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ULTRAPOWERS OF $L_1(\mu)$ AND THE SUBSEQUENCE SPLITTING PRINCIPLE*

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ABSTRACT

We study the ultrapowers $L_1(\mu)_{\mathfrak{U}}$ of a $L_1(\mu)$ space, by describing the components of the well-known representation $L_1(\mu)_{\mathfrak{U}} = L_1(\mu_{\mathfrak{U}}) \oplus_1 L_1(\nu_{\mathfrak{U}})$, and we give a representation of the projection from $L_1(\mu)_{\mathfrak{U}}$ onto $L_1(\mu_{\mathfrak{U}})$. Moreover, the subsequence splitting principle for $L_1(\mu)$ motivates the following question: if $\mathfrak{V}$ is an ultrafilter on $\mathbb{N}$ and $[f_i] \in L_1(\mu)_{\mathfrak{V}}$, is it possible to find a weakly convergent sequence $(g_i) \subset L_1(\mu)$ following $\mathfrak{V}$ and a disjoint sequence $(h_i) \subset L_1(\mu)$ such that $[f_i] = [g_i] + [h_i]$? If $\mathfrak{V}$ is a selective ultrafilter, we find a positive answer by showing that $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{V}}$ belongs to $L_1(\mu_{\mathfrak{V}})$ if and only if its representatives $\{f_i\}$ are weakly convergent following $\mathfrak{V}$, and $\mathbf{f} \in L_1(\nu_{\mathfrak{V}})$ if and only if it admits a representative consisting of pairwise disjoint functions. As a consequence, we obtain a new proof of the subsequence splitting principle. If $\mathfrak{V}$ is not a p-point then the above characterizations of $L_1(\mu_{\mathfrak{V}})$ and $L_1(\nu_{\mathfrak{V}})$ fail and the answer to the question is negative.

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1. Introduction

The ultraproduct construction was introduced in Banach space theory by Dacunha-Castelle and Krivine [5], stimulated by the development of the local theory of Banach spaces. Besides the solution of several open problems, the ultraproducts of Banach spaces shed new light upon many known results and made the relation between finite dimensional and infinite dimensional results more transparent. We refer to the partially expository paper by Heinrich [8] for more details.

Among other things, the authors of [5] proved that the ultraproduct of a $L_1(\mu)$ space is another L_1 space, and gave a representation of this ultrapower as

$$L_1(\mu)_{\mathfrak{U}} = L_1(\mu_{\mathfrak{U}}) \oplus_1 L_1(\nu_{\mathfrak{U}}).$$

Here we characterize the vectors $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$ which belong to the components $L_1(\mu_{\mathfrak{U}})$ and $L_1(\nu_{\mathfrak{U}})$, in terms of the representatives $\{f_i\}$. Namely, $\mathbf{f} \in L_1(\mu_{\mathfrak{U}})$ if and only if it admits a relatively weakly compact representative $\{f_i\}$, and $\mathbf{f} \in L_1(\nu_{\mathfrak{U}})$ if and only if it admits a representative $\{f_i\}$ so that $\lim_{\mathfrak{U}} \mu(\{t\colon f_i(t) \neq 0\}) = 0$. Moreover, we give a concrete expression for the projection $g_{\mathbf{f}}$ of $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$ onto the first component $L_1(\mu_{\mathfrak{U}})$, by showing that $g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) = \lim_{\mathfrak{U}} f_i(t_i)$, $\mu_{\mathfrak{U}}$-a.e.

As an application, we give in the last Section a proof of the fact that the reflexive subspaces of $L_1(\mu)$ are superreflexive. Other applications can be found in [7] where, using some results of [6], we characterized the tauberian operators acting on $L_1(\mu)$ spaces.

