 0 \leqq x < 1$, $0 \leqq y < 1$, in the $z = x + iy$ plane with opposite sides identified. Similarly, let S be represented by a rectangle $S_0 : 0 \leqq u < K$ $(K > 1)$, $0 \leqq v < 1$ in the $w = u + iv$ plane. The once punctured tori are denoted by $\dot{R}$, $\dot{S}$ resp., and the punctures are supposed to correspond to the four vertices of R_0 resp. S_0. The horizontal stretching f of R_0 by the factor K evidently is an extremal quasiconformal mapping of $\dot{R}$ onto $\dot{S}$, since it is extremal for the torus and maps $\dot{R}$ onto $\dot{S}$.

The universal covering surface $\hat{\dot{R}}$ of $\dot{R}$ is constructed in the following way: To each side of the square R_0 is attached a new square. Each of the four new squares has three free sides, and to every one of these sides we attach a new square, a.s.f. It is important that we always use new squares: We get a simply connected relatively unbounded and unbranched covering surface of the plane punctured at the points $\mu + i\nu$, $\mu, \nu \in \mathbf{Z}$. A similar construction leads to the universal covering surface $\hat{\dot{S}}$ of $\dot{S}$.

The lift $\hat{f}$ of the extremal mapping f is again the horizontal stretching f (of $\hat{\dot{R}}$ onto $\hat{\dot{S}}$). We now construct a variation $\tilde{f}$ of f in the following way: We start with any one of the squares and its image, say R_0, S_0. Instead of mapping R_0 onto S_0, we map it onto a polygon arising from S_0 by attaching two triangles along the horizontal sides and cutting out two triangles along the vertical sides (see Fig. 1, dotted lines).

The boundary correspondence on every side is made linear. Both types of changes will have the same effect, namely a lowering of the dilatation.

Let now R_1, R_2 be attached squares, along a vertical resp. horizontal side, with S_1 and S_2 the corresponding rectangles. The change we have made causes a bad heritage: S_1, which is attached to S_0 along a vertical side, is stretched a little bit

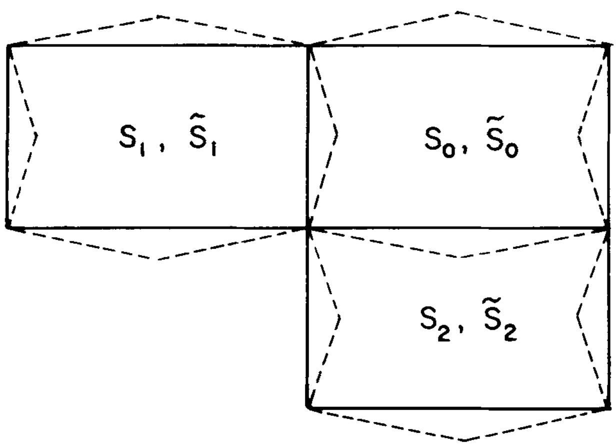

Fig.1

more, whereas S_2, which is attached to S_0 along a horizontal side, is pressed from the side a little bit. Both changes have a magnifying effect for the maximal dilatation. But we have three sides left to make up for it. Therefore, apart from the first rectangle S_0, we evidently have to consider the following two cases:

A) Make worse on one vertical side, improve on the three other sides (Fig. 1, S_1).

B) Make worse on a horizontal side, improve on the three other sides (Fig. 1, S_2).

Each of the rectangles attached to one of the three free sides now inherits a loss (= increase of the maximal dilatation, namely on the attached side) but can compensate it on the three remaining sides, with new rectangles attached.

We only consider mappings of type A) and B), leaving the case $R_0 \to S_0$ aside, which of course follows along the same lines. But it is in fact not necessary to construct this mapping, as with a slight change of the construction we can as well start with a mapping of type $R_1 \to S_1$. Only some of the symmetry is lost.

We have to show that the mappings induced by $\hat{f}$ and $\tilde{f}$ on the unit disk have the same boundary values. Let $h : \hat{S} \to |\zeta| < 1$ be the conformal mapping of the universal covering surface onto the unit disk. The rectangle S_0, taken half open without the four vertices (or likewise any other rectangle of the lattice) covers the surface S exactly once. It is mapped onto a fundamental domain $h(S)$ of the group of cover transformations. Such a fundamental domain is bounded by four circles orthogonal to the circumference $|\zeta| = 1$. The Euclidean diameters of any sequence of pairwise different fndamental domains evidently tend to zero.

