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ON THE IDENTITIES OF THE GRASSMANN ALGEBRAS IN CHARACTERISTIC $p > 0$

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ABSTRACT

In this note we exhibit bases of the polynomial identities satisfied by the Grassmann algebras over a field of positive characteristic. This allows us to answer the following question of Kemer: Does the infinite dimensional Grassmann algebra with 1, over an infinite field K of characteristic 3, satisfy all identities of the algebra $M_2(K)$ of all 2×2 matrices over K? We give a negative answer to this question. Further, we show that certain finite dimensional Grassmann algebras do give a positive answer to Kemer's question. All this allows us to obtain some information about the identities satisfied by the algebra $M_2(K)$ over an infinite field K of positive odd characteristic, and to conjecture bases of the identities of $M_2(K)$.

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1. Introduction

The Grassmann algebras and their identities play an important role in the theory of PI algebras. Over a field of zero characteristic, these algebras are the "building blocks" for the T-prime T-ideals (see [9]) and, when the base field is of positive characteristic, they turned out to be crucial too (see for instance [10], [11], [12], [13]).

Let $X = \{x_1, x_2, \ldots\}$ be a countable infinite set of symbols (variables), and let $K_1(X)$ and $K(X)$ be the free associative algebra with 1 and without 1, respectively, over a field K. In 1962, Latyshev [16] proved that if $\operatorname{char} K = 0$ then the T-ideal of $K_1(X)$ generated by the commutator $[x_1, x_2, x_3]$ is Spechtian. Here $[a, b] = ab - ba$, and $[a, b, c] = [[a, b], c]$ for every a, b, $c \in K_1(X)$. In 1973, Krakowski and Regev [15] proved that the polynomial $[x_1, x_2, x_3]$ forms a basis of the polynomial identities of the infinite dimensional unitary Grassmann algebra G over a field of characteristic 0. In 1991, Di Vincenzo [4] gave a different proof of this result and he also exhibited, for any k, finite bases of the identities of G_k, the Grassmann algebra of a k-dimensional vector space.

Concerning positive characteristic, in 1980, Stojanova-Venkova (see [22]) found finite bases of the identities satisfied by the non-unitary finite dimensional Grassmann algebras, over an arbitrary field K. In 1981, Siderov [3] did the same in the infinite dimensional case. Further, he proved that for an arbitrary field K, every T-ideal of $K(X)$, the free associative algebra without 1, containing $[x_1, x_2, x_3]$ is finitely based. In 1991, Regev [19] studied properties of the multilinear parts of the T-ideal of the infinite dimensional Grassmann algebra (unitary as well as non-unitary), over a finite field.

Further results on the Grassmann algebras, their polynomial identities and related topics have been obtained in recent years by several authors (see for instance [7], [14], [2], [17], [5]).

In this paper we deal with algebras over a field of positive characteristic $p > 2$. In the next section we give some preliminary notation and results. Section 3 deals with the identities of the infinite dimensional Grassmann algebra G with 1. We give a basis of the polynomial identities of G over an infinite field and we describe the corresponding relatively free algebra. As a corollary we exhibit a basis of the polynomial identities of the Grassmann algebras G_k defined by a k-dimensional vector space, $k < \infty$, and we prove the Specht property for the T-ideals of G and of G_k. In section 4 we show that the Grassmann algebra G does not satisfy all the identities of the matrix algebra of order 2 when the base field K is of characteristic 3. This answers in the negative for the case $p = 3$ a question of Kemer (see [12]) asking whether G satisfies all the identities of the matrix algebra of order $(p+1)/2$ in characteristic p. We feel that with some "adjustment" the question remains still open in its generality. The main tool in this section is Specht's reduction of the study of T-ideals to the study of their commutator polynomials. Finally, in section 5 we give a brief account of the results of [22] and [3] since it seems they are not readily accessible.

