Bell's Theorem and the EPR Paradox.

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1. – Early developments.

1·1. *The genesis.* – «Anyone who is not shocked by quantum theory has not understood it», once wrote Niels Bohr. The first shock that came from quantum physics was generated by the lack of separability of the quantum system from the measuring apparatus. Yet this lack of separability is understandable, since after all there is concrete physical interaction between the microsystem and the measuring apparatus. Instead, much more shocking is the lack of separability between two widely separated and noninteracting quantum systems, a feature that persists undiminished even when their relative separation becomes infinitely large.

1·1.1. The Einstein-Podolsky-Rosen argument. During the spring of 1935 Einstein, Podolsky and Rosen (henceforth referred as EPR) wrote their classic paper «*Can Quantum Mechanical Description of Physical Reality be considered complete?*» [1]. This seminal work has continued to sprout animated discussions and vigorous debates over the past five decades. According to a recent book devoted to the EPR paradox [2], at least 600 papers have been written on various aspects of the EPR argument. In this review we make an attempt to provide an up-to-date overview of the germane and significant studies, starting from the original formulation of the argument.

The key ingredient in the EPR paper was their *reality criterion*: «If, without in any way disturbing a system, we can predict with certainty (*i.e.*, with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity». We begin by analysing the meaning of this criterion with reference to the particular situation dealt with in the EPR treatment.

Position in quantum mechanics is described by a linear Hermitian operator Q whose action on the wave function is equivalent to its multiplication by the position parameter. The eigenvalue equation

$$Q\,u(q;\,x) = q\,u(q;\,x)$$

is solved by an arbitrary real value of q with the corresponding eigenfunction (normalized on unit length in the wave number space)

$$u(q;\,x) = \delta(x-q) \tag{1.1}$$

which is the Dirac δ-function. The wave function (1.1) predicts a fixed position

q, with all possible values of momentum having equal probability (since $\delta(x-q)$ can be written as a superposition of all possible plane waves with constant weight factor). If the EPR reality criterion is applied to the wave function (1.1), it follows that an element of physical reality corresponds to the predicted value q of position which belongs to the physical system described by (1.1).

Considering next the momentum operator

$$P = -i\hbar\frac{\partial}{\partial x},$$

the eigenvalue equation

$$Pv(p;x) = pv(p;x)$$

is solved by an arbitrary real value of p with the corresponding eigenfunction (plane wave, normalized on unit length)

$$v(p;x) = \exp[ipx/\hbar] \tag{1.2}$$

which leads to an exact prediction of momentum but with a completely undefined position, in the sense that all conceivable positions can be found with equal probability. Application of the EPR reality criterion to the wave function (1.2) leads to the attribution of an element of physical reality corresponding to the fixed value p of momentum. The notations $u(q;x)$ and $v(p;x)$ show then explicitly the objective physical properties q and p of the respective wave functions at one particular time, say t_0.

At this stage it is important to emphasize that on the basis of the EPR reality criterion it is not possible to attribute to a single quantum system the two elements of reality corresponding to P and Q simultaneously. This is of course consistent with the nonvanishing commutator of P and Q: $[P, Q] = i\hbar$. It can be presumed that when Q is measured and a definite value is obtained, the element of reality corresponding to P is destroyed. It seems also natural to assume that this destruction is brought about by the inevitable *disturbance*, or the action quanta exchanged between the macroscopic measuring apparatus and the observed quantum system. Viewed in this way, ascribing the elements of reality to quantum systems becomes a rather innocuous exercise in conformity with the formalism of quantum mechanics.

This state of affairs, however, changes dramatically when two correlated quantum objects (α and β) are considered. Possible fixed-time wave functions for the system $\alpha+\beta$ are

$$\Phi(q_0; x_1, x_2) = \int \mathrm{d}q' c(q')\, u_\alpha(q'; x_1)\, u_\beta(q_0+q'; x_2), \tag{1.3}$$

$$\tilde{\Phi}(p_0; x_1, x_2) = \int \mathrm{d}p' \tilde{c}(p')\, v_\alpha(p'; x_1)\, v_\beta(p_0-p'; x_2), \tag{1.4}$$

where the notation for fixed-position and fixed-momentum wave functions is the same as before, the only change being the specification of the quantum system (α or β) to which they refer.

The standard interpretation of $\Phi(q_0; x_1, x_2)$ is as follows: a position measurement on α will yield the result q' with probability $|c(q')|^2$; if q' has been found for α, then it can be predicted with certainty that a position measurement for β will give the result $q_0 + q'$. In other words, correlated position measurements made on α and β will lead to results whose difference equals q_0 with probability equal to unity. It can then be inferred that there is an element of reality corresponding to q_0 which belongs to $(\alpha + \beta)$.

A similar argument applied to $\tilde{\Phi}$ leads to the conclusion that one can ascribe an element of reality to the sum p_0 of the momenta belonging to $(\alpha + \beta)$.

Now it needs to be noted that the simultaneous attribution of q_0 and p_0 to a pair of quantum systems is not prohibited by the quantum formalism – unlike the case of q and p for a single quantum system – since the difference of positions and the sum of momenta are represented by commuting operators:

$$[Q_\alpha - Q_\beta, P_\alpha + P_\beta] = 0. \tag{1.5}$$

This fact was used by Einstein, Podolsky and Rosen to formulate their example. They considered the wave function given by

$$\Psi(q_0, p_0; x_1, x_2) = \frac{1}{h}\int \mathrm{d}p' \exp[i(x_1 - x_2 + q_0)p'/\hbar] \tag{1.6}$$

which can be written in the form (1.4) with $p_0 = 0$ and with $\tilde{c}(p') = h^{-1}$ (apart from a constant phase factor). It can also be written in the form (1.3) (with $c(q') = 1$):

$$\Psi(q_0, p_0; x_1, x_2) = \delta(x_1 - x_2 + q_0) = \int \mathrm{d}q' \delta(x_1 - q')\,\delta(q' - x_2 + q_0). \tag{1.7}$$

Consequently, one can invoke the EPR criterion of physical reality and conclude that there is an element of physical reality corresponding to the position of β. EPR presumed that this element of physical reality exists regardless of whether or not a measurement on α has been made because otherwise one would have to invoke *spooky* action at a distance to contend that the measurement on α creates instantaneously the element of reality corresponding to the spatially separated system β. It is therefore concluded that there corresponds an element of reality to the position of β for all pairs of the entire ensemble E.

A similar argument can be made for momenta: taking a subset E_2 of E, let us consider momentum measurement on all the α's of E_2; $p'_1, p''_1, \ldots$ denote the obtained results. Since it can be predicted with certainty that subsequent measurements of the momentum of β will give $-p'_1$ for the first pair, $-p''_1$ for the second pair and so on, it can be concluded that an element of reality corresponds to the momentum of β for all β's of E_2. Excluding the possibility that this element of reality could be created instantaneously by the measurements made on α, one can then extend the above conclusion to the entire ensemble E (also the standard 'principle of induction' enters here).

Obviously, the choice of the system (α or β) on which measurements are performed is arbitrary. A symmetrical reasoning therefore leads to the inference that there are also simultaneous elements of reality corresponding to the position and momentum of the particle α. The upshot of the entire argument is that individual positions and momenta can be considered to be in a sense 'real' before measurements for all objects (α and β) comprising the ensemble E. The underlying sense being that there exist some inherent elements of physical reality associated with α and β that lead necessarily to preassigned results if and when a measurement of one or the other of the two observables is made.

Since the wave function (1.6) implies that these quantities are *a priori* indeterminate, EPR concluded that the description of the physical reality provided by (1.6) is not complete. The EPR paper was, thus, in essence an argument claiming the incompleteness of the existing quantum theory. It is interesting that EPR themselves never used the term paradox in their paper (which was, incidentally, also unique in that it did not contain a single reference).

A little known but curious historical fact is that an embryo of the EPR argument can be found in an earlier paper by von Weizsäcker in 1931 [3]. Weizsäcker was analysing Heisenberg's gamma-ray microscope thought experiment in the case of a photon being scattered by an electron, the initial momenta of both being assumed to be known before their collision. By measuring the momentum of the scattered photon one could therefore infer the final momentum of the electron after the collision. Alternatively, one could ascertain the position of the electron at the moment of its collision with the photon by directing the scattered photon to the image plane of an optical system. This foreshadows at least the spirit of the EPR argument; however, von Weizsäcker did not elaborate its conceptual implications. Max Jammer [4] has referred to this as an example of «how often in the history of science, a slight critical turn may open a new vista with far-reaching consequences».

1·1.2. Bohr's response. The EPR paper marked the climax of the historic Bohr-Einstein exchanges. Reflecting on the impact of the EPR paper Bohr [5] had remarked: «Due to the lucidity and apparently incontestable character of their argument, the EPR paper created a stir among the physicists». Rosenfeld, Bohr's close associate at that time, recalls [6]: «This onslaught came down upon us as a bolt from the blue... A new worry could not have come at a less propitious time. Yet, as soon as Bohr had heard my report of Einstein's arguments, everything else was abandoned».

In spite of different claims it is not too difficult to comprehend Bohr's ideas, the cautiousness of his writings notwithstanding. The paper which he wrote as an answer to EPR [7] contains a rather interesting analysis. To put it succinctly, Bohr did not question the correctness of the EPR reasoning once all its premises were accepted. But it is precisely these premises which Bohr refused to accept. His contention was that the assumptions underlying the EPR argument could be invalidated within the framework of *complementarity* which was for him «a new feature of natural philosophy». The notion of *complementarity* implies

a) renouncement of the classical idea of causality, and

b) radical revision of our concept of physical reality.

In order to grasp the crux of Bohr's objection to the EPR argument, it is first necessary to clarify the roots of *complementarity* which can be expressed by the assertion that it is in principle impossible to describe the processes in quantum mechanics in a realist way as developing causally in space and time. By *causal description* Bohr meant the process to be described according to well-defined rules among which he considered the most important one being the law of conservation of energy and momentum. In quantum mechanics the two possibilities of space-time description and causality are seen to be mutually compatible because position and momentum observables are represented by noncommuting operators. The measurement of one of them necessarily destroys all previous knowledge pertaining to the other one. It is this peculiar feature of quantum formalism which led Bohr to his conclusion that acts of observation for any two noncommuting operators have to be considered as mutually exclusive. One can then apply this reasoning to the analysis of the EPR argument based on the wave function (1.6).

Let us consider two apparatuses Q_1 and P_1 (Q_2 and P_2) performing, respectively, position and momentum measurements on the system α (β). If one chooses to use Q_1 and Q_2, the wave function (1.6) predicts that the results x_1 and x_2 will be precisely correlated: $x_2 - x_1 = q_0$. If instead one chooses to use P_1 and P_2, there is a precise correlation between the results p_1 and p_2: $p_1 + p_2 = 0$. The crucial point here is that the two apparatuses Q_1 and P_1 are mutually incompatible: one can choose to employ either Q_1 or P_1, but never the two of them simultaneously; the same holds for Q_2 and P_2. From this point of view one cannot therefore conclude that position and momentum correspond to two simultaneously existing elements of physical reality, because it is not possible to perform simultaneous measurements of position and momentum. According to Bohr, the expression «without in any way disturbing a system» as used by EPR in their criterion of reality contains «an essential ambiguity» because the conditions (or the experimental arrangements) defining the possible types of prediction concerning a system constitute an inseparable part of the phenomenon (unanalysable whole) to which the term *physical reality* can be associated («no elementary quantum phenomenon is a phenomenon until it is registered»). Since these conditions depend on whether Q_1 or P_1 is being rmeasured, the EPR conclusion appears to be unjustified.

In fact, EPR had themselves anticipated in their paper the possibility of such a refutation. They wrote: «Indeed, one would not arrive at our conclusion if one insisted that one or more physical quantities can be regarded as simultaneous elements of reality only when they can be simultaneously measured or predicted. From this point of view, since either one or the other, but not both simultaneously, of the quantities P and Q can be predicted, they are not simultaneously real. This makes the reality of P and Q depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality could be expected to do this».

The crux of the debate, therefore, boils down to the irreconciliability between the EPR reality criterion and Bohr's tenet that a particle on which no measurement is being performed cannot be considered as an independent seat of *physically real* attributes.

The completeness of quantum-mechanical description, challenged by EPR, was thus saved according to Bohr's view by the feature of *wholeness*. This is the way he himself summarized his position [8]: «The unambiguous account of proper quantum phenomena must, in principle, include a description of all relevant features of the experimental arrangement ... the feature of wholeness typical of proper quantum phenomena finds its logical expression in the circumstance that any attempt at a well-defined subdivision would demand a change in the experimental arrangement incompatible with the definition of the phenomena under investigation». Here we should add the remark that Bohr's reaction to the EPR paper was particularly significant from the viewpoint of the philosophy of science because it signalled the revival of that notion of *unanalysable wholeness* which had no place in the rationalistic approach of classical physics.

It is relevant to note that many leading contemporaries of Einstein, such as Max Born, had considerable difficulty in grasping the essence of the EPR argument. Here is Born's summing up which clearly reflects his erroneous understanding of the EPR paradox: «The root of the difference between Einstein and me was the axiom that the events which happen in different places A and B are independent of one another, in the sense that an observation on the state of affairs at B cannot teach us anything about the state of affairs at A» [9]. It is interesting to recall Bell's comment on it: «Misunderstanding could hardly be more complete. Einstein had no difficulty accepting that affairs in different places could be correlated. What he could not accept was that an intervention at one place could *influence*, immediately, affairs at the other. These references to Born are not meant to diminish one of the towering figures of modern physics. They are meant to illustrate the difficulty of putting aside preconceptions and listening to what is actually being said» [10].

It needs to be stressed that the EPR idea of separability hinges on the notion that two objects which are sufficiently separated in space (in such a way that the spatial separation is large compared to the ranges of all known physical interactions between them) should be incapable of influencing one another. One may object to this idea on the ground that the quantum formalism does not represent «real state of affairs» in ordinary space and time because the Schrödinger equation for n-particles describes a wave function propagating in configuration space. It can be argued that separability in ordinary space is not sufficient to ensure separability in configuration space. However, the main point is that while configuration space is a theoretical construct, four-dimensional space-time, on the other hand, is more fundamental and in some sense has physical reality independent of us. Separability in ordinary space is therefore a physically relevant objective criterion, particularly because all known interactions decrease rapidly with distance in ordinary space.

There is also considerable misunderstanding about the point calling in question the EPR reality criterion on the ground that it associates elements of reality with observables that cannot be measured simultaneously. In this context it may be useful to illustrate the meaning of the EPR reality criterion with the help of a classical example: one can predict with certainty that water below 0 °C will solidify to form ice; hence the EPR reality criterion allows us to associate an element of reality λ_1 with this property of water. Another fundamental property of water is that it becomes vapour above 100 °C and since this also can be predicted with certainty, the EPR reality criterion associates a second element of reality λ_2 with this property of water. In this case the the elements of reality λ_1 and λ_2 are understood as physically relevant features of the interaction potential between water molecules giving rise to freezing and boiling processes at the appropriate temperatures. The fact that one cannot observe at the same time the freezing and boiling of water does not preclude the simultaneous existence of these elements of reality λ_1 and λ_2. It is this same realistic approach that is applied to the microphysical phenomena by the EPR reality criterion.

1.2. *The subsequent formulations.* – Sparse but important results on the EPR paradox were obtained during the thirty years following 1935. The most interesting ones are reviewed in this section, starting from the incisive analysis by Erwin Schrödinger in 1935 and ending with the 1957 Bohm-Aharonov paper.

1.2.1. Schrödinger's version. Like Einstein, Schrödinger was another prominent physicist who had contributed substantially to the development of quantum theory, and yet, at the same time, was a severe critic of its standard interpretation. It was the EPR paper which motivated him to present his critical views about the quantum-mechanical treatment of two correlated quantum systems [11].

Schrödinger's approach was in many ways an antithesis of Bohr's response to the EPR paper. Schrödinger concentrated on generalizing the EPR argument in terms of a rigorous analysis based on the mathematical formalism and regarded its weird conceptual implication as an indication of a serious inadequacy of quantum mechanics. The paradoxical aspect of the EPR example was clearly brought out in his analysis (the term *paradox* is used here in the sense of a «plausible argument leading from plausible premises to an implausibile conclusion» [12]).

It should be interesting to mention here that though the term paradox was not used in the original EPR paper, Einstein used it for the first time (in the context of the EPR argument) in his letter to Schrödinger on August 8, 1935, and Schrödinger picked it up in his response on August 19. Subsequently, Schrödinger referred to the EPR argument as a paradox in his paper [11] and Einstein also used the term in his 1936 article titled *Physics and Reality* [13]. For a detailed account of the illuminating correspondences between Einstein and Schrödinger during the summer of 1935, see Fine [14]. These exchanges served as a prelude to the important paper [11] by Schrödinger on the EPR problem which we are now going to discuss.

Schrödinger emphasized that the issue of *entanglement* lay at the heart of the EPR argument. When two particles emerge from a temporary interaction and get separated, the wave function of the system of two particles is no longer the product of separate wave functions of the individual particles. Hence the knowledge of the total wave function would not enable us to ascribe an individual wave function to each of the particles; in other words, the two-particle system S in a pure state has to be regarded as a single whole even after the particles cease to interact. In this notion of *entanglement* Schrödinger recognised the characteristic trait of quantum mechanics – «the one that enforces its entire departure from classical lines of thought». Quantum mechanics does not allow to perceive S as comprising of two separate and individual particles. Such a perception is valid only after the phase relations involved in the superposition of states are destroyed by performing measurements on any one of them; that is, the *entanglement* between the two subsystems is then broken. In a sense, the measuring apparatus *interacts* with the whole system even though we may perceive it to tinker with only one of the particles. This *disentanglement* is, according to Schrödinger, of *sinister importance*, particularly in its form underlying the EPR argument.

To give a sharper orientation to this reasoning Schrödinger formulated the following theorem (in the treatment that follows we adopt the same notation as in paragraph 1.1.1).

Statement: To every Hermitian operator $F(Q_\alpha, P_\alpha)$ of the particle α in an EPR pair there corresponds another Hermitian operator $G(Q_\beta, P_\beta)$ of the other particle β such that

$$[F(Q_\alpha, P_\alpha) - G(Q_\beta, P_\beta)]\, \Psi(x_1, x_2) = 0\,, \tag{1.8}$$

i.e. $\Psi(x_1, x_2)$ is an eigenfunction of $(F - G)$ with eigenvalue zero. Here $\Psi(x_1, x_2)$ satisfies the two eigenvalue equations

$$Q\Psi(x_1, x_2) = q_0\Psi(x_1, x_2)\,, \qquad P\Psi(x_1, x_2) = p_0\Psi(x_1, x_2)\,, \tag{1.9}$$

where $Q = Q_\alpha - Q_\beta$ and $P = P_\alpha + P_\beta$.

Proof: We consider the operator

$$F_{mn}(Q_\alpha, P_\alpha) = Q_\alpha^m P_\alpha^n + \text{h.c.} \tag{1.10}$$

and

$$\begin{aligned} G_{mn}(Q_\beta, P_\beta) &= (Q_\beta + q_0)^m (p_0 - P_\beta)^n + \text{h.c.} = \\ &= (Q_\alpha - Q + q_0)^m (p_0 + P_\alpha - P)^n + \text{h.c.} \end{aligned} \tag{1.11}$$

Now let us figure out the result of application of G_{mn} to $\Psi(x_1, x_2)$. First note that the factor $(p_0 + P_\alpha - P)^n$ becomes P_α^n because of (1.9). Then since P_α commutes with $(Q_\alpha - Q + q_0)$, one can shift P_α^n to the left of the factor $(Q_\alpha - Q + q_0)^m$ which when applied to $\Psi(x_1, x_2)$ gives Q_α^m because of (1.9). Finally one obtains

$$G_{mn}\Psi(x_1, x_2) = [P_\alpha^n Q_\alpha^m + \text{h.c.}]\, \Psi(x_1, x_2) = F_{mn}\Psi(x_1, x_2)$$

using (1.10). Equation (1.8), therefore, holds for the operators F_{mn} and G_{mn} defined by (1.10) and (1.11). The previous result is obviously generalizable to functions of the type

$$F(Q_\alpha, P_\alpha) = \sum_{m,n} c_{mn} Q_\alpha^m P_\alpha^n + \text{h.c.}\,, \tag{1.12}$$

where the c_{mn}'s are numerical coefficients. It is then easy to extend Schrödinger's theorem to an arbitrary analytic function F.

An immediate corollary of the theorem is that since measurements pertaining to an F operator and its corresponding G operator must always yield equal results if α and β are described by the wave function $\Psi(x_1, x_2)$, a measurement of F on α has to be interpreted as steering β into an eigenstate of G. It is precisely this strange counterintuitive aspect which, according to Schrödinger, is one of the deep mysteries of quantum mechanics and endows the EPR argument with an enigmatic character. This thesis is eloquently captured in his following assertion: «It is rather discomforting that the theory should allow a system to be steered or piloted into one or the other type of state at the experimenter's mercy in spite of his having no access to it. This paper does not aim at a solution of the paradox, it rather adds to it, if possible». It is fair to add that even today noboby knows how to measure the observables corresponding to F and G except for a few very simple cases.

Schrödinger elaborated this viewpoint and its relation with his general philosophical objection to the standard interpretation of quantum mechanics in a series of papers published in *Die Naturwissenschaften* [15]. As an illustration of the EPR argument, Schrödinger compared the member of the two-particle pair which is assumed not to be measured with «a scholar in examination» who is questioned as to the value of his position or momentum coordinate. The scholar is always prepared to give the correct answer to the first question he is asked, although thereafter he «is invariably so disconcerted or tired out that all the following answers are wrong». But since he always provides the right answer to the first question without knowing which of the two questions – position or momentum – he is going to be asked first, «he must know both the answers».

Schrödinger's conviction, as reflected in these papers, was that the paradox exhibited in the EPR argument could not be resolved within the framework of conventional quantum mechanics. In contrast to EPR, for Schrödinger it was not only a matter of incompleteness of the quantum theory, but a manifestation of a fundamental deficiency in its very foundation. His suspicion was that a possible source of this limitation laid in the treatment of *time* in quantum mechanics and the intriguing way it appeared in the problem of measurement. To summarize, Schrödinger extended the EPR argument in a way which appeared to him to provide substantial support to his claim that «the reigning doctrine is born of distress».

1·2.2. Furry's analysis. Schrödinger's extension of the EPR argument was soon supplemented by Furry's treatment [16] whose starting point can be best appreciated by posing the following problem: given a general state vector $|\eta\rangle$ for the two-particle system $\varepsilon = (\alpha, \beta)$, can one always write it as a direct product of two state vectors separately describing the particles composing ε?

That the answer is negative can be easily seen as a necessary consequence of the superposition principle: every possible vector $|\Psi_i\rangle|\Phi_j\rangle$ is a conceivable state vector for ε (it describes α in the state $|\Psi_i\rangle$ and β in the state $|\Phi_j\rangle$). Therefore, all possible linear combinations of these vectors are also permissible state vectors of ε. To write this in a most general way, let $\{|\Psi_i\rangle\}$ and $\{|\Phi_j\rangle\}$

be two orthonormal and complete sets of states for α and β, respectively. Carrying out a (double) expansion of $|\eta\rangle$ over the sets $\{|\Psi_i\rangle\}$ and $\{|\Phi_j\rangle\}$, we get

$$|\eta\rangle = \sum_{i,j} c_{ij} |\Psi_i\rangle |\Phi_j\rangle, \tag{1.13}$$

where the c_{ij}'s constitute a set of generally complex coefficients. It is important to note that in the absence of superselection rules, every vector $|\eta\rangle$ of the form (1.13) represents a possible state for ε. The coefficients c_{ij} are unrestricted, except for the normalization condition

$$\sum_{i,j} |c_{ij}|^2 = 1. \tag{1.14}$$

In particular, one *cannot* restrict the c_{ij}'s to only of the following type:

$$c_{ij} = d_i e_j, \tag{1.15}$$

where the d_i's and the e_j's are suitable numerical factors.

The superposition principle, therefore, forces one to consider vectors $|\eta\rangle$ of the type (1.13) for which (1.15) does *not* hold. This implies the existence of at least some states for ε such that neither α nor β is in a well-defined quantum state. States of this kind are known as *the second type*, while those for which (1.15) holds are called *the first type*. To put it more precisely, state vectors of the first (second) type are the factorizable (nonfactorizable) state vectors. Since the concept of state vector is the crucial link which connects the quantum-mechanical formalism with the microphysical reality, it follows that quantum theory does not ascribe any separate reality to the particles α and β whose complex (α, β) is described by a state vector of the second type. Furry interpreted the essence of the EPR example as illustrating this particular feature of quantum formalism: «... the assumption [that] a system free from mechanical interference necessarily has independent real properties, is contradicted by quantum mechanics».

To elaborate his thesis, Furry made use of a theorem proved by von Neumann according to which the state vector (1.13) can always be written in the form

$$|\eta\rangle = \sum_i c_i |\Psi_i\rangle |\Phi_i\rangle \tag{1.16}$$

if the complete orthonormal sets $\{|\Psi_i\rangle\}$ and $\{|\Phi_j\rangle\}$ (in general, different from those entering in (1.13)) are suitably chosen. If $|\eta\rangle$ is of the first (second) type, only one (more than one) of the coefficients c_i will be different from zero. Let these two new sets of state vectors constitute eigenstates of two linear Hermitian operators A and B, respectively, so that the relations

$$A|\Psi_i\rangle = a_i|\Psi_i\rangle; \quad B|\Phi_i\rangle = b_i|\Phi_i\rangle \tag{1.17}$$

are valid for all values of the index i. One can then say that A (B) represents an observable of the particle α (β) and that the possible values of such an observable are the eigenvalues a_i (b_i). The state vector (1.16), whose form is maintained unchanged during time evolution, implies in general the following EPR type correlation: if an observer measures at time t_1 the observable A on the particle α and finds the value a_k, then a second observer will necessarily find the value b_k for a measurement of B made at time $t \geq t_1$ on the other spatially separated member β of the same pair of particles. Now invoking the EPR reality criterion, it follows that the result pertaining to measurement of B on β must have been fixed and equal to b_k already before the measurement on α was performed. To adhere to the quantum formalism, one has to then assert that the state of β must have been already the eigenvector $|\Phi_k\rangle$ before the time t_1. Similar argument can also be made for the state of α and one would conclude that the state vector for this particular pair (α, β) before the time t_1 was $|\Psi_k\rangle|\Phi_k\rangle$. Repeating this reasoning for all members of the statistical ensemble of pairs (α, β) leads to the inference that the state vectors describing this ensemble are actually incoherent *mixtures* of the state vectors of the type

$$|\eta_k\rangle = |\Psi_k\rangle \Phi_k\rangle \tag{1.18}$$

with the probabilities $|c_k|^2$.

Now, eqs. (1.16) and (1.18) are obviously different mathematical descriptions of the ensemble. Furry pointed out that they are not only mathematically different but also that these are not equivalent physical descriptions. At this stage, instead of following Furry's original reasoning, we shall present an elegant way of formulating this incompatibility due to Fortunato [17].

Consider the projection operator

$$P_\eta = |\eta\rangle\langle\eta|, \tag{1.19}$$

which is a bounded Hermitian operator and hence, at least in principle, can be assumed to correspond to an observable. Its expectation value for the state (1.16) is given by

$$\langle\eta|P_\eta|\eta\rangle = 1. \tag{1.20}$$

The expectation value of the same operator for the mixed state (1.18) is instead

$$\langle P_\eta\rangle = |c_1|^2\langle\Psi_1\Phi_1|P_\eta|\Psi_1\Phi_1\rangle + |c_2|^2\langle\Psi_2\Phi_2|P_\eta|\Psi_2\Phi_2\rangle + \\ + \ldots + |c_k|^2\langle\Psi_k\Phi_k|P_\eta|\Psi_k\Phi_k\rangle + \ldots$$

Since $\langle\Psi_k\Phi_k|P_\eta|\Psi_k\Phi_k\rangle = |c_k|^2$, it follows that

$$\langle P_\eta\rangle = |c_1|^4 + |c_2|^4 + \ldots + |c_k|^4 + \ldots \tag{1.21}$$

But from the normalization condition we have

$$|c_1|^2 + |c_2|^2 + \ldots + |c_k|^2 + \ldots = 1. \tag{1.22}$$

Using (1.22) it follows from (1.21) that

$$\langle P_\eta \rangle < 1,$$

provided that there are at least two coefficients different from zero. The latter condition is, however, precisely that of having a state vector of the second type. Therefore the observable corresponding to P_η has an expectation value equal to unity (less than unity) for the state vector of the second type (1.16) (for the mixture of state vectors of the first type (1.18)).

Though it may appear that the paradoxical element in the EPR argument can be avoided by assuming that, after spatial separation between the two particles α and β, the phase relations involved in the state vector of the second type (1.16) are somehow destroyed leading to a mixture of factorizable state vectors of the first type (1.18), this supposition (henceforth referred to as Furry's hypothesis) leads to an incompatibility between the statistical predictions derived from (1.16) and (1.18). The importance of Furry's analysis, therefore, lays not in its attempt to «refute» the EPR argument, as sometimes alleged, but in the fact that it indicated the possibility of relating the EPR paradox to empirical probing.