Moreover, if $\mathfrak{V}$ is a selective ultrafilter on $\mathbb{N}$ (namely, a rare p-point on $\mathbb{N}$ [CN]), whose existence may be obtained from the Martin axiom, we show that $\mathbf{f} \in L_1(\mu)_{\mathfrak{V}}$ belongs to $L_1(\mu_{\mathfrak{V}})$ if and only if all its representatives $\{f_i\}$ are weakly convergent following the ultrafilter $\mathfrak{V}$, and $\mathbf{f} \in L_1(\nu_{\mathfrak{V}})$ if and only if it admits a representative consisting of pairwise disjoint functions. Thus, for every bounded sequence (f_n) in $L_1(\mu)$, there are a pairwise disjoint sequence (z_n) and a sequence (y_n) which is weakly convergent following the ultrafilter $\mathfrak{V}$, so that $[f_n] = [y_n] + [z_n]$ in $L_1(\mu)_{\mathfrak{V}}$. The **subsequence splitting principle** states that every bounded sequence in $L_1(\mu)$ admits a subsequence which can be decomposed as the sum of a pairwise disjoint sequence and a weakly convergent sequence. It was obtained by H. Rosenthal (see [2, Proof of Corollary 2.3]) as a refinement of an argument of Kadec and Pełczyński [9], which showed that a bounded sequence in $L_1(\mu)$ has a subsequence which can be written as the sum of a sequence converging to 0 in probability and a weakly convergent sequence. Our result can be seen as an ultrapower version of the classical subsequence splitting principle. Moreover, this principle is easily derived from our result.

Finally, we show that if the ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is not a p-point, then the results obtained using the ultrafilter $\mathfrak{V}$ are not valid in $L_1(\mu)_{\mathfrak{U}}$. We observe that the existence of rare p-points was proved by Choquet [3] by means of the Continuum Hypothesis, and further, by Booth [1] by just using Martin's axiom. In addition, Shelah (see 15) showed that the existence of p-points on $\mathbb{N}$ is undecidable in the Zermelo–Fraenkel set theory plus the Axiom of Choice. In the last sections we describe these facts and give a direct proof of the existence of selective ultrafilters on $\mathbb{N}$, using the Continuum Hypothesis.

We use standard notations: X is a Banach space and X^* the dual of X. We identify X with a subspace of X^{**}. We denote the set of all positive integers by $\mathbb{N}$, and the set of all real numbers by $\mathbb{R}$.

Let (Ω, Σ, μ) be a finite measure space. For a function $f\colon \Omega \longrightarrow \mathbb{R}$, we write $D(f) := \{x \in \Omega\colon f(x) \neq 0\}$. We also use sometimes the short form $\{f > \alpha\}$ to represent $\{x\colon f(x) > \alpha\}$. A sequence $(f_n) \subset L_1(\mu)$ is said to be **disjoint** if $f_k(x) \cdot f_m(x) = 0$ a.e. for $k \neq m$. Note that, since μ is a finite measure, for each disjoint sequence $(f_n) \subset L_1(\mu)$ we have $\lim_n \mu(D(f_n)) = 0$. We denote by χ_A the characteristic function of $A \in \Sigma$.

2. Preliminaries

We recall some concepts about ultrafilters and ultrapowers. See [4] for more complete information. We denote by $\wp(I)$ the family of all subsets of a set I.

Let $\mathfrak{U}$ and $\mathfrak{V}$ be ultrafilters on I and J respectively. Given $A \subset I \times J$, for every $i \in I$ we write $A_i := \{j \in J\colon (i,j) \in A\}$ and $I_A := \{i \in I\colon A_i \in \mathfrak{V}\}$. The product of $\mathfrak{U}$ by $\mathfrak{V}$ is defined as the subset of $\wp(I \times J)$ given by

$$\mathfrak{U} \times \mathfrak{V} := \{A \subset I \times J\colon I_A \in \mathfrak{U}\}.$$

The product $\mathfrak{U} \times \mathfrak{V}$ is an ultrafilter on $I \times J$ [13, Chapter 13, Prop. 1].

LEMMA 1: *Given the ultrafilters $\mathfrak{U}$ and $\mathfrak{V}$ on I and J respectively, and a bounded family $(a_{ij})_{(i,j)\in I\times J}$ of real numbers, we have that*

$$\lim_{\mathfrak{U}\times\mathfrak{V}} a_{ij} = \lim_{\mathfrak{U}(i)} \lim_{\mathfrak{V}(j)} a_{ij}.$$

Notice that the iterated limit in the identity of Lemma 1 does not commute.