Let now $\zeta_n \to \zeta_0$ ($|\zeta_0| = 1$) be a convergent sequence of points of the unit disk, with $\tilde{\zeta}_n$ their variations. Let P_n, $\tilde{P}_n$ be the corresponding points on $\hat{S}$. $\tilde{P}_n$ lies in the same rectangle as P_n, or else in one of the four rectangles attached to it along its sides. One concludes:

If a certain fundamental domain, say $h(S_0)$, contains infinitely many points ζ_n, the points $\tilde{\zeta}_n$ corresponding to this subsequence evidently tends to the same boundary point ζ_0. But then this must be true for the whole sequence $(\tilde{\zeta}_n)$, as the correspondence is quasiconformal. If every fundamental domain contains only a finite number of points ζ_n, the diameter of the fundamental domains $h(S_n) \ni \zeta_n$ tends to zero. The same is true for the diameter of the sets consisting of $h(S_n)$ and the fundamental domains attached to its four sides. Thus $\tilde{\zeta}_n \to \zeta_0$, q.e.d.

In the second example below the surface S is compact, and the same is true for the closure of any fundamental domain. Therefore only the second of the above two cases can happen. Again the two mappings coincide on $|\zeta| = 1$.

4. In order to get piecewise affine mappings we will subdivide the polygons into quadrangles and these into triangles. The triangles can be mapped by affine mappings

$$w = pz + q\bar{z} + c,$$

which gives a complex dilatation of the form

$$\kappa = \frac{q}{p} = k\frac{1 + \delta a}{1 + \delta b} = k(1 + \delta(a - b) + O(\delta^2)).$$

Here, a and b are constants depending only on K and some fixed parameter of the construction, while δ is a small positive parameter. If we can show that $\Delta = \mathrm{Re}(a - b) < 0$, the ray $k(1 + \delta(a - b))$ cuts the circle $|z| = k$, and therefore $|\kappa| < k$ for all sufficiently small $\delta > 0$.

5. Mapping of a square onto a polygon of type A

The heights of the triangles are chosen to be $\delta > 0$ resp. $\eta = \delta/K$. We claim that for every sufficiently small δ there is a piecewise affine mapping of the unit square onto the polygon $\tilde{S}_1$ (Fig. 2, dotted line) with a maximal dilatation $\tilde{K} < K$.

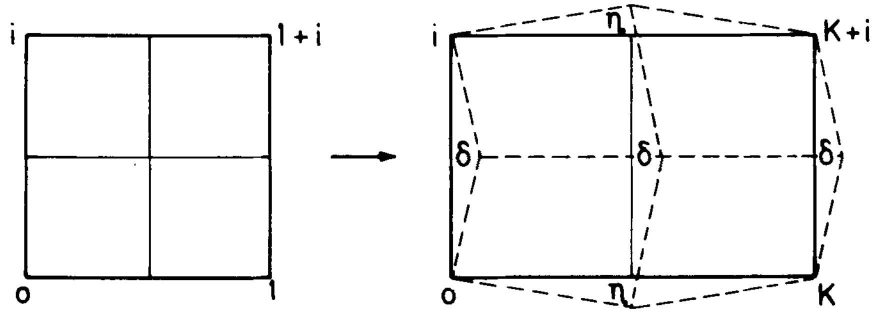

Fig. 2

Proof. Because of the symmetry of the configuration we only have to consider the two upper quadrangles.

a) *Upper left quadrangle* (magnified by 2):

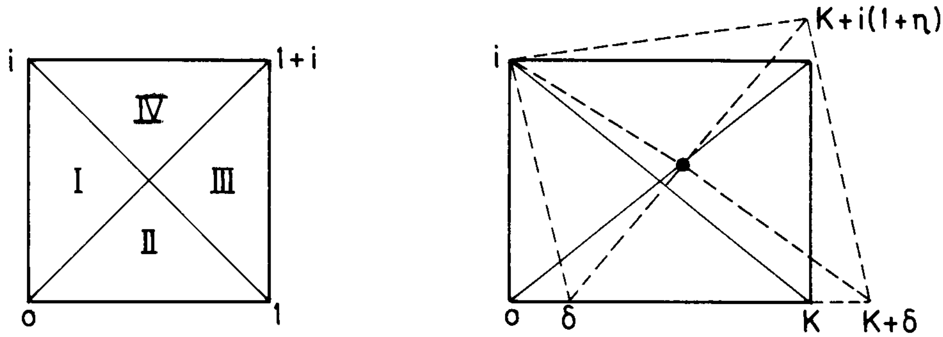

Fig. 3

As new center (common vertex of the four triangles) we choose the point $\frac{1}{2}K+\frac{1}{2}\delta(1-\varepsilon)+i(\frac{1}{2}+\frac{1}{4}\eta)$, $0<\varepsilon<1$. The complex dilatations and the crucial numbers Δ now become in turn

I) $$\kappa=\frac{K-1-\varepsilon\delta-i\delta(1-1/2K)}{K+1-\varepsilon\delta+i\delta(1+1/2K)},\qquad \Delta=-\frac{2\varepsilon}{K^2-1}$$