2. Preliminaries

All algebras and vector spaces considered here are over a field K of positive characteristic p. If A is an algebra and a, $b \in A$ then $[a,b] = ab - ba$ stands for the commutator of a and b, and $a \circ b = ab + ba$. All higher commutators are left normed, i.e., $[a,b,c] = [[a,b],c]$ etc. If $L(X)$ is the free Lie algebra freely generated by X then $L(X)$ can be naturally embedded in $K_1(X)$. Denote by $B(X)$ the associative subalgebra of $K_1(X)$ generated by all homogeneous elements of $L(X)$ of degree ≥ 2. Then $B(X)$ is spanned over K by all products of (long) commutators in the free generators X. Call the elements of $B(X)$ commutator (or proper) polynomials.

If we deal with infinite fields then every polynomial identity is equivalent (as an identity) to a finite collection of multihomogeneous identities. Hence, when the base field is infinite we may consider multihomogeneous polynomials only. It is well known that in this case, every T-ideal T in $K_1(X)$ is generated as a T-ideal by its commutator polynomials, i.e., by $T \cap B(X)$, see [21].

If A is an associative algebra over K, we denote by $\operatorname{var}(A)$ the variety of algebras generated by A and by $T(A)$ the T-ideal of A and of $\operatorname{var}(A)$. The variety $\operatorname{var}(A)$ is Spechtian if its T-ideal $T(A)$ is finitely based and every T-ideal containing $T(A)$ is finitely based too. Sometimes we say that A or even $T(A)$ is Spechtian. A celebrated theorem of Kemer (see, e.g., [9]) states that when $\operatorname{char} K = 0$ every T-ideal is Spechtian. There exist examples of varieties of associative algebras over fields of positive characteristic that are not finitely based. When the base field is finite, such an example is given by Belov in [1]; and for an infinite field, examples are due to Grishin [8] and to Shchigolev [20].

Let V be a vector space over K of countable infinite dimension with basis e_1, e_2, $\ldots$, and denote by V_k the subspace spanned by e_1, e_2, $\ldots$, e_k. The Grassmann algebra G of V is the associative algebra with K-basis consisting of 1 and all products of the form

$$\{e_{i_1} e_{i_2} \cdots e_{i_m} \mid i_1 < i_2 < \cdots < i_m, \quad m = 1, 2, \ldots\}$$

and with multiplication induced by $e_i^2 = 0$, $e_ie_j = -e_je_i$. The algebra G_k is the subalgebra of G generated by 1 and V_k. Analogously, one defines the non-unitary Grassmann algebra H as the subalgebra of G generated by the products $e_{i_1}e_{i_2}\cdots e_{i_m}$, $i_1 < i_2 < \cdots < i_m$, $m \geq 1$, and also $H_k = H \cap G_k$. Denote by G^0 the subspace of G spanned by 1 and by all basic elements of the form $e_{i_1}e_{i_2}\cdots e_{i_{2m}}$, $m \geq 1$, and let G^1 be the subspace spanned by all elements of the form $e_{i_1}e_{i_2}\cdots e_{i_{2m+1}}$, $m \geq 0$. Then G^0 is the centre of G, and $ab = -ba$ for every $a, b \in G^1$.

When $\operatorname{char} K = p = 2$, then obviously all these algebras are commutative and hence they are not very "interesting" from the PI point of view. Therefore, we restrict our attention to the case $p > 2$.

3. Unitary Grassmann algebras

Throughout this section K will be an infinite field. We now derive some useful polynomial identities for the Grassmann algebras G and G_k. Since $G_k \subset G$, $H \subset G$ and $H_k \subset G_k$ these identities must hold for G_k, H and H_k, respectively.

The Grassmann algebra G satisfies the identity

$$[x_1, x_2, x_3] \equiv 0. \tag{1}$$

Let T be the T-ideal of the free associative algebra $K_1(X)$ generated by the polynomial $[x_1, x_2, x_3]$, and denote by $F = K_1(X)/T$ the corresponding relatively free algebra of countable rank in the variety determined by the identity (1).

Denote by $s_n = s_n(x_1, x_2, \ldots, x_n) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$ the standard polynomial of degree n. Here S_n stands for the symmetric group acting on the set $\{1, 2, \ldots, n\}$, and $(-1)^\sigma$ is the sign of the permutation $\sigma \in S_n$. The following fact is well known; we give its proof for completeness.