An ultrafilter $\mathfrak{U}$ on I is said to be **countably incomplete** if there is a countable partition $\{I_n\}_{n=1}^{\infty}$ of I verifying $I_n \notin \mathfrak{U}$ for all $n \in \mathbb{N}$. This partition $\{I_n\}_{n=1}^{\infty}$ is said to be **disjoint with** $\mathfrak{U}$.

Evidently, non-trivial ultrafilters on $\mathbb{N}$ are countably incomplete. Moreover, every infinite set I admits a countably incomplete ultrafilter [8]. Henceforth, if there is not an explicit mention, all ultrafilters which appear here are countably incomplete.

Let $\mathfrak{U}$ be an ultrafilter on I, and X a Banach space. Consider the Banach space $\ell_\infty(I,X)$ which consists of all bounded families $(x_i)_{i\in I}$ in X endowed with the norm $\|(x_i)\|_\infty := \sup\{\|x_i\|: i \in I\}$. Let $N_{\mathfrak{U}}(X)$ be the closed subspace of all families $(x_i) \in \ell_\infty(I,X)$ which converge to 0 following $\mathfrak{U}$. The **ultrapower of X following** $\mathfrak{U}$ is defined as the quotient

$$X_{\mathfrak{U}} := \frac{\ell_\infty(I,X)}{N_{\mathfrak{U}}(X)}.$$

The element of $X_{\mathfrak{U}}$ including the family $(x_i) \in \ell_\infty(I,X)$ as a representative is denoted by $[x_i]$. Its norm in $X_{\mathfrak{U}}$ is given by

$$\big\|[x_i]\big\| = \lim_{\mathfrak{U}} \|x_i\|.$$

Sometimes we use bold letters $\mathbf{x}, \mathbf{y}, \ldots$ to denote elements of $X_{\mathfrak{U}}$.

3. Ultrapowers of $L_1(\mu)$

The ultrapower $L_1(\mu)_{\mathfrak{U}}$ has been extensively studied in [5]. For the convenience of the reader, we give a description of it. Details and complete proofs can be found in [8] and [13].

Let (Ω, Σ, μ) be a finite measure space, and let $\mathcal{B}(I,\Omega)$ denote the class of all families $(x_i)_{i\in I}$ contained in Ω. The ultrafilter $\mathfrak{U}$ induces an equivalence relation $\sim$ on $\mathcal{B}(I,\Omega)$ given by $(x_i)_{i\in I} \sim (y_i)_{i\in I}$ if $\{i \in I: x_i = y_i\} \in \mathfrak{U}$. We write

$$\Omega^{\mathfrak{U}} := \frac{\mathcal{B}(I,\Omega)}{\sim},$$

and $(x_i)^{\mathfrak{U}}$ denotes the element of $\Omega^{\mathfrak{U}}$ whose representative is $(x_i)_{i\in I}$. If $\{A_i: i \in I\}$ is a family of subsets of Ω, we write $(A_i)^{\mathfrak{U}} := \{(x_i)^{\mathfrak{U}}: x_i \in A_i\}$. We consider the family $\Sigma_{\mathfrak{U}} := \{(A_i)^{\mathfrak{U}}: A_i \in \Sigma\}$, which is a Boolean algebra on $\Omega^{\mathfrak{U}}$ [8, Proposition 5.1]. The least σ-algebra containing the algebra $\Sigma_{\mathfrak{U}}$ will be denoted by $\sigma(\Sigma_{\mathfrak{U}})$. The measure μ induces a measure $\mu_{\mathfrak{U}}$ on $\sigma(\Sigma_{\mathfrak{U}})$ univocally defined by its value on the elements $(A_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$, given by $\mu_{\mathfrak{U}}((A_i)^{\mathfrak{U}}) := \lim_{\mathfrak{U}} \mu(A_i)$. Thus, given $A \in \sigma(\Sigma_{\mathfrak{U}})$, its measure is

$$\mu_{\mathfrak{U}}(A) := \inf\{\mu_{\mathfrak{U}}(C): A \subset C,\ C \in \Sigma_{\mathfrak{U}}\}.$$