1'2.3. Bohm's formulation. There were a few unsatisfactory features associated with the wave function (1.6), (1.7) used by EPR in their original reasoning. Epstein [18] raised the question of time dependence of the wave function, which was ignored in the EPR treatment. The wave function (1.6), (1.7) is valid only at the particular instant $t = 0$ (when the two particles α and β interacted before getting spatially separated); subsequently on time evolution, it is no longer a stable solution of the Schrödinger equation. A similar argument against the appropriateness of the EPR wave function was also made by Piccioni *et al.* [19]. Another disadvantage of the EPR wave function (1.6) is that since it is based on plane waves it describes the two correlated particles α and β as present with constant probability at all points in space; hence it is not appropriate for describing completely separated particles in space. Cooper [20] pointed out that the assumption of spatial separation between the particles after their mutual interaction implies that they no longer have self-adjoint representations for their momenta, suggesting that for completely separated quantum systems (confined to spatially bounded regions) their momenta cannot be treated as observables in the usual quantum-mechanical sense. However, these difficulties were circumvented in the formulation presented by Bohm [21] in 1951, who made use of dichotomic, discrete observables to make the argument mathematically sound and conceptually more transparent.

Consider (α, β) pairs of two spin-(1/2) particles with the following wave function:

$$\Psi(x_1, x_2) = \eta_0 \Psi_\alpha(x_1)\, \Psi_\beta(x_2). \tag{1.23}$$

Here $\Psi_\alpha(x_1)$, $\Psi_\beta(x_2)$ are the space parts of the wave functions for α and β, respectively, and η_0 is the singlet state given by

$$\eta_0 = \frac{1}{\sqrt{2}}[u_\alpha(+)u_\beta(-) - u_\alpha(-)u_\beta(+)], \tag{1.24}$$

where $u_\alpha(+)$, $u_\alpha(-)$ are eigenvctors corresponding to the eigenvalues $+1$ and -1, respectively, of the Pauli matrix $\sigma_z(\alpha)$ representing the z-component of the spin angular momentum for α, and $u_\beta(+)$ and $u_\beta(-)$ are the corresponding eigenvectors of the Pauli matrix $\sigma_z(\beta)$ for β. Note that the space-dependent part of the wave function (1.23) allows for incorporating time dependence, while the spin part is considered to be time independent. Therefore the singlet state η_0 preserves its form over the passage of time.

We now suppose that $\Psi_\alpha(x_1)$ is a Gaussian function with modulus appreciably different from zero only in a region R_1 of width Δ_1, centred around the point x_1'. Similarly, let $\Psi_\beta(x_2)$ be a Gaussian function localized in the region R_2 of width Δ_2, centred around x_2'. As a sufficient condition for separability of the particles α and β, we invoke the following stipulation:

$$|x_2' - x_1'| \gg \Delta_1,\ \Delta_2 . \tag{1.25}$$

If the particles α and β are now supposed to move to the left and to the right, respectively, so that the distance between the centres of the two wave packets increases linearly with time, it can be shown that the Schrödinger equation allows condition (1.25) to be satisfied during time evolution even though Δ_1 and Δ_2 increase with time. One can then assert that α and β are located within the two localized small regions R_1 and R_2, respectively, well separated from one another, so that all known physical interactions between them become negligibly small. In such a condition one would be naturally inclined to infer that a measurement performed on α cannot give rise to any effect on β, and vice versa. However, the presence of η_0 in the wave function (1.23) leads to a puzzling nonlocal effect (the EPR paradox).

The following important characteristics of η_0 are used in Bohm's version of the EPR argument:

P1) η_0 is not a factorizable state.

P2) It is rotationally invariant.

P3) It predicts opposite results for measurements of the components (along any arbitrary direction $\hat{n}$) of the spins of the particles α and β.

P4) It predicts the result zero for a measurement of the total squared spin of the particles α and β.

To develop the EPR-type reasoning in this case, consider a large set E of (α, β) pairs in the state (1.23). Measure $\sigma_z(\alpha)$ at time t_0 on all α's of a subset E_1 of E. If $+1$ (-1) is found, a future measurement at time $t\,(t > t_0)$ of $\sigma_z(\beta)$ will certainly give -1 $(+1)$. Invoking the EPR reality criterion, we can then assign to the β's of E_1 an element of reality λ_1 (λ_2) fixing *a priori* the result -1 $(+1)$ of the $\sigma_z(\beta)$ measurement. Excluding the possibility that λ_1 (λ_2) be created by action at a distance due to the measurement of $\sigma_z(\alpha)$, it follows that

λ_1 (λ_2) actually belongs to all β's of the entire ensemble E. Now if we apply the *completeness* assumption of EPR, we have to describe the particle β with predetermined value of $\sigma_z(\beta)$ by assigning to it the eigenstate $u_\beta(-)\,[u_\beta(+)]$. The strict correlation implied by P3), applied to the z-axis, leads to the inference that the ensemble E has to be described in spin space by a mixture of the factorizable state vectors with the same weight factor

$$u_\alpha(+)\,u_\beta(-) \quad \text{and} \quad u_\alpha(-)\,u_\beta(+) \tag{1.26}$$

even for $t < t_0$.

At this stage it is important to note that there is an empirically meaningful difference between an ensemble whose members are described by η_0 (1.24) and an ensemble which is a mixture of states (1.26). This is most easily seen by observing that measurements of the total squared spin on a set of (α, β) pairs described as a mixture of the factorizable state vectors (1.26) will yield with equal probability the results 0 and $2\hbar^2$, which contradicts the property P4) implied by η_0 (1.24). This completes Bohm's formulation of the EPR paradox as a testable contradiction between the consequences derived from the state vector of the nonfactorizable type and those from the notion (Furry's hypothesis) that the nonfactorizable state vector may spontaneously decompose into mixture of factorizable state vectors, once the particles get spatially separated (to recall once again, the latter possibility was envisaged to satisfy the assumption that particles once freed from mutual dynamical interference could be regarded as possessing independent real properties).

Greenberger and YaSin [22] have introduced a new variation in Bohm's version of the EPR example. They consider a gedanken example of a spin-0 system decaying into two neutrons flying out in opposite directions, one of them (say, α) directed into a neutron interferometer and split into two sub-beams. Then using a magnetic field, each of these two sub-beams is envisaged to be separated into spin-up and spin-down components. A *double-neutron mirror* device is assumed to be inserted into each of these four separated sub-beams such that it enables one to observe which of these four sub-beams the neutron is in (by noting the displacement of a double-mirror system during the passage of a neutron through it). This in turn specifies the spin component of the neutron α, and one can therefore infer that its partner β must certainly have the opposite spin. If, however, one does not observe the recoil of the device before the neutron gets out of it, the net displacement of the system disappears, and one can no longer know the spin of the neutron α and that of β. Invoking the EPR reality criterion, this leads to the following interesting situation. Even though one never tinkers with the neutron β, it suddenly acquires an element of reality corresponding to its spin component when its spatially separated partner α enters the double-mirror device and then suddenly loses it when α leaves the device. Greenberger and YaSin interpret this example as indicating the queer fact that «remote measurements can not only bestow properties on a particle, but they can equally as well remove them». They conclude: «Reality should be made of sterner stuff ... the EPR criterion does not possess the simple qualities it appears to at first sight». This example has been called «a haunted version for the EPR paradox». Its implications call for further probing.

Now, going back to the historical perspective on Bohm's formulation, the next logical step was to look for actual experimental results to shed further light on this issue, and it was precisely this analysis which was taken up by Bohm and Aharonov [23] in 1957. As a motivation for such a study, Bohm and Aharonov stated that even Einstein in his later years entertained a point of view sympathetic to Furry's hypothesis: «... Einstein has (in a private communication) actually proposed such an idea, namely, that the current formulation of the many-body problem in quantum mechanics may break down when particles are far enough apart».

1.2.4. The Bohm-Aharonov investigation. Bohm and Aharonov [23] began by emphasizing that in the absence of any clear-cut experimental inference related to the physical situation considered in the EPR paradox, one could adhere to Furry's hypothesis without getting into conflict with the available experimental results. They then pointed out that if the two spin-(1/2) particles used in Bohm's formulation were replaced by the two photons produced in the annihilation of a positron-electron pair, such a case did correspond to an experiment that had already been perfomed: the measurement of the polarization correlation in the pairs of annihilation photons (not directly but through polarization-dependent joint distribution for Compton scattering) by Wu and Shaknov in 1950 [24].

Consider the quantum state

$$|0^-\rangle = \frac{1}{\sqrt{2}}\{|x\rangle_\alpha |y\rangle_\beta - |y\rangle_\alpha |x\rangle_\beta\}, \tag{1.27}$$

which is the zero angular-momentum negative-parity state, where x and y denote the mutually orthogonal directions of linear polarization of the photons α and β. Note that the state $|0^-\rangle$ is rotationally invariant. This means that each photon is always found in a state of linear polarization orthogonal to that of the other, no matter what may be the choice of axes with respect to which the state of polarization is expressed.

To establish connection with the relevant experiment, Bohm and Aharonov considered the ratio $R = \Gamma_1/\Gamma_2$, where Γ_1 is the rate of double scattering of the two photons through a fixed angle θ, when the planes π_1 and π_2 formed by the lines of motion of the scattered first and second photon with their common original direction of motion are mutually perpendicular; Γ_2 is the same rate when the planes π_1 and π_2 are parallel to each other. The value of R derived from the $|0^-\rangle$ state is given by

$$R = \frac{(\gamma - 2\sin^2\theta)^2 + \gamma^2}{2\gamma(\gamma - 2\sin^2\theta)}, \tag{1.28}$$

where

$$\gamma = (k_0/k) + (k/k_0). \tag{1.29}$$

Here k_0 is the wave number of the incident photon, k that of the scattered photon. In the actual experiment by Wu and Shaknov, the scattered photons were detected with an angular spread around 82°. Corresponding to such a situation, eq. (1.28) predicts using standard quantum electrodynamics

$$R = 2.00 \tag{1.30}$$

obtained with a suitable angular average. This value agrees very well with the experimental result

$$R = 2.04 \pm 0.08\,. \tag{1.31}$$

Bohm and Aharonov then proceeded to demonstrate that the hypothesis of a breakdown of the quantum description (1.27) with the increasing separation between the two photons, implying its substitution by a mixture of factorizable states, leads necessarily to the constraint

$$R \leq 1.5 \tag{1.32}$$

which is clearly incompatible with the experimental value given by (1.31). Bohm and Aharonov, therefore, concluded that the Wu-Shaknov experiment provided empirical evidence against Furry's hypothesis, that is against a simple-minded breakdown of the quantum formalism that avoided the EPR paradox.

This contention was, however, contested by Peres and Singer [25] who argued that the spin of photons is *physically different* from the spin of fermions. The photon spin is always oriented along the direction of its propagation; the components of the spin orthogonal to the direction of propagation are not gauge-invariant and therefore have no physical significance.

Hence, Peres and Singer asserted that it could not be used in the context of Bohm's example of the EPR paradox. Countering this objection, Bohm and Aharonov [26] pointed out that they did not use in their treatment explicitly the spin operator of photons. The analogy with the spin case of fermions is that when linear polarization of photon is measured, the circular polarization is indeterminate and vice versa; while with spin, when one component is measured, the others are indeterminate and vice versa. For the purpose of formulating the EPR argument, the fact that there are three components of the spin and only two for polarization is not at all relevant.

Accepting the Bohm-Aharonov conclusion on the basis of the Wu-Shaknov experiment, it would, however, not be correct to interpret it as an empirical evidence against *local realism* per se, since there are well-known local realist models capable of reproducing the quantum-mechanical predictions for the examples of the type studied in the Wu-Shaknov experiment. A simple illustration is the model proposed by Kasday [27]. Suppose there are two *hidden* vectors $\hat{\lambda}_\alpha$ and $\hat{\lambda}_\beta$ associated with the photons α and β, respectively, and let the photons be ultimately scattered in the directions of these vectors. It is assumed that $\hat{\lambda}_\alpha$ and $\hat{\lambda}_\beta$ have the same probability distribution as that of the momenta $\hat{k}_\alpha$ and $\hat{k}_\beta$ of the scattered photons, as predicted by quantum mechanics. This *realist* assumption obviously implies that the photons have *decided in advance* (at the time of annihilation) in which directions they would ultimately scatter.

The model is manifestly *local*: altering the position of the detector 1 does not affect the parameter $\hat{\lambda}_\beta$, and therefore it does not change the response of the detector 2. Furthermore, it can be shown that the model reproduces the quantum-mechanical results derived from eq. (1.27) for all the measurements that can be made on the scattered photons.

It is interesting to note that, notwithstanding the claim by Bohm and Aharonov to have found empirical evidence against Furry's hypothesis, several authors still pursued the idea. Jauch [28] tried to argue on the basis of an «algebra of propositions» that the states of the type eq. (1.27) do not exist for the EPR pairs. Similarly, de Broglie [29] strongly suspected that the standard quantum formalism would be inapplicable for particle separation larger than the coherence length of the wave packets. Ghirardi, Rimini and Weber [30] proposed a modification of the quantum formalism such that the time evolution is governed by the Schödinger equation provided the two correlated quantum systems are close together, while a continuous transition to a mixed state takes place with increasing spatial separation between the systems. However, Furry's hypothesis has now certainly lost much of its original interest, not only because its empirical validity has been shown untenable, but also because there are simple *local realist* models of correlated spins which clearly contradict the results derived from Furry's hypothesis. As emphasized in the analysis by Garuccio, Scalera and Selleri [31], it is now increasingly realized «how narrow is a reduced quantum mechanics» obtained by applying Furry's hypothesis to standard quantum mechanics.

2. – Testable incompatibility between quantum mechanics and local realism.

2·1. *Bell's inequality*. – The paradox of Einstein, Podolsky and Rosen was advanced as an argument that quantum mechanics could not be a complete theory but should be supplemented by additional (*hidden*) variables. These additional variables were thought by EPR to introduce in the theory a causal and local description. The 1935 paper by Schrödinger pointed out that the paradox was much stronger: There was not only a problem with completeness, but a clear indication that the reality of one of a pair of correlated quantum systems depended on the type of measurement performed on the other system.

A crucial development of Schrödinger's conclusion was obtained thirty years later by Bell [32]. After giving a mathematical proof of the incompatibility between quantum theory and a broad class of local hidden-variable theories, Bell wrote:

«In a theory in which parameters are added to quantum mechanics to determine results of individual measurements, without changing the statistical predictions, there must be a mechanism whereby the setting of one measuring device can influence the reading of another instrument, however remote. Moreover, the signal involved must propagate instantaneously, so that such a theory could not be Lorentz invariant».

Three proofs of Bell's inequality will be reviewed in this section, the first one being essentially equivalent to the original (1965) one. The second one (Wigner's proof) introduces for the first time probabilities in the study of Bell's theorem. The third proof gives rise to a dichotomy in the quantum-mechanical treatment of correlated systems, corresponding to what d'Espagnat defined as *proper* and *unproper* mixtures. Finally, we will review a recent study of quantum nonlocality not based on Bell's inequality.

2·1.1. Deterministic proof. A proof of Bell's theorem will now be given which is essentially equivalent to the original proof. We start by giving the definition of *correlation function* and by recalling some simple and well-known quantum-mechanical results.

Consider an ensemble formed by a very large number N of pairs of correlated quantum systems (α, β), for example, coming from the decay of an unstable system. Suppose that α and β fly away from one another, for example, α flies to the right and β to the left of some observer. Suppose then that an observer O_α measures on α in the space region R_α a dichotomic observable $A(a)$, while a second observer O_β measures on β in the space region R_β another dichotomic observable $B(b)$. The regions R_α and R_β are taken very far from one another and with no overlap.

The observables $A(a)$ and $B(b)$ are taken to be dichotomic, meaning that

$$A(a) = \pm 1 \quad \text{and} \quad B(b) = \pm 1$$

and they depend on the arguments a and b, respectively, which are assumed to be experimental parameters, fixed in the structure of the apparatuses in any given experiment, but possibly variable over different experiments. Examples of such dichotomic observables are those represented by the spin matrices $\boldsymbol{\sigma}(\alpha) \cdot \hat{a}$ and $\boldsymbol{\sigma}(\beta) \cdot \hat{b}$, where the experimental parameters are the unit vectors $\hat{a}$ and $\hat{b}$, defined by the directions of inhomogeneous magnetic fields. In practice, any physical quantity can be used to define a dichotomic observable: One could say, for example, that $A(a) = \pm 1$ if the energy of an atom is above or below a certain level E_a.

When measurements of such observables are made on all the N pairs of the given ensemble, O_α will obtain a set of results $\{A_1, A_2, ..., A_N\}$, while O_β will collect a similar set $\{B_1, B_2, ..., B_N\}$, all with respect to the fixed values of the parameters a and b. The results of the two sets are correlated in the sense that A_1 and B_1 pertain to the particles α and β respectively arising from the first decay; A_2 and B_2 are similarly associated with the second decay; and so on. By definition, these results are, in every case, equal to ± 1.

The *correlation function* $P(a, b)$ of the results A_i and B_i is defined as the average product of the results obtained by O_α and O_β from the same decays:

$$P(a, b) = \frac{1}{N} \sum_{i=1}^{N} A_i B_i . \tag{2.1}$$

Since every product $A_i B_i$ is ± 1, it follows that

$$-1 \leq P(a, b) \leq +1 . \tag{2.2}$$

As a particular example of correlation function let us consider the case of two spin-(1/2) particles in the «singlet» state $|\eta_0\rangle$ for which quantum mechanics predicts

$$P(\hat{a}, \hat{b}) = \langle \eta_0 | \boldsymbol{\sigma}(\alpha) \cdot \hat{a} \otimes \boldsymbol{\sigma}(\beta) \cdot \hat{b} | \eta_0 \rangle = -\hat{a} \cdot \hat{b}. \tag{2.3}$$

This result is, however, incompatible with local realism as we will soon see.

Let us define the quantity

$$\Delta = |P(\hat{a}, \hat{b}) - P(\hat{a}, \hat{b}')| + |P(\hat{a}', \hat{b}) + P(\hat{a}', \hat{b}')|. \tag{2.4}$$

Consider two orthogonal unit vectors $\hat{a}$ and $\hat{a}'$ associated with particle α and two orthogonal unit vectors $\hat{b}$ and $\hat{b}'$ associated with particle β, and suppose that the relative orientation of these vectors is such that they can be found by clockwise rotations of $\frac{\pi}{4}$ in the order $\hat{a}$, $\hat{b}$, $\hat{a}'$, $\hat{b}'$. One can then easily see that the substitution of (2.3) in (2.4) leads to

$$\Delta = |\hat{a} \cdot \hat{b} - \hat{a} \cdot \hat{b}'| + |\hat{a}' \cdot \hat{b} + \hat{a}' \cdot \hat{b}'| = 2\sqrt{2}.$$

It can also be shown that $2\sqrt{2}$ is the maximum value of Δ for all conceivable orientations of the vectors $\hat{a}$, $\hat{b}$, $\hat{a}'$, $\hat{b}'$. This result is of great interest, because, as we will see next, Δ is a physical quantity for which local realism allows a maximum value of 2. The inequality $\Delta \leq 2$ is *Bell's inequality*. It has been called «the most profound discovery of science» [33].

In a theory developed according to the EPR reality criterion there are elements of reality λ which determine all the observables. They can be expected to vary with density $\rho(\lambda)$ over a set Λ. Of course, the following condition has to be satisfied

$$\int_\Lambda d\lambda \rho(\lambda) = 1. \tag{2.5}$$

The role of the variable λ is that of fixing the values of the dichotomic observables, for example

$$\boldsymbol{\sigma}(\alpha) \cdot \hat{a} \to A(a, \lambda), \qquad \boldsymbol{\sigma}(\beta) \cdot \hat{b} \to B(b, \lambda), \tag{2.6}$$

where the real discontinuous functions $A(a, \lambda)$ and $B(b, \lambda)$ can assume only the values ± 1. The correlation function as defined in (2.1) (average product of the two observables) can obviously be written in the form

$$P(a, b) = \int d\lambda \rho(\lambda) A(a, \lambda) B(b, \lambda). \tag{2.7}$$

This is a local expression, in the sense that neither A depends on b, nor B on a.

It is a simple exercise to show that

$$|P(a, b) - P(a, b')| \leq \int d\lambda \, \rho(\lambda) |B(b, \lambda) - B(b', \lambda)|, \tag{2.8}$$

since $|A(a, \lambda)| = 1$, and that

$$|P(a', b) + P(a', b')| \leq \int d\lambda \, \rho(\lambda) |B(b, \lambda) + B(b', \lambda)|. \tag{2.9}$$

By adding (2.8) and (2.9) and using the result

$$|B(b, \lambda) - B(b', \lambda)| + |B(b, \lambda) + B(b', \lambda)| = 2,$$

which is a consequence of $|B(b, \lambda)| = |B(b', \lambda)| = 1$, one obtains from (2.5) Bell's inequality in the following form:

$$\Delta \equiv |P(a, b) - P(a, b')| + |P(a', b) + P(a', b')| \leq 2. \tag{2.10}$$

The above proof of the inequality is *deterministic* because the variable λ fixes *a priori* the results of the measurements that can be made on α and/or on β. In fact it determines the values of $A(a, \lambda)$, $A(a', \lambda)$, $B(b, \lambda)$ and $B(b', \lambda)$.

2·1.2. Wigner's proof. The proof of Bell's inequality given by Wigner [34] was based on two assumptions. The interest of this proof is in its simplicity and in the theoretical approach which has a deterministic background but uses at the same time probabilities.

The first assumption was that the results of all conceivable measurements on the correlated pairs (α, β) are simultaneously preassigned, even in the case of the incompatible observables. This deterministic standpoint does not contradict Heisenberg's relations, because the latter can be taken to imply only that an actual measurement made on a given object modifies the preassigned values of other nonmeasured observables incompatible with the observed one. Wigner's picture is fully consistent with the realistic idea that *before the action of the instrument* all conceivable observables have preassigned values.

The second assumption was locality. A measurement made on α (β) does not modify the prefixed values of the observables $B(b)$, $B(b')$, $(A(a), A(a'))$ of β (α). If one writes

$$A(a) = s, \quad A(a') = s'; \quad B(b) = t, \quad B(b') = t', \tag{2.11}$$

where s, s', t, t' are all equal to ± 1, locality means that these four parameters, preassigned by the first assumption, are not modified due to *action at a distance* by remote measurements. Therefore, if $A(a)$ is measured on an α object, for example, and the value s is found, the preassigned values t and t' associated with the correlated β object are in no way modified (while s' in general could be).

As a consequence of these assumptions a set E of N (α, β) pairs splits into 2^4 subsets with well-defined populations in which the outcomes of the four possible measurements are predetermined. Let $E(s, s', t, t')$ be a subset of E with prefixed values of the four observables (2.11) and $n(s, s', t, t')$ be its population. Naturally

$$\sum n(s, s', t, t') = N, \tag{2.12}$$

where $\sum$ denotes the sum over the 2^4 different sets of values of the dichotomic parameters. As an example consider $E(+1, -1, -1, -1)$: it is that subset of E for which are *a priori* determined the following results:

$$A(a) = +1, \quad A(a') = -1, \quad B(b) = -1, \quad B(b') = -1.$$

Of course, only one of the observables $A(a)$ and $A(a')$, and only one of the observables $B(b)$ and $B(b')$ can be measured on a given pair.

By virtue of the locality assumption the actual performance of the measurement of $A(a)$ or of $A(a')$ on the α objects does not modify in any way the prefixed values of $B(b)$ and $B(b')$ of the corresponding β objects. In other words, there is no action at a distance modifying $B(b)$ or $B(b')$ arising from the measurements of $A(a)$ or $A(a')$ (and vice versa).

The *a priori* probabilities

$$\omega(s, s', t, t') = \frac{n(s, s', t, t')}{N}, \tag{2.13}$$

which satisfy the normalization condition

$$\sum \omega(s, s', t, t') = 1 \tag{2.14}$$

as a consequence of (2.12), can be used for the calculation of correlation functions of actually performed experiments. One has

$$\begin{cases} P(a, b) = \sum \omega(s, s', t, t')\, st, \\ P(a, b') = \sum \omega(s, s', t, t')\, st', \\ P(a', b) = \sum \omega(s, s', t, t')\, s't, \\ P(a', b') = \sum \omega(s, s', t, t')\, s't', \end{cases} \tag{2.15}$$

where $\sum$ denotes again a sum over all the dichotomic variables. It is now trivial to show that from $|s| = 1$ and $|s'| = 1$ the following inequalities follow, respectively,

$$\begin{cases} |P(a, b) - P(a, b')| \le \sum \omega(s, s', t, t')\, |t - t'|, \\ |P(a', b) + P(a', b')| \le \sum \omega(s, s', t, t')\, |t + t'|. \end{cases} \tag{2.16}$$

By adding the last two inequalities and using $|t - t'| + |t + t'| = 2$, a consequence of $|t| = |t'| = 1$, the inequality (2.10), which is Bell's inequality, follows readily.

With Wigner's proof probabilities entered for the first time in the EPR paradox. Probabilities were, however, deduced from a deterministic background, much in the same way as done originally by Laplace in his formulation of probability calculus.

2·1.3. Local quantum-mechanical correlations. Given the set of all possible quantum-mechanical state vectors for the correlated (α, β) pairs, Bell's inequality has the important property of classifying it into two subsets, local and nonlocal. In paragraph **1**·2.2 we defined state vectors of the first type (factorizable) and of the second type (nonfactorizable). Belonging to the latter type is the singlet state of two spin-(1/2) particles which, as we saw, violates Bell's inequality. It will be shown that the violations of locality are due only to the vectors of the second type. On the other hand, a remarkable property of mixtures of factorizable state vectors is that they satisfy Bell's inequality in all cases, as was first shown by Capasso, Fortunato and Selleri [35].

Consider an ensemble E of N quantum pairs (α, β) and suppose they are described by factorizable state vectors $|\Psi_i\rangle|\Phi_i\rangle$ with frequency N_i/N $(i = 1, 2, \ldots)$. In the ensemble E one has

$$(2.17)\qquad \begin{cases} \text{the vector } |\Psi_1\rangle|\Phi_1\rangle \text{ applying to } N_1 \text{ pairs;} \\ \text{the vector } |\Psi_2\rangle\Phi_2\rangle \text{ applying to } N_2 \text{ pairs;} \\ \qquad\ldots\ldots\ldots\ldots\ldots\ldots \\ \text{the vector } |\Psi_i\rangle|\Phi_i\rangle \text{ applying to } N_i \text{ pairs;} \\ \qquad\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

where

$$\sum_i N_i = N .$$

Suppose that the dichotomic observables to be measured on α and β are described quantum mechanically by the operators $\tilde{A}(a)$ and $\tilde{B}(b)$, respectively, so that the operator corresponding to the product of the joint measurements on the two systems is $\tilde{A}(a) \otimes \tilde{B}(b)$. The correlation function predicted by quantum theory is precisely the average of the latter observable over the mixture (2.17), so that

$$(2.18)\qquad P(a, b) = \sum_i p_i \langle \Psi_i | \langle \Phi_i | \tilde{A}(a) \otimes \tilde{B}(b) | \Psi_i \rangle | \Phi_i \rangle ,$$

where

$$(2.19)\qquad p_i = N_i/N, \qquad \sum_i p_i = 1 .$$

If one writes

$$(2.20)\qquad \begin{cases} \bar{A}_i = \langle \Psi_i | \tilde{A}(a) | \Psi_i \rangle, & \bar{A}'_i = \langle \Psi_i | \tilde{A}(a') | \Psi_i \rangle , \\ \bar{B}_i = \langle \Phi_i | \tilde{B}(b) | \Phi_i \rangle, & \bar{B}'_i = \langle \Phi_i | \tilde{B}(b') | \Phi_i \rangle , \end{cases}$$

the four correlation functions entering in Bell's inequality can be written as follows:

$$(2.21)\qquad \begin{cases} P(a, b) = \sum_i p_i \bar{A}_i \bar{B}_i , & P(a', b) = \sum_i p_i \bar{A}'_i \bar{B}_i , \\ P(a, b') = \sum_i p_i \bar{A}_i \bar{B}'_i , & P(a', b') = \sum_i p_i \bar{A}'_i \bar{B}'_i . \end{cases}$$

Expectation values of operators having eigenvalues ± 1 have moduli not exceeding unity. Therefore

$$|\bar{A}_i| \leq 1; \quad |\bar{A}'_i| \leq 1; \quad |\bar{B}_i| \leq 1; \quad |\bar{B}'_i| \leq 1. \tag{2.22}$$

By inserting (2.21) in (2.4), one easily obtains

$$\Delta \leq \sum_i p_i \Delta_i, \tag{2.22}$$

where

$$\Delta_i = |\bar{A}_i \bar{B}_i - \bar{A}_i \bar{B}'_i| + |\bar{A}'_i \bar{B}_i + \bar{A}'_i \bar{B}'_i|. \tag{2.24}$$

By using (2.22) one can immediately deduce

$$\Delta_i \leq |\bar{B}_i - \bar{B}'_i| + |\bar{B}_i + \bar{B}'_i|, \tag{2.25}$$

whence it follows

$$\Delta_i \leq 2, \tag{2.26}$$

since any two real numbers x and y such that $|x| \leq 1$ and $|y| \leq 1$ always satisfy $|x - y| + |x + y| \leq 2$.

If (2.26) is inserted in (2.23), one gets finally $\Delta \leq 2$, that is Bell's inequality. Given the arbitrariness of the population N_i and of the state vectors $|\Psi_i\rangle|\Phi_i\rangle$ we conclude that *an arbitrary mixture of state vectors of the first type (factorizable state vectors) always satisfies Bell's inequality.*

2·1.4. Incompatibility between nonfactorizable state vectors and local realism. Let us begin by considering the general form for a nonfactorizable state vector of a composite system given by

$$|\eta\rangle = \sum_{i,j} c_{ij} |\Psi_i\rangle \Phi_j\rangle, \tag{2.27}$$

where $|\Psi_i\rangle$'s and $|\Phi_j\rangle$'s denote complete orthonormal sets of eigenstates corresponding to the correlated subsystems I and II, respectively.