II) $$\kappa=\frac{K-1-\delta/2K-i\delta(1+\varepsilon)}{K+1+\delta/2K+i\delta(1+\varepsilon)},\qquad \Delta=-\frac{1}{K^2-1}$$

III) $$\kappa=\frac{K-1+\delta(\varepsilon-1/K)-i\delta(1-1/2K)}{K+1+\delta(\varepsilon+1/K)+i\delta(1+1/2K)},\qquad \Delta=-\frac{2(1-\varepsilon)}{K^2-1}$$

IV) $$\kappa=\frac{K-1-\delta/2K+i\delta(1/K-1+\varepsilon)}{K+1+\delta/2K+i\delta(1/K+1-\varepsilon)},\qquad \Delta=-\frac{1}{K^2-1}.$$

b) *Upper right quadrangle* (magnified by factor 2):

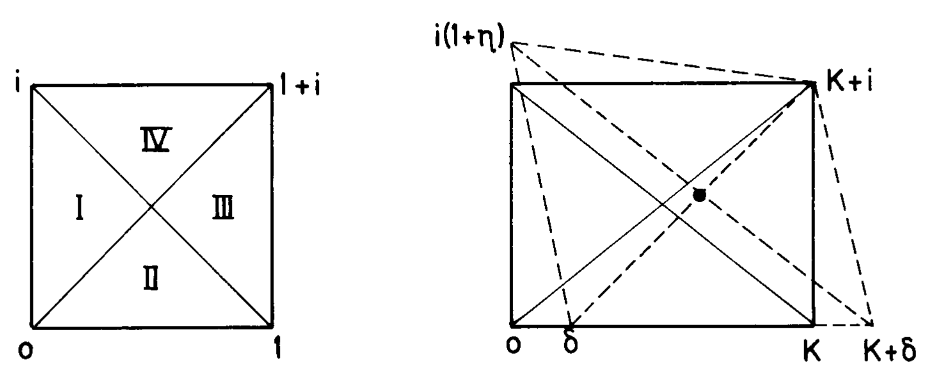

Fig. 4

The new center is chosen to be $\frac{1}{2}K + \frac{1}{2}\delta(1+\varepsilon) + i(\frac{1}{2} + \frac{1}{4}\eta)$. We now get for the different triangles

I) $$\kappa = \frac{K - 1 + \delta(\varepsilon - 1/K) - i\delta(1 + 1/2K)}{K + 1 + \delta(\varepsilon + 1/K) + i\delta(1 - 1/2K)}, \qquad \Delta = -\frac{2(1-\varepsilon)}{K^2 - 1}$$

II) $$\kappa = \frac{K - 1 - \delta/2K - i\delta(1-\varepsilon)}{K + 1 + \delta/2K + i\delta(1-\varepsilon)}, \qquad \Delta = \frac{-1}{K^2 - 1}$$

III) $$\kappa = \frac{K - 1 - \varepsilon\delta - i\delta(1 + 1/2K)}{K + 1 - \varepsilon\delta + i\delta(1 - 1/2K)}, \qquad \Delta = \frac{-1}{K^2 - 1}$$

IV) $$\kappa = \frac{K - 1 - \delta/2K - i\delta(1 + \varepsilon + 1/K)}{K + 1 + \delta/2K + i\delta(1 + \varepsilon - 1/K)}, \qquad \Delta = \frac{-1}{K^2 - 1}.$$

6. Mapping of a square onto a polygon of type B

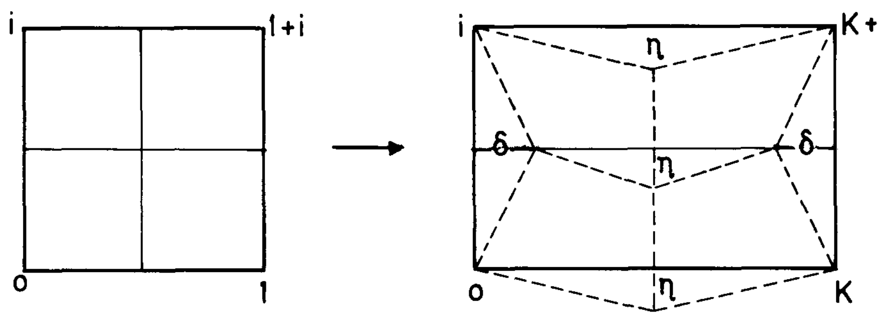

Fig. 5

In this case we have a vertical line of symmetry and thus only have to consider the two left hand quadrangles.

a) *Upper left quadrangle* (magnified):