Analogously, given $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$ with $f_i \geq 0$ for every $i \in I$, we consider the measure $\nu_{\mathbf{f}}$, defined on $\Sigma_{\mathfrak{U}}$ by

$$\nu_{\mathbf{f}}(A) := \lim_{\mathfrak{U}} \int_{A_i} f_i \, d\mu, \quad \text{where } A = (A_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}},$$

and extended to the whole of $\sigma(\Sigma_{\mathfrak{U}})$ by

$$\nu_{\mathbf{f}}(A) := \inf\{\nu_{\mathbf{f}}(C) \colon A \subset C, \ C \in \Sigma_{\mathfrak{U}}\}, \quad \text{where } A \in \sigma(\Sigma_{\mathfrak{U}}).$$

For the general case $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$, we write $\mathbf{f} = \mathbf{f}^+ - \mathbf{f}^-$, with $\mathbf{f}^+ := [f_i^+]$, and define $\nu_{\mathbf{f}} := \nu_{\mathbf{f}^+} - \nu_{\mathbf{f}^-}$.

Next we give the description of $L_1(\mu)_{\mathfrak{U}}$, which is the suitable context to introduce some additional notation.

PROPOSITION 2: *Given a finite measure space (Ω, Σ, μ) and an ultrafilter $\mathfrak{U}$, there exists a measure $\nu_{\mathfrak{U}}$ such that*

$$L_1(\mu)_{\mathfrak{U}} = L_1(\mu_{\mathfrak{U}}) \oplus_1 L_1(\nu_{\mathfrak{U}}). \tag{1}$$

Sketch of the Proof: Let us define an isometry $J \colon L_1(\mu_{\mathfrak{U}}) \longrightarrow L_1(\mu)_{\mathfrak{U}}$. First, we define J on the characteristic functions χ_A, where $A = (A_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$, by

$$J(\chi_A) := [\chi_{A_i}] \in L_1(\mu)_{\mathfrak{U}}.$$

Since $\{\sum_{k=1}^n \alpha_k \chi_{A^k} \colon n \in \mathbb{N}, \ \alpha_k \in \mathbb{R}, \ A^k \in \Sigma_{\mathfrak{U}}\}$ is dense in $L_1(\mu_{\mathfrak{U}})$, J can be extended to the whole of $L_1(\mu_{\mathfrak{U}})$.

Now, we describe the projection P from $L_1(\mu)_{\mathfrak{U}}$ onto $L_1(\mu_{\mathfrak{U}})$. Let $\nu_{\mathbf{f}}$ be the measure associated to $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$. By the Radon–Nikodym Theorem, there exist unique measures $w_{\mathbf{f}}$, $m_{\mathbf{f}}$ and a function $g_{\mathbf{f}} \in L_1(\mu_{\mathfrak{U}})$ such that $m_{\mathbf{f}} \perp \mu_{\mathfrak{U}}$, $w_{\mathbf{f}} \ll \mu_{\mathfrak{U}}$, $\nu_{\mathbf{f}} = w_{\mathbf{f}} + m_{\mathbf{f}}$, and

$$w_{\mathbf{f}}(A) = \int_A g_{\mathbf{f}} \, d\mu_{\mathfrak{U}}, \ A \in \sigma(\Sigma_{\mathfrak{U}}).$$

For each $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$, we define $P\mathbf{f} := Jg_{\mathbf{f}}$. Clearly, $J(\chi_A) \in R(P)$ for every $A = (A_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$; thus $R(P) = J(L_1(\mu_{\mathfrak{U}}))$.