It was shown by von Neumann (1955) that by suitable choice of the complete orthonormal sets

$$\{|\Psi_i\rangle\}; \quad \{|\Phi_j\rangle\} \tag{2.28}$$

one can write

$$|\eta\rangle = \sum_i \sqrt{\omega_i} |\Psi_i\rangle |\Phi_i\rangle, \tag{2.29}$$

where the ω_i's are real and nonnegative. The normalization condition of $|\eta\rangle$ implies

$$\sum_i \omega_i = 1. \tag{2.30}$$

In the present paper only the orthonormality conditions of the sets (2.28) will be used:

$$\begin{cases} \langle \Psi_i | \Psi_j \rangle = \delta_{ij}, \\ \langle \Phi_i | \Phi_j \rangle = \delta_{ij}. \end{cases} \tag{2.31}$$

Now, to show that at least in principle observables can always be chosen in such a way that Bell's inequality is violated for any state vector of the type given by eq. (2.29), we consider pairs of noncommuting dichotomic observables (with eigenvalues ± 1) D_S, D'_S pertaining to I and D_T, D'_T pertaining to II, with the following definition for them:

$$\begin{cases} D_S = 2P_S - 1, \\ D'_S = 2P'_S - 1, \\ D_T = 2P_T - 1, \\ D'_T = 2P'_T - 1, \end{cases} \tag{2.32}$$

where the projection operators are bounded Hermitian operators corresponding to normalized wave functions

$$\begin{cases} P_S = |\Psi_1\rangle\langle\Psi_1|, \\ P_T = |\Phi_1\rangle\langle\Phi_1|, \\ P'_S = [\alpha_1|\Psi_1\rangle + \alpha_2|\Psi_2\rangle][\alpha_1^*\langle\Psi_1| + \alpha_2^*\langle\Psi_2|], \\ P'_T = [\beta_1|\Phi_1\rangle + \beta_2|\Phi_2\rangle][\beta_1^*\langle\Phi_1| + \beta_2^*\langle\Phi_2|], \end{cases} \tag{2.33}$$

where $|\alpha_1|^2 + |\alpha_2|^2 = 1 = |\beta_1|^2 + |\beta_2|^2$. Note that, for the sake of simplicity, the observables above have been defined with respect to a two-dimensional subspace of the multidimensional Hilbert space spanned by the state vector (2.29), but it is sufficient for the purpose of proving our theorem. We assume that both ω_1 and ω_2 are nonzero, as it is possible to do without loss of generality, since in (2.29) there are certainly at least two ω_i's $\neq 0$.

Now, using eqs. (2.29), (2.32) and (2.33) it is straightforward algebra to obtain the following results:

$$\begin{cases} \langle\eta|D_S \otimes D_T|\eta\rangle = 1, \\ \langle\eta|D'_S \otimes D_T|\eta\rangle = (1-\Sigma) + \Sigma\,\Delta\alpha, \\ \langle\eta|D_S \otimes D'_T|\eta\rangle = (1-\Sigma) + \Sigma\,\Delta\beta, \\ \langle\eta|D'_S \otimes D'_T|\eta\rangle = (1-\Sigma) + \Sigma\,\Delta\alpha\,\Delta\beta + \\ \qquad + [(\Sigma^2 - \Delta\omega^2)(1-\Delta\alpha^2)(1-\Delta\beta^2)]^{1/2}\cos(\phi_\alpha + \phi_\beta), \end{cases} \tag{2.34}$$

where

$$\text{(2.35)}\qquad \begin{cases} \Sigma \;\; = \omega_1 + \omega_2\,, & \Delta\alpha = |\alpha_1|^2 - |\alpha_2|^2\,,\\ \Delta\omega = \omega_1 - \omega_2\,, & \Delta\beta = |\beta_1|^2 - |\beta_2|^2 \end{cases}$$

and ϕ_α, ϕ_β are the relative phases of α_1, α_2 and of β_1, β_2, respectively.

Now, recalling that the standard form of Bell's inequality is given by

$$-2 \le \langle D_S \otimes D_T \rangle - \langle D'_S \otimes D_T \rangle + \langle D_S \otimes D'_T \rangle + \langle D'_S \otimes D'_T \rangle \le 2\,,$$

we have from eqs. (2.34) the following possibility for the violation of Bell's inequality:

$$\text{(2.36)}\qquad 1 - (1-\Sigma) - \Sigma\,\Delta\alpha + (1-\Sigma) + \Sigma\,\Delta\beta + (1-\Sigma) + \Sigma\,\Delta\alpha\,\Delta\beta + \\ + [(\Sigma^2 - \Delta\omega^2)(1-\Delta\alpha^2)(1-\Delta\beta^2)]^{1/2}\cos(\phi_\alpha + \phi_\beta) > 2\,,$$

which reduces to the following

$$\text{(2.37)}\qquad \cos(\phi_\alpha + \phi_\beta) > \frac{\Sigma}{(\Sigma^2 - \Delta\omega^2)^{1/2}}\,\frac{(1+\Delta\alpha)}{(1-\Delta\alpha^2)^{1/2}}\,\frac{1-\Delta\beta}{(1-\Delta\beta^2)^{1/2}}\,.$$

Now, to show that condition (2.37) can be satisfied by an appropriate choice of observables, we consider that ϕ_α and ϕ_β are chosen such that

$$\text{(2.38)}\qquad \cos(\phi_\alpha + \phi_\beta) = 1\,.$$

Then (2.37) reduces to the form

$$\text{(2.39)}\qquad 1 - \frac{\Delta\omega^2}{\Sigma^2} > \frac{1+\Delta\alpha}{1-\Delta\alpha}\,\frac{1-\Delta\beta}{1+\Delta\beta}\,.$$

Remembering that $\Delta\omega^2 < \Sigma^2$ because in (2.35) both ω_1 and ω_2 are nonzero, it is evident that condition (2.39) can easily be satisfied by choosing $\Delta\alpha$ and $\Delta\beta$ appropriately, because the left-hand side of (2.39) lies between 0 and 1, and $-1 \le \Delta\alpha \le 1$, $-1 \le \Delta\beta \le 1$.

This completes the proof of the theorem whose statement can be formulated as follows: *For any given nonfactorizable state vector of correlated quantum systems it is always possible to choose observables in such a way that Bell's inequality is violated by quantum-mechanical predictions.*

This resut is important in reinforcing the notion that the incompatibility between quantum mechanics and local realism is rather deep-rooted and though Bell's inequality does not contain fully all the restrictions implied by local realism it is nevertheless sufficient for the purpose of displaying in great generality the incompatibility between nonfactorizable state vectors and local realism.

2·1.5. The GHZ argument. Recently, Greenberger, Horne and Zeilinger (GHZ) [36] have formulated an interesting new theorem that has attracted wide attention [37]. They consider a system consisting of three mutually well-separated and correlated spin-(1/2) particles, in the context of which an incompatibility is demonstrated between quantum mechanics and the conjunction of the EPR reality criterion with the locality condition. Their demonstration, unlike Bell's theorem, concerns only perfect correlations rather than statistical correlations and dispenses with the inequalities.

Let us begin by recapitulating the essence of the GHZ argument. The argument is developed by considering the eight-dimensional space of three spin-(1/2) particles and the system of these spatially separated particles in a particular state $|\Psi\rangle$ which satisfies the following conditions for the three observables O_1, O_2, O_3:

$$O_i|\Psi\rangle = +|\Psi\rangle \tag{2.40}$$

with $i = 1, 2, 3$ where $O_1 = \sigma_x^1\sigma_y^2\sigma_y^3$, $O_2 = \sigma_y^1\sigma_x^2\sigma_y^3$, $O_3 = \sigma_y^1\sigma_y^2\sigma_x^3$. An explicit form for $|\Psi\rangle$ can be written as (referring to the z-axis components)

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\{|1, 1, 1\rangle - |-1, -1, -1\rangle\}. \tag{2.41}$$

In then follows that one can predict with *certainty* the result S_x of measuring the x component of the spin of any one particle by *far away* measurements of the y components of the other two. Similar is the case for the result S_y of measuring the y component of any one of them. Applying the EPR reality criterion it is clear that we can consider $S_{x,y}^{1,2,3}$ (S_x, S_y components pertaining to the particles 1, 2, 3) to be elements of reality having pre-assigned values ± 1. From the locality condition it follows that these values are independent of whichever of the different sets of three single-particle spin measurements one might choose to make on these spatially separated particles. Consistent with (2.40) we can therefore write the following relations:

$$S_x^1 S_y^2 S_y^3 = +1, \tag{2.42a}$$

$$S_y^1 S_x^2 S_y^3 = +1, \tag{2.42b}$$

$$S_y^1 S_y^2 S_x^3 = +1. \tag{2.42c}$$

Remembering that $S_{x,y}^{1,2,3} = \pm 1$, one obtains from (2.42a), (2.42b) and (2.42c)

$$S_x^1 S_x^2 S_x^3 = +1. \tag{2.43}$$

However it follows from (2.40) that the operator $O_4 = \sigma_x^1\sigma_x^2\sigma_x^3$ satisfies the following eigenvalue equation:

$$\sigma_x^1\sigma_x^2\sigma_x^3|\Psi\rangle = -|\Psi\rangle \tag{2.44}$$

which is in contradiction with (2.43). This completes the GHZ proof of an incompatibility between predictions derived from the quantum-mechanical wave function (2.41) and *local realism* (EPR reality criterion plus the locality condition).

Note that the above argument, which is technically correct, hinges on the following implicit assumptions:

A) The wave function given by (2.41) represents a «physically real» state.

B) The Hermitian operators O_1, O_2, O_3 and O_4 correspond to observables which are actually measurable.

Both the assumptions *A*) and *B*) are unverified. In fact, for those believing in local realism, the GHZ argument leads to the conlusion that either the wave function (2.41) does not correspond to physical reality or the Hermitian operators O_1, O_2, O_3 and O_4 do not correspond to actual observables. It is important to emphasize that such a conclusion is not *a priori* ruled out even within the framework of quantum mechanics. The so-called superselection rules discussed by Wick, Wightman and Wigner (WWW) [38] provide examples of Hermitian operators that are not associated with any observables. As an illustration, we consider the superselection rule for electric charge. If $|g_1\rangle$ and $|g_2\rangle$ are two different eigenstates of the charge operator for which all other quantum numbers are equal, then one can construct, for instance, the state vector

$$|\eta\rangle = \frac{1}{\sqrt{2}}(|g_1\rangle + |g_2\rangle) \tag{2.45}$$

and the operator

$$P_\eta = |\eta\rangle\langle\eta| . \tag{2.46}$$

Note that since P_η is Hermitian, it is expected to represnt an observable O_η with possible values 1 and 0, the measurement of which would necessarily leave the system in the state $|\eta\rangle$ whenever the result $O_\eta = +1$ is found. However, it is well known that the state $|\eta\rangle$ does not correspond to any physical reality and the operator P_η does not represent any measurable physical property—in fact, nobody knows how to produce a state like $|\eta\rangle$ or measure an operator like P_η.

Even letting aside the WWW superselection rules, Lamb [39] has pointed out operational difficulties associated with the occasionally used quantum-mechanical axiom of one-to-one correspondence between Hermitian operators and observables. From his discussion it is evident how hard it is to devise physically realizable measuring procedures for even apparently simple Hermitian operators such as $xp + px$. As Penrose [40] has put it, it is one of the peculiarities of the formalism of quantum mechanics that it cannot in itself specify which Hermitian operators are actually measurable and which are not. Moreover, if one considers the classical limit of quantum mechanics it is evident that not all possible superpositions of quantum states occur in nature. For example, a macroscopic ball confined in a box with reflecting walls is always found in a state described by a localized wave packet and never, for instance, in a state described by a plane wave which can be viewed as a superposition of the localized states.

It needs to be stressed here that we are not subscribing to the viewpoint that all nonfactorizable state vectors leading to incompatibility with local realism are physically unrealizable and that one cannot in practice discriminate between quantum mechanics and local realism. Some nonfactorizable state vectors containing EPR-type correlations are consequences of fundamental principles in quantum mechanics, like the conservation of angular momentum or other relevant invariance conditions like that of charge conjugation. We have discussed in paragraph **2·1**.4 the proof of a general argument showing that *all nonfactorizable state vectors necessarily lead to violation of Bell's inequalities.* This reinforces the strength of Bell's theorem. The key point is to search for actual cases where the quantum mechanically predicted violation of Bell's inequalities can be tested in practice.

Another limitation of the GHZ argument is that it is strictly restricted to the deterministic form of local realist theories. On the other hand, Bell's theorem holds good for probabilistic local realist theories as well. On this point some may refer to a so-called «equivalence theorem» [41] to the effect that «the predictions of any stochastic local theory can be duplicated by an appropriate deterministic local theory» [42]. If this were true, any argument showing an incompatibility between quantum mechanics and deterministic local realist theories could be construed as implying incompatibility with probabilistic theories as well. However, such a general conclusion is not justified.

To see this, we note that the demonstration of the above-mentioned «equivalence theorem» is essentially with reference to the EPR-type *two-particle* correlation experiments involving experimentally measurable distribution functions for *four* compatible pairs of observables (AB, AB', $A'B$, $A'B'$). A crucial element in the proof of this theorem is the proposition that the existence of joint probabilities for noncommuting observables is equivalent to the validity of Bell-type inequalities. Logical tenability of this proposition was contested by Redhead [43] and Stapp [44] who pointed out that it was possible to derive Bell-type inequalities by avoiding any commitment to joint distributions for noncommuting observables. Subsequently, Svetlichny *et al.* [45] have shown that Bell-type inequalities do not necessarily imply the existence of joint distributions for noncommuting observables, if such probabilities are interpreted in the physically relevant *relative frequency* sense. In particular, it has been shown that corresponding to the validity of Bell's inequality one can have a situation in which the joint distributions of the noncommuting (incompatible) observables AA' and BB', defined in terms of limiting relative frequencies on product sequences, do not exist, while limiting relative frequencies do exist with respect to the individual sequences and compatible product sequences. In view of these serious criticisms about the so-called «equivalence theorem» we contend that it cannot be invoked to extend the range of validity of the GHZ formulation.

We disagree with the contention by Mermin [46] that the GHZ formulation «is an altogether more powerful refutation of the existence of elements of reality than the one provided by Bell's theorem». The reasons are, first of all that one can talk of *refutation* only after an unambiguous experimental verdict, and moreover:

A) Unlike Bell's theorem, the GHZ argument is based on unverified assumptions concerning the physical reality of a particular state vector and measurability of certain Hermitian operators pertaining to a system of three correlated spin-(1/2) particles.

B) Unlike Bell's theorem, the GHZ formulation is limited to deterministic local theories.

C) A direct experimental test of the GHZ argument is probably impossible.

2.2. *The strong inequalities.* – Bell's inequality has never been tested experimentally and can be considered a *weak* inequality if compared with much stronger (*i.e.* more restrictive) inequalities that can be deduced from local realism if some suitable *additional assumptions* are made. The present subsection is devoted to a review of the most important published proofs of the strong inequalities.

2.2.1. The CHSH additional assumptions. A way of testing experimentally could be as follows. A source is built in such a way that it emits the correlated objects α and β in opposite directions where two analysers can transmit them or absorb them depending on their physical properties. The dichotomic choice forced in this way upon each atomic object can be used for defining corresponding dichotomic observables, by saying that $A(a) = \pm 1$ ($B(b) = \pm 1$), depending on the choice, transmission $(+1)$ or absorption (-1), taken by $\alpha(\beta)$.

In 1969 Clauser, Horne, Shimony and Holt (CHSH) [47] suggested the use of pairs of optical photons emitted in atomic cascades. For such photons the binary choice was between transmission and absorption *in a polarizer*. For given orientations a and b of the two polarizers they introduced four probabilities $T(a_\pm, b_\pm)$ where, for example, $T(a_+, b_-)$ is the probability that $A(a) = +1$ (photon α transmitted by the first polarizer) and $B(b) = -1$ (photon β absorbed in the second polarizer). The correlation function can be written in the form

$$P(a, b) = T(a_+, b_+) - T(a_+, b_-) - T(a_-, b_+) + T(a_-, b_-) . \tag{2.47}$$

The double-transmission probabilities $T(a_\pm, b_\pm)$ must satisfy

$$T(a_+, b_+) + T(a_+, b_-) + T(a_-, b_+) + T(a_-, b_-) = 1 . \tag{2.48}$$

Considering now the case in which the second polarizer has been removed (the symbol ∞ is used to indicate this), one has

$$T(a_+, b_+) + T(a_+, b_-) = \mathrm{T}(a_+, \infty) . \tag{2.49}$$

If instead the first polarizer is removed, one has

$$T(a_+, b_+) + T(a_-, b_+) = T(\infty, b_+) . \tag{2.50}$$

Finally, if both polarizers have been removed both photons will certainly be transmitted, so that

$$T(\infty, \infty) = 1. \tag{2.51}$$

By introducing in (2.47) the last four relations one easily gets

$$P(a, b) = 4T(a_+, b_+) - 2T(a_+, \infty) - 2T(\infty, b_+) + 1. \tag{2.52}$$

Only cases of double transmission appear in the above expression, which thus refers more directly to the experimental observations, since it is impossible to detect absorption of a photon in a polarizer.

The only way to know that a photon has been transmitted is however to detect its presence. But photon detectors have efficiencies of $(10 \div 20)\%$ only, meaning that one cannot really measure a double-transmission probability, but only a *joint probability for double transmission and double detection.* This is not the probability entering in (2.52)!

This problem has traditionally been «solved» by means of *ad hoc* assumptions concerning the nature of the transmission/detection process. The additional assumption of CHSH is the following: *given that a pair of photons emerge from two regions of space where two polarizers can be located, the probability of their joint detection from two photomultipliers* (D_0) *is independent of the presence and the orientation of the polarizers.*

Denoting with the symbol Ω the joint probability for transmission *and* detection, one obtains from the previous assumption

$$\begin{cases} \Omega(a, b) = D_0 T(a_+, b_+), & \Omega(\infty, b) = D_0 T(\infty, b_+), \\ \Omega(a, \infty) = D_0 T(a_+, \infty), & \Omega(\infty, \infty) = D_0 T(\infty, \infty), \end{cases} \tag{2.53}$$

since the double-detection probability (D_0) has been assumed in all cases to be the same. In (2.53), $\Omega(a, b)$ is the joint probability for double transmission and double detection in the case of polarizers with orientations a and b, $\Omega(a, \infty)$ is the same probability with the second polarizer removed, and so on.

The *rates* of double detections are proportional to the number N_0 of photon pairs entering per second in the solid angles defined by the optical apparatuses. One has

$$\begin{cases} R(a, b) = N_0 \Omega(a, b), & R(\infty, b) = N_0 \Omega(\infty, b), \\ R(a, \infty) = N_0 \Omega(a, \infty), & R_0 = N_0 D_0, \end{cases} \tag{2.54}$$

where $R(a, b)$ is the number of photon pairs detected per second (*detection rate*) when the polarizers have orientations a and b, and the meaning of the other symbols is obvious. Notice that $R(\infty, \infty)$ has been called R_0. If one obtains the T probabilities using (2.53) and (2.54) and substitutes them in (2.52) one obtains

$$P(a, b) = 4\frac{R(a, b)}{R_0} - 2\frac{R(a, \infty)}{R_0} - 2\frac{R(\infty, b)}{R_0} + 1. \tag{2.55}$$

Only the coincidence rates enter in (2.55): *by virtue of the* CHSH *additional assumption the correlation function has become measurable!*

Since Bell's inequality can be written in the form

$$(2.56)\qquad -2 \le P(a, b) - P(a, b') + P(a', b) + P(a', b') \le +2$$

the substitution of the expressions of the type (2.55) for the four correlation functions leads to

$$(2.57)\qquad -1 \le \frac{R(a, b)}{R_0} - \frac{R(a, b')}{R_0} + \frac{R(a', b)}{R_0} + \frac{R(a', b')}{R_0} - \frac{R(a', \infty)}{R_0} - \frac{R(\infty, b)}{R_0} \le 0 .$$

Only (measurable) coincidence rates enter in the previous inequalities, which can therefore be checked experimentally. A useful simplification is obtained if two predictions of quantum theory are accepted that are not at all paradoxical and can anyway be checked directly in every experiment:

i) the prediction that $R_1 = R(a', \infty)$ does not depend on a', and that $R_2 = R(\infty, b)$ does not depend on b.

ii) the prediction that every R function should depend only on the relative angle between the axes of the polarizers.

One then gets

$$(2.58)\qquad -1 \le \frac{R(a-b)}{R_0} - \frac{R(a-b')}{R_0} + \frac{R(a'-b)}{R_0} + \frac{R(a'-b')}{R_0} - \frac{R_1}{R_0} - \frac{R_2}{R_0} \le 0 .$$

If the axes are chosen according to

$$(2.59)\qquad a - b = a' - b = a' - b' = \phi; \qquad a - b' = 3\phi$$

it follows from (2.58)

$$(2.60)\qquad -1 \le \frac{3R(\phi)}{R_0} - \frac{R(3\phi)}{R_0} - \frac{R_1 + R_2}{R_0} \le 0 .$$

Considering the previous inequalities for $\phi = 22.5°$ and for $\phi = 67.5°$, where the maximum quantum-mechanical violations exists, one obtains the so-called Freedman's inequality

$$(2.61)\qquad \left| \frac{R(22.5°)}{R_0} - \frac{R(67.5°)}{R_0} \right| - \frac{1}{4} \le 0 ,$$

which does not contain R_1 and R_2. Notice that all the new inequalities obtained from (2.57) to (2.61) could be deduced *only because the* CHSH *additional assumption has been made.* It is therefore misleading to confuse the original Bell's inequality with the *much stronger* inequalities now deduced. We will therefore adopt the following definitions:

Weak inequality: An inequality exclusively deduced from local realism and violated by quantum mechanics in the case of nearly perfect instruments.

Strong inequality: An inequality deduced from local realism and from certain additional assumptions, such as the CHSH one stated above, and violated by quantum mechanics in the case of actual instruments.

A full justification of these definitions is left for the end of the next section where it will be shown that in a typical experimental situation the strong inequalities limit the crucial (paradoxical) observable to a n•merical interval 100 times narrower than that implied by the corresponding weak inequalities.

2'2.2. The CH additional assumption. In 1974 Clauser and Horne [48] used a variable λ to represent the physical state of a pair of correlated quantum objects within a general probabilistic scheme in which

$p(a, \lambda)$ is the probability that the object α in the state λ crosses the analyser with parameter a and is subsequently detected.

$q(b, \lambda)$ is the similar probability for β.

$\Omega(a, b, \lambda)$ is the probability that both α and β cross their respective analysers with parameters a and b, and that they are both detected.

Furthermore, Clauser and Horne assumed that the locality condition could be expressed by the following conditions:

CH1) Factorizability: $\Omega(a, b, \lambda) = p(a, \lambda)\, q(b, \lambda)$;

CH2) Neither the probability density $\rho(\lambda)$, nor the set Λ of values of λ depend on a or on b.

It is not obvious (and in fact is not true) that these definitions should exhaust all possible local and realistic situations. This important problem will be discussed in detail in the next subsection (**2**'3, probabilistic local realism).

In the Clauser-Horne (CH) approach the ensemble probabilities are written as weighted averages of individual probabilities:

$$(2.62)\qquad \begin{cases} p(a) = \int \mathrm{d}\lambda\, \rho(\lambda)\, p(a, \lambda)\,; \\ q(b) \;= \int \mathrm{d}\lambda\, \rho(\lambda)\, q(b, \lambda)\,; \\ \Omega(a, b) = \int \mathrm{d}\lambda\, \rho(\lambda)\, p(a, \lambda)\, q(b, \lambda)\,. \end{cases}$$

In order to deduce inequalities, Clauser and Horne considered the following theorem. Given six real numbers x, x', X, y, y', Y, such that

$$0 \le x,\ x' \le X; \qquad 0 \le y,\ y' \le Y\,,$$

one must always have

$$(2.63)\qquad -XY \le xy - xy' + x'y + x'y' - x'Y - Xy \le 0\,.$$

The proof is straghtforward, since the intermediate quantity in (2.63) is linear in each of the four variables x, x', y, y' so that its extremes must be looked for on the boundary of these variables.

The CH inequalities can be applied to the EPR paradox by taking

$$\begin{cases} x = p(a, \lambda), & y = q(b, \lambda), \\ x' = p(a', \lambda), & y' = q(b', \lambda). \end{cases} \tag{2.64}$$

Introducing (2.64) in (2.63), multiplying the result by $\rho(\lambda)$, and integrating over λ, one obtains

$$-XY \le \Omega(a, b) - \Omega(a, b') + \Omega(a', b) + \Omega(a', b') - Yp(a') - Xq(b) \le 0. \tag{2.65}$$

What are the appropriate values for X and Y in (2.65)? The straightforward answer is of course $X = Y = 1$, since the probabilities (2.64) might equal unity for some values of λ. This leads to inequalities of the *weak* type (no additional assumptions):

$$-1 \le \Omega(a, b) - \Omega(a, b') + \Omega(a', b) + \Omega(a', b') - p(a') - q(b) \le 0, \tag{2.66}$$

which could also have been deduced from Bell's inequality. Inequalities like the previous one that we decided to call «of the weak type» are sometimes also called «inhomogeneous inequalities» since they are based both on double and single detection probabilities. «Homogeneous inequalities» will next be deduced which are based on double detection probabilities only.

The problem with (2.66) is that the quantum-mechanical predictions do not violate it for the available detectors. For this reason Clauser and Horne introduced the following additional assumption (α and β are now photons): *for every photon in the state λ the probability of detection with a polarizer placed on its trajectory is less than or equal to the detection probability with the polarizer removed.*

This additional assumption leads to the following inequalities:

$$\begin{cases} p(a, \lambda) \le p(\infty, \lambda); & p(a', \lambda) \le p(\infty, \lambda), \\ q(b, \lambda) \le q(\infty, \lambda); & q(b', \lambda) \le q(\infty, \lambda) \end{cases} \tag{2.67}$$

where the symbol ∞ indicates, as usual, that a polarizer has been removed. One can now use (2.63) and (2.64) with

$$X = p(\infty, \lambda); \qquad Y = q(\infty, \lambda) \tag{2.68}$$

in order to obtain

$$-D_0 \le \Omega(a, b) - \Omega(a, b') + \Omega(a', b) + \Omega(a', b') - \Omega(a', \infty) - \Omega(\infty, b) \le 0, \tag{2.69}$$

where D_0 is the same as in eqs. (2.53) and the meaning of the new symbols is obvious.

Since a ratio of double-detection probabilities coincides with the corresponding ratio of the detection rates, (2.69) can easily be seen to coincide with (2.57). From this observation it follows that all the results obtained in the CHSH approach from (2.57) – that is inequalities (2.58), (2.60) and (2.61) – are true also in the present (CH) approach.

A numerical example will help in understanding the difference between weak and strong inequalities. If one defines

$$\Gamma = \Omega(a, b) - \Omega(a, b') + \Omega(a', b) + \Omega(a', b')\,, \tag{2.70}$$

the weak inequalities (2.66) and the strong inequalities (2.69) can, respectively, be written as follows:

$$-1 + p(a') + q(b) \le \Gamma \le p(a') + q(b) \quad \text{(weak)} \tag{2.71}$$

and

$$-D_0 + \Omega(a', \infty) + \Omega(\infty, b) \le \Gamma \le \Omega(a', \infty) + \Omega(\infty, b) \quad \text{(strong)}\,. \tag{2.72}$$

These inequalities should be compared with the quantum-mechanical predictions:

$$p(a') = \varepsilon_+^1 \eta_1/2; \qquad q(b) = \varepsilon_+^2 \eta_2/2; \qquad D_0 = \eta_1 \eta_2\,, \tag{2.73}$$

$$\Omega(a', \infty) = \varepsilon_+^1 \eta_1 \eta_2/2; \qquad \Omega(\infty, b) = \varepsilon_+^2 \eta_1 \eta_2/2\,, \tag{2.74}$$

and

$$\Omega(a, b) = \frac{1}{4}[\varepsilon_+^1 \varepsilon_+^2 + \varepsilon_-^1 \varepsilon_-^2 \cos 2(a - b)]\eta_1 \eta_2\,. \tag{2.75}$$

It should be noted that the crucial quantity containing the paradoxical features of the quantum-mechanical predictions is Γ. There is instead nothing paradoxical in the quantities given in (2.73) and (2.74), while *the* EPR *paradox is fully contained in the predicted general validity of* (2.75), which leads to Γ via (2.70).

In (2.73)-(2.75) $\varepsilon_\pm^1$ and $\varepsilon_\pm^2$ are well-known parameters related to the transmittances of the two polarizers, while η_1 and η_2 are the quantum efficiencies of the two photodetectors.