LEMMA 1: *The polynomial $[x_1, x_2][x_1, x_3]$ belongs to the T-ideal T.*

Proof: The polynomial $[x_1, x_2, x_1, x_3]$ belongs to T, and the equality

$$\begin{aligned}[x_1, x_2, x_1, x_3] =& [x_1, x_2][x_1, x_3] + [x_1, x_2, x_3]x_1 \\ & - x_1x_1x_2x_3 + x_1x_2x_1x_3 - x_3x_1x_2x_1 + x_3x_1x_1x_2\end{aligned}$$

holds in the free associative algebra. On the other hand,

$$x_1x_1x_2x_3 - x_1x_2x_1x_3 + x_3x_1x_2x_1 - x_3x_1x_1x_2 = [x_1, x_1x_2, x_3] \in T,$$
$$[x_1, x_2][x_1, x_3] = [x_1, x_2, x_1, x_3] + [x_1, x_1x_2, x_3] - [x_1, x_2, x_3]x_1 \in T. \qquad \blacksquare$$

The following corollary was proved in [16, Lemma 1] using a slightly different approach.

COROLLARY 2: *The polynomials:*

$$[x_1,x_2][x_3,x_4]+[x_1,x_3][x_2,x_4],\quad [x_1,x_2][x_3,x_4]+[x_3,x_2][x_1,x_4],$$
$$[x_1,x_2][x_3,x_4]+[x_1,x_4][x_3,x_2],\quad [x_1,x_2][x_3,x_4]+[x_4,x_2][x_3,x_1]$$

belong to T.

Proof: Linearise the polynomials $[x_2,x_1][x_1,x_3]$, $[x_1,x_2][x_1,x_3]$, $[x_2,x_1][x_3,x_1]$, $[x_1,x_2][x_3,x_1]$ (all of them belong to T), and afterwards change appropriately the indices. Since $\operatorname{char} K = p > 2$ and the degree of x_1 equals 2, the linearisation is harmless. ∎

PROPOSITION 3: *The equality*

$$2^n s_{2n}(x_1,x_2,\ldots,x_{2n}) = (2n)!\,[x_1,x_2][x_3,x_4]\ldots[x_{2n-1},x_{2n}] \tag{2}$$

holds in the algebra F.

Proof: Apply the previous corollary, using the representation

$$2^n s_{2n}(x_1,x_2,\ldots,x_{2n}) = \sum_{\sigma\in S_{2n}} (-1)^\sigma [x_{\sigma(1)},x_{\sigma(2)}]\ldots[x_{\sigma(2n-1)},x_{\sigma(2n)}].$$ ∎

This proposition shows that if $p < 2n$ then s_{2n} is an identity on G. One can use the polynomial $[x_1,x_2][x_3,x_4]\cdots[x_{2n-1},x_{2n}]$ instead; if $\operatorname{char} K = 0$ or $\operatorname{char} K > 2n$, then the latter polynomial and s_{2n} are equivalent as identities modulo the identity (1).

LEMMA 4: *The polynomials $t_{2n} = [x_1,x_2][x_3,x_4]\cdots[x_{2n-1},x_{2n}]$ do not vanish on the Grassmann algebra G, $n = 1, 2, \ldots$.*

Proof: Observe that $t_{2n}(e_1,e_2,\ldots,e_{2n}) = 2^n e_1e_2\cdots e_{2n} \neq 0$. ∎

LEMMA 5: *Let $\operatorname{char} K = p$. The standard polynomial s_k is a polynomial identity of G if and only if $k \geq p+1$.*

Proof: By Proposition 3, s_{p+1} vanishes in G and

$$s_p(e_1e_2,e_3,\ldots,e_{p+1}) = e_1e_2 s_{p-1}(e_3,\ldots,e_{p+1}) = (p-1)!e_1e_2\cdots e_{p+1}.$$

(Note that $p-1$ is even.) ∎

Hence s_{p+1} is the standard identity of minimal degree satisfied by the Grassmann algebra G when $\operatorname{char} K = p$.

THEOREM 6: *Over an infinite field K of characteristic $p \neq 2$ all polynomial identities of the infinite dimensional Grassmann algebra G are consequences of the single identity $[x_1, x_2, x_3]$.*

Proof: As mentioned above in the case of $\operatorname{char} K = 0$, the theorem was proved in [16] and in [15]. The proof in our case is similar to that in [16]. We may assume that $\operatorname{char} K = p > 2$. Since $1 \in G$ it is sufficient to prove that all *commutator* identities of G follow from the above identity.