That $N(P)$ is isometric to a L_1-space is a consequence of the Bohnenblust–Nakano Theorem. ∎

Observe that, for each $\mathbf{f} = [f_i] \in L_1(\mu_{\mathfrak{U}})$, we have $|\mathbf{f}| = [|f_i|]$ and $\nu_{|\mathbf{f}|} = |\nu_{\mathbf{f}}|$. Hence, $\nu_{\mathbf{f}} \perp \mu_{\mathfrak{U}}$ is equivalent to $\nu_{|\mathbf{f}|} \perp \mu_{\mathfrak{U}}$. Moreover, $\mathbf{f} \in R(P)$ if and only if $m_{\mathbf{f}} = 0$.

One of our goals is to give a more practical and pleasant representation of the elements of the components $L_1(\mu_{\mathfrak{U}})$ and $L_1(\nu_{\mathfrak{U}})$. We recall that each measure space (Ω, Σ, μ) can be enlarged to its completion, denoted by $(\Omega, \overline{\Sigma}, \overline{\mu})$. First we give a technical Lemma based on the fact that $\Sigma_{\mathfrak{U}}$ is a boolean algebra.

LEMMA 3: *Let $\mathfrak{U}$ and $\mathfrak{V}$ be a pair of ultrafilters on I and J respectively. For $A \in \sigma(\Sigma_{\mathfrak{U}})$ and $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$, we have:*

(a) *if $f_i \geq 0$ for all i, then*

$$\nu_{\mathbf{f}}(A) = \sup\{\nu_{\mathbf{f}}(C): C \in \Sigma_{\mathfrak{U}}, A \supset C\};$$

(b) *if $\nu_{\mathbf{f}}(C) = 0$ for every $C \in \Sigma_{\mathfrak{U}}$ contained in A, then $\nu_{\mathbf{f}}(A) = 0$;*

(c) *$\mu_{\mathfrak{U}\times\mathfrak{V}}$ and $(\mu_{\mathfrak{V}})_{\mathfrak{U}}$ have the same completion.*

Proof: Parts (a) and (b) are trivial. In order to prove (c), we only need check that $((\Omega^{\mathfrak{V}})^{\mathfrak{U}}, \overline{\sigma(\sigma(\Sigma_{\mathfrak{V}})_{\mathfrak{U}})}, \overline{(\mu_{\mathfrak{V}})_{\mathfrak{U}}}) = (\Omega^{\mathfrak{V}\times\mathfrak{U}}, \overline{\sigma(\Sigma_{\mathfrak{V}\times\mathfrak{U}})}, \overline{\mu_{\mathfrak{V}\times\mathfrak{U}}})$.

First, note that $(\Omega^{\mathfrak{V}})^{\mathfrak{U}} = \Omega^{\mathfrak{U}\times\mathfrak{V}}$ by identifying each $((t_{ij})_{j\in J})_{i\in I}$ with $(t_{ij})_{(i,j)\in I\times J}$. So, we may identify $(\Sigma_{\mathfrak{V}})_{\mathfrak{U}} = \Sigma_{\mathfrak{U}\times\mathfrak{V}}$, and the inclusion $\overline{\sigma(\Sigma_{\mathfrak{U}\times\mathfrak{V}})} \subset \overline{\sigma(\sigma(\Sigma_{\mathfrak{V}})_{\mathfrak{U}})}$ is clear. For the converse inclusion, it is enough to realize that each $(A_i)^{\mathfrak{U}} \in \sigma(\Sigma_{\mathfrak{V}})_{\mathfrak{U}}$ (here $A_i \in \sigma(\Sigma_{\mathfrak{V}})$) is contained in a set of $\Sigma_{\mathfrak{U}\times\mathfrak{V}}$ with the same measure. Indeed, let $\{I_n: n \in \mathbb{N}\}$ be a partition of I disjoint with $\mathfrak{U}$. For each positive integer n and $i \in I_n$, there is $B_i \in \Sigma_{\mathfrak{V}}$ so that $B_i \supset A_i$ and $\mu_{\mathfrak{U}}(B_i \setminus A_i) < n^{-1}$. Write $B_i = (B_{ij})^{\mathfrak{V}(j)}$, with $B_{ij} \in \Sigma$. It is immediate that $(A_i)^{\mathfrak{U}} \subset (B_{ij})^{\mathfrak{U}(i)\times\mathfrak{V}(j)}$; besides, since Lemma 1 leads to $(\mu_{\mathfrak{V}})_{\mathfrak{U}}\,|_{\Sigma_{\mathfrak{U}\times\mathfrak{V}}} = \mu_{\mathfrak{U}\times\mathfrak{V}}\,|_{\Sigma_{\mathfrak{U}\times\mathfrak{V}}}$, we have that $(\mu_{\mathfrak{V}})_{\mathfrak{U}}((A_i)^{\mathfrak{U}}) = (\mu_{\mathfrak{V}})_{\mathfrak{U}}((B_i)^{\mathfrak{U}})$. ∎