By substituting (2.75) in (2.70) it is easy to show [49] that

$$\begin{cases} (\Gamma)_{\max} = \dfrac{1}{2}\left[\varepsilon_+^1 \varepsilon_+^2 + \varepsilon_-^1 \varepsilon_-^2 \sqrt{2}\right]\eta_1 \eta_2\,, \\[2ex] (\Gamma)_{\min} = \dfrac{1}{2}\left[\varepsilon_+^1 \varepsilon_+^2 - \varepsilon_-^1 \varepsilon_-^2 \sqrt{2}\right]\eta_1 \eta_2\,. \end{cases} \tag{2.76}$$

Typical numerical values of the experimental parameters are

$$\varepsilon_+^1 = \varepsilon_+^2 = 1; \qquad \varepsilon_-^1 = \varepsilon_-^2 = 0.95; \qquad \eta_1 = \eta_2 = 0.1\,. \tag{2.77}$$

If they are substituted in (2.71) and in (2.72) these relations, respectively, become

$$-0.9 \le \Gamma \le 0.1 \quad \text{(weak)}, \tag{2.78}$$

$$0 \le \Gamma \le 0.01 \quad \text{(strong)}, \tag{2.79}$$

while (2.76) gives

$$(\Gamma)_{\min} = -0.00138; \quad (\Gamma)_{\max} = 0.01138. \tag{2.80}$$

One can thus see that while the strong inequality is violated, the weak one is fully compatible with the quantum-theoretical predictions and therefore with the experimental results.

One can also observe that the role of local realism is to give the set of possible values of Γ a spread of 1.0, while the additional assumptions bring down this figure to only 0.01, just enough to generate a violation of the quantum-theoretical predictions. The upper limit of (2.79) is, for example, violated by (2.80) by about 14%. It appears so that the disagreement is generated mostly by the additional assumptions themselves. The spread of admitted values of Γ is reduced by *two orders of magnitude* in the previous example, and this feature is common to all experiments on the EPR paradox hitherto performed.

2·2.3. The GR additional assumption. In 1981 Garuccio and Rapisarda (GR) [50] studied an experiment in which two-ways polarizers (*e.g.*, calcite prisms), monitored by two detectors put on the ordinary and on the extraordinary rays were used as analysers for each one of the two photons in an EPR-type experiment. Garuccio and Rapisarda adopted the same approach as Clauser and Horne with the density $\rho(\lambda)$ and with factorizable probabilities and applied it to the case of four simultaneously measurable coincidence rates.

Denoting a photon detection of the ordinary or on the extraordinary ray with + and −, respectively, one has the following generalizations of (2.62):

$$\begin{cases} p(a_\pm) = \int \mathrm{d}\lambda\, \rho(\lambda)\, p(a_\pm, \lambda); \\ q(b_\pm) = \int \mathrm{d}\lambda\, \rho(\lambda)\, q(b_\pm, \lambda); \\ \Omega(a_\pm, b_\pm) = \int \mathrm{d}\lambda\, \rho(\lambda)\, p(a_\pm, \lambda)\, q(b_\pm, \lambda), \end{cases} \tag{2.81}$$

where $p(a_+, \lambda)$ is the probability that the photon α with state λ emerges and is detected in the ordinary beam when the calcite axis has orientation a, and so on.

Garuccio and Rapisarda proposed a new definition of correlation function, based on all the available experimental information:

$$E(a, b) \equiv \frac{\Omega(a_+, b_+) - \Omega(a_+, b_-) - \Omega(a_-, b_+) + \Omega(a_-, b_-)}{\Omega(a_+, b_+) + \Omega(a_+, b_-) + \Omega(a_-, b_+) + \Omega(a_-, b_-)}. \tag{2.82}$$

Substituting the third of eqs. (2.81) in the above definition, one gets

$$E(a,b)=\frac{\int \mathrm{d}\lambda\,\rho(\lambda)f(a,\lambda)g(b,\lambda)}{\int \mathrm{d}\lambda\,\rho(\lambda)F(a,\lambda)G(b,\lambda)}, \tag{2.83}$$

where

$$\begin{cases} f(a,\lambda) \equiv p(a_+,\lambda)-p(a_-,\lambda)\,,\\ g(b,\lambda) \equiv q(b_+,\lambda)-q(b_-,\lambda)\,,\\ F(a,\lambda) \equiv p(a_+,\lambda)+p(a_-,\lambda)\,,\\ G(b,\lambda) \equiv q(b_+,\lambda)+q(b_-,\lambda)\,. \end{cases} \tag{2.84}$$

No inequality violated by quantum theory can be obtained from (2.83) for the actually feasible experiments. Therefore, in the present case also an additional assumption is needed. The GR assumption is that: *For every photon in the state λ the sum of detection probabilities in the ordinary and in the extraordinary beams emerging from a two-way polarizer does not depend on the polarizer's orientation.*

The practical implications of the GR additional assumption are that F does not depend on a, G does not depend on b and the denominator of $E(a,b)$ does not depend on either a or b. A better notation is then $F(\lambda)$ (instead of $F(a,\lambda)$), $G(\lambda)$ (instead of $G(,\lambda)$) and

$$H_0=\int \mathrm{d}\lambda\,\rho(\lambda)F(\lambda)G(\lambda)\,. \tag{2.85}$$

One can then obtain an inequality *of the strong type* violated by the quantum-mechanical predictions in the case of real experiments. In fact

$$\begin{aligned}&|E(a,b)-E(a,b')+E(a',b)+E(a',b')|\leq\\ &\qquad\leq H_0^{-1}\int \mathrm{d}\rho(\lambda)\left(|f(a,\lambda)|\,|g(b,\lambda)-g(b',\lambda)|+|f(a',\lambda)|\,|g(b,\lambda)+g(b',\lambda)|\right\},\end{aligned}$$

whence, using the obvious inequalities

$$|f(a,\lambda)|,\ |f(a',\lambda)|\leq F(\lambda)\,,\qquad |g(b,\lambda)|,\ |g(b',\lambda)|\leq G(\lambda)\,,$$

one obtains

$$S\equiv|E(a,b)-E(a,b')+E(a',b)+E(a',b')|\leq 2\,, \tag{2.86}$$

since any two numbers g and g' satisfying $|g|\leq G$, $|g'|\leq G$, must also satisfy: $|g-g'|+|g+g'|\leq 2G$. Garuccio and Rapisarda could show that the quantum-mechanical predictions violate the strong inequality (2.86) by as much as 50%.

2.2.4. Strong inequalities do not hold in nature. While the review of the experimental situation is discussed in subsect. **2.5** we can here anticipate that all the experiments performed with atomic photon pairs have found a very good agreement with the quantum-theoretical predictions (with an isolated exception) and violations *of the strong inequalities* by several standard deviations. It thus seems very likely that strong inequalities do not hold in Nature and it must be concluded that some of the assumptions used in order to derive them are not valid. The choice is to get rid either of a fundamental pysical principle (local realism) or of arbitrary and experimentally untestable additional assumptions.

2'3. *Probabilistic local realism.* – Both the original proof of Bell's inequality and Wigner's proof were based on a deterministic formulation of local realism. In fact, the EPR reality criterion itself used a deterministic approach. However, this results in a limited validity of the reality criterion. This can be easily seen by noting that the EPR reality criterion defines as real only those properties which can be predicted with certainty. However, in the quantum domain it is well known that it is only in the case of an eigenstate of some Hermitian operator that on can make predictions about the relevant observable with certainty. EPR were, of course, aware of the limitation; they wrote: «... this criterion while far from exhausting all possible ways of recognizing a physical reality, at least provides us with one such way». Even without reference to the quantum-mechanical formalism, a fundamental criticism of the EPR reality criterion concerns the fact that predictability with certainty of a physical quantity presupposes a complete specification of the state of a system. This is an extremely idealized notion because it is exceedingly difficult, if not impossible, to be absolutely certain of all the relevant factors in a given experimentally verifiable situation. A probabilistic formulation of local realism is, therefore, essentially required. A well-known probabilistic approach is the Clauser-Horne (CH) formulation [48] of «objective local theories». However, the CH approach is not general enough and a more general formulation of probabilistic local realism has recently been given [51]. The present subsection is devoted to a review of these aspects.

2'3.1. Are «objective local theories» general enough? Clauser and Horne [48] defined the «objective local theories» by starting from the standard EPR situation: independent pairs (α, β) of correlated quantum systems are emitted from a source with system α (β) flying to the right (left) where the analyser 1 and the detector 1 (analyser 2 and the detector 2) are placed. The «state» of the two systems is denoted by λ which is given, over a statistical ensemble of similar pairs, with a positive-definite and normalized density function $\rho(\lambda)$. The variable λ is taken to vary in a «space of states» Λ. The two analysers have each an adjustable parameter, a (b) being the value of the parameter of the apparatus 1 (2). The value of λ fixes the probabilities $p(a, \lambda)$, $q(b, \lambda)$, $\Omega(a, b, \lambda)$ discussed in paragraph **2**'2.2.

The above formulation is sometimes considered to be the most general one, encompassing all conceivable situations that can be described in terms of probabilistic local realism. However, a serious criticism of the CH formulation is that the probabilities $p(a, \lambda)$, $q(b, \lambda)$ and $\Omega(a, b, \lambda)$ refer to a single EPR pair and that an individual definition of probability is necessarily adopted. This is unsatisfactory because no known formulation of probability calculus uses the notion of probability referred to an individual object. Any operationally meaningful definition of probability must be in terms of relative frequencies referring to an ensemble of systems. Some papers have also pointed out other difficulties with the CH approach:

a) Deterministic models have been found that are factorizable as such, but lose factorizability as soon as they become probabilistic, owing to averaging over one of the hidden variables [52].

b) A concrete physical model based on local realism has been proposed in the macroscopic domain which does not satisfy (CH1) and (CH2) simultaneously [53].

c) It has been shown that a probabilistic model assumed to satisfy (CH1) and (CH2) in n variables does not, in general, satisfy them anymore as soon as an averaging is done over even one of these variables [51].

In what follows we discuss in some detail the difficulty *c*).

Let $\{\lambda\}$ be the set of variables specifying the state of (α, β) and λ_0 be just one of these variables. $\{\lambda'\}$ be the set of remaining variables after λ_0 has been taken away from $\{\lambda\}$. We assume that factorizability holds for $\{\lambda\} = \{\lambda, \lambda_0\}$ and then if $\rho(\lambda', \lambda_0)$ is the density function one can write

$$\Omega(a, b) = \int \mathrm{d}\lambda_0 \int \mathrm{d}\lambda' \rho(\lambda', \lambda_0) p(a, \lambda', \lambda_0) q(b, \lambda', \lambda_0).$$

Now, suppose that both (CH1) and (CH2) hold after the averaging over the variable λ_0 has been performed. Then one should have

$$\rho_1(\lambda')\bar{p}(a, \lambda')\bar{q}(b, \lambda') = \int \mathrm{d}\lambda_0 \rho(\lambda', \lambda_0) p(a, \lambda', \lambda_0) q(b, \lambda', \lambda_0),$$

where ρ_1 is the new probability density and a bar denotes a probability after averaging over λ_0. However, that this is not possible in general can be easily seen by dividing the previous relation by its partial derivative with respect to the argument a:

$$\frac{\bar{p}(a, \lambda')}{\bar{p}'_a(a, \lambda')} = \frac{\int \mathrm{d}\lambda_0 \rho(\lambda', \lambda_0) p(a, \lambda', \lambda_0) q(b, \lambda', \lambda_0)}{\int \lambda_0 \rho(\lambda', \lambda_0) p'_a(a, \lambda', \lambda_0) q(b, \lambda', \lambda_0)}. \tag{2.87}$$

While the left-hand side of (2.87) depends only on a (and not on b), the right-hand side in general depends both on a and on b. Therefore, only for those very special cases in which the right-hand side of (2.87) does not depend on b one can maintain the validity of (CH1) and (CH2) after averaging over λ_0. This shows that, in general, (CH1) and (CH2) can be satisfied only if a perfectly complete description of the state of the correlated microsystems is provided by a whole set of λ's. This is again an untenable assumption because there is no way one can ensure that one has «a perfectly complete description of the state» by a given set of λ's.

The above difficulties of the CH formulation have been overcome by a new approach (using definition of probability in terms of relative frequency) discussed in the following paragraph **2**'3.2. This new approach uses a very general definition of local realism in probabilistic terms. This not only provides a logically satisfactory starting point, but also clarifies that it is (CH2) which is not valid in general though (CH1) remains valid. Here it is relevant to note that even the first probabilistic formulation of local realism, given by Wigner in 1970 (see paragraph 2.1.2), does not satisfy (CH2).

2.3.2. Probabilistic formulation of local realism. A general formulation of local realism in probabilistic terms has recently been proposed [51]. Its key ingredient is a *probabilistic reality criterion* resting on the idea that probabilities which can be predicted correctly before they are measured prove the existence of real physical properties of those statistical ensembles for which they are predicted.

More precisely, let S and T be two sets of physical objects of the same type (*e.g.*, photons)

$$S = \{\alpha_1, \ldots, \alpha_N\}, \qquad T = \{\beta_1, \ldots, \beta_N\}.$$

These objects are supposed to be produced in pairs, α_1 with β_1, α_2 with β_2, and so on, but different pairs are assumed to be totally independent.

Suppose that it is possible to measure a dichotomic physical quantity $A(a)$ on the objects of S and let ± 1 be the two possible outcomes. If one can predict correctly that in a certain subensemble S' of S the results $+1$ and -1 will be found with respective probabilities $\omega(a+)$ and $\omega(a-)$ $[\omega(a+) + \omega(a-) = 1]$, then it is natural to conclude that the latter probabilities belong to S', in the sense that they are necessary consequences of some concrete physical property of S'. We can therefore assume that

If it is possible:

1) *to predict the existence of subset S' of S:*

$$S' = \{\alpha_{i1}, \ldots, \alpha_{il}\} \tag{2.88}$$

such that future measurements of $A(a)$ on S' will give the results $+1$ and -1 with the frequencies $\omega(a+)$ and $\omega(a-)$, respectively;

2) *to predict the population N' of S' $(0 < N' \leq N)$;*

3) *to make the previous predictions without disturbing in any way the α objects of S and S';*

then it will be said that a physical property Λ' belongs to S' that fixes the probabilities:

$$\omega(a+) = \Omega(a+, \Lambda'); \qquad \omega(a-) = \Omega(a-, \Lambda'). \tag{2.89}$$

The previous *probabilistic reality criterion* (PRC) provides a natural generalization of the famous criterion of (deterministic) reality proposed by Einstein, Podolsky and Rosen in 1935.

In EPR experiments S' will be discovered by performing measurements on the set T of the (distant) β particles individually correlated with the α particles of S: The observable $B(b)$ defines a subset $T' \subset T$, by means of a constant value, *e.g.*, $P(b) = +1$ (result of measurement). In such conditions it is often possible to predict that another observable $A(a)$ will be found to assume the values $+1$ and -1, upon measurement, with respective probabilities $\omega(a+)$ and $\omega(a-)$ in the subset S' of the α particles born together with those β particles that compose T'.

The PRC goes naturally together with the following *probabilistic locality principle* (PLP): *Measurements performed on the β systems do not modify the physical property Λ' belonging to the subset S' of S.*

A concrete measurement of $B(b)$ splits the ensemble E of (α, β) pairs

$$E = \{(\alpha_1, \beta_1), \ldots, (\alpha_N, \beta_N)\} \tag{2.90}$$

into two parts, $E(b+)$ and $E(b-)$, which contain all β particles for which $B(b) = +1$ and $B(b) = -1$ has, respectively, been found. Correspondingly, also the ensembles S and T will be split into two parts: $S(b+)$ and $T(b+)$, which compose physically $E(b+)$, and $S(b-)$ and $T(b-)$ which compose physically $E(b-)$. With the usual notation for conditional probabilities we write

$$\begin{cases} \omega(a+|b+): \text{probability of obtaining } A(a) = +1 \text{ in } E(b+), \\ \omega(a-|b+): \text{probability of obtaining } A(a) = -1 \text{ in } E(b+), \\ \omega(a+|b-): \text{probability of obtaining } A(a) = +1 \text{ in } E(b-), \\ \omega(a-|b-): \text{probability of obtaining } A(a) = -1 \text{ in } E(b-). \end{cases} \tag{2.91}$$

We assume that the previous probabilities can be predicted correctly either because we have a theory which we trust (*e.g.*, quantum theory) or because we have a large number of previous experiments that have taught us what values these probabilities will assume. We can then apply the PRC and attribute to $S(b+)$ and to $S(b-)$ physical properties, Λ_+ and Λ_- respectively, that fix the probabilities (2.91):

$$\begin{cases} \omega(a+|b+) = \Omega(a+, \Lambda_+), & \omega(a-|b+) = \Omega(a-, \Lambda_+); \\ \omega(a+|b-) = \Omega(a+, \Lambda_-), & \omega(a-|b-) = \Omega(a-, \Lambda_-). \end{cases} \tag{2.92}$$

The previous notation is not inconsistent because the physical properties Λ_+ and Λ_- are expected to depend on the observable $B(b)$ which determines the boundary between $S(b+)$ and $S(b-)$. This dependence has obviously nothing to do with a breakdown of locality, in spite of the fact that the probabilities (2.92) are ultimately measured on the α systems.

If a different observable $B(b')$ is considered, a different splitting of S in $S(b'+)$ and $S(b'-)$ is generated, and two new physical properties Λ'_+ and Λ'_- are attributed, using the PRC, to $S(b'+)$ and $S(b'-)$, respectively. One has, for example,

$$\omega(a+|b'+) = \Omega(a+, \Lambda'_+). \tag{2.93}$$

If the sets $S(b+)$, $S(b-)$ and $S(b+)$ were homogeneous, in the sense that the probabilities of $A(a) = \pm 1$ were the same for every possible subset of each of them, then one could reason as follows.

The set $S(b'+)$ is composed of pairs necessarily belonging either to $S(b+)$ or to $S(b-)$, with respective fractions γ and $(1-\gamma)$, say. The probability (2.93) should then be a weighted average of the first two probabilities in (2.92):

$$\omega(a+|b'+) = \gamma\,\omega(a+|b+) + (1-\gamma)\,\omega(a+|b-)\,. \tag{2.94}$$

Linear relations of this type should obviously hold (with a different γ) for all possible choices of b and b'. But (2.94) implies that the left-hand side, weighted average of $\omega(a+|b+)$ and $\omega(a+|b-)$, should in no case be external to the numerical interval of the two previous quantities, a generally unacceptable conclusion! (For example the quantum-mechanical probabilities deduced from the *singlet* state do not satisfy (2.94) for all possible choices of b and b'.)

We have so concluded that the subensembles $S(b\pm)$ cannot be homogeneous and that the observed probabilities must result of averages of probabilities in general different for different subsets composing $S(b\pm)$, and therefore S.

This conclusion can easily be extended to $T(a\pm)$ and T, and must therefore hold also for E which is the physical union of S and T.

In the previous reasoning locality has been used in an essential way because it has been assumed that the probabilities (2.93) emerge in case of measurement of $A(a)$ in some *unknown* subensembles of S also if no measurement of $B(b)$ is performed. The opposite assumption would imply that measurements on T create at a distance the properties $\Lambda_\pm$ of $S(b\pm)$.

2.3.3. Probabilities for EPR-type experiments. We have concluded that if PRC and PLP are correct assumptions an ensemble E of (α, β) pairs canot be homogeneous as far as the probabilities of finding $A(a) = \pm 1$ are concerned. A symmetrical conclusion holds for $B(b) = \pm 1$. The most general situation in a local realistic world is that E splits into a certain number of *homogeneous subensembles* [51]. Let us call these subsets

$$\sigma_1(a),\ \sigma_2(a),\ \dots,\ \sigma_m(a)$$

for $A(a) = \pm 1$, and

$$\tau_1(b),\ \tau_2(b),\ \dots,\ \tau_n(b)$$

for $B(b) = \pm 1$. Obviously

$$\sigma_1(a) \cup \sigma_2(a) \cup \dots \cup \sigma_m(a) = \tau_1(b) \cup \tau_2(b) \cup \dots \cup \tau_n(b) = E\,. \tag{2.95}$$

The previous notation reflects the fact that a subset like $\sigma_i(a)$ that is homogeneous for $A(a) = \pm 1$ is not in general expected to be homogeneous for $A(a') = \pm 1$, if $a' \neq a$.

One can easily find subensembles which are homogeneous for two observables. If $A(a)$ and $B(b)$ are considered, these subensembles are

$$E_k(a, b) = \sigma_i(a) \cap \tau_j(b)\,, \tag{2.96}$$

where the index k has been chosen to correspond to the pair of indices i, j. One can easily generalize to an arbitrary number of observables. Consider the following subensembles of E:

$$(2.97)\quad \begin{cases} \sigma_1(a_1),\ \sigma_2(a_1),\ \ldots,\ \sigma_{m_1}(a_1) & \text{homogeneous for } A(a_1)\,, \\ \sigma_1(a_2),\ \sigma_2(a_2),\ \ldots,\ \sigma_{m_2}(a_2) & \text{homogeneous for } A(a_2)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \sigma_1(a_r),\ \sigma_2(a_r),\ \ldots,\ \sigma_{m_r}(a_r) & \text{homogeneous for } A(a_r)\,, \\ \tau_1(b_1),\ \tau_2(b_1),\ \ldots,\ \tau_{n_1}(b_1) & \text{homogeneous for } B(b_1)\,, \\ \tau_1(b_2),\ \tau_2(b_2),\ \ldots,\ \tau_{n_2}(b_2) & \text{homogeneous for } B(b_2)\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \tau_1(b_s),\ \tau_2(b_s),\ \ldots,\ \tau_{n_s}(b_s) & \text{homogeneous for } B(b_s)\,. \end{cases}$$

The union of the ensembles of every line must in all cases give E, similarly to (2.95). By means of intersections one can generate smaller subensembles in which all the considered observables have constant probabilities. One has

$$(2.98)\quad E_k(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s) = \\ = \sigma_{i_1}(a_1) \cap \sigma_{i_2}(a_2) \cap \ldots \cap \sigma_{i_r}(a_r) \cap \tau_{j_1}(b_1) \cap \tau_{j_2}(b_2) \cap \ldots \cap \tau_{j_s}(b_s)$$

for a typical subensemble having homogeneous probabilities for all the considered dichotomic observables (in number of $r + s$). In the previous definition the single index k has been chosen, for simplicity, to correspond in a one-to-one way to the set of indices $(i_1, i_2, \ldots, i_r, j_1, j_2, \ldots, j_s)$. The notation can be simplified if one introduces a «vector» V having $r + s$ components in the following way:

$$(2.99)\quad V = (a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s)\,.$$

The homogeneous subensemble and its population can then, respectively, be written as

$$(2.100)\quad E_k(V) = E_k(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s)\,,$$

$$(2.101)\quad N_k(V) = N_k(a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s)\,.$$

Probabilities which are simultaneously constant for all the pairs of the subensemble $E_k(V)$ are

$$(2.102)\quad p_k(a\,\pm) \qquad (\text{with } a = a_1, a_2, \ldots, a_r)$$

constant for $A(a) = \pm 1$, and

$$(2.103)\quad q_k(b\,\pm) \qquad (\text{with } b = b_1, b_2, \ldots, b_s)$$

constant for $B(b) = \pm 1$.

It can be shown [51] that within every homogeneous subensemble the joint probability for fixed values of $A(a)$ and $B(b)$ is given by

$$\Omega_k(a\pm, b\pm) = p_k(a\pm)\, q_k(b\pm). \tag{2.104}$$

The same joint probability over E is then given by

$$\Omega(a\pm, b\pm) = \sum_k \rho_k(V)\, p_k(a\pm)\, q_k(b\pm), \tag{2.105}$$

where

$$\rho_k(V) = \frac{N_k(V)}{N}, \qquad \sum_k \rho_k(V) = 1, \tag{2.106}$$

where $\rho_k(V)$ is the *a priori* probability of the subensemble $E_k(V)$.

Equation (2.105) is in a way similar to the Clauser-Horne factorizability formula, the role of the hidden variable λ being here played by the index k. There is however also an important difference, because now the probability «density» (2.106) *depends on the values of the parameters of the considered observables.*

Equation (2.105) provides a satisfactory formulation of probabilistic local realism, since it is a consequence of the two basic assumptions of reality (PRC) and locality (PLP). It is a very simple exercise to show that the validity of Bell's inequality is a consequence of (2.105). The proof is straightforward and will not be repeated here.

2˙4. *Consequences of local realism.* – Local realism is certainly a simple and fundamental physical idea, but one is faced with the puzzling fact that it does *not* admit of an equally simple formulation at the empirical level: Bell's inequality is but one of an infinite set of equalities that can be deduced from local realism. All the meaningful inequalities can of course be obtained by using the formulation of probabilistic local realism presented in subsect. **2˙3**. We will, however, not discuss the proofs of the inequalities for the sake of brevity.

In paragraph **2˙4.1**, the inequalities valid for arbitrary linear combinations *of correlation functions* will be reviewed, while those holding for arbitrary linear combinations of *joint probabilities* will be reviewed in paragraph **2˙4.2**. The two sets are not equivalent, and the latter is more general than the former, as shown by the Garg-Mermin model for joint probabilities (paragraph **2˙4.3**).

2˙4.1. Inequalities for correlation functions. The first examples of inequalities providing physical restrictions not contained in Bell's inequality were given by Roy and Singh in 1978 [54]. They could show, for example, that local realism implies

$$\sum_{i=1}^{4}\sum_{j=1}^{5} C_{ij}\, P(a_i, b_j) \le 6, \tag{2.107}$$

where

$$(2.108) \qquad C_{ij} = \begin{pmatrix} 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & -1 & 1 & 0 \\ 1 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

Supposing, for example, that

$$(2.109) \qquad P(a_i, b_j) = \frac{1}{3}[2 + c_{ij}(1 - c_{ij})],$$

so that $P(a_i, b_j) = 0$ if $c_{ij} = -1$, and $P(a_i, b_j) = \frac{2}{3}$ otherwise, one can easily see that (2.107) is violated since its left-hand side equals $\frac{20}{3}$.

Any Bell's combinations of four of these correlation functions can instead take only the values $\frac{6}{3}$, $\frac{4}{3}$, 0, and thus none of the corresponding inequalities is violated. It is then clear that the set of 20 values (2.109) of $P(a, b)$ violate local realism in spite of the fact that it satisfies all conceivable Bell's inequalities.

That additional inequalities can be deduced from local realism was also noticed by Pearle [55] and by d'Espagnat [56]. Garuccio and Selleri [57] obtained an inequality for an arbitrary linear combination of correlation functions. They could show that, given the numerical coefficients c_{ij} to be real but otherwise arbitrary, and the correlation functions $P(a_i, b_j)$ with $i = 1, ..., n$ and $j = 1, ..., m$, local realism implies

$$(2.110) \qquad -M_0 \leq \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} P(a_i, b_j) \leq M_0$$

with

$$(2.111) \qquad M_0 = \operatorname*{Max}_{\xi, \eta} \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \xi_i \eta_j \right\},$$

where among all possible choices of the sign factors $\xi_i = \pm 1$ $(i = 1, ..., n)$ and $\eta_j = \pm 1$ $(j = 1, ..., m)$ one has to take the one giving the maximum value to the quantity within parenthesis in (2.111).

The whole story of this development, whith the various methods of proofs, is told in Garuccio's review paper [58]. Here only the four main points will be recalled:

1) Bell's inequality is a particular case of (2.110) with $m = n = 2$, with three c_{ij}'s equal to $+1$, and with the fourth one equal to -1.

2) All the physical restrictions of the set of the possible inequalities with $n = m = 2$ are given by Bell's inequality.

3) An inequality is *trivial* (it does not provide physical restrictions) if the c_{ij}'s have factorizable signs, that is, writing

$$c_{ij} = |c_{ij}|\,\sigma_{ij},$$

if

$$\sigma_{ij} = \mu_i \nu_j$$

with $\mu_i = \pm 1$ and $\nu_j = \pm 1$ ($i = 1, ..., n$; $j = 1, ..., m$).

4) There are inequalities providing physical restrictions on the $P(a_i, b_j)$'s which cannot be deduced by any Bell's inequality. The so-called *superinequalities* are of this type.

2.4.2. Inequalities for joint probabilities. The most general set of inequalities following from local realism has been found by Lepore [59]. Consider m dichotomic observables $A(a_1), ..., A(a_m)$ for the S ensemble and n dichotomic observables $B(b_1), ..., B(b_n)$ for the T ensemble. Let

$$\Omega(a_\mu\sigma, b_\nu\tau)$$

be the joint probability of measuring $A(a_\mu)$ on α and $B(b_\nu)$ on β and obtaining

$$A(a_\mu) = \sigma; \qquad B(b_\nu) = \tau,$$

where $\sigma, \tau = \pm 1$; $\mu = 1, ..., m$; $\nu = 1, ..., n$.

As we have seen in subsect. **2.3** one can write

$$\Omega(a_\mu\sigma, b_\nu\tau) = \sum_{i\in I} \frac{N_i}{N} p_i(a_\mu\sigma)\, q_i(b_\nu\tau), \tag{2.112}$$

where

$$N_i = N_i(a_1, ..., a_m, b_1, ..., b_n) \tag{2.113}$$

and where the set of integers I depends on the relevant observables

$$I = I(a_1, ..., a_m, b_1, ..., b_n). \tag{2.114}$$

Naturally, one has

$$\sum_{i\in I} \frac{N_i}{N} = 1. \tag{2.115}$$

The following conditions hold good:

$$0 \le p_i(a_\mu\sigma) \le 1, \qquad 0 \le q_i(b_\nu\tau) \le 1, \tag{2.116}$$

$$(2.117) \qquad p_i(a_\mu +) + p_i(a_\mu -) = 1, \qquad q_i(b_\nu +) + q_i(b_\nu -) = 1,$$

where $\mu = 1, \ldots, m$; $\nu = 1, \ldots, n$; $i \in I$.