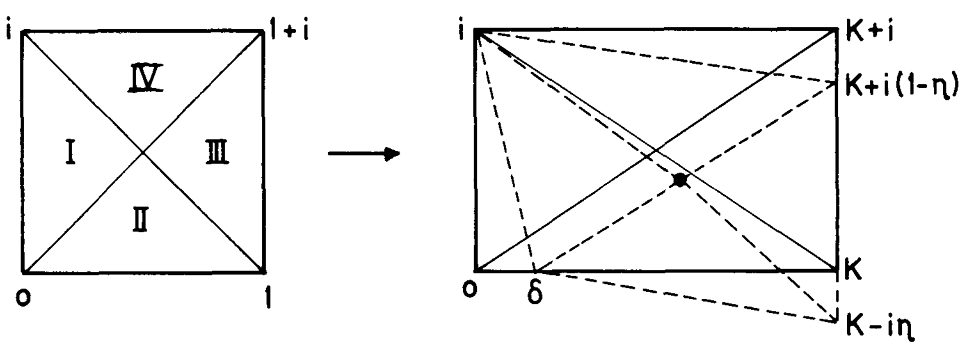

Fig. 6

The new center is $\frac{1}{2}K + \frac{1}{2}\delta(1-\varepsilon) + i(\frac{1}{2} - \frac{3}{4}\eta)$. We then get:

I) $$\kappa = \frac{K-1-\varepsilon\delta - i\delta(1+3/2K)}{K+1-\varepsilon\delta + i\delta(1-3/2K)}, \qquad \Delta = \frac{-2\varepsilon}{K^2-1}$$

II) $$\kappa = \frac{K-1-\delta(1-1/2K) - i\delta(\varepsilon+1/K)}{K+1-\delta(1+1/2K) + i\delta(\varepsilon-1/K)}, \qquad \Delta = \frac{-1}{K^2-1}$$

III) $$\kappa = \frac{K-1-\delta(1-\varepsilon) - i\delta/2K}{K+1-\delta(1-\varepsilon) - i\delta/2K}, \qquad \Delta = -\frac{1}{K^2-1}$$

IV) $$\kappa = \frac{K-1-\delta/2K - i\delta(1-\varepsilon+1/K)}{K+1+\delta/2K + i\delta(1-\varepsilon-1/K)}, \qquad \Delta = \frac{-1}{K^2-1}.$$

b) *Lower left quadrangle* (magnified):

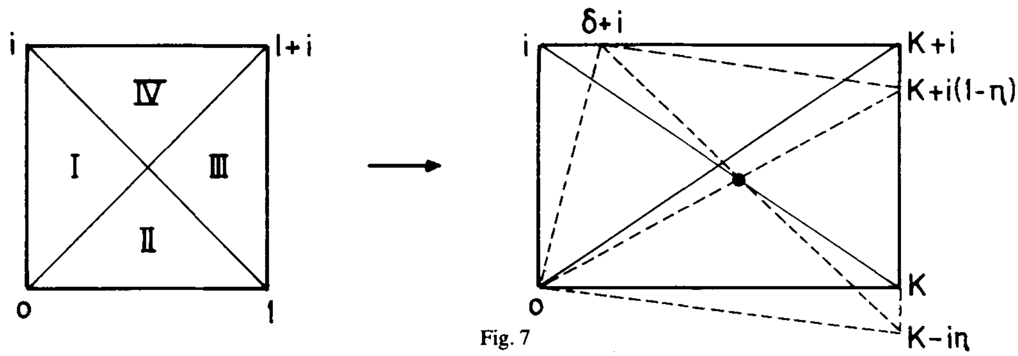

Fig. 7

The new center is $\frac{1}{2}K + \frac{1}{2}\delta(1-\varepsilon) + i(\frac{1}{2} - \frac{1}{4}\eta)$. We then get:

I) $$\kappa = \frac{K-1-\varepsilon\delta + i\delta(1-1/2K)}{K+1-\varepsilon\delta - i\delta(1+1/2K)}, \qquad \Delta = \frac{-1}{K^2-1}$$

II) $$\kappa = \frac{K-1-\delta/2K + i\delta(1-\varepsilon-1/K)}{K+1+\delta/2K - i\delta(1-\varepsilon+1/K)}, \qquad \Delta = \frac{-1}{K^2-1}$$

III) $$\kappa = \frac{K-1-\delta(1-\varepsilon) - i\delta 3/2K}{K+1-\delta(1-\varepsilon) - i\delta 3/2K}, \qquad \Delta = \frac{-1}{K^2-1}$$

IV) $$\kappa = \frac{K-1-\delta(1-1/2K) + i\delta(\varepsilon-1/K)}{K+1-\delta(1+1/2K) - i\delta(\varepsilon+1/K)}, \qquad \Delta = \frac{-1}{K^2-1}.$$

Choosing $\varepsilon = \frac{1}{2}$, we get $\Delta = -1/(K^2-1)$ in all cases. Thus, for all sufficiently small values of δ we have constructed a variation of the lift with a smaller maximal dilatation.