Let $f(x_1, x_2, \ldots, x_n) \in B(X)$ be an identity on G. Write $f = \sum \alpha_u u_1 u_2 \cdots u_k$ where u_j are commutators of length ≥ 2. Due to the identity (1) we may assume that all commutators u_i are of the form $[x_a, x_b]$, and they are central. Using Lemma 1 we may consider f multilinear. Applying Corollary 2, the polynomial f can be reduced to the form

$$f(x_1, x_2, \ldots, x_n) = \alpha[x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]$$

where $\alpha \in K$ and n is even. Hence if $\alpha \neq 0$ in K then Lemma 4 yields that f is not an identity on G. Therefore $\alpha = 0$, and we are done. ∎

Denote by $T(G)$ the T-ideal of the Grassmann algebra G in $K_1(X)$, hence $T = T(G)$, and $F \cong K_1(X)/T$ is the relatively free algebra of countable rank in the variety of unitary algebras generated by G.

COROLLARY 7: *The T-ideal $T = T(G)$ is Spechtian.*

Proof: Let T_1 be a T-ideal containing $T(G)$. According to the proof of Theorem 6, every homogeneous polynomial $f \in T_1$ can be reduced, modulo $T(G)$, to the form $\alpha[x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n]$. Choose the least n such that

$$\alpha[x_1, x_2][x_3, x_4] \cdots [x_{n-1}, x_n] \in T_1, \quad \alpha \neq 0.$$

Then T_1 is generated as a T-ideal by the above product of commutators and by $[x_1, x_2, x_3]$. ∎

Now we describe bases of the identities for the Grassmann algebras G_k. Notably certain difference arises in the cases $k \leq p$ and $k > p$.

COROLLARY 8: *Let $p \le k < \infty$ and $t = [k/2] + 1$, $[a]$ being the integer part of the number a. Then the identities*

$$[x_1, x_2, x_3], \quad [x_1, x_2][x_3, x_4] \cdots [x_{2t-1}, x_{2t}]$$

form a basis of the identities of the algebra G_k.

If $k < p$, one substitutes the last identity for the standard identity s_{k+1} when k is odd, and for s_{k+2} when k is even.

Proof: If $s < t$ then obviously $[e_1, e_2][e_3, e_4] \cdots [e_{2s-1}, e_{2s}] \neq 0$ in G_k. It can be verified easily that $[x_1, x_2][x_3, x_4] \cdots [x_{2t-1}, x_{2t}]$ is indeed an identity of G_k. Since the latter polynomial is multilinear and alternating modulo the identity $[x_1, x_2, x_3]$, then one can substitute in it only distinct elements from the basis of G_k. But in this polynomial every variable appears in a commutator, hence we cannot substitute central elements for x_i. Now the statement for $k \ge p$ is obvious.

A similar reasoning proves the statement when $k < p$, using the fact that when k is even then

$$s_{k+1}(1, e_1, e_2, \ldots, e_k) = s_k(e_1, e_2, \ldots, e_k) = k!\, e_1 e_2 \cdots e_k. \quad \blacksquare$$

Remark: The last Corollary implies that one cannot distinguish the Grassmann algebra G_{2k} from G_{2k+1} by means of polynomial identities since they satisfy the same identities, $T(G_{2k}) = T(G_{2k+1})$.

4. The algebra of 2×2 matrices

Denote by M_2 the algebra of 2×2 matrices over K and by sl_2 the Lie algebra of all traceless 2×2 matrices over K. In [18, Section 4], Razmyslov proved the following theorem.