Consider next the linear combination of joint probabilities

$$(2.118) \qquad L = \sum_{\substack{\sigma, \tau \\ \mu, \nu}} c_{\mu\nu}^{\sigma\tau} \Omega(a_\mu \sigma, b_\nu \tau),$$

where $c_{\mu\nu}^{\sigma\tau}$ are $4mn$ real coefficients, otherwise arbitrary.

The linear function associated with (2.118) is defined to be the function $F(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$ obtained from (2.118) with the substitutions

$$(2.119) \qquad \begin{cases} \Omega(a_\mu +, b_\nu +) \rightarrow \varepsilon_\mu \eta_\nu, \\ \Omega(a_\mu +, b_\nu -) \rightarrow \varepsilon_\mu (1 - \eta_\nu), \\ \Omega(a_\mu -, b_\nu +) \rightarrow (1 - \varepsilon_\mu) \eta_\nu, \\ \Omega(a_\mu -, b_\nu -) \rightarrow (1 - \varepsilon_\mu)(1 - \eta_\nu). \end{cases}$$

That is

$$(2.120) \qquad F(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n) = \sum_{\mu, \nu} [C_{\mu\nu}^{++} \xi_\mu \eta_\nu + C_{\mu\nu}^{+-} \xi_\mu (1 - \eta_\nu) + \\ + C_{\mu\nu}^{-+} (1 - \xi_\mu) v_\eta + C_{\mu\nu}^{--} (1 - \xi_\mu)(1 - \eta_\nu)].$$

It is defined in the hypercube $\mathscr{C} \subseteq \boldsymbol{R}^{m+n}$

$$\mathscr{C} = \{(\xi_1, \ldots, \eta_n) \boldsymbol{R}^{m+n} | 0 \leq \xi_1, \ldots, \eta_n \leq 1\}.$$

The points at which all the variables $\xi_1, \ldots, \eta_n$ take the values 1 or 0 are the vertices of the hypercube $\mathscr{C}$.

Obviously, one has

$$\sum_i \frac{N_i}{N} F[p_i(a_1 +), \ldots, p_i(a_m +), q_i(b_1 +), \ldots, q_i(b_n +)] = L.$$

F is linear in the variables $\xi_1, \ldots, \eta_n$ and therefore its maximum and minimum can be found on the vertices of $\mathscr{C}$. Therefore, setting

$$m = \min_{\xi_1, \ldots, \eta_n = 0,1} \{F(\xi_1, \ldots, \eta_n)\}, \qquad M = \operatorname{Max}_{\xi_1, \ldots, \eta_n = 0,1} \{F(\xi_1, \ldots, \eta_n)\},$$

one has

$$(2.121) \qquad m \leq F(p_i(a_1 +), \ldots, q_i(b_n +)) \leq M.$$

The function F can also be used for writing down the linear combinations of the joint probabilities. From (2.118) one has

$$\sum_{\mu,\nu} C^{\sigma\tau}_{\mu\nu}\Omega(a_\mu\sigma,\ b_\nu\tau) = \sum_{i\in I}\frac{N_i}{N}F(p_i(a_1+),\ \dots,\ q_i(b_n+)) \tag{2.122}$$

and from (2.121), (2.122) and using (2.115) one obtains the set of inequalities

$$m \leq \sum_{\substack{\sigma,\tau\\ \mu,\nu}} C^{\sigma\tau}_{\mu\nu}\Omega(a_\mu\sigma,\ b_\nu\tau) \leq M \tag{2.123}$$

The previous set gives inequalities for all possible linear combinations of joint probabilities. For every choice of the coefficient $C^{\sigma\tau}_{\mu\nu}$, eq. (2.123) provides the most stringent inequality that can be deduced from local realism [59].

Consider the case of three observables $A(a_1)$, $A(a_2)$, $A(a_3)$ for the set S of α particles and of three observables $B(b_1)$, $B(b_2)$, $B(b_3)$ for the set T of β particles and consider the following linear combination of joint probabilities:

$$\begin{aligned}\Omega(a_3+,b_3+) &- \Omega(a_1-,b_2-) + \Omega(a_1-,b_1+) +\\ &+\Omega(a_2+,b_2-) + \Omega(a_3-,b_2-) - \Omega(a_3-,b_1+) +\\ &+2\Omega(a_1+,b_2+) + \Omega(a_2-,b_3-) - \Omega(a_1+,b_3-)\,.\end{aligned} \tag{2.124}$$

For such a linear combination the associated F function is

$$\begin{aligned}F(\xi_1,\xi_2,\xi_3,\eta_1,\eta_2,\eta_3) &= \xi_3\eta_3 - (1-\xi_1)(1-\eta_2) + (1-\xi_1)\eta_1 + \xi_2(1-\eta_2) +\\ &+(1-\xi_3)(1-\eta_2) - (1-\xi_3)\eta_1 + 2\xi_1\eta_2 + (1-\xi_2)(1-\eta_3) - \xi_1(1-\eta_3)\,.\end{aligned}$$

By calculating the value of F over the 2^6 vertices of the hypercube one gets

$$\min F = 0\,.$$

Therefore, one particular inequality of the set (2.123) is given by

$$\begin{aligned}\Omega(a_3+,b_3+) &- \Omega(a_1-,b_2-) + \Omega(a_1-,b_1+) +\\ &+\Omega(a_2+,b_2-) + \Omega(a_3-,b_2-) - \Omega(a_3-,b_1+) +\\ &+2\Omega(a_1+,b_2+) + \Omega(a_2-,b_3-) - \Omega(a_1+,b_3-) \geq 0\,.\end{aligned} \tag{2.125}$$

2·4.3. A nonlocal model. Inequality (2.125) contains physical restrictions not deducible from Bell's inequality. In fact a numerical example of probabilities satisfying the inequalities of Bell's type but violating (2.125) was given by Garg and Mermin [60]. If one writes

$$\Omega(a_{i\pm},\ b_{k\pm}) \equiv \Omega_{ik}(\sigma,\ \tau)\,, \tag{2.126}$$

where $\sigma = \pm$ and $\tau = \pm$ are sign factors, the model proposed by Garg and

Mermin is

$$\Omega_{ik}(\sigma, \tau) = \frac{1}{4}[1 - c(\sigma + \tau) + P_{ik}(\sigma\tau)], \tag{2.127}$$

where $i, k = 1, 2, 3$; $0 < c \leq \frac{1}{3}$; $P_{11} = P_{22} = 1$ and $P_{ik} = -\frac{1}{3}$ in all other cases. It is very easy to show that all the inequalities of the type (2.66) that can be written with the nine quantities (2.127) reduce to

$$|P_{ik} - P_{il} + P_{jk} + P_{jl}| \leq 2 \tag{2.128}$$

so that they are always satisfied. Notice that the coefficient c has disappeared from the locality conditions (2.128).

By substituting in (2.125) the probabilities of the Garg-Mermin model one gets

$$-c \geq 0\,. \tag{2.129}$$

This contradicts the definition of c: hence, the Garg-Mermin probabilistic model violates the particular inequality (2.125). This shows that this model is nonlocal. However, the same model satisfies all the inequalities for linear combinations of correlation functions, as it is easy to check by noticing that in all cases the correlation function associated with (2.127) coincides with the quantity P_{ik}.

Other interesting consequences of local realism were found by Garg and Mermin [61] who were able to deduce Bell-type inequalities for two spin-j particles (with arbitrary j). They could show that the singlet state for two particles with spin j leads to violations of local realism for arbitrarily large values of j right up and above the threshold of the classical world. But in the classical domain it is always possible to assign *a priori* well-defined values to all observable quantities. This result of Garg and Mermin is disturbing for the coherence and rationality of the existing quantum theory, which seems to extend its «magic» predictions also to the macroscopic domain where classical physics had successfully banished all «magical» approaches.

2'5. *The experimental evidence.* – The present subsection is devoted to the discussion of those experiments on the EPR paradox which have been (or can be) performed with atomic photon pairs. A related interesting topic is the possibility to perform experiments on the EPR paradox with pairs of subatomic particles (*e.g.*, with K^0-$\bar{K}^0$ pairs). To this the following paragraph is devoted.

Paragraph **2**'5.1 reviews the quantum-mechanical predictions for the physical cases in which excited atoms can emit EPR correlated pairs of photons. A concise review of the published experiments is contained in paragraph **2**'5.2, while **2**'5.3 deals with ideas of experiments to be performed in future with pairs of atomic photons.

2'5.1. Quantum states and probabilities. Actual experiments on the EPR paradox have nearly all been carried out with photons. The quantum-mechanical treatment of photon polarization is similar to that of spin-(1/2) in one important respect: both the observables are dichotomic. The absence of photon rest mass has the effect of eliminating from the theoretical scheme the longitudinal polarizations. Only the linear polarization states perpendicular to the direction of propagation are left, similar to the case of classical electromagnetic waves whose tranverse nature is well known. Considering also the states of circular polarization one can define

$|R\rangle$: single-photon state with right-handed circular polarization;

$|L\rangle$: single-photon state with left-handed circular polarization;

$|x\rangle$: single-photon state with linear polarization along the x-axis;

$|y\rangle$: single-photon state with linear polarization along the y-axis.

Elementary quantum theory gives

$$\begin{cases} |R\rangle = \dfrac{1}{\sqrt{2}}\{|x\rangle + i|y\rangle\}, \\[2ex] |L\rangle = \dfrac{1}{\sqrt{2}}\{|x\rangle - i|y\rangle\}. \end{cases} \tag{2.130}$$

The existence of dichotomic observables for photons has the important implication that the Bell-type inequalities can be formulated for pairs of photons. There are situations were quantum theory describes the polarization of two correlated photons with nonfactorizable state vectors analogous to the singlet state of two spin-(1/2) objects which lead to violations of local realism.

In the case of photons the parity quantum number plays an important role and it is necessary to distinguish, for example, the $J^P = 0^+$ from the $J^P = 0^-$ states, represented respectively by the state vectors

$$\begin{cases} |0^+\rangle = \dfrac{1}{\sqrt{2}}\{|R_\alpha\rangle|R_\beta\rangle + |L_\alpha\rangle|L_\beta\rangle\}, \\[2ex] |0^-\rangle = \dfrac{1}{\sqrt{2}}\{|R_\alpha\rangle|R_\beta\rangle - |L_\alpha\rangle|L_\beta\rangle\}. \end{cases} \tag{2.131}$$

These states can also be expressed in terms of linear polarizations by using (2.130) both for the photon α and for the photon β. One then obtains

$$\begin{cases} |0^+\rangle = \dfrac{1}{\sqrt{2}}\{|x_\alpha\rangle|x_\beta\rangle + |y_\alpha\rangle|y_\beta\rangle\}, \\[2ex] |0^-\rangle = \dfrac{1}{\sqrt{2}}\{|x_\alpha\rangle|y_\beta\rangle - |y_\alpha\rangle|x_\beta\rangle\}. \end{cases} \tag{2.132}$$

The basic states with respect to which the linear polarization is expressed are arbitrary: Using the rotated x'- and y'-axes one obtains results identical to (2.132) for both states, with x' and y' in place of x and y. This property is due to the invariance under rotations around the z-axis of the zero angular momentum states.

All the inequalities of the Bell type (weak and strong) discussed in the previous sections clearly apply also to photon pairs, since they were deduced from the dichotomic nature of the measured quantities, besides, of course, from locality and realism. In order to check that the quantum-mechanical predictions violate those inequalities we will give the theoretical formulae for the most important probabilities and correlation functions. This will allow us to stress once more the very important distinction between weak inequalities (*e.g.*, Bell's original inequality) and the strong ones.

Widely used is the $(J=0)\rightarrow(J=1)\rightarrow(J=0)$ cascade of calcium. The quantum-mechanical predictions, following from the state $|0^+\rangle$, which apply to this case are for the double-transmission probabilities:

$$(2.133)\qquad \begin{cases} T(a_+, b_+) = \dfrac{1}{4}[\varepsilon^1_+ \varepsilon^2_+ + \varepsilon^1_- \varepsilon^2_- F_1(\theta)\cos 2(a-b)]\,, \\[2ex] T(a_+, \infty) = \dfrac{1}{2}\varepsilon^1_+\,, \\[2ex] T(\infty, b_+) = \dfrac{1}{2}\varepsilon^2_+\,, \\[2ex] T(\infty, \infty) = 1\,. \end{cases}$$

These relations give the correlation function $P(a, b)$ through

$$(2.134)\qquad P(a, b) = (1-\varepsilon^1_+)(1-\varepsilon^2_+) + \varepsilon^1_- \varepsilon^2_- F_1(\theta)\cos 2(a-b)\,.$$

In these relations $F_1(\theta)$ is a function of the half-angle θ subtended by the primary lenses representing a depolarization due to noncollinearity of the photons and

$$(2.135)\qquad \varepsilon^1_\pm = \varepsilon^1_M \pm \varepsilon^1_m\,; \qquad \varepsilon^2_\pm = \varepsilon^2_M \pm \varepsilon^2_m\,.$$

Here $\varepsilon^1_M\,(\varepsilon^1_m)$ is the transmittance of the first polarizer for light polarized parallel (perpendicular) to the polarizer axis; and a similar notation has been used for the second polarizer. All these transmittances are usually very close to the ideal case, with ε^i_M close to unity and ε^i_m close to zero ($i = 1, 2$). The depolarization factor F_1 is also usually very close to unity, so that $P(a, b)$ as given by (2.134) violates Bell's inequality. As already stressed, the difficulty is, however, that transmission probabilities *are not measurable*, so that Bell's inequality cannot be tested.

If the CHSH additional assumption of subsect. **2·2** is made, the double detection probability D_0 becomes a crucial quantity, which is assumed to be independent of the presence and the orientation of the polarizers. Quantum theory predicts

$$D_0 = \eta_1 \eta_2 , \tag{2.136}$$

where η_1 (η_2) is the quantum efficiency of the first (second) photomultiplier. In the performed experiments η_1 and η_2 were of the order ot 10%, so that D_0 was of the order of 10^{-2}. The latter quantity relates the double-transmission probabilities T to the measurable double transmission *and* detection probabilities Ω, *once the* CHSH *additional assumption has been made.*

In the usual quantum theory the CHSH assumption does not need to be explicitly made, its validity being always taken for granted. The quantum-mechanical expressions for the Ω probabilities defined in (2.53) are

$$\begin{cases} \Omega(a, b) = \frac{1}{4}[\varepsilon_+^1 \varepsilon_+^2 + \varepsilon_-^1 \varepsilon_-^2 F_1(\theta)\cos 2(a-b)]\eta_1\eta_2 , \\ \Omega(a, \infty) = \frac{1}{2}\varepsilon_+^1 \eta_1\eta_2 , \\ \Omega(\infty, b) = \frac{1}{2}\varepsilon_+^2 \eta_1\eta_2 , \\ \Omega(\infty, \infty) = \eta_1\eta_2 . \end{cases} \tag{2.137}$$

These double-detection and transmission probabilites are obviously proportional to the respective coincidence rates R (see eq. (2.54)), the proportionality factor being N_0, the number of photon pairs entering, per second, in the appropriate solid angles defined by the optical apparatus. The strong inequality (2.57) can therefore be written in the form

$$-1 \le \frac{\Omega(a, b)}{D_0} - \frac{\Omega(a, b')}{D_0} + \frac{\Omega(a', b)}{D_0} + \frac{\Omega(a', b')}{D_0} - \frac{\Omega(a', \infty)}{D_0} - \frac{\Omega(\infty, b)}{D_0} \le 0 , \tag{2.138}$$

and it can easily be shown to be violated by the quantum-mechanical predictions (2.137). Experimentally it has been found to be violated. One should remember that (2.57), as well as (2.138), are consequences of local realism in *conjunction with* the additional assumption. Its violation can only mean that one of these tenets is wrong, it cannot decide which one. It is, for example, possible to build explicit local realistic models that do not satisfy the CHSH additional assumption, and that violate (2.138).

Notice that (2.138) essentially coincides with (2.69) deduced with the help of the CH additional assumption. If instead the inhomogeneous inequality (2.66) is considered, which is deduced only from local realism, one can see that the quantum-theoretical predictions for the single-photon transmission and detection probabilities are

$$p(a') = \frac{1}{2}\varepsilon_+^1\eta_1 ; \quad q(b) = \frac{1}{2}\varepsilon_+^2\eta_2 . \tag{2.139}$$

Owing to the presence of a single η-factor these probabilities are an order of magnitude larger than the double-transmission and detection probabilities written in (2.138). This implies that (2.66) is never violated in actual experiments.

Coming now to Rapisarda-type experiments with two-ways polarizers, the quantum-mechanical predictions for the Ω probabilities defining $E(a, b)$ (see (2.82)) are

$$(2.140)\quad \begin{cases} \Omega(a_+, b_+) = \dfrac{1}{4}[T^1_+ T^2_+ + T^1_- T^2_- F_1(\theta)\cos 2(a-b)]\,\eta_1\eta_2\,, \\ \Omega(a_+, b_-) = \dfrac{1}{4}[T^1_+ R^2_+ - T^1_- R^2_- F_1(\theta)\cos 2(a-b)]\,\eta_1\eta_2\,, \\ \Omega(a_-, b_+) = \dfrac{1}{4}[R^1_+ T^2_+ - R^1_- T^2_- F_1(\theta)\cos 2(a-b)]\,\eta_1\eta_2\,, \\ \Omega(a_-, b_-) = \dfrac{1}{4}[R^1_+ R^2_+ + R^1_- R^2_- F_1(\theta)\cos 2(a-b)]\,\eta_1\eta_2\,, \end{cases}$$

where

$$T^i_+ = T^i_\parallel + T^i_\perp; \qquad T^i_- = T^i_\parallel - T^i_\perp\,,$$

and

$$R^i_+ = R^i_\parallel + R^i_\perp; \qquad R^i_- = R^i_\parallel - R^i_\perp\,,$$

with $i = 1, 2$. The T and R parameters are transmittances defined in the following way. There are two prisms, denoted with the index $i = 1, 2$ above. From each prism two beams emerge, a reflected one and a transmitted one.

$T_\parallel$ ($T_\perp$) is the prism transmittance along the transmitted path for incoming light polarized parallel (perpendicular) to the polarization plane of the transmitted channel;

$R_\parallel$ ($R_\perp$) is the prism transmittance along the reflected path for incoming light polarized parallel (perpendicular) to the polarization plane of the reflected channel.

A measurement [62] gave, for example,

$$T_\parallel = 0.9095 \pm 0.0023; \qquad T_\perp = 0.0044 \pm 0.0002\,,$$
$$R_\parallel = 0.7625 \pm 0.0024; \qquad R_\perp = 0.0041 \pm 0.0003\,.$$

Insertion of (2.140) in (2.82) gives

$$(2.141)\qquad E(a, b) = \frac{f + g\cos 2(a-b)}{f' + g'\cos 2(a-b)}\,,$$

where

$$f = (T^1_+ - R^1_+)(T^2_+ - R^2_+); \qquad g = (T^1_- + R^1_-)(T^2_- + R^2_-),$$

$$f' = (T^1_+ + R^1_+)(T^2_+ + R^2_+); \qquad g' = (T^1_- - R^1_-)(T^2_- - R^2_-).$$

Garuccio and Rapisarda [50] showed that the predictions (2.140) violate the inequality (2.86) if $E(a, b)$ is defined as in (2.82). The violation can be as large as 50%.

2'5.2. Experiments with pairs of atomic photons. In the present paragraph we will review the ten published experiments on the EPR paradox which were performed by using atomic photon pairs.

1) Freedman and Clauser [63].

In this experiment the $3d4p\,^1P_1$ state of calcium in a beam was excited by radiation from a deuterium arc lamp. About 10% of the atoms go to the $4p^2\,^1S_0$ state which is the initial state of the (0, 1, 0) cascade emitting photons of wavelengths 551.3 nm and 422.7 nm. Since the natural calcium used in this experiments was an almost pure sample of the isotope with zero nuclear spin, there was no significant reduction expected in the polarization correlation due to hyperfine structure. On each side of the source the photons were collected and collimated by a lens, then passed through a filter and a linear polarizer to a photomultiplier. Freedman and Clauser used pile-of-plates polarizers, each of which was about 1 m in length and consisted of ten glass sheets inclined nearly at Brewster's angle.

The photomultiplier pulses were fed to a coincidence circuit and coincidence measurements were made for 100 s time intervals, the intervals during which all the plates were removed alternating with intervals in which the plates were inserted. The results obtained, as the relative orientation of the transmission axes of the polarizers was varied, were found to be in good agreement with the quantum-mechanical predictions. The strong inequality (2.61) («Freedman's inequality») can be written as

$$\delta \le 0.250,$$

where

$$\delta = \left| \frac{R(22.5°)}{R_0} - \frac{R(67.5°)}{R_0} \right|. \tag{2.142}$$

The results for $R(22.5°)$ and for $R(67.5°)$, combined with those for R_0 with both the polarizers removed, yielded $\delta = 0.300 \pm 0.008$, in clear violation of Freedman's strong inequality and in agreement with the quantum-mechanical prediction $\delta_{qm} = 0.301 \pm 0.007$.

2) Holt and Pipkin [64]

In the second experiment, the 567.6 nm and 404.7 nm photons emitted in the (1, 1, 0) cascade of the zero nuclear spin isotope ^{198}Hg of mercury were observed. Since the final cascade level is not the ground state of the atom, no precaution had to be taken to avoid the effects of resonance trapping observed in the Freedman-Clauser experiment [65]. To produce the required radiation, mercury vapour was excited to the 9^1P_1 state by a 100 eV electron beam, both the beam and the vapour were contained in an encapsulated source made by Pirex glass. Calcite polarizers were used. These polarizers have a much better extinction ratio than pile-of-plates polarizers, but the values of ε_M are somewhat low (see table I).

TABLE I. – *Optical transmittance of the two polarizers.*

Reference	ε_M^1	ε_m^1	ε_M^2	ε_m^2
[63]	0.97 ± 0.01	0.038 ± 0.004	0.96 ± 0.01	0.037 ± 0.04
[64]	0.910 ± 0.001	$< 10^{-4}$	0.880 ± 0.001	$< 10^{-4}$
[67]	$\simeq 0.965$	$\simeq 0.011$	$\simeq 0.972$	$\simeq 0.008$
[69]	0.98 ± 0.01	0.02 ± 0.005	0.97 ± 0.01	0.02 ± 0.005
[70]	0.971 ± 0.005	0.029 ± 0.005	0.968 ± 0.005	0.028 ± 0.005
[72]	0.950 ± 0.005	0.007 ± 0.005	0.930 ± 0.005	0.007 ± 0.005
[73]	0.96 ± 0.01	0.005 ± 0.005	0.93 ± 0.01	0.007 ± 0.005
[87]	0.9095 ± 0.0023	0.0044 ± 0.0002	0.7625 ± 0.0024	0.0041 ± 0.000

Experimentally it was found that $\delta = 0.216 \pm 0.013$, a result which disagrees with the quantum-mechanical prediction $\delta_{qm} = 0.266$ and clearly does *not* violate the strong inequality. This discrepancy has never been completely explained. It has been suggested [66] that there may be some significance attached to the use of calcite polarizers.

3) Clauser [67]

Clauser repeated the Holt-Pipkin experiment using the same cascade but involving the ^{202}Hg isotope of mercury. Also, instead of calcite polarizers, he used pile-of-plates polarizers. This experiment gave $\delta = 0.2885 \pm 0.0093$, violating Freedman's strong inequality and in close agreement with the quantum-mechanical prediction $\delta_{qm} = 0.2841$.

In an extension of the previous experiment Clauser [68] measured the *circular* polarization correlation by inserting quarter-wave plates between each linear polarizer and the source. The quarter-wave plates were obtained by applying pressure to bars of commercial grade quartz. Assuming ideal quarter-wave plates, quantum mechanics predicts that the zero angular-momentum state vectors (2.131) should remain unmodified after the two photons have crossed the plates. Therefore (2.142) also remains a valid form of Freedman's strong inequality. From the experimental results Clauser found $\delta = 0.235 \pm 0.025$, while, taking into account the transmission efficiencies of the polarizers, and the assumed lack of stability of the quarter-wave plates, he obtained from the theory $\delta_{qm} = 0.252$ (which almost does not violate (2.142).

Within the limit of experimental errors these circular-polarization results were in agreement with quantum mechanics, but the agreement was not very satisfactory.

4) Fry and Thompson [69]

These authors used the 435.8 nm and 253.7 nm photons emitted in the (1, 1, 0) cascade using the zero nuclear spin isotope ^{200}Hg of mercury (see table II). The 7^3S_1 state of a mercury beam was populated in a two-step process and the electron bombardment excitation of the 6^3P_2 metastable state followed downstream, where all short-lived states had decayed, by absorption of resonant radiation from a tunable dye laser. The laser band width was narrow enough so that the ^{200}Hg isotope could be selectively excited. The polarizers used in this experiment were of the pile-of-plates variety, and the magnetic field in the interaction volume was reduced to less than 5 mG.

TABLE II. – *Experiments using atomic cascades to test the strong inequalities.*

Reference	Atom	Cascade	λ_1	λ_2
[63]	^{40}Ca	$4p^2\,{}^1S_0 \to 4p4s\,{}^1P_1 \to 4s^2\,{}^1S_0$	551.3	422.7
[64]	^{198}Hg	$9^1P_1 \to 7^3S_1 \to 6^3P_0$	567.6	404.7
[67]	^{202}Hg	$9^1P_1 \to 7^3S_1 \to 6^3P_0$	567.6	404.6
[68]	^{202}Hg	$9^1P_1 \to 7^3S_1 \to 7^3P_0$	567.6	404.6
[69]	^{200}Hg	$7^3S_1 \to 6^3P_1 \to 6^1S_0$	435.8	253.7
[70]	^{40}Ca	$4p^2\,{}^1S_0 \to 4p4s\,{}^1P_1 \to 4s^2\,{}^1S_0$	551.3	422.7
[72]	^{40}Ca	$4p^2\,{}^1S_0 \to 4p4s\,{}^1P_1 \to 4s^2\,{}^1S_0$	551.3	422.7
[73]	^{40}Ca	$4p^2\,{}^1S_0 \to 4p4s\,{}^1P_1 \to 4s^2\,{}^1S_0$	551.3	422.7
[87]	^{40}Ca	$4p^2\,{}^1S_0 \to 4p4s\,{}^1P_1 \to 4s^2\,{}^1S_0$	551.3	422.7

Since the initial state of the cascade had $J = 1$, it was necessary to take into account the possibility of unequal population of, and coherence between, the initial Zeeman sublevels, which Fry and Thompson did by measuring the polarization of the 435.8 nm radiation. Allowing for such effects and considering the transmission efficiencies of table I it was predicted that $\delta_{qm} = 0.294 \pm 0.007$, while the experiment gave $\delta = 0.296 \pm 0.014$, in agreement with quantum theory, but in violation of Freedman's strong inequality.

5) Aspect, Grangier and Roger [70]

Aspect, Grangier and Roger (AGR) used the 551.3 nm and 422.7 nm photons from the (0, 1, 0) cascade of calcium. In their case the calcium atoms were excited to the $4p^2\ {}^1S_0$ state by a nonresonant two-photon absorption process using a krypton-ion laser beam of wavelength 406 nm and a dye laser beam tuned to 581 nm, both laser beams being at right angles to the calcium atomic beam emitted from a tantalum oven. The laser beams were focused at the interaction region to provide a source of about 60 m in diameter by 1 mm long. The density varied between $3 \cdot 10^{10}$ cm^{-3} and 10^{11} cm^{-3}, resulting in cascade rates equal to or higher than $4 \cdot 10^{7}$ s^{-1}. Selective excitation of the ^{40}Ca isotope of calcium prevented the polarization correlation from being reduced by hyperfine-structure effects. The photons from the cascade were analysed by polarizers of the pile-of-plates type and filters in much the same way as in the previous experiments.

The high atomic density of the source generated coincidence rate of up to 100 s^{-1}, allowing a 1% statistical accuracy in only 100 s counting time. Measuring $R(22.5°)$, $R(67.5°)$ and R_0, AGR obtained $\delta = 0.3072 \pm 0.0043$, in agreement with the quantum-mechanical prediction of $\delta_{qm} = 0.308 \pm 0.002$, but in violation of Freedman's (strong) inequality, by more than 13 standard deviations. An even stronger violation of 40 standard deviations was reported by Aspect at the Perugia conference [71].

6) Aspect, Grangier and Roger [72]

In 1982 AGR performed an experiment, originally suggested and analysed by Garuccio and Rapisarda [50], using two-channel polarizers instead of the previous one-channel pile-of-plates type. Each polarizer was a polarizing cube, built using the properties of dielectric thin films and antireflection coated and was rotatable about the observation axis. The arrangement allowed the quantity $E(a, b)$ defined in eq. (2.82) to be measured directly in a single run, using a fourfold coincidence technique for each of the four relative orientations of the polarizers: (a, b), (a, b'), (a', b) and (a', b').

If the left-hand side of the (strong) inequality (2.86) is called S, AGR found experimentally $S = 2.697 \pm 0.015$ in violation of (2.86) itself, but in full agreement with the quantum-mechanical prediction $S_{qm} = 2.70 \pm 0.05$.