Second example: compact surface of genus two

7. Description of the model. We start with the rectangle $0 \leqq x < 3$, $-1 \leqq y \leqq 1$, slit along the real axis by two cuts of length one (see Fig. 8). This slit rectangle is sewn along its horizontal edges to another replica lying underneath: We thus get a flat tube, which splits into two tubes on both ends. These ends are then identified by a horizontal translation to get the desired surface R of genus 2. Of course one can introduce local conformal parameters. For interior points of the upper sheet the local homeomorphisms are projections to the underlying complex plane. At interior points of the lower sheet we take the projection with a subsequent reflection. At interior points of the horizontal edges we have to reflect the lower sheet on the edge and then use projection. Finally, at the two endpoints of the slits we first take the square root and then apply the same procedure.

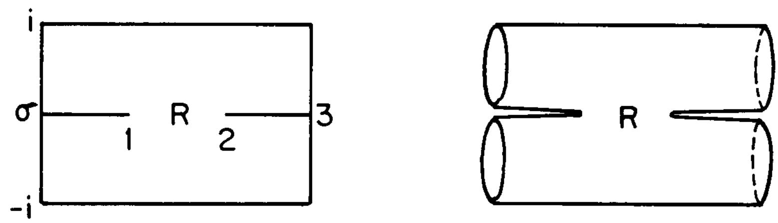

Fig. 8

In order to get the desired Schottky covering surface S we attach, instead of identification, new replicas of the model to its four vertical boundaries. Each of the four added pieces has three free vertical boundary curves, to each of which we attach a new replica a.s.f. The resulting surface evidently is a planar covering surface of R, since the whole construction can be performed in the plane if we start with the Riemann sphere with four circular holes.

A horizontal stretching of the rectangular surface R by a factor $K > 1$ onto a surface R' of the same kind is an extremal quasiconformal mapping of R onto R'. This can be shown in the usual way by the length area method. But it is also a consequence of the general uniqueness theorem and the fact that the constant $\varphi = 1$ is a holomorphic quadratic differential on R: The two endpoints of the slits are second order zeroes, as four horizontal trajectories are ending at these points; all the other points are non-critical.

8. Construction of a mapping. We are now going to construct a piecewise affine mapping $\tilde{f}$ of the covering surface S onto the corresponding Schottky covering surface S' of R', which has a maximal dilatation $\tilde{K} < K$. The basic idea is the same as in the first example: Let R_0 be the first rectangular piece of which S is composed. Instead of stretching it by the factor K onto the first rectangular piece R_0' of S', we map it onto a rectangular subsurface $\tilde{R}_0$ of R_0' by pushing in each of its four boundary curves. Each of the four rectangular pieces $R_1, \cdots, R_4$ attached to R_0 along its four boundary curves has to be streched more at this particular boundary. But it has three other boundary curves to make up for it. It is evidently enough to construct a mapping $\tilde{f}_0$ for the following situation (Fig. 9):

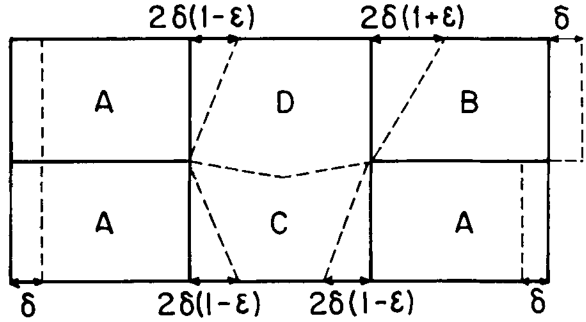

Fig. 9

(In fact, a mapping of S onto S' can be composed by mappings of this kind, as is easily seen, but this construction is less symmetric.) The rectangle $0 \leqq x < 3$, $-1 \leqq y \leqq 1$ of Fig. 8 is mapped onto the hexagon of Fig. 9, bounded, at its two vertical ends, by the dotted lines. At the upper right end we have an extension by the quantity δ, whereas the three other ends are shrinked by the same amount. By the symmetry of the surface, the same construction can be applied to the lower sheet.

We now have to find mappings of the unit square onto quadrangles and pentagons of the four different types indicated in Fig. 9.

9. Mapping of a square onto a quadrangle of type A

(Figs. 9 and 10):

We claim: For every $K > 1$ and $0 < \varepsilon < 1$ there exists a number $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ the square can be mapped onto the quadrangle of Fig. 10 (dotted line) by a piecewise affine mapping with a dilatation $\tilde{K} < K$.