THEOREM 9 ([18, Theorem 4]): *Let K be an infinite field and let $\operatorname{char} K \neq 2$. If all multilinear identities of the Lie algebra sl_2 follow from a finite number of them, then the multilinear identities of the associative algebra M_2 also follow from a finite number of such identities.*

On the other hand, Vasilovsky in [23, Theorem 1] proved that whenever K is an infinite field of characteristic $\neq 2$ then the identities of the Lie algebra sl_2 follow from the single identity

$$[x_1, x_2, [x_3, x_4], x_5] + [x_1, x_2, [x_3, x_5], x_4] + [x_1, x_4, [x_2, x_5], x_3] + [x_1, x_5, [x_2, x_4], x_3]. \tag{3}$$

Hence the multilinear identities of M_2 when $\operatorname{char} K \neq 2$ follow from some finite collection of multilinear identities of M_2.

In [12] Kemer posed the following question: Does the infinite dimensional Grassmann algebra G satisfy all polynomial identities of the matrix algebra $M_{(p+1)/2}$ where $p = \operatorname{char} K$?

We can now answer this question in case $\operatorname{char} K = 3$. In fact we prove

THEOREM 10: *Let K be an infinite field of characteristic 3. Then G does not satisfy all polynomial identities of the matrix algebra M_2 over K but G_4 (and G_5) does.*

Proof: First, since both G and M_2 are unitary algebras we may consider commutator identities only. On the other hand, we have already proved that every commutator polynomial that is not multilinear is an identity on G.

Now we trace Razmyslov's proof of Theorem 9 given in [18, Section 4], in order to prove our theorem. Essentially, Razmyslov's theorem proves the following. All multilinear identities of M_2 follow from the identities

$$4[x_1,x_2](v_3 \circ v_4) - [x_1,v_3,v_4,x_2] - [x_1,v_4,v_3,x_2] + [x_2,v_3,x_1,v_4] + [x_2,v_4,x_1,v_3] = 0, \tag{4}$$

$$[v_1 \circ v_2, x_3] = 0, \tag{5}$$

$$[x_1,x_2] \circ [x_3,x_4] - [x_1,x_3] \circ [x_2,x_4] + [x_1,x_4] \circ [x_2,x_3] = 0, \tag{6}$$

$$[x_1,x_2,[x_3,x_4],x_5] + [x_1,x_2,[x_3,x_5],x_4] + [x_1,x_4,[x_2,x_5],x_3] + [x_1,x_5,[x_2,x_4],x_3] = 0, \tag{7}$$

$$\sum_i \alpha_i (u_i \circ [v_i, x_6]) = 0. \tag{8}$$

Here u_i and v_i are commutators of lengths ≥ 2. The last but one identity is Vasilovsky's identity. The last identities are obtained by the following procedure. Write Vasilovsky's polynomial as $\sum_i \alpha_i [u_i, v_i]$ for $\alpha_i \in K$, u_i, v_i commutators and $\deg(u_i) \geq 2$, and substitute them in the left-hand side of the last expression.

Now clearly the identities (7) and (8) hold in G. The identity (6) also holds in G since it is equal to the standard identity s_4. The identity (5) is satisfied by G, too, since the commutators are central in G. Now, the identity (4) is not satisfied by the Grassmann algebra G since the summands except for the first vanish on G, and the first does not:

$$4[e_1,e_2]([e_3,e_4][e_5,e_6] + [e_5,e_6][e_3,e_4]) = 64 e_1 e_2 e_3 e_4 e_5 e_6 \neq 0.$$

Since in G_4 and in G_5 the polynomial (4) is an identity, this completes the proof of the theorem. ∎

Remark: Note that when $\operatorname{char} K = 0$ it was shown in [6] that the standard identity s_4 and Hall's identity $[[x_1, x_2]^2, x_3]$ form a (minimal) basis of the identities of M_2 over a field of characteristic 0. This does not contradict our last theorem since the result of [6] does *not* imply that all multilinear identities of M_2 can be obtained as consequences of these two identities using only multilinear consequences if the characteristic of K does not equal 0.

Now we state as an open question a weak version of Kemer's problem.

Open question: Does the Grassmann algebra G_{2p-2} satisfy all polynomial identities of the matrix algebra $M_{(p+1)/2}$ where $p = \operatorname{char} K$? When $p = 3$ the answer is "Yes" due to our last theorem.

LEMMA 11: *The equality*

$$[y, z, [t, x], x] + [y, x, [z, x], t] = s_4(z, y, x, tx) + x s_4(z, y, x, t) - s_4(xz, y, x, t) - s_4(z, xy, x, t)$$

holds for every x, y, z, t in the free associative algebra.