7) Aspect, Dalibard and Roger [73]

In all experiments described so far the transmission axes of the polarizers were held fixed during every set of measurements. Thus, there was the possibility of an exchange of information between the two polarizers with a velocity not exceeding that of light. Such a possibility, although very unlikely, given the known nature of interactions, could be ruled out if the settings of the polarizers were changed in a time shorter than the time of flight of the photons from the source to the polarizers. In the experiment performed by Aspect, Dalibard and Roger (ADR) an optical switch rapidly redirected the light incident from the source to one of two polarizing cubes on each side of the source. In contrast to the previous experiment only the transmitting channels of the polarizing cubes were used. The switching of the light was obtained by a Bragg reflection from an ultrasonic standing wave in water. The light was completely transmitted when the amplitude of the standing wave was zero, and was almost fully deflected through 10 mrad when the amplitude was a maximum. Switching between the two channels occurred about once every 10 ns and since this time, as well as the 5 ns lifetime of the intermediate level of the cascade was smaller than L/c (40 ns), where $L = 12$ m was the separation between the two switches and c the speed of light, a detection event on one side and the corresponding change of orientation on the other side were separated by a spacelike interval.

In the ADR experiment the coincidence rates were only a few per second, with an accidental background of about one per second. If U is the intermediate quantity in the (strong) inequality (2.57), ADR found experimentally $U = 0.101 \pm 0.020$, in clear violation of (2.57) itself, but in agreement with the quantum-mechanical prediction $U = 0.112$.

Although the switching was in practice periodic rather than random, the switches on the two sides were driven by two different generators at different frequencies and it was *assumed* that they functioned in an uncorrelated way. That this situation can instead hide a conceptual difficulty has been shown by Zeilinger [74].

Finally it should be noted that some criticisms [75, 76] have been made of the AGR and ADR experiments on the ground that there may have been significant resonance trapping due to the high density of the calcium source. A reply to this criticism has been given by Aspect and Grangier [77].

8) Perrie, Duncan, Beyer and Kleinpoppen [78]

Perrie, Duncan, Beyer and Kleinpoppen (PDBK) measured for the first time the polarization correlation of the two photons emitted simultaneously by metastable atomic deuterium in a true second-order decay process. Single-photon decay from the $2S_{1/2}$ state of deuterium is forbidden and the main channel for the de-excitation is by the simultaneous emission of two photons which can have any wavelength consistent with energy conservation for the pair. However, in practice, because of the absorption in oxygen, the observation window was limited between 185 nm and 355 nm.

In the PDBK experiment a 1 keV metastable atomic deuterium beam of density of about 10^4 cm^{-3} was produced by charge exchange, in cesium vapour, of deuterons extracted from a radiofrequency ion source. Electric field pre-quench plates upstream from the observation region allowed the $2S_{1/2}$ component of the beam to be switched on and off by Stark mixing the $2S_{1/2}$ and $2P_{1/2}$ states and, at the and of the apparatus, the beam was fully quenched so that the resulting Lyman-signal could be used to normalize the two-photon coincidence signal. The two-photon radiation was collected and collimated by a pair of lenses and the polarizers were of the pile-of-plates type.

Measuring $R(22.5°)$, $R(67.5°)$ and R_0, PBDK obtained $\delta = 0.268 \pm 0.010$, in agreement with the quantum-theoretical prediction $\delta_{qm} = 0.272 \pm 0.008$, but in violation of Freedman's strong inequality by slightly more than two standard deviations.

In an extension of the previous experiment [79] the circular polarization correlation was measured by placing achromatic quarter-wave plates in each detection arm between the linear polarizer and the source. The obtained results did *not* violate Freedman's inequality and would have disagreed with the quantum-theoretical predictions *if* the quarter-waves plates had been assumed to be perfectly achromatic. With the introduction of a considerable degree of achromaticity PDBK could reconcile their observation with theory, considering also imperfect parallelism of the incoming photons. It is surprising however that the only two measurements of circular polarizations in EPR experiments (the present one and the already discussed Clauser [68] experiment) led to results of difficult interpretation.

9) Hassan, Duncan, Perrie, Beyer and Kleinpoppen [80]

These authors inserted in the apparatus of the previous experiment a $\lambda/2$ plate in one detection arm between the polarizer and the photomultiplier. By rotating the fast axis of the half-wave plate through half the angle of rotation of the transmission axis of the linear polarizer, it was possible to ensure that the planes of polarization of the two photons were always parallel just prior to detection in the photomultipliers. The results of this experiment led to $\delta = 0.271 \pm 0.021$, in violation of Freedman's strong inequality and in agreement with the quantum-mechanical prediction $\delta_{qm} = 0.272 \pm 0.008$.

Garuccio and Selleri [81] had proposed a mechanism accounting for the observed violation of the CHSH additional assumption, which was based on the idea of polarization-dependent «enhanced» detection of the photon pairs. The results of this experiment are inconsistent with the Garuccio-Selleri mechanism which is, therefore, ruled out. There remain other possible mechanisms, as shown in the final section of this review.

10) Hassan, Duncan, Perrie, Beyer and Kleinpoppen [80]

In a further experiment an additional linear polarizer was inserted on one arm of the detection systems: in this way the first photon crossed *two* polarizers (a, a') while the second one crossed just one polarizer (b). The orientation of polarizer a was held fixed, while polarizer b was rotated through an angle $a - b$ in a clockwise sense and polarizer a' through an angle $a' - a$ in the opposite sense. The ratio $R(a-b;\ a'-a)/R(a-b;\ \infty)$ was measured, where $R(a-b;\ a'-a)$ was the coincidence rate with all polarizers in place and $R(a-b;\ \infty)$ the coincidence rate with polarizer a' removed. The results are in good agreement with the quantum-mechanical predictions thus indicating once more that a polarization-dependent «enhanced» photon detection was not responsible for the violations of strong inequalities.

A summary of some of the experimental results on the EPR paradox discussed above is given in table III.

TABLE III. – *Weak and strong limits of Bell's inequality. The quantities Γ, δ, S have been defined in eqs.* (2.70), (2.142), (2.86), *respectively.*

Experimental tests	Weak inequalities	Strong inequalities	Quantum min. values	Quantum max. values
[63]	$-0.80 \le \Gamma \le 0.20$	$10^{-6} \le \Gamma \le 0.036$	-0.0036	0.040
	$\delta \le 6.9$	$\delta \le 0.25$		0.301
[64]	$-0.85 < \Gamma \le 0.15$	$-0.002 \le \Gamma \le 0.018$	-0.0029	0.019
	$\delta \le 12.02$	$\delta \le 0.25$		0.269
[67]	$-0.84 \le \Gamma \le 0.16$	$-10^{-4} \le \Gamma \le 0.017$	-0.0016	0.019
	$\delta \le 13.74$	$\delta \le 0.25$		0.284
[70]	$-0.85 \le \Gamma \le 0.15$	$-10^{-5} \le \Gamma \le 0.014$	-0.0017	0.016
	$\delta \le 16.67$	$\delta \le 0.25$		0.308
[72]	$S \le 2$	$S \le 0.027$		0.036
[78]	$-0.82 \le \Gamma \le 0.18$	$-0.002 \le \Gamma \le 0.037$	-0.0034	0.038
	$\delta \le 6.25$	$\delta \le 0.25$		0.263

2·5.3. Experiments using correlated two-photons interference. Following a proposal by Franson [82], a new form of fourth-order interference experiment with two photons has recently been studied with a view to testing Bell-type inequalities. The basic idea of the experiment is as follows: two photons in an entangled quantum state, produced by a common source, travel along arms A and B to two detectors D_A and D_B (fig. 1). The photons can travel either directly along the shorter path or via a longer path involving reflections from two beam splitters and two mirrors, as shown in fig. 1. The difference in propagation time or length between the longer and the shorter paths is the same in both arms and is much greater than the coherence time or length of each photon wave packet. Now, if one looks for simultaneous detections by both D_A and D_B, quantum mechanics predicts interference effects between the different two-photon probability amplitudes, though the single detection rates registered by D_A and D_B are not expected to show any interference effect. Quantum nonlocality in such an experiment is considered to arise from the fact that the detectors are widely separated and the wave packets of the photons along the two arms do not overlap.

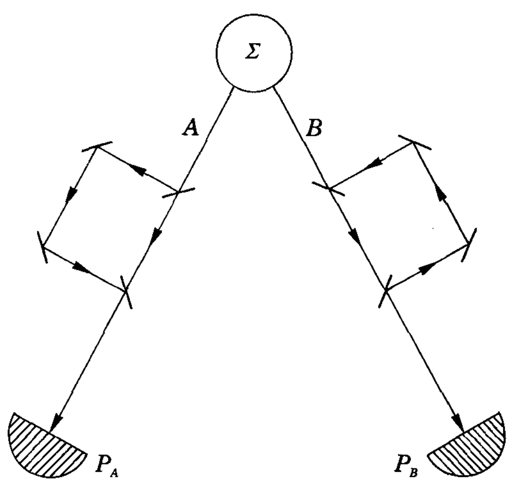

Fig. 1. – The Franson experiment with fourth-order interference. Σ is a source of correlated photon pairs, P_A and P_B are detectors.

To see in more detail the violation of Bell-type inequalities predicted by quantum mechanics for this case, let us consider a typical outline of the set-up for the experiment as shown in fig. 2. The interferometer basically consists of the single-photon interferometers for observing the correlated pairs of photons γ_1 and γ_2 produced by appropriate atomic-cascade transition or in parametric down conversion using a nonlinear crystal. Since the difference ΔT in transit times along the longer and shorter paths are assumed to be the same for both interferometers, the total probability amplitude A_{T} for photon 1 (ω_1) to be detected in D_1 and photon 2 (ω_2) to be detected in D_2 at the same time is given by

$$A_T = A_{SS} + \exp[i(\phi_1 + \phi_2)]\exp[i(\omega_1 + \omega_2)\Delta T]A_{LL} + \exp[i\phi_1]\exp[i\omega_1\Delta T]A_{LS} + \exp[i\phi_2]\exp[i\omega_2\Delta T]A_{SL}. \tag{2.143}$$

Here A_{LS} is the joint probability amplitude for γ_1 and γ_2 to have travelled along paths L_1 and S_2, and similarly for A_{SL}, A_{SS}, and A_{LL}; ϕ_1 and ϕ_2 are the relative phase shifts introduced in the arms L_1 and L_2, respectively. The coincidence rate registered in D_1 and D_2 is proportional to $A_T^* A_T$.

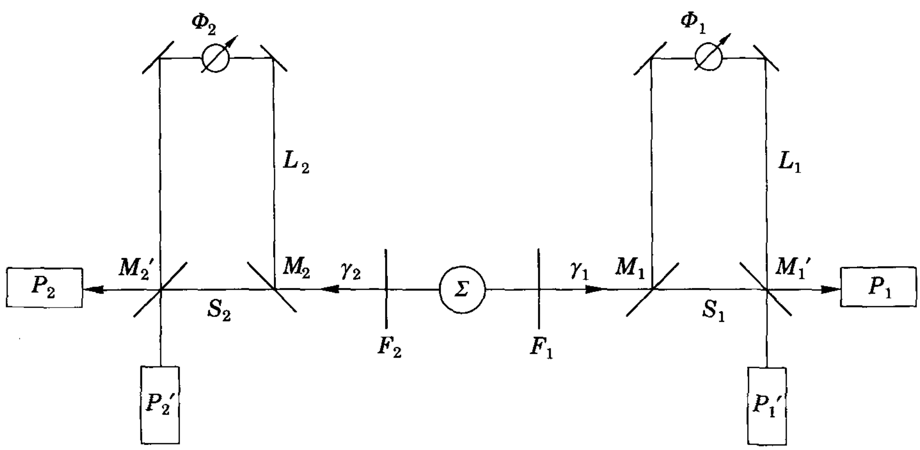

Fig. 2. – Bell-type experiment for fourth-order interference. For photon γ_i ($i = 1, 2$) M_i and M_i' are semi-transparent mirrors, P_i and P_i' are photodetectors, Φ_i is a phase-shifting device and S_i and L_i are the short and the long path, respectively.

Now, let ΔT be chosen much larger than the first-order coherence times, but much smaller than the lifetime of the initial state and such that it exceeds the width of the coincidence resolving time. Then we have

$$A_{LS} = A_{SL} = 0 \tag{2.144}$$

and eq. (2.143) reduces to

$$A_T = A_{SS} + \exp[i(\phi_1 + \phi_2)]\exp[i(\omega_1 + \omega_2)\Delta T]A_{LL}. \tag{2.145}$$

In ref. [82] it is shown that eq. (2.145) can be approximated by the following formula:

$$A_T = A(1 + \exp[i(\phi_1 + \phi_2)]\exp[i(\omega_{10} + \omega_{20})\Delta T]), \tag{2.146}$$

where ω_{10} and ω_{20} are the centre frequencies (about which spreads in frequencies are considered) and $A_{SS} = A_{LL} = A$. From eq. (2.146) one obtains the coincidence rate given by

$$R_C = 4A^2 \cos^2(\phi'_1 - \phi'_2), \tag{2.147}$$

where

$$\phi'_1 = \frac{1}{2}\phi_1, \quad \phi'_2 = -\frac{1}{2}[\phi_2 + (\omega_{10} + \omega_{20})\Delta T]. \tag{2.148}$$

One immediately notices that the coincidence rate of the form given by eq. (2.147) violates Bell-type inequalities where relative angles between polarizer settings are replaced by differences in relative phase shift.

A number of experiments [83-86] have been performed to check the quantum mechanically predicted oscillation effects in the coincidence rate given by (2.147) and the results are consistent with quantum predictions. However, mere verification of (2.147) is not sufficient to claim a true nonlocal effect; for that one has to demonstrate an unambiguous violation of Bell-type inequality without any subsidiary assumption. So far, this type of experiments have only shown violation of Bell-type inequalities derived with a subsidiary assumption, *viz.* the no-ehnancement postulate. As acknowledged in one of the papers [85]: «In principle, detection efficiences approaching 100% could be achieved in this type of experiment and it may be possible to demonstrate a violation of the strong form of Bell's equation». In this connection it may be noted that technological developments related to the solid-state photon counting detectors in the near infrared look promising and efficiences of the order of (50 ÷ 60)% are now feasible.

2'5.4. New experimental proposals. New ideas for experiments on the EPR paradox in elementary particle physics will be discussed in the next subsection. Here we will instead present some important improvements of the experiments performed with pairs of atomic photons which have been suggested.

The first idea, due to Falciglia, Garuccio, Iaci and Pappalardo [87], is based on the introduction of the source of atomic-photon pairs into a $(10^2 \div 10^3)$ G magnetic field. For a (0, 1, 0) cascade the intermediate $J = 1$ atomic level splits into three levels with $m_J = 0, \pm 1$, the energy separation being $\mu_B B$ (μ_B is the Bohr magneton, B the magnetic field). As is well known, the $m_J = 0$ state does not contribute if the photons propagate in directions parallel/antiparallel to that of $\boldsymbol{B}$. There remain two types of atomic transitions for the emission of the photon pair:

1) $(J = 0,\ m_J = 0) \rightarrow (J = 1,\ m_J = +1) \rightarrow (J = 0,\ m_J = 0)$. This leads to the emission of a photon pair in the polarization state $|R_\alpha\rangle|R_\beta\rangle$.

2) $(J = 0,\ m_J = 0) \rightarrow (J = 1,\ m_J = -1) \rightarrow (J = 0,\ m_J = 0)$. This leads to the emission of a photon pair in the polarization state $|L_\alpha\rangle|L_\beta\rangle$.

Therefore, if the photons are detected along $\pm \boldsymbol{B}$ and if the energy separation is large enough to allow one to detect, at least in principle, the energy difference between process 1) and process 2), one expects (paradoxical) state vectors (2.131) to reduce themselves to a (nonparadoxical) mixture of state vectors $|R_\alpha\rangle|R_\beta\rangle$ and $|L_\alpha\rangle|L_\beta\rangle$. The correlation function should make a sudden jump when the magnetic field is switched on, since the state vectors (2.131) violate while the mixture satisfies Bell's inequality. Therefore, the *reduction of the wave packet*, a typical quantum-mechanical feature, finds in the case of the EPR paradox a striking possibility of experimental control.

The zero angular-momentum states (2.131) satisfy a peculiar invariance condition: If on the trajectories of the two photons are placed quarter-wave plates *with parallel optical axes*, it is easy to show that the state vectors (2.131) are, respectively, transformed into the vectors (2.132). But the latter are mathematically identical with the former and the action of the plate is thus seen to have no physical implication. This means that the photon polarization correlations remain exactly the same, at least for ideal plates. This prediction can be checked experimentally in an EPR experiment, by inserting a quarter-wave plate on the trajectory of each photon before the interaction with the analyser: the coincidence counting rates should remain the same for all relative orientations of the analysers' axes.

Two experiments of the previous type have been performed and surprisingly both gave results in agreement with the strong inequalities. In Clauser's experiment [68] the retardation induced by the plates had a measured drift of $\pm 8.5°$ between different calibrations of the plates. The probability for transmission plus detection of both photons for the (1, 1, 0) cascade used by Clauser is predicted to be

$$\Omega(a, b) = \frac{1}{4}[\varepsilon_+^1\varepsilon_+^2 - \varepsilon_-^1\varepsilon_-^2 F_2(\theta)\cos(\xi_1 - \xi_2)\cos 2(a-b)]\,\eta_1\eta_2\,, \tag{2.149}$$

where θ is the half-angle substended by the primary lenses, $F_2(\theta)$ is a numerical parameter measuring the polarization decorrelation due to photon lack of collinearity; ξ_1 and ξ_2 are retardations induced by the quarter-wave plates (in the ideal case $\xi_1 = \xi_2 = \pi/2$); the remaining symbols are the same as in (2.137). Clauser tried to check Freedman's inequality $\delta \leq 0.250$, with δ defined in (2.142), and found experimentally $\delta_{ex} = 0.235 \pm 0.025$. He claimed that the previous result was in agreement with the quantum-theoretical prediction. The previous prediction is however incompatible with the quoted values of the retardations: even if one assumed a very pessimistic value of $\xi_1 - \xi_2 = 17°$, one would get from Clauser's experimental data $\delta = 0.274$ which deviates from the experimental result by 1.5 standard deviations.

In the Stirling experiment [88] some interesting data have been collected with polarizers kept with fixed and parallel axes but with varying relative angle ϕ of the quarter-wave plates (one of them had fast axis rotated by $\pi/4$ with respect to the common direction of the polarizers axes, and kept fixed, the other one was rotated during the experiment). The quantum formula for the double-transmission and detection probability for this experiment is

$$\Omega(\Phi) = \frac{1}{4}[\varepsilon_+^1\varepsilon_+^2 + \varepsilon_-^1\varepsilon_-^2 F\cos(\xi_1 - \xi_2)\cos 2\phi]\,\eta_1\eta_2\,, \tag{2.150}$$

where the geometrical factor $F = 0.996$. Experimentally it was found $\delta = 0.195 \pm 0.016$. The quantum predictions can reproduce such a value of δ, which does *not* disagree with Freedman's inequality, only by assuming a highly pathological value of $\xi_1 - \xi_2 \simeq 41°$. A more reasonable value of $\xi_1 - \xi_2$ of about 7° that one can deduce from the plate retardations as a function of photon wavelength would lead to a prediction for δ at variance with its measured value.

In conclusion we want to call attention to the fact that the insertion of quarter-wave plates in EPR experiments has been attempted twice and that both the times:

i) the predicted invariance of the relative counting rate was not observed;

ii) Freedman's (strong) inequality was satisfied;

iii) very strange effects had to be invoked in order to explain the data.

The calculation of the quantum-mechanical predictions for an EPR experiment with two plates of arbitrary orientation and with two polarizers has recently been performed by De Caro [89]. Several interesting cases emerge from this general calculation, *e.g.*, the possibility to violate the strong inequalities simply by varying the plates' redardations with fixed directions of their axes.

Several experiments have been proposed in order to overcome the distinction between weak and strong inequalities or, in other words, in order to have a concrete physical situation in which the disagreement between local realism (*Einstein locality*) and the quantum-mechanical predictions is fully realized and is not vitiated by arbitrary additional assumptions. An interesting proposal of this type was made by Lo and Shimony [90] who suggested to measure spin correlations of two Na atoms emerging from the dissociation of a Na_2 molecule in a singlet state. For the detection of Na atoms better instruments are available than for optical photons (see, however, the critical remarks made by Santos [91]). Another idea was suggested by Drummond [92] who considered the cooperative emission of photons from excited atoms and showed that in principle one can obtain quantum-mechanical correlations of two wave packets each containing N photons that violate Bell's inequality. The efficiency of a photodector for detecting at least one of the N photons is obviously much larger than the single-photon detection probability (for N high enough) and one can thus approach the high-detection region where the distinction between strong and weak inequalities disappears. A partly similar situation was studied by Reid and Walls [93].

Chubarov and Nikolayev [94] considered the standard experimental arrangement used in the observations of the Hanbury Brown-Twiss effect: Radiation from a source is split by a semi-transparent mirror into two beams and registered by two photon-counting detectors D_1 and D_2. Two polarizers are inserted in front of the detectors as in the standard EPR experiments. These authors showed that the coincidence counting rate has to obey inequalities of the strong type (deduced with the help of the usual additional assumptions), for different orientations of the polarizers. If the quantum states so analysed are photonic states with sub-Poissonian statistics, one can instead have cases where the quantum-mechanical predictions lead to violations of the strong inequalities.

The previous idea was extended by Ou, Hong and Mandel [95] who showed that violations of the strong inequalities can be obtained even for the nearly Poissonian but highly correlated signals of photons pairs that are created in the process of spontaneous parametric down conversion.

A proposal for an experimental study of the EPR paradox by using nuclear beta decay has been advanced by Skalsey [96]. Beta decay electrons, in cascade with the conversion electrons, are longitudinally polarized due to parity violation in the weak decay. Therefore the detection of the electron direction is equivalent to a spin measurement.

Study of the EPR paradox by using Rydberg atoms has been suggested by Oliver and Stroud [97]. Nearly 100% efficient state-selective ionizers can be used as detectors in Rydberg-atom experiments, and this could provide a decisive advantage over the usual EPR experiments with pairs of atomic photons. In this proposal the photon polarization is replaced by the two-level-atom Bloch vector. An interesting method for producing pairs of Rydberg atoms with the appropriate quantum-mechanical correlation (*state-vector of the second type*) is also proposed by these authors.

2·6. *New tests of the EPR paradox from particle physics.* – The EPR paradox, originally discovered in nonrelativistic quantum mechanics, is also applicable in the relativistic domain. The simplest way to realize this is perhaps to consider the rare but allowed decay

$$\pi^0 \rightarrow e^+ + e^- ,$$

which accounts for a fraction of about $2 \cdot 10^{-7}$ of all π^0 decays. After the application of C and P conservation to the decay process one can show that in the centre of mass of the electron-positron pair the only possible spin state is the singlet state (1.24). As a consequence, the EPR paradox exists also for this relavistic process where, for example, the two final particles have velocities very near to that of light.

Other examples are discussed below.

2·6.1. Preliminary remarks. Törnqvist [98] pointed out an example involving the decay of a $J^P = 0^-$ meson η_c of mass 2980 MeV/c^2 into a Λ-hyperon plus $\bar{\Lambda}$-antihyperon pair (with the subsequent decays $\Lambda \rightarrow p + \pi^-$ and $\bar{\Lambda} + \bar{p} + \pi^+$) was one such candidate. If $\hat{a}$ and $\hat{b}$ are unit vectors in the directions of the emitted π^- and π^+ mesons in the p and $\bar{p}$ rest frames, respectively, then the suitably normalized decay rate summed over p and $\bar{p}$ spins is, according to quantum theory, given by

$$r_0(a, b) = 1 + \alpha_\Lambda^2 \hat{a} \cdot \hat{b} ,$$

where $\alpha_\Lambda = -0.642 \pm 0.013$ is the Λ-decay asymmetry parameter. Törnqvist showed that the above prediction disagreed up to 10% from a limit deducible from certain types of local realist models. However, the generality of this result for all possible local realist models is very doubtful.

Another example, investigated by various authors, is the decay of a $J^{PC} = 1^{--}$ vector meson into a pair of neutral pseudoscalar mesons. Lee and Yang [99] were the first to point out the EPR-type features of this case (pertaining to the pair of kaons K^0-$\bar{K}^0$ resulting from a $p\bar{p}$ annihilation), followed by d'Espagnat [100], Six [101] and Selleri [102]. Here we seek to review the present status of this example. We begin with a résumé of its salient features. We consider specifically the decay of a spin-1 Φ (1020) resonance, by strong interaction, into a pair of neutral kaons K^0-$\bar{K}^0$.

2.6.2. EPR-type situation for $\Phi \to K^0$-$\bar{K}$. Invoking charge conjugation invariance of strong interaction, the wave function of the K^0-$\bar{K}^0$ pairs at the time of production ($t = 0$) from the decay of the $J^{PC} = 1^{--}$ state is given by

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}[|K^0\rangle_L|\bar{K}^0\rangle_R - |\bar{K}^0\rangle_L|K^0\rangle_R], \tag{2.151}$$

where L (R) refers to the left (right) hemisphere. The subsquent time development of the K^0-$\bar{K}^0$ pair is described in terms of the eigenstates of the effective Hamiltonian which includes weak interactions. In the situation under consideration, the weak interactions induce decays of both $K^0, \bar{K}^0$ and also give rise to K^0-$\bar{K}^0$ transition (a pure $|K^0\rangle$ ($|\bar{K}^0\rangle$) state, evolving in time, becomes a superposition of $|K^0\rangle$, $|\bar{K}^0\rangle$, and decay products. This gives rise to K^0-$\bar{K}^0$ oscillations). The effective Hamiltonian is written as $H = M - i\Gamma/2$ where M and Γ are the Hermitian mass and decay matrices, respectively. The eigenstates of this H are $|K_L\rangle$ and $|K_S\rangle$ with the eigenvalues $\lambda_L = m_L - i\gamma_L/2$ and $\lambda_S = m_S - i\gamma_L/2$, respectively, where m_L (m_S) and γ_L (γ_S) are, respectively, the mass and the decay width of $|K_L\rangle$ ($|K_S\rangle$); $m_L - m_S = 0.53 \cdot 10^{10}\ \hbar\,s^{-1}$ and $\gamma_S = 582 \cdot 10^{10}\ \hbar\,s^{-1}$, $\gamma_L = 1.12 \cdot 10^{10}\ \hbar\,s^{-1}$. We assume CP invariance; the implications of CP violation will be discussed later (paragraph **2.6.4**). Note that

$$|K_L\rangle = \frac{1}{\sqrt{2}}[|K^0\rangle + |\bar{K}^0\rangle], \qquad |K_S\rangle = \frac{1}{\sqrt{2}}[|K^0\rangle - |\bar{K}^0\rangle].$$

They time evolve as

$$U(t, 0)|K_L\rangle = \exp[-i\lambda_L t]|K_L\rangle + |\Phi_L(t)\rangle \tag{2.152}$$

with a corresponding equation for $|K_S\rangle$. Here $|\Phi_L(t)\rangle$ ($|\Phi_S(t)\rangle$) represents the decay products from $|K_L\rangle$ ($|K_S\rangle$); $|\Phi_L\rangle$ ($|\Phi_S\rangle$) is taken orthogonal to the state $|K_L\rangle$ ($|K_S\rangle$). CP invariance requires that $\langle K_L|K_S\rangle = 0$. In terms of the states $|K_L\rangle$ and $|K_S\rangle$, the wave function $|\Psi_0\rangle$ given by eq. (2.151) can be written as

$$|\Psi_0\rangle = \frac{1}{\sqrt{2}}[|K_S\rangle_L|K_L\rangle_R - |K_L\rangle_L|K_S\rangle_R]. \tag{2.153}$$

The time evolution of the nonseparable form of the two-particle wave function

$|\Psi_0\rangle$ correlates the oscillations between the $|K^0\rangle$ and the $|\bar{K}^0\rangle$ states such that it carries the essence of nonlocal correlation, reminiscent of the EPR-type situation. If the left (right) kaon is observed to be a K^0 (strangeness $S = +1$) at a particular instant then the right (left) kaon can be predicted, with certainty, to be observed as a $\bar{K}^0$ ($S = -1$) at that same instant. Alternatively, if the left (right) kaon decays in the K_S mode ($CP = +1$), then the right (left) kaon is bound to decay as a K_L ($CP = -1$) at some future instant. It is to be noted that there is a subtle distinction between the K^0-$\bar{K}^0$ and K_L-K_S correlations; while the former holds only for equal proper times, the latter is a time-independent consequence of the nonseparable form of the wave function. This aspect was discussed by Selleri [102].

Six [101] suggested that an experimental test of this EPR-type situation would be the measurement of the joint probability $\Omega_{--}(t_1, t_2)$ of a double $\bar{K}^0$ observation (*i.e.* on two sides), at times t_1 and t_2 on the left and right, respectively. The quantum-mechanical prediction for $\Omega_{--}(t_1, t_2)$ is given by

$$\Omega_{--}(t_1, t_2) = |\langle \bar{K}^0_L \bar{K}^0_R | \Psi(t_1, t_2)\rangle|^2,$$

where $|\Psi(t_1, t_2)\rangle$ is the state evolved from $|\Psi_0\rangle$ at $t = 0$:

$$(2.154)\qquad |\Psi(t_1, t_2)\rangle = \frac{1}{\sqrt{2}}\{|K_S\rangle_L |K_L\rangle_R \exp[-i(\lambda_S t_1 + \lambda_L t_2)] - |K_L\rangle_L |K_S\rangle_R \exp[-i(\lambda_L t_1 + \lambda_S t_2)]\}$$

whence on obtains

$$(2.155)\qquad \Omega_{--}(t_1, t_2) = \frac{1}{8}\{\exp[-(\gamma_S t_1 + \gamma_L t_2)] + \exp[-(\gamma_L t_1 + \gamma_S t_2)] - 2\exp[-\gamma(t_1 + t_2)]\cos\Delta m(t_1 - t_2)\},$$

where $\gamma = (\gamma_L + \gamma_S)/2$ and $\Delta m = m_L - m_S$.