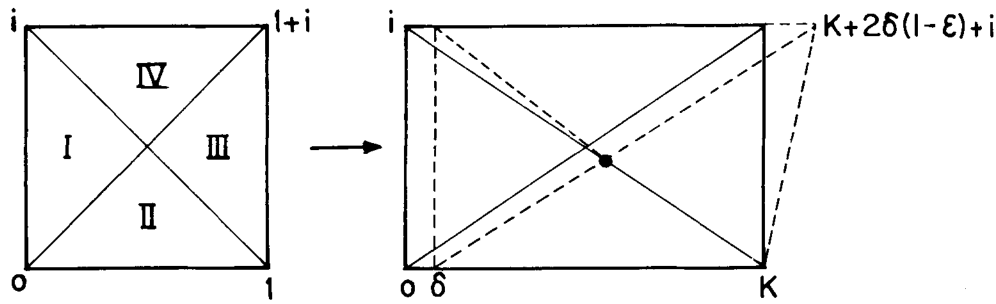

Fig. 10

Proof. We choose the new center (common vertex of the four triangles) to be $\frac{1}{2}K+\delta(1-\frac{1}{2}\varepsilon)+i(\frac{1}{2}-\eta)$, where $\eta>0$ will be determined later. Now the mappings of the four triangles have the following complex dilatations κ and variational coefficients Δ:

I) $$\kappa=\frac{K-1-\varepsilon\delta-2i\eta}{K+1-\varepsilon\delta-2i\eta},\qquad \Delta=-\frac{2\varepsilon}{K^2-1}$$

II) $$\kappa=\frac{K-1-\delta+2\eta+i\delta(1+\varepsilon)}{K+1-\delta-2\eta-i\delta(1+\varepsilon)},\qquad \Delta\delta=\frac{4K\eta-2\delta}{K^2-1}$$

III) $$\kappa=\frac{K-1-\delta\varepsilon+i2(\eta+\delta(1-\varepsilon))}{K+1-\delta\varepsilon+i2(\eta-\delta(1-\varepsilon))},\qquad \Delta=-\frac{2\varepsilon}{K^2-1}$$

IV) $$\kappa=\frac{K-1+\delta-2\varepsilon\delta-2\eta-i\delta(1-\varepsilon)}{K+1+\delta-2\varepsilon\delta+2\eta-i\delta(1-\varepsilon)},\qquad \Delta\delta=\frac{-4K\eta+2\delta(1-2\varepsilon)}{K^2-1}.$$

Choosing $\eta=\frac{1-\varepsilon}{2K}\delta$, we get $\Delta=\frac{-2\varepsilon}{K^2-1}$ for all four triangles.

10. Mapping of a square onto a rectangle of type B

(Figs. 9 and 11):

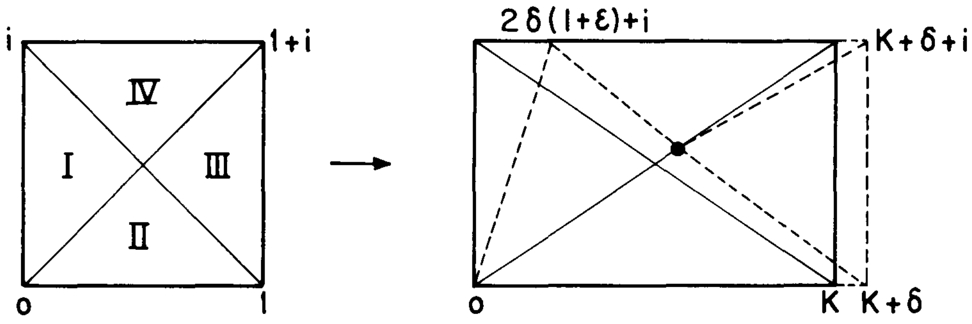

Fig. 11

With the new center $\frac{1}{2}K+\delta(1+\frac{1}{2}\varepsilon)+i(\frac{1}{2}+\eta)$ the mappings of the four triangles have the following complex dilatations and variational coefficients Δ:

I) $$\kappa=\frac{K-1-\varepsilon\delta+i2(\delta(1+\varepsilon)+\eta)}{K+1-\varepsilon\delta-i2(\delta(1+\varepsilon)-\eta)},\qquad \Delta=-\frac{2\varepsilon}{K^2-1}$$

II) $$\kappa=\frac{K-1+\delta-2\eta+i\delta(1+\varepsilon)}{K+1+\delta+2\eta-i\delta(1+\varepsilon)},\qquad \Delta\delta=\frac{-4K\eta+2\delta}{K^2-1}$$