Proof: Expand both sides and cancel all terms... ∎

Remark: Note that the linearisation of the polynomial on the left-hand side is Vasilovsky's identity (7). The above expression does not depend on the characteristic of the field and thus Vasilovsky's identity belongs to the T-ideal generated by the standard polynomial s_4.

We have already proved that the identity (4) does not follow from (5) and (6) if the characteristic of the field K equals 3. It is not difficult to see that it does follow from these two identities when $\operatorname{char} K \geq 5$. These considerations lead us to the following conjecture.

Conjecture: The identities (5) and (6) form a basis of the polynomial identities in the matrix algebra $M_2(K)$ when K is an infinite field of characteristic ≥ 5. When $\operatorname{char} K = 3$ a basis of the identities for $M_2(K)$ consists of these two identities and the identity (4).

5. Non-unitary Grassmann algebras

In this section we briefly describe the results of the papers [22] and [3] concerning the identities of the non-unitary Grassmann algebras H and H_k. Note that in this case one cannot use commutator identities. But, on the other hand, even over a finite field K, the T-ideal $T(H)$ is multihomogeneous, see [19, Lemma 5.1(b)]. This means that it is sufficient to consider multihomogeneous polynomials even when K is finite.

LEMMA 12 ([19, Lemma 1.2(b)], [3, Lemma 2.9]): *The Grassmann algebra H without 1 satisfies the identity x^p, where* $\operatorname{char} K = p$.

Remark: It is worth mentioning that all partial linearisations of x^p are consequences of $[x_1, x_2, x_3]$. In fact, consider $\sum_{i=0}^{p-1} x^{p-1-i}yx^i$, the first partial linearisation of x^p. An easy induction shows that the following equality holds in every characteristic:

$$[y, x, \ldots, x] = \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} x^i y x^{n-1-i}$$

where the length of the commutator equals n. Thus if $n = p = \operatorname{char} K$ the binomial coefficients satisfy the congruence $\binom{n-1}{i} \equiv (-1)^i \pmod{p}$. It follows that $[y, x, \ldots, x]$ equals the first partial linearisation of x^p when $\operatorname{char} K = p$. Since $p \geq 3$, the p-th commutator follows from $[x_1, x_2, x_3]$ and therefore the first linearisation (and, so, every linearisation) of x^p is also a consequence of the identity $[x_1, x_2, x_3]$. Notice that the complete linearisation of x^p equals $\sum_{\sigma \in S_p} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(p)}$.

Denote $v_k = (\cdots((x_1 \circ x_2) \circ x_3) \cdots \circ x_{k-1}) \circ x_k$.

LEMMA 13 ([22, Corollary 3]): (a) *The algebra H_{2k} satisfies the identity v_{k+1}.*
(b) *The algebra H_{2k+1} satisfies the identities $x_{k+2}v_{k+1}$ and $v_{k+1}x_{k+2}$.*

Denote by Q the T-ideal in $K\langle X\rangle$ generated by the polynomials x_1^p and $[x_1, x_2, x_3]$. Let $T(H)$ stand for the T-ideal of H in $K\langle X\rangle$.

THEOREM 14 ([3, Theorem 3]): *For every field K of characteristic $p > 0$, one has $Q = T(H)$.*

Let Q_{2k} be the T-ideal generated by Q and by the polynomial v_{k+1}. Denote by Q_{2k+1} the T-ideal generated by Q and by the polynomials $x_{k+2}v_{k+1}$, $v_{k+1}x_{k+2}$, and in case $s = (k+1)/(2(2p-1))$ is an integer, add the polynomial $\prod_{j=1}^{s} [x_{2j-1}, x_{2j}] x_{2j-1}^{p-1} x_{2j}^{p-1}$. Let $T(H_k)$ stand for the T-ideal of H_k in $K\langle X\rangle$.

THEOREM 15 ([22, Theorem 1]): *If K is an arbitrary field of characteristic $p > 2$ then $Q_k = T(H_k)$.*

THEOREM 16 ([3, Theorem 2]): *The variety generated by the non-unitary Grassmann algebra H is Spechtian.*

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