Selleri derived an upper bound on $\Omega_{--}(t_1, t_2)$ for the $K^0 - \bar{K}^0$ system using a general argument based on the notion of local realism:

$$(2.156)\qquad \Omega_{--}(t_1, t_2) \le \frac{1}{8}\{\exp[-(\gamma_S t_1 + \gamma_L t_2)] + \exp[-(\gamma_L t_1 + \gamma_S t_2)]\}.$$

This local realistic upper bound differs from the quantum-mechanical prediction (2.155) by the absence of the interference term. Quantum mechanics, therefore, leads to a prediction that violates eq. (2.156) whenever the interference term is positive, that is, whenever $\cos\Delta m(t_1 - t_2) < 0$. The maximum possible discrepancy is calculated to be about 12% for $\gamma_S(t_1 - t_2) \simeq 5$.

It is important to note that the experimental study suggested in the context of the eq. (2.155) and (2.156) has an intrinsic handicap: for meaningful results, t_1 and t_2 must be shorter than the lifetimes of K_L and K_S, *i.e.* one requires t_1, $t_2 \le 10^{-10}$ s. The uncertainties involved in ensuring that the observations are at the specified instants t_1 and t_2 would be quite appreciable within such a small time interval. This difficulty may, however, be circumvented by considering the time-integrated joint probabilities. This aspect has recently been examined by Datta and Home [103] for the case of the B^0-$\bar{B}^0$ system. This system is almost identical to the K^0-$\bar{K}^0$ system, the only difference being that $\gamma_L = \gamma_S$ $(= \gamma \approx 10^{12}\ \hbar\, s^{-1})$ for the eigenstates of the B^0-$\bar{B}^0$ system which are analogous to the $|K_L\rangle$, $|K_S\rangle$ states. They are denoted by $|B_H\rangle$ and $|B_L\rangle$ with masses m_H and m_L, respectively ($m_H > m_L$).

2·6.3. EPR-type test using B^0-$\bar{B}^0$ system. Recent experiments on the decay of the spin-1 $\Upsilon(4s)$ vector meson into a pair of neutral pseudoscalar mesons B^0-$\bar{B}^0$ have attracted considerable attention in view of the search for evidence of the B^0-$\bar{B}^0$ mixing. Datta and Home [103] have analysed the possibility of investigating experimentally the EPR-type quantum nonlocal correlations within the framework of the current experiments on the decay $\Upsilon(4s) \rightarrow B^0$-$\bar{B}^0$. Here one considers the time-integrated joint probabilities, remembering that B^0 and $\bar{B}^0$ can be identified by their decay channels: $B^0 \rightarrow \ell^- \bar{\nu} X$; $\bar{B}^0 \rightarrow \ell^+ \nu X$ where ℓ and X denote lepton and hadron, respectively. From the decay kinematics of $\Upsilon(4s) \rightarrow B^0$-$\bar{B}^0$ it can be shown that the spatial separation between B^0-$\bar{B}^0$ is of the order of 0.1 mm (much larger than the de Broglie wavelength of the particles involved) after a time-interval of the order of the life-time of the decaying particles.

The experimental arrangement currently in use to study $\Upsilon(4s) \rightarrow B^0$-$\bar{B}^0$ is designed to measure the parameter R defined as follows:

$$R = \frac{N_{++} + N_{--}}{N_{+-} + N_{-+}}, \tag{2.157}$$

where

N_{++} = total number of double-$\bar{B}^0$ decays (correspoding to the observation of double ℓ^+ decay products on both sides);

N_{--} = total number of double-B^0 decays (corresponding to the observation of double ℓ^- decay products on both sides);

N_{+-} = total number of $\bar{B}^0$ decays on the left associated with B^0 decays on the right (corresponding to the observation of ℓ^+ decay products on the left associated with ℓ^- decay products on the right);

N_{-+} = total number of B^0 decays on the left associated with $\bar{B}^0$ decays on the right (corresponding to the observation of ℓ^- decay products on the left associated with ℓ^+ decay products on the right).

The parameter R is calculated by evaluating the quantities N_{ij} $(i, j = \pm)$. The general expression for N_{ij} is given by

$$N_{ij} = 2N_0\lambda^2 \int_0^\infty \mathrm{d}t_1 \int_{t_1}^\infty \mathrm{d}t_2 \Omega_{ij}(t_1, t_2), \tag{2.158}$$

where $\Omega_{ij}(t_1, t_2)$ is the joint probability for observing the decay products ℓ^i, ℓ^j on two sides at times t_1 and t_2, respectively; N_0 is the total number of $\Upsilon(4s)$ decays, and λ is the semi-leptonic decay width of B^0 decaying into a ℓ^- (wich is equal to the semi-leptonic decay width of $\bar{B}^0$ decaying into a ℓ^+). The quantum-mechanical expressions for $\Omega_{ij}(t_1, t_2)$ derived from the nonseparable form of the wave function (2.154) are given by

$$\Omega_{++}(t_1, t_2) = \Omega_{--}(t_1, t_2) = \frac{1}{8}\{2\exp[-\gamma(t_1 + t_2)] - 2\exp[-\gamma(t_1 + t_2)]\cos\Delta m(t_1 - t_2)\}, \tag{2.159}$$

$$\Omega_{+-}(t_1, t_2) = \Omega_{-+}(t_1, t_2) = \frac{1}{8}\{2\exp[-\gamma(t_1 + t_2)] + 2\exp[-\gamma(t_1 + t_2)]\cos\Delta m(t_1 - t_2)\}, \tag{2.160}$$

where $\Delta m = m_H - m_L$. Using eqs. (2.159) and (2.160) we obtain from (2.158) the following quantum-mechanical values for N_{ij}:

$$N_{++} = N_{--} = (N_0\lambda^2)[(1/4\gamma^2) - (1/4\alpha^2)], \tag{2.161}$$

$$N_{+-} = N_{-+} = (N_0\lambda^2)[(1/4\gamma^2) + (1/4\alpha^2)], \tag{2.162}$$

where $\alpha^2 = \gamma^2 + (\Delta m)^2$. This leads to the following quantum-mechanical prediction for the parameter R defined by (2.157):

$$R_{QM} = \frac{x^2}{2 + x^2}, \tag{2.163}$$

where $x = \Delta m/\gamma$. The result given by eq. (2.163) hinges on the quantum nonseparability which is built into the wave function (2.151) and is assumed to be maintained even after the particles get well separated in space. The experimental verification of (2.163) will, therefore, constitute an interesting test for quantum nonseparability in this EPR-type situation.

It should be noted that R_{QM} is model dependent, since no independent experimental data is available at present. Confining our attention within the ambit of the Glashow-Weinberg-Salam standard model of electroweak interactions, we make the following observation. There are two types of B^0 mesons: B^0_d ($b\bar{d}$ quark-antiquark bound state) and B^0_s ($b\bar{s}$ quark-antiquark bound state). $\Upsilon(4s)$ decays into the B^0_d-$\bar{B}^0_d$ system only (the B^0_s-$\bar{B}^0_s$ channel is forbidden by the kinematics considerations). In this case, the precise prediction of the standard model as regards $\Delta m/\Gamma$ suffers from certain inherent uncertainties which are now being debated.

The most recent experimental study indicates strong evidence for substantial B^0_d-$\bar{B}^0_d$ mixing and the value of R is claimed to be 0.21 ± 0.08 [104]. Datta and Home [103] have shown that in this case Furry's hypothesis leads to the prediction $R = 1$, which is obviously ruled out by the experimental result.

2.6.4. Quantum nonlocality and CP nonconservation. In a 1987 paper [105] Datta, Home and Raychaudhuri (DHR) have pointed out a curious gedanken example of the EPR paradox using CP noninvariance, which has evoked considerable controversy. The example involves a pair of correlated neutral pseudo-scalar mesons (M^0-$\bar{M}^0$) originating from the decay of a $J^{PC} = 1^{--}$ vector meson—typical intances are the decays of the $\Phi(1020)$ resonance into K^0-$\bar{K}^0$ and $\Upsilon(4s)$ into B^0-$\bar{B}^0$. The exponentially decaying states, with the associated masses and lifetimes, are $|M_L\rangle$ and $|M_S\rangle$ (which are certain linear combinations of $|M^0\rangle$ and $|\bar{M}^0\rangle$); in the standard formalism (outlined in the paragraph **2**.6.2) these eigenstates of the effective Hamiltonian are used to describe the quantum-mechanical time-evolution of the system. In the presence of CP nonconservation, $|M_L\rangle$ and $|M_S\rangle$ are nonorthogonal. This nonorthogonality of the physically relevant states (unique characteristic of the quantum-mechanical treatment of CP nonconservation) gives a new twist to the EPR paradox.

Taking the cue from paragraph **2**.6.2 and ref. [105], the two-particle wave function at the time of production ($t = 0$) of the M^0-$\bar{M}^0$ pairs is given by

$$|\Psi_0\rangle = \frac{1}{N}[|M_S M_L\rangle - |M_L M_S\rangle], \tag{2.164}$$

where N is a normalization factor and the first (second) member of each pair refers to the left (right) hemisphere.

Following the treatment in paragraph **2**.6.2, the time-evolved wave function can be written in the form

$$|\Psi(t)\rangle = c_1|M_L\Phi_S\rangle + c_2|M_S\Phi_L\rangle + c_3|\chi\rangle, \tag{2.165}$$

where c_1, c_2, c_3 are time-dependent constants, and

$$|\chi\rangle \sim |M_S M_L\rangle - |M_L M_S\rangle$$

represents the undecayed piece with $\langle\chi|\chi\rangle = 1$. $|\Phi_L\rangle$ ($|\Phi_S\rangle$) corresponds to the decay products on the right from $|M_L\rangle$ ($|M_S\rangle$). The important difference between eq. (2.165) and the EPR-type correlations in other standard examples lies in the fact that $|M_L\rangle$ and $|M_S\rangle$ are nonorthogonal eigenstates of the effective Hamiltonian $H = M - i\Gamma/2$, where M and Γ are noncommuting. In writing eq. (2.165) we have not considered those components of the wave function which contain decay products on the left, as they are not relevant for our subsequent discussion which is focused on the flux of, say, undecayed $|M^0\rangle$ on the left.

It can be easily seen from eq. (2.165) that the above flux on the left would involve a contribution due to the overlap between the decay product states $|\Phi_L\rangle$ and $|\Phi_S\rangle$ on the right, which is nonvanishing in the presence of CP noninvariance. Note that $\langle\Phi_L(t)|\Phi_S(t)\rangle$ is proportional to $\langle M_L|M_S\rangle$, contribution to this nonorthogonality comes essentially from the common decay products of M_L and M_S (in the presence of CP violation).

That the statistical property of the particles on the one side has some formal dependence pertaining to the overlap between the physical states of the particle on the other side is the key feature of the example suggested by DHR. Whether this overlap can be physically tampered (even in principle) by suitable «measurement» is the controversial issue raised by this example.

In the absence of CP violation, the mutually orthogonal $|\Phi_L\rangle$ and $|\Phi_S\rangle$ states can be unambiguously distinguished. However, in the presence of a small but nonvanishing *CP* violation, if one can partially discriminate between $|\Phi_L\rangle$ and $|\Phi_S\rangle$, on the one side, by exploiting the differences in physical attributes, there arises a possibility to affect the above overlap, thereby leading to a nonlocal effect on the flux of undecayed kaons, on the other side. Such a scheme, of course, envisages measurements partially destroying the coherence of the original pure state and leading to statistical mixtures of nonorthogonal states. Concept of this type of nonorthodox measurement (*partial collapse*), though unconventional, is not inadmissible and can be dealt with, in principle, by a suitable generalization of the standard quantum theory of measurement, as has been shown by various authors [106]. To digress a bit, we observe that there are various examples of realistic measurements such as approximate measurements or nonideal measurements which cannot be described by the standard scheme based on orthogonal projections only. Ivanovic [106] and Dieks [106] have analysed the viabilities of possible schemes to differentiate between nonorthogonal states. *Partial distinction* between nonorthogonal states, in a sense, involves *unsharp* or *imprecise* simultaneous measurement of noncommuting observables, a concept whose tenability has been explicitly analysed by various authors [107].

It needs to be noted that in their original treatment, DHR had first assumed the collapse to a mixed state comprising of nonorthogonal components to be *total* and then the error involved (due to an overlap between the probability distributions of the invariant masses of the decay products corresponding to the nonorthogonal states) was estimated. It was argued by DHR that the error could, in principle, be made small compared to a suitably defined measure of the nonlocal effect. However, the scheme for estimating this error has certain ambiguity, depending upon the choice of the parameter used as a measure of the error. This *ambiguity* can be avoided by directly incorporating the notion of *partial collapse* and by properly taking the probability conservation into account through a formal density matrix treatment, as claimed by DHR in their subsequent paper [108]. Of course, if one chooses to confine one's attention only to standard quantum measurements involving, for example, the invariant masses of the individual and mutually orthogonal decay product components of $|\Phi_L\rangle$ and $|\Phi_S\rangle$, then there will be no nonlocal effect at the statistic level [109]. However, the new feature in this case is whether one can envisage nonorthodox or generalized measurements in the sense of selection processes to tinker with the overlap between $|\Phi_L\rangle$ and $|\Phi_S\rangle$ which contributes to the flux of undecayed kaons on the other side. One such possibility, albeit at the gedanken level, is to exploit the difference in the lifetimes of the states $|M_L\rangle$ and $|M_S\rangle$ to select out partially the decay products corresponding to, say, the $|\Phi_L\rangle$ state. This may be done, for example, by using an apparatus, on the one side, which registers only the decay products originating within the specified time interval (say, ΔT) around the time of the order of the lifetime of $|M_L\rangle$ (*). These decay products will come predominantly from $|M_L\rangle$. There will also be a small but nonvanishing probability (which can be made as small

(*) This can be viewed as a nonorthodox measurement of the decay times within a specific time interval; a simple model of the measuring apparatus which can register the decay products along with their time of origin has been discussed by Sudbery [110].

as we like at the level of a thought experiment by assuming the lifetimes of $|M_L\rangle$ and $|M_S\rangle$ to be widely different) of the decay products from $|M_S\rangle$ being registered during ΔT. But the relevant point is whether the overlap between the common decay products from $|M_S\rangle$ (which originated well before ΔT) and those from $|M_L\rangle$ during ΔT can get affected by this process.

Hall [111] and Ghirardi *et al.* [112] have argued on the basis of the operation-effect formalism (using the first representation theorem [113]) that the type of wave function collapse envisaged in the DHR example necessarily corresponds to a *measurement* which simultaneously affects the two separated subsystems; this would ensure that no action at a distance is involved here. Relationship of this abstract mathematical treatment with specific selection processes like the one mentioned above and its generality to cover all possible realistic models of nonorthodox measurements need to be carefully examined for further clarification.

To sum up, the DHR example raises the issue of *nonorthodox* measurements and of their compatibility with locality in the context of the EPR paradox, in arena hitherto left unexplored. Since *CP* noninvariance implies time-reversal asymmetry (given the *CPT* theorem) one may also wonder whether time-irreversible interaction, in general, may introduce a new element in the quantum-mechanical treatment of the EPR-type situations.

2·6.5. EPR correlations and nonlocal signalling. There are formal proofs of a no-go theorem in quantum mechanics which rules out the possibility of signalling (nonlocal effect at the statistical level) using EPR-type correlations. These proofs are of two types, one based on the use of the operation-effect formalism [114] and the other on the standard formalism of quantum mechanics [115].

The first type has recently been critically analysed by Home and Srinivas [116]. Since the operation-effect formalism is too abstract, we will confine our attention to the second type, with reference to an elegant version given by Eberhard and Ross [117].

The proof uses the usual formalism of nonrelativistic quantum mechanics. Let the initial pure state be

$$|\Psi(t)\rangle = \sum_a c_a |a\rangle \tag{2.166}$$

with $\langle\Psi(t)|\Psi(t)\rangle = 1$. Then, for any observable A we have

$$\langle A\rangle = \langle\Psi(t)|A|\Psi(t)\rangle = \langle\Psi(0)|\tilde{A}(t)|\Psi(0)\rangle \tag{2.167}$$

with $\tilde{A}(t) = U^\dagger(t)AU(t)$ and $|\Psi(t)\rangle = U(t)|\Psi(0)\rangle$. Let a measurement M be performed at S on $|\Psi\rangle$ at time t_s. After an infinitesimal time lapse t, this will turn $|\Psi\rangle$ into a mixed state with components

$$|a\rangle = \frac{P_a}{\sqrt{p(a)}}|\Psi(t)\rangle, \tag{2.168}$$

where $P_a = |a\rangle\langle a|$ is the projection operator for the state $|a\rangle$ and

$p(a) = |c_a|^2$. Note that here one assumes ideal measurement and applies the standard form of the collapse postulate. The conditional probability of an outcome $A(a)$ for a measurement of A performed by R at time t_r on the collapsed state $|a\rangle(t_s + t)$ is given by

$$p_a = \langle a(t_s + t) | U^{\dagger}_{t_r,t_s} A U_{t_r,t_s} | a(t_s + t) \rangle, \tag{2.169}$$

where U_{t_r,t_s} is the evolution operator between times t_s and t_r:

$$U_{t_r,t_s} = U(t_r)\, U^{\dagger}(t_s). \tag{2.170}$$

Using eq. (2.168), we obtain

$$p_a = \frac{1}{p(a)} \langle \Psi(t_s) | P_a U^{\dagger}_{t_r,t_s} A U_{t_r,t_s} P_a | \Psi(\Psi(t_s) \rangle. \tag{2.171}$$

Since R does not know the result of the measurement made at S, the relevant expectation value of A is

$$\begin{aligned} \langle A \rangle' &= \sum_a p(a)\, p_a = \\ &= \sum_a \langle \Psi(t_s) | P_a U(t_s) U^{\dagger}(t_r) A\, U(t_r) U^{\dagger}(t_s) P_a | \Psi(t_s) \rangle = \\ &= \sum_a \langle \Psi(0) | \tilde{P}_a(t_s) \tilde{A}(t_r) \tilde{P}_a(t_s) | \Psi(0) \rangle, \end{aligned} \tag{2.172}$$

where

$$\tilde{A}(t_r) = U^{\dagger}(t_r)\, A\, U(t_r) \tag{2.173}$$

and

$$\tilde{P}_a(t_s) = U^{\dagger}(t_s)\, P_a U(t_s). \tag{2.174}$$

Using the commutation relations

$$[\tilde{A}(t_r),\ \tilde{P}_a(t_s)] = 0, \tag{2.175}$$

which are valid whenever the spatial separation between the subsystems is such that they are no-interacting and their Hilbert spaces are disjoint, we get

$$\langle A \rangle' = \langle \Psi(0) | \left(\sum_a \tilde{P}_a(t_s) \right) \tilde{A}(t_r) | \Psi(0) \rangle . \tag{2.176}$$

Notice that spacelike separations are sufficient but not necessary to guarantee eq. (2.175): while in relativistic field theory eq. (2.175) is, in fact, not valid for timelike separations, in nonrelativistic quantum mechanics eq. (2.175) is valid also for timelike separations as long as the subsystems are noninteracting, which is indeed the case in the EPR-type examples.

Comparing eqs. (2.167) and (2.176), we can see that $\langle A \rangle' = \langle A \rangle$ provided

$$\sum_a \tilde{P}_a(t_s) = 1 . \tag{2.177}$$

This completes the proof given by Eberhard and Ross.

A critical analysis of this proof has been given by Ghose and Home [118] who have argued that condition (2.177) can be circunvented for timelike separated *orthodox* or *ideal* measurements within the framework of nonrelativistic quantum mechanics or for nonorthodox measurements. Ghose and Home have pointed out that condition (2.177) holds for *orthodox* measurements provided the post-measurement ensemble comprises entirely the incoherent components given by eq. (2.168). However, measurements are possible which do not necessarily conform to this ideal situation. For example, one can conceive of an *incomplete* measurement (recently designated as *incomplete collapse* by Peres and Ron [119]) which does not completely destroy the coherence of the original state. For such measurements, the summation over the collapsed states occurring in eq. (2.177) does not include the entire post-measurement state and so eq. (2.177) does not hold good. To what extent the no-go theorem in its usual form can be reformulated to cover such nonideal cases requires further careful studies.

One can, of course, take the attitude that there is nothing to worry about having superluminal signals in a nonrelativistic theory. As Pearle [120] puts it: «It is surprising that nonrelativistic quantum theory does not allow superluminal communication to take place via correlated particles (*e.g.*, EPR phenomena) when it can take place by other mechanisms (*e.g.*, wave packet travel or spread)». What prevents such superluminal signals in the usual EPR-type situations (subject to *ideal* measurements) in ordinary nonrelativistic linear quantum mechanics is the commutativity of the measurement operators for noninteracting subsystems. In this context it should be mentioned that the formulation of hidden-variable theories satisfying the no-signalling condition, «a condition weaker than the Einstein-Bell locality condition», has been discussed by Roy and Singh [121]. An interesting step in the direction of developing realistic treatments of field theory has also been taken by Roy and Singh [122] who have analysed a possible framework of what they call «generalized beable quantum field theory».

Here it is important to stress that the issue of superluminal signalling in quantum-mechanical treatment of the EPR-type examples and its incompatibility with special relativity can be meaningfully discussed on the basis of a rigorous formulation of the EPR problem in quantum field theory. Bohm *et al.* [123] have analysed the EPR problem in quantum field theory using the Schrödinger picture. However, in the Schrödinger as well as in the Heisenberg pictures, the theory is divided into two sections, one giving the kinematical relations between various quantities at the «same instant of time» (for example, equal-time commutation relations) and the other determining causal relations between quantities at different instans of time (*e.g.*, the Schrödinger equation). This way of separating the theory into two sections is nonrelativistic in spirit. The situation becomes more acute in the case of *collapse* involving the notion of a universal time, an inherently nonrelativistic concept. Ghose and Home [124] have suggested that the Tomonaga-Schwinger formalism in the *interaction picture* should be more appropriate in this context. In this formalism the earlier-mentioned separation can be carried out in a manifestly covariant fashion and as shown by Ghose and Home it provides a manifestly covariant description of an EPR correlated state defined on a curved spacelike surface. This avoids the notion of a universal time, and clearly demarcates between the completion of the measuring process on a member of an EPR pair, and its nonlocal effect on the state of its distant partner.

Further investigations are called for to analyse the EPR problem in more detail and critically within the framework of quantum field theory.

3. – Attempted solutions of the EPR paradox.

3'1. *The quantum potential approach.* – In the quantum-potential model, originally proposed by Bohm [125] and later elaborated by Bohm an Hiley [126], an attempt is made to formulate a self-consistent causal interpretation of quantum mechanics on the assumption that an individual microphysical entity, such as an electron, follows a casually determined trajectory and has an associated wave field which satisfies the Schrödinger equation. The underlying motivation has been expressed by Bohm [127] as follows: «We should say that quantum mechanics does not explain anything; it merely gives a formula for certain results. And I'm trying to give an explanation». The central idea is that quantum mechanics can be understood in a *realistic* spirit and conceptually clear way by assuming that an individual electron, for example, is subject not only to the classical potential V but also to the quantum potential Q, which depends on the wave field $\Psi(\boldsymbol{x}, t)$ causally determined by the Schrödinger equation in any particular case:

$$Q = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}, \tag{3.1}$$

where $R^2 = |\Psi|^2$ and m is the mass of the particle. The action of the quantum potential Q is regarded as the key source of difference between classical and quantum theories. $|\Psi|^2$ is interpreted as the probability density for the particle to *be* at a certain position, in contrast with the standard interpretation that it corresponds to the probability of *finding* the particle there in a suitable measurement. It should be noted that Q is not changed when Ψ is multiplied by an arbitrary constant. This means that the effect of Q depends only on the *form* of the guiding wave field Ψ and is independent of its strength (*i.e.* the amplitude) unlike the case of classical waves. Another striking feature of the quantum potential Q is that in the case of the many-body system it cannot be expressed as a fixed and pre-assigned function of the individual particle coordinates; rather it depends on the quantum state Ψ of the system *as a whole*. To see this explicitly, let us consider the case of two particles with mass m. The wave function $\Psi(\boldsymbol{x}_1, \boldsymbol{x}_2, t)$ satisfies the Schrödiner equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}[\nabla_1^2 + \nabla_2^2]\Psi + V\Psi\,, \tag{3.2}$$

where ∇_1 and ∇_2 refer to particles 1 and 2, respectively. Writing

$$\Psi = R\exp[iS/\hbar] \tag{3.3}$$

and defining

$$P = R^2 = \Psi^*\Psi\,, \tag{3.4}$$

we obtain from (3.2)

$$\frac{\partial P}{\partial t} + \nabla_1\cdot\left(\frac{P}{m}\nabla_1 S\right) + \nabla_2\cdot\left(\frac{P}{m}\nabla_2 S\right) = 0 \tag{3.5}$$

and

$$\frac{\partial S}{\partial t} + \frac{(\nabla_1 S)^2}{2m} + \frac{(\nabla_2 S)^2}{2m} + V + Q = 0\,, \tag{3.6}$$

where

$$Q = Q(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = -\frac{\hbar^2}{2m}\left(\frac{\nabla_1^2 R}{R} + \frac{\nabla_2^2 R}{R}\right). \tag{3.7}$$

Evidently eq. (3.5) describes the conservation of probability with density $P = \Psi^*\Psi$ in the configuration space of the two particles. Equation (3.6) is the Hamilton-Jacobi equation for the system of two particles, acted on not only by the classical potential V, but also by the quantum potential Q. The latter depends on the coordinates ($\boldsymbol{x}_1$ and $\boldsymbol{x}_2$) of both particles in such a way that it does not fall off as $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \to \infty$. One, thus, obtains the possibility of a *nonlocal interaction* between the two particles. Bohm [128] presents his outlook as follows: «This sort of nonlocality would, for example, give a simple and direct explanation of the paradox of Einstein, Podolsky and Rosen, because in measuring some property of one of a pair of particles with correlated wave functions, one will alter the «non-local» quantum potential so that the other particle responds in a corresponding way».

However, the above response is instantaneous and hence it would appar at first sight to contradict the theory of relativity, which requires that no signals be transmitted faster than the speed of light. To annul this possibility of a casual paradox within the framework of the principle of relativity, it is argued that the state dependence and fragility of the instantaneous connection between the two spatially separated particles is such that it is not controllable in the way required for transmitting a signal (or, coded information). The *nonlocal* connection is manifested only in the correlations (revealed through a comparison of the experimental data independently gathered at each of the two particles) and not at the level of statistical properties of the particles at each end of the connection. It is generally believed that these *nonlocal* correlations can never be used as a signalling device. Mermin [129] has commented: «While it is wrong to suggest that EPR correlation will replace sonar, it seems to me something is lost by ignoring them or shrugging them off». The proponents of the quantum potential school maintain the following standopoint, as expounded by Hiley [130]: «... relativity in the quantum-potential approach comes out as a statistical effect, not as an absolute effect... The problem is how are we going to design experiments which will go beyond the statistical level to see these instantaneous connections. That's not clear at the moment». Here it may be noted parenthetically that Einstein was not sympathetic to the quantum-potential point of view, primarily because he insisted on a description of physical reality in space-time with only *local* interactions. According to Bohm and Hiley [131], the «most fundamentally new ontological feature» implied by the quantum theory and illustrated in the EPR paradox is that an independent dynamical significance can be attributed to the whole system, which is *not reducible* to the properties of its components and their interrelationships. It is this notion which Bohm refers to as *unbroken wholeness* that is associated with a system of two correlated quantum particles. This may seem similar to Bohr's idea of *individed wholeness* but there is an important difference. Bohm's approach implies that the «whole» is *analysable* in thought (*e.g.*, through the concept of nonclassical but causal trajectory of a particle acted on by the quantum potential); in constrast, Bohr maintained that the entire experimentally relevant situation was inherently an *unalysable whole* about which nothing could be said at all. It is interesting to recall here the comment made by Gell-Mann [132]: «Niels Bohr brainwashed a whole generation of physicists into believing that the problem had been solved fifty years ago».

Finally, we should like to mention the simple but instructive example of a hologram used by Bohm [133] to illustrate the essence of his interpretation of the EPR paradox. The hologram of two spheres, for instance, contains the information of each sphere over the entire hologram. It can, therefore, be said that in the hologram the two spheres are really amalgamated in a way that it is impossible to separate them (this is related to the fact that by illuminating only a part of the hologram one can get information about the entire object, even if less detailed). Similarly, Bohm views the EPR paradox as a manifestation of a truly interconnected wholeness characterizing a quantum system of two *apparently* separate yet correlated particles.

3·2. *Other nonlocal solutions.* – Many of those who believe that the quantum-mechanical predictions for the situation dealt with in the EPR paradox have been conclusively confirmed «under experimental conditions essentially equivalent to those needed for the EPR argument» tend to conclude, as articulated by d'Espagnat [134]: «To the extent that the notion an independent reality has a meaning, such a reality must be nonlocal». However, the advocates of this outlook have widely divergent views as regards how to formulate the nonlocal picture of microphysical reality. This state of affairs carries reminiscence of the remark made by Pauli [135]: «I think the important and extremely difficult task of our time is to try to build up a fresh idea of reality», although of course Pauli's view of reality was the opposite of the one advocated by the present authors.