III) $$\kappa=\frac{K-1-\varepsilon\delta-i2\eta}{K+1-\varepsilon\delta-i2\eta},\qquad \Delta=-\frac{2\varepsilon}{K^2-1}$$

IV) $$\kappa = \frac{K-1-\delta-2\varepsilon\delta+2\eta+i\delta(1+\varepsilon)}{K+1-\delta-2\varepsilon\delta-2\eta-i\delta(1+\varepsilon)}, \qquad \Delta\delta = \frac{4K\eta-2\delta-4\varepsilon\delta}{K^2-1}.$$

Putting $\eta = \frac{1+\varepsilon}{2K}\delta$, we get $\Delta = -\frac{2\varepsilon}{K^2-1}$ for all four triangles. We have proved that for all sufficiently small δ the complex dilatation of the mapping in Fig. 11 will have an absolute value smaller than k.

11. Mapping of a square onto a pentagon of type C

(Figs. 9 and 12): The configuration has a vertical axis of symmetry. We therefore only have to construct a mapping of the half square onto the left half of the pentagon.

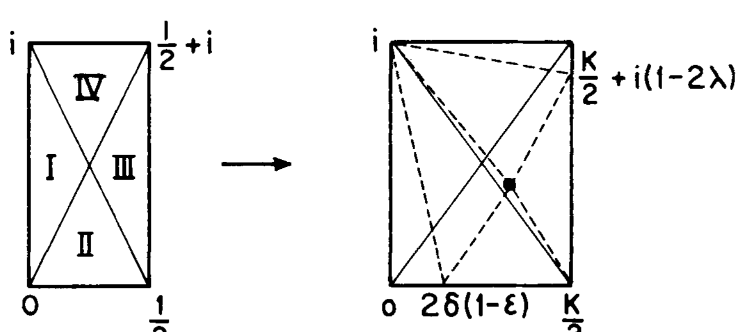

Fig. 12

As new center we choose $\frac{1}{4}K+\delta(1-2\varepsilon)+i\frac{1}{2}(\frac{1}{2}-\eta)$. We claim: With $\lambda = 2\delta(1-3\varepsilon)/K$, $\eta = 2\delta(1-2\varepsilon)/K$ there is, for all sufficiently small δ, a piecewise affine mapping with a maximal dilatation $\tilde{K} < K$.

Proof. The complex dilatations of the four affine mappings and their variational coefficients are as follows:

I) $$\kappa = \frac{K-1-4\varepsilon\delta-i2(\delta(1-\varepsilon)-2\eta)}{K+1-4\varepsilon\delta+i2(\delta(1-\varepsilon)-2\eta)}, \qquad \Delta = -\frac{8\varepsilon}{K^2-1}$$

II) $$\kappa = \frac{K-1-4\delta(1-\varepsilon)+2\eta-i2\delta\varepsilon}{K+1-4\delta(1-\varepsilon)-2\eta+i2\delta\varepsilon}, \qquad \Delta\delta = \frac{4K\eta-8\delta(1-\varepsilon)}{K^2-1}$$

III) $$\kappa = \frac{K-1-4\delta(1-2\varepsilon)+2\lambda+i4(\eta-\lambda)}{K+1-4\delta(1-2\varepsilon)-2\lambda+i4(\eta-\lambda)}, \qquad \Delta\delta = \frac{4K\lambda-8\delta(1-2\varepsilon)}{K^2-1}$$

IV) $$\kappa = \frac{K-1-2\eta+2\lambda+i2(\delta(1-2\varepsilon)-2\lambda)}{K+1+2\eta-2\lambda-i2(\delta(1-2\varepsilon)+2\lambda)}, \qquad \Delta\delta = \frac{-4K\eta+4K\lambda}{K^2-1}.$$

With the given choices of λ and η we get $\Delta = -8\varepsilon/(K^2-1)$ for all four triangles.

12. Mapping of a square onto a pentagon of type D

(Figs. 9 and 13):