D'Espagnat [136] observed that in the usual approach the Bell-type inequalities could not be derived from the locality condition alone, but only from the union of that assumption with another one, namely the one according to which counterfactual assertions are meaningful at least in some well-defined cases. One may, therefore, infer that the empirical violation of these inequalities would not hinge on the violation of the locality assumption, but on the denial that any counterfactual extrapolation of a meaningful factual statement can itself be universally valid. This is the standpoint which, according to d'Espagnat's conjecture, Bohr would have taken with respect to Bell's theorem. Relevant to this issue, the model proposed by Stapp [137] is particularly instructive. Stapp concludes: «...neither determinism, nor counterfactual definiteness, nor any idea of reality incompatible with orthodox quantum thinking need be assumed in order to prove the incompatibility of the empirical predictions of quantum theory with the EPR idea that no influence can propagate faster than light». Let us now look at the details of Stapp's model. It contains certain *hidden variables* which represent all the deterministic and stochastic quantities which are *not* used to provide the basis for a Clauser-Horne factorization structure of probabilities. Stapp writes $\lambda = (\lambda', \lambda'')$, where λ' is strictly predetermined, and λ'' is any stochastic variable.

Furthermore in this theory it is assumed that every act of measurement involves a *choice*. This choice «picks the actual from among what had previously been mere possibilities: the choice renders fixed and settled something that had prior to the choice been undetermined». A *choice* variable Z is also introduced and written $Z = (x, y)$, where x and y represent the choices of experiment in the regions R_α and R_β, respectively, where two correlated observations of the EPR type are made. The *choices* x and y are treated as independent free variables. Each of them can assume an infinite number of different values.

Now, suppose there are two observables A and A' that can be measured in R_α and other two, B and B', that can be measured in R_β. *The choice variable picks one observable before an act of measurement is made.* More precisely, in R_α, the chosen observable is

$$A, \text{ if } x \in X, \qquad A', \text{ if } x \in X', \tag{3.8}$$

where $X \cup X'$ is the set of possible values of x. Furthermore, in R_β, the chosen observable is

$$B, \text{ if } y \in Y, \qquad B', \text{ if } y \in Y', \tag{3.9}$$

where $Y \cup Y'$ is the set of possible values of y. Depending on the values of x and y there are so four possible experiments that can be chosen to be performed in R_α and in R_β, corresponding to the four pairs of observables

$$(3.10) \qquad (A, B); \quad (A, B'); \quad (A', B); \quad (A', B').$$

Now, in a general nonlocal model, the results of the measurements are assumed to be

$$(3.11) \qquad r_\alpha(x, y, \lambda) \text{ in } R_\alpha; \quad r_\beta(x, y, \lambda) \text{ in } R_\beta,$$

while if instead locality is assumed, r_α does not depend on y and r_β does not depend on x.

Stapp could prove that the local choice in (3.11) directly leads to incompatibility with the empirical predictions of quantum mechanics, which justifies his conclusion quoted earlier. An important point to be noted is that in Stapp's model the fixing of x, y and λ fixes the value only of the observable that is actually measured — the values of the other three observables remain completely indefinite. This indicates the absence of a «counterfactual definiteness» in the above model.

However, as regards this type of hidden-variable models, d'Espagnat [138] raises a pertinent question: «Through what process are the supplementary variables directly connected with perception and determine it causally while, at the same time — still according to the model — these variables are completely dependent on fields of forces, classical and quantum, which they in turn do not influence at all?» It seems that in interpreting these hidden-variable models one arbitrarily bestows on the hidden variables an *ontological status* which is superior to that of fields, whereas their complete dependence on the latter suggest the opposite. It is argued that only the explicit introduction of a postulate according to which «we perceive supplementary variables and not fields» would allow us to justify such a difference in status.

D'Espagnat [139] takes the standpoint that there is a sign of some «great truth» in the mismatch between the quantum rules and the notion of locality: «...it would mean that we are wrong when we believe the notions we have of space, of time, of spacetime, of the positions of things and events, are faithful descriptions of features possessed by independent reality». He advocates that a distinction between *empirical reality* (the set of phenomena to which we have a strictly cognitive access) and knowable or unknowable *independent reality* is conceptually relevant and useful. While the notion of *empirical reality* involves all that is precisely knowable, d'Espagnat cites the example of Bell's model (in terms of *beables*) [140] as indicating the possibility of a self-consistent description of *independent reality* which reproduces all the observed features of *empirical reality* correctly predicted by quantum mechanics. However, d'Espagnat believes that the conventional notions (such as that the space-time) reflect *something* of the *independent reality* in such an inherently incomplete way that it is impossible to reconstruct on their basis, with full clarity, what «independent reality» really is. In short, according to d'Espagnat, some aspects of *independent reality* will inevitably remain *veiled* to us. Detailed elaboration of this viewpoint can be found in his recent book [141].

Penrose [142, 143] interprets the EPR paradox as a manifestation of the difficulty associated with the idea of ascribing a certain physical objectivity to the state vector and of taking the state vector reduction to be a *real physical* process. If we have an EPR-type system of two particles A and B with a spacelike separation between them, the observation at either A or B results in a state vector collapse, but the collapse which takes place *earlier* — say, that at A — provides the state which is to be observed in the other measurement — say at B. However, the temporal order in which these collapses take place — *i.e.*, whether A or B is considered to be the *earlier* — depends upon the overall reference frame and, therefore, according to relativity, is not an objective property. Penrose points out that if the collapses were to take place along, say, the future light-cones of A and B, then there would be an inconsistency with the correlations predicted by quantum mechanics. This impies an inconsistency with the *spirit* of relativity if we wish to regard the collapse of state vector to be *physically real*. On the other hand, Penrose argues that so long as the rules of quantum mechanics are presumed to hold rigorously and the formalism does not in itself specify which measurement is possible in practice, the state vector does represent some *objective* property of an individual system in the following counterfactual sense: the state of an individual system is characterized by the results of experiments that one *might* perform on it. This contention is in sharp contrast to the *ensemble interpretation* of quantum mechanics [144] which asserts that the state vector describes only statistical properties of ensembles of systems and does not represent any objective aspect of physical reality associated with a single system; the latter viewpoint was reflected in Born's succinct remark [145]: «To say that Ψ describes the «state» of one single system is just a figure of speech». This outlook is, however, contested by Penrose using the following argument.

Suppose the state vector is $|\Psi(t)\rangle$. At any istant we can consider, at least in principle, making an observation on a single system (in a state represented by $|\Psi(t)\rangle$) corresponding to the observable represented by a (bounded) Hermitian opeator

$$Q = |\Psi(t)\rangle\langle\Psi(t)|\,.$$

The state $|\Psi(t)\rangle$ is (up to a phase) the only state pertaining to the single system for which the observable Q yields the result unity with certainty. Penrose infers: «The state must "know" that it has to produce this result in the event that the observation Q is actually performed. This is a completely objective property». This in turn suggests that one must have, in some appropriate sense, an objective physical description of the state vector reduction process. What the EPR paradox shows, according to Penrose, is that such a description «must be nonlocal in a way that fundamentally affects even the very fabric of space-time... The space-time must itself become subject to this nonlocal description».

How to reconcile such a counterintuitive nonlocal picture of physical reality with the *spirit* of relativity is a delicate issue. How can one develop a nonlocal quantum space-time theory with the relativistic invariance built-in? Penrose holds the opinion that the formalism of twistor theory, which is essentially a nonlocal treatment of space-time, ought to provide an important input in this direction [146]. It is conjectured that the theory of spin-networks [147] may also be relevant. However, as Penrose himself points out, neither the twistor theory nor the spin-network formalism has any time-asymmetrical ingredient, whereas the realistic view of the state vector reduction necessarily entails time asymmetry. This indicates the necessity of new physical inputs to such theoretical models in order to provide a satisfactory understanding of the issues related to the EPR paradox.

As a possible solution of the EPR paradox one can develop the idea, originally proposed by Dirac [148], that the aether, with appropriate properties, can be a permissible concept even by the special theory of relativity, if one takes into account the probabilistic nature of quantum phenomena. The basic assumption, underlying this approach, is that the velocity distribution of the particles constituting the aether has a constant value over the hyperboloid

$$v_0^2 - v_1^2 - v_2^2 - v_3^2 = 1 . \tag{3.12}$$

In this case the velocity distribution is the same for all the observers and there is perhaps no physical effect on moving bodies. In such a model one presumes that this aetherlike physical vacuum is made of extended rigid particles which can support signals, with superluminal velocity. The real random fluctuations of the aether are reflected in the statistical properties of quantum objects.

Within the framework of this theory one also postulates quantum waves which propagate as real physical collective excitations (*i.e.* as density waves) on the top of the foregoing Dirac's aether. In this way, information originating on the boundary of the Ψ wave acts with superluminal velocity (via the quantum potential) on the particle motions which propagate with subluminal group velocities along the flow lines of the quantum-mechanical Ψ waves [149].

Now one may raise the following objection against such a theory: consider two particles propagating in two widely separated regions of space R_1 and R_2 and having EPR-type correlation. Let their propagation take place according to precise deterministic equations containing nonlocal potentials like, for instance, Bohm's quantum potential. Then each particle *knows* instantaneously what the other particle is doing. It is therefore tempting to infer that the switching on or off of a magnetic field in R_1 must have instantaneous consequences on the particle located in R_2. One can then set up an arrangement such that the particle in R_2 enters a detector D_1 (a detector D_2) if the magnetic field in R_1 is off (is on). The observer in R_2 can therefore instantaneously learn what the other observer is doing in R_1. Using ensembles of such pairs it then seems possible to transmit instantaneous information from R_1 to R_2.

The above objection was tackled by Cufaro Petroni [150]. His contention was that it does not make any sense to consider *modifications* of the properties of our world (such as the one introduced above through the switching on and off of a magnetic field) because we live «in a completely deterministic world».

3'3. *Action of the future on the past.* – A solution of the EPR paradox based on the idea that it is possible to modify past events by means of retroactions from the future was developed in detail by Costa de Beauregard [151-153]. He noted that twice in classical physics, contradictions were discovered between *factlike* irreversible processes and *lawlike* reversibility of the physical theory:

i) When Boltzmann used statistical mechanics for deducing the Second Law of thermodynamics: the paradox inherent in extracting time symmetry from a theory like Newtonian mechanics that is intrinsically time symmetric was exposed by Loschmidt and Zermelo.

ii) When the principle of retarded waves was used in physical optics and in classical electrodynamics in order to exclude one-half of the mathematically permissible solutions of the wave equations.

Costa de Beauregard contends that retroactions in time do play a role and should not be discarded like in i) and ii). One way to see this is to consider modern cybernetics. In computers and other information-processing machines the chain

$$\text{information} \xrightarrow{(1)} \text{negentropy} \xrightarrow{(2)} \text{information}$$

means that a concept is coded and sent as a message, before being decoded and received. Negentropy is entropy with a minus sign. The step (2) above is the *learning transition*, where information shows up as an increase in knowledge, while step (1) is the *willing transition*, where the concept of information manifests as an organizing power.

In the theoretical framework (*de jure*) there is a complete symmetry between the two transitions. In spite of this, there is an asymmetry in practice (*de facto*) because of the fact that irreversibility is generated by misprints in the coding, noise along the line, mistakes in decoding, and so on. The relationship between the variation of negentropy ΔN and the variation of information ΔI is

$$\Delta N = k \ln (2\, \Delta I) . \tag{3.13}$$

If N and I are both expressed in *practical* units, it turns out that the factor multiplying ΔI is very small, of the order of 10^{-16}. It is therefore very difficult to produce significant increases of negentropy (decreases of entropy) by increasing the information. Vice versa, even a very small increase of negentropy can give rise to a large gain of information. If one lets $k \to 0$, one obtains a situation where gaining knowledge is absolutely costless, but producing order is utterly impossibile. In this limit, consciousness is made totally passive.

While the roots of this idea go deep into classical physics, it is in quantum theory that de Beauregard thinks the most important effects of retroaction can be seen. Here the theory is completely time symmetrical, but only until the idea of collapse of wave function is introduced. As regards the EPR paradox, de Beauregard argues that the problem is essentially that of tayloring the wording of the EPR situation after the mathematics. In his opinion, there has been in our century an irreversible victory of formalism over modelism.

Costa de Beauregard's analysis entails acceptance of the EPR paradox as a true fact and he attempts to formalize it within the framework of relativistic quantum theory using Jordan-Pauli propagators. In this formalism, the completeness of the basis for expanding the wave function at any instant in terms of orthogonal propagators requires the presence of both retarded and advanced waves. This is shown by de Beauregard to imply that the wave function collapse in a certain space-time region produces consequences propagating both towards the future and towards the past; in the latter case, the propagation is through negative energies. However, in view of some unresolved basic problems associated with the formulation of a fully consistent relativistic quantum theory, one is inclined to view this type of approach with caution.

The conclusion that one can draw from this type of analysis is that the elements of reality introduced in the formulation of the EPR paradox are created by acts of measurement, and they propagate backward in time with one of the two correlated quantum objects from the place of measurement to the source.

In particular, this approach dismisses any question of associating elements of reality with observables that are not actually measured. In this sense, the solution of the EPR paradox using the idea of retroaction towards the past has some philosophical similarity with that of Bohr. For the sake of completeness it should be noted that several other authors have also proposed similar solutions of the EPR paradox: Stapp [154], Davidon [155], Rayski [156], Rietdijk [157], Cramer [158] and Sutherland [159].

3·4. *The nonergodic interpretation.* – The key idea underlying the nonergodic interpretation of quantum mechanics is that a sequence of quantum systems, even if separated by large time intervals from one another, do not behave independently, while interacting with the measuring apparatus. To illustrate this idea let us consider the double-slit experiment.

The indirect interaction postulated above is such that a particle passing throung a slit knows whether the other slit is open through information coded in the intervening medium between the two slits. The particles which emerged earlier from the other slit modified the physical properties of space and gave rise to the recording of the relative information. Interference happens after a sufficiently large number of particles have passed through the apparatus and conditioned the medium. This is how the particles are pictured to interfere with other particles indireclty through the medium [160].

As a more general situation, consider a quantum experiment being repeated a large number of times; let us call every repetition a *run*. The number of runs be denoted by R and N be the number of quantum objects in every run; for simplicity, we take N to be the same for every run. We represent the state of the n-th particle in the r-th run by λ_{rn} and s_{rn} denotes the state of the measuring apparatus just before interacting with the n-th particle of the r-th run. Once λ_{rn} and s_{rn} are given, the result of the measurement, A_{rn}, is assumed to be completely fixed. We have therefore

$$A_{rn} = A(\lambda_{rn}, s_{rn}) .$$

Now, two types of averages are possible:

$$\bar{A}_r = \frac{1}{N} \sum_{n=1}^{N} A_{rn}; \qquad \bar{A}_n = \frac{1}{R} \sum_{r=1}^{R} A_{rn} . \tag{3.14}$$

$\bar{A}_r$ is called the run average, while $\bar{A}_n$ is the ensemble average at *time* as Buonomano observed that the ergodic assumption implies

$$\bar{A}_r = \bar{A}_n \tag{3.15}$$

but that one should not take for granted the validity of such a strong assumption, but rather check it experimentally. However, to do this it is necessary to avoid the medium polarization effects which means that one has to keep the runs separated in time from one another, and eventually also in different regions of space where no experiments have been carried out previously. It is therefore clear that the ensemble average for $n = 1$, $\bar{A}_{n=1}$, should represent events placed in conditions where there is no medium effect on the particles. $\bar{A}_{n=1}$ represents a situation where there is no quantum effect and classical physics holds good. In contrast, $\bar{A}_r$ describes quantum-mechanical situations for all r. The case of $\bar{A}_n$ for not too large values of n, but with $n \neq 1$, represents situations where transition between classical and quantum physics takes place.

It is claimed that in principle the nonergodic interpretation of quantum mechanics can solve the EPR paradox. It explains the apparent violations of local realism as due to nonergodic effects within the framework of a strictly local theory. As an illustrative case let us consider the left-hand side of an EPR-type polarization correlation experiment and divide the space between polarizer and source into M cells, numbering them from left to right; the polarizer is in cell 1 and the source in cell M. One then presumes that the state of the cell M depends on the previous state of the neighbouring cells. It is then clear that after the passage of one photon, the state of cell 2 depends on that of the polarizer. After the passage of two photons, the state of cell 3 depends on that of the polarizer. Hence after the passage of M photons, the cell M, that is the source, depends on the state of the polarizer.

Similarly one can treat the right-hand side of the polarization-correlation experiment. Then one obtains a situation where the source produces pairs of photons in a state which is dependent on the configuration of the analysing-detecting apparatus. Bell's type inequality cannot be obtained in such a case and hence it is in the sense that one claims to avoid the EPR paradox.

In all cases Buonomano's ideas can be put to an empirical test as shown in a recent paper by Buonomano and Bartmann [161] who propose an experiment with a laser for studying experimentally the validity of the ergodic assumption. The experiment has been performed and has given a negative result, as acknowledged by Buonomano [162].

3'5. *Negative probabilities.* – The notion of negative probabilities was discussed by Dirac in 1942 [163]. Dirac commented: «Negative energies and probabilities should not be considered as nonsense. They are well-defined concepts mathematically, like a negative sum of money, since the equations which express the important properties of energies and probabilities can still be used when they are negative. Thus negative energies and probabilities should be considered simply as things which do not appear in experimental results».

Feynman [164] asserted that the essential difference between a probabilistic classical world and the quantum world «is that somehow or other it appears as if the probabilities would have to go negative...»

These ideas were developed by Mückenheim in 1982 [165] in order to propose a solution of the EPR paradox. In this context it is important to note that in the proofs of Bell's inequality one always assumes that probabilities are positive and not larger than one. If this condition is relaxed, Bell's inequality cannot be deduced anymore.

Let the spin vector be $\boldsymbol{S}$ for one member of the EPR-type pair of particles, and $-\boldsymbol{S}$ for its partners, where $\boldsymbol{S}$ is assumed to have a random distribution over the sphere of radius $\frac{\sqrt{3}}{2}\,\hbar$ pertaining to a statistical ensemble of such pairs. The length $\frac{\sqrt{3}}{2}\,\hbar$ is chosen in order to reproduce the quantum-mechanical eigenvalue of $\boldsymbol{S}^2$, which is $\frac{3}{4}\,\hbar^2$. Note that if $\hat{a}$ is a unit vector, the projection of $\boldsymbol{S}$ over a satisfies the following condition:

$$-\frac{\sqrt{3}}{2}\hbar \leq \boldsymbol{S}\cdot\hat{a} \leq +\frac{\sqrt{3}}{2}\hbar\,. \tag{3.16}$$

In this model it is assumed that the probabilities $\omega(\hat{a}_+, \boldsymbol{S})$ and $\omega(\hat{a}_-, \boldsymbol{S})$ of measuring $\boldsymbol{S}\cdot\hat{a}$ and finding the positive and negative eigenvalue, respectively, are linear functions of $\boldsymbol{S}\cdot\hat{a}$, and that their expressions satisfying

$$\omega(\hat{a}_+, \boldsymbol{S}) + \omega(\hat{a}_-, \boldsymbol{S}) = 1 \tag{3.17}$$

are given by

$$\begin{cases}\omega(\hat{a}_+, \boldsymbol{S}) = 0.5 + \boldsymbol{S}\cdot\hat{a}\,,\\ \omega(\hat{a}_-, \boldsymbol{S}) = 0.5 - \boldsymbol{S}\cdot\hat{a}\,.\end{cases} \tag{3.18}$$

It should be noted that these probabilities can assume negative values because of (3.16).

Now, considering the case of correlated spin measurements along $\hat{a}$ and $\hat{b}$ for the first pair, respectively, we have the correlation function given by

$$P(\hat{a}, \hat{b}) = \frac{\hbar^2}{16\pi}\int \mathrm{d}\Omega\,[\omega(\hat{a}_+, \boldsymbol{S}) - \omega(\hat{a}_-, \boldsymbol{S})][\omega(\hat{b}_+, -\boldsymbol{S}) - \omega(\hat{b}_-, -\boldsymbol{S})]\,. \tag{3.19}$$

Using eq. (3.18) one obtains from the above expression

$$P(\hat{a}, \hat{b}) = -\frac{\hbar^2}{4}\hat{a}\cdot\hat{b}\,, \tag{3.20}$$

which agrees with the quantum-mechanical correlation function for the singlet state. It is therefore demonstrated that a local realist model can reproduce the quantum-mechanical violations of Bell's inequality provided one allows for negative probabilities. Ivanovic [166] has furthermore shown that by incorporating complex probabilities as well one can obtain agreement between the quantum-mechanical predictions and local realism in the context of the EPR-type examples.

The possibility of a negative-probability solution of the EPR paradox has also been studied by Polubarinov [167].

We will now give a general argument, following Home, Lepore and Selleri [168], showing that one can always reproduce the quantum-mechanical results for nonfactorizable state vectors of correlated systems by hidden-variable models using nonphysical probabilities.

In order to develop the argument, let us show that any state vector

$$|\Psi\rangle = \sum_{nl} f_{nl} |\varphi_n\rangle |\xi_l\rangle , \tag{3.21}$$

where $|\varphi_n\rangle$'s and $|\xi_l\rangle$'s are orthonormal sets of eigenstates corresponding to the correlated systems I and II, respectively, can be written in the form

$$|\Psi\rangle = \sum_l \sqrt{\omega_l} |\Psi_l\rangle |\xi_l\rangle . \tag{3.22}$$

From the normalization condition of $|\Psi\rangle$ as given in (3.21) one has

$$\sum_{nl} |f_{nl}|^2 = 1 , \tag{3.23}$$

whence, by defining

$$\omega_l = \sum_n |f_{nl}|^2 , \tag{3.24}$$

one sees that $\omega_l = 0$ only if $f_{nl} = 0$ for all n, a possibility that can be excluded from the beginning in (3.21), by limiting $|\Psi\rangle$ to that part of the Hilbert space where its coefficients f_{nl} are not all zero for fixed n and variable l. With this restriction we have, for all l

$$\omega_l > 0 \tag{3.25}$$

and

$$\sum_l \omega_l = 1 . \tag{3.26}$$

Defining

$$|\Psi_l\rangle = \frac{1}{\sqrt{\omega_l}} \sum_n f_{nl} |\varphi_n\rangle , \tag{3.27}$$

one easily obtains eq. (3.22) from eq. (3.21).

To proceed further, we take up two dichotomic observables $\hat{A}$ (pertaining to I) and $\hat{B}$ (pertaining to II) with eigenvalues $= \pm 1$ and eigenvectors $\{|A_+\rangle, |A_-\rangle\}$ and $\{|B_+\rangle, |B_-\rangle\}$, respectively. Considering joint probabilities

$$p_{AB}(A = \pm 1, B = \pm 1) = |\langle A_\pm B_\pm | \Psi\rangle|^2, \tag{3.28}$$

we obtain by using eq. (3.22)

$$p_{AB}(A = \pm 1, B = \pm 1) = \sum_l \sqrt{\omega_l}\langle A_\pm | \Psi_l\rangle\langle B_\pm | \xi_l\rangle \cdot \sum_{l'} \sqrt{\omega_{l'}}\langle \Psi_{l'} | A_\pm\rangle\langle \xi_{l'} | B_\pm\rangle$$

or

$$p_{AB}(A = \pm 1, B = \pm 1) = \sum_l \sum_{l'} \sqrt{\omega_l}\sqrt{\omega_{l'}}\langle A_\pm | \Psi_l\rangle\langle \Psi_{l'} | A_\pm\rangle \cdot \langle B_\pm | \xi_l\rangle\langle \xi_{l'} | B_\pm\rangle. \tag{3.29}$$

Defining

$$\rho(l, l') = \frac{\sqrt{\omega_l}\sqrt{\omega_{l'}}}{\left(\sum_k \sqrt{\omega_k}\right)^2} \tag{3.30}$$

and the single probabilities

$$p(A = \pm 1; l, l') = \left(\sum_k \sqrt{\omega_k}\right)\langle A_\pm | \Psi_l\rangle\langle \Psi_{l'} | A_\pm\rangle \tag{3.31}$$

and

$$q(B = \pm 1; l, l') = \left(\sum_k \sqrt{\omega_k}\right)\langle B_\pm | \xi_l\rangle\langle \xi_{l'} | B_\pm\rangle. \tag{3.32}$$

One can write eq. (3.29) in the form

$$p_{AB}(A = \pm 1, B = \pm 1) = \sum_{ll'} \rho(l, l')\, p(A = \pm 1; l, l')\, q(B = \pm 1; l, l') \tag{3.33}$$

with

$$\sum_{ll'} \rho(l, l') = 1, \tag{3.34}$$

which follows from eq. (3.30).

Equation (3.33) shows that the joint probabilities obtained from a nonfactorizable state of the type given by eq. (3.21) can always be written in forms similar to the expressions used in local realistic models based on the Clauser-Horne factorizability condition. It is, however, evident from eq. (3.26) that

$$\sum_k \sqrt{\omega_k} > 1 \tag{3.35}$$

if at least two ω_k's are nonzero. If only one $\omega_k \neq 0$ (and then $\omega_k = 1$) the state vector (3.22) is factorizable and is fully compatible with local realism, as is well known. It follows, in general, from (3.31) and (3.32) that $p(A = \pm 1;\, l,\, l')$ and $q(B = \pm 1;\, l,\, l')$ may be complex and even if they are real they do not, in general, satisfy the conditions

$$0 \leq p(A = \pm 1;\, l,\, l'),\ q(B = \pm 1;\, l,\, l') \leq 1\,. \tag{3.36}$$

One is therefore led to the conclusion that the general validity of eq. (3.33) ensures that we can always reproduce the quantum-mechanical results for nonfactorizable state vectors with probabilities of the Clauser-Horne type provided one allows for nonphysical probabilities violating the conditions (3.36).

3·6. *Variable probabilities.* – The notion of *variable probabilities* as a possible solution of the EPR paradox is provided by the results of the performed experiments with atomic photon pairs, which indicate that the inequalities of the strong type (deduced from local realism and from additional assumptions) are violated. This result seems very likely to be correct even though there is a protracted debate on the role of rescattering in the atomic source (see Sanz and Sanchez Gomez [77] and the references quoted therein).

It perfectly logical to adopt the viewpoint that it is not local realism but the additional assumptions which should be blamed for the violations of the strong inequalities. To follow this outlook one must then study local realistic models in which the logical negation of the additional assumptions is explicitly taken into account. The relevant models should then satisfy the following conditions:

1) If a pair of photons emerge from the regions of space where two polarizers can be located the probability of their joint detection from two photomultipliers depends on the presence and/or on the orientations of the polarizers (CHSH property).

2) Pertaining to a photon in the state λ, the probability of its detection with a polarizer interposed on its path can be larger than the detection probability with the polarizer removed (CH property).

3) For a photon in the state λ the sum of the detection probabilities in the *ordinary* and in the *extraordinary* beams emerging from a two-way polarizer depends on the polarizer's orientation (GR property).

A comprehensive overview of this line of research can be found in the recent review articles by Pascazio [169] and by Ferrero, Marshall and Santos [170].

From this point of view it follows that the issue of quantum theory *vs.* local realism can be settled only if weak inequalities (deduced from local realism alone, without any auxiliary assumption) can be experimentally tested. This appears not to be possible in near future in the context of the experiments involving pairs of photons. This situation can be better in the case of some proposed particle physics experiments and in the case of experiments using pairs of atoms where the detectors have higher efficiency.

The interesting point to be stressed here is that it is possible to do meaningful studies, even for the case of low-efficiency detectors, by replacing the usual additional assumptions (called CHSH, CH and GR) by more physically appropriate conditions. It is very much unlikely that the large disagreement between quantum theory and local realism for the ideal high-efficiency detectors would reduce to perfect agreement for the low-efficiency detectors.

In this connection it should be interesting to study the use of symmetrical functions for describing the detection processes of the two photons in view of the demonstration by Caser [171] that the quantum-mechanical predictions for such a case cannot agree with the factorizable probabilities of Clauser and Horne.

Along this line of thought there remains a lot of work to be done beyond the proof that such models can indeed reproduce exactly the quantum-mechanical predictions for low-efficiency detectors. What is still missing is a great physical idea about the true nature of photon pairs and of their interactions (local or nonlocal) with polarizers and detectors. An attempt of this type was made by Garuccio and Selleri [81], but their predictions turn out to be incompatible with the recent experiments.

Selleri and Zeilinger [172] generalized the deterministic model proposed by Wigner [34] by incorporating additional variables which determine whether a specific photon will trigger the detector or not. In this approach, each individual photon pair is described by the set of ten variables

$$(s, s', \sigma, \sigma', \delta, t, t', \tau, \tau', \varepsilon)$$

with the first five pertaining to photon α and the second five pertain to photon β. Each of these variables can only be either zero or unity: $s = 1$ $(s' = 1)$ $[t = 1$ $(t' = 1)]$ determines that photon α [photon β] will traverse its polarizer oriented along direction $a\,(a')\,[b\,(b')]$, $\sigma = 1\,(\sigma' = 1)\,[\tau = 1\,(\tau' = 1)]$ determines that photon α [photon β] will be registered by its detector after having passed the polarizers oriented along direction $a\,(a')\,[b\,(b')]$, $\delta = 1\,[\varepsilon = 1]$ determines that photon α [photon β] will be registered by its detector if no polarizer is in its beam path, $s, s', \sigma, \sigma', \delta = 0\,[t, t', \tau, \tau', \varepsilon = 0]$ determines that photon α [photon β] will not pass its polarizers or will not be registered by the detector. This model leads naturally to the violation of the strong inequalities, but, being local, naturally always satisfies the weak ones.

* * *

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