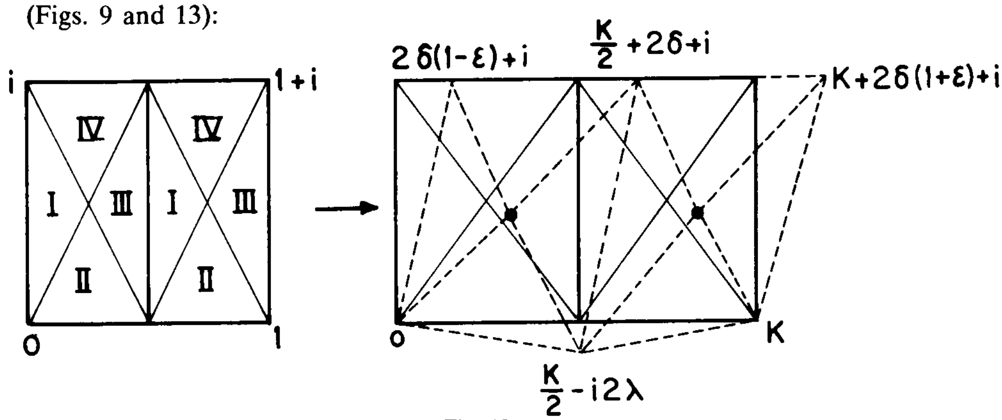

Fig. 13

The pentagon is split into two quadrangles. For the first part we choose the new center equal to $\frac{1}{4}K + \delta(1-2\varepsilon) + i(\frac{1}{2} - \eta)$. With this, we get the following complex dilatations and variational coefficients:

I) $$\kappa = \frac{K-1-4\delta\varepsilon + i2(\delta(1-\varepsilon)-2\eta)}{K+1-4\delta\varepsilon - i2(\delta(1-\varepsilon)-2\eta)}, \qquad \Delta = -\frac{8\varepsilon}{K^2-1}$$

II) $$\kappa = \frac{K-1+2\eta-2\lambda + i2(\delta(1-2\varepsilon)-2\lambda)}{K+1-2\eta+2\lambda - i2(\delta(1-2\varepsilon)+2\lambda)}, \qquad \Delta\delta = \frac{4K\eta-4K\lambda}{K^2-1}$$

III) $$\kappa = \frac{K-1+8\varepsilon\delta-2\lambda + i2(\delta+2\eta-2\lambda)}{K+1+8\varepsilon\delta+2\lambda - i2(\delta-2\eta+2\lambda)}, \qquad \Delta\delta = \frac{-4K\lambda+8\varepsilon\delta}{K^2-1}$$

IV) $$\kappa = \frac{K-1+4\varepsilon\delta-2\eta + i2\delta(1+\varepsilon)}{K+1+4\varepsilon\delta+2\eta - i2\delta(1+\varepsilon)}, \qquad \Delta\delta = \frac{-4K\eta+8\varepsilon\delta}{K^2-1}.$$

With $\lambda = 2\delta(1-3\varepsilon)/K$ as before, $\eta = \lambda/2 = \delta(1-3\varepsilon)/K$ we get the following four variational coefficients:

$$-\frac{8\varepsilon}{K^2-1}, \quad \frac{-4(1-3\varepsilon)}{K^2-1}, \quad \frac{-8(1-2\varepsilon)}{K^2-1}, \quad \frac{-4(1-5\varepsilon)}{K^2-1}.$$

For the second half of Fig. 13 we choose as new center the point $\frac{3}{4}K + \delta(1+2\varepsilon) + i(\frac{1}{2} - \eta)$. The complex dilatations and variational coefficients then become:

I) $$\kappa = \frac{K-1+8\varepsilon\delta-2\lambda + i2(\delta-2\eta+2\lambda)}{K+1+8\varepsilon\delta+2\lambda - i2(\delta+2\eta-2\lambda)}, \qquad \Delta\delta = \frac{-4K\lambda+16\varepsilon\delta}{K^2-1}$$

II) $$\kappa = \frac{K-1+2\eta-2\lambda+i2(\delta(1+2\varepsilon)+2\lambda)}{K+1-2\eta+2\lambda-i2(\delta(1+2\varepsilon)-2\lambda)}, \qquad \Delta\delta = -\frac{4K(\lambda-\eta)}{K^2-1}$$

III) $$\kappa = \frac{K-1-4\varepsilon\delta+i2(\delta(1+\varepsilon)+2\eta)}{K+1-4\varepsilon\delta-i2(\delta(1+\varepsilon)-2\eta)}, \qquad \Delta = \frac{-8\varepsilon}{K^2-1}$$

IV) $$\kappa = \frac{K-1+4\varepsilon\delta-2\eta+i\delta(2-\varepsilon)}{K+1+4\varepsilon\delta+2\eta-i\delta(2-\varepsilon)}, \qquad \Delta\delta = \frac{-4K\eta+8\varepsilon\delta}{K^2-1}.$$

Choosing $\eta = 2\delta(1-4\varepsilon)/(K^2-1)$ we get, with the earlier value $\lambda = 2\delta(1-3\varepsilon)/K$, for the variational coefficients the values

$$-\frac{8(1-5\varepsilon)}{K^2-1}, \frac{-8\varepsilon}{K^2-1}, \frac{-8\varepsilon}{K^2-1}, \frac{-8(1-5\varepsilon)}{K^2-1}.$$

Therefore, for $\varepsilon < 1/5$, we have constructed a piecewise affine mapping of the square onto the pentagon D with a maximal dilatation $\tilde{K}$ which is smaller than K for all sufficiently small δ.

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