Weis [14, Proposition 11] obtained a characterization for the elements of the first component of $L_1(\mu)_{\mathfrak{U}}$ for ultrafilters on $\mathbb{N}$. It is possible to adapt his proof to the case of the countably incomplete ultrafilter $\mathfrak{U}$. However, we prefer a different proof.

THEOREM 4: *An element $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$ belongs to $L_1(\mu_{\mathfrak{U}})$ if and only if it has a relatively weakly compact representative.*

Proof: Let $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$, having a relatively weakly compact representative $(f_i)_{i\in I}$, and take $\varepsilon > 0$. Since $\{f_i: i \in I\}$ is equiintegrable, there exists $\delta > 0$ so that $\int_A |f_i|\, d\mu < \varepsilon$ for all $i \in I$ and all $A \in \Sigma$ with $\mu(A) < \delta$. Let $C = (C_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$ with $\mu_{\mathfrak{U}}(C) = 0$. Then we have that $J := \{i \in I: \mu(C_i) < \delta\} \in \mathfrak{U}$; hence

$$\int_{C_i} |f_i|\, d\mu < \varepsilon \quad \text{for all } i \in J.$$

Therefore $\nu_{|\mathbf{f}|}(C) = \lim_{\mathfrak{U}} \int_{C_i} |f_i|\, d\mu = 0$, and an application of Lemma 3 shows that $\nu_{\mathbf{f}} \ll \mu_{\mathfrak{U}}$ so $m_{\mathbf{f}} = 0$, hence $\mathbf{f} \in L_1(\mu_{\mathfrak{U}})$.

For the converse, take $\mathbf{f} = [f_i] \in L_1(\mu_{\mathfrak{U}})$ with $\|f_i\| = 1$ for all i, and denote $f_i^k := f_i \cdot \chi_{f_i^{-1}([-k,k])}$, $\mathbf{f}^k := [f_i^k]$ and $A_i^k := \{t\colon |f_i(t)| > k\}$. Then $((A_i^k)^{\mathfrak{U}(i)})_k$ is a decreasing sequence in $\Sigma_{\mathfrak{U}}$. So, denoting $A := \bigcap_{k=1}^{\infty} (A_i^k)^{\mathfrak{U}(i)} \in \sigma(\Sigma_{\mathfrak{U}})$, we have

$$\mu_{\mathfrak{U}}(A) = \lim_k (\lim_{\mathfrak{U}(i)} \mu(A_i^k)) \le \lim_k k^{-1} = 0.$$

Now, by hypothesis, we have that $\nu_{|\mathbf{f}|} \ll \mu_{\mathfrak{U}}$, so

$$0 = \nu_{|\mathbf{f}|}(A) = \lim_k \nu_{|\mathbf{f}|}((A_i^k)^{\mathfrak{U}(i)}) = \lim_k \|\mathbf{f} - \mathbf{f}^k\|.$$

Let $\{I_n\}_{n=1}^{\infty}$ be a partition of I disjoint with $\mathfrak{U}$. We write $r_k := \|\mathbf{f} - \mathbf{f}^k\|$ and take

$$H_1 := \{i\colon \|f_i - f_i^1\| < 2r_1\} \in \mathfrak{U},$$
$$H_k := (\bigcup_{i=k}^{\infty} I_i) \cap \{i \in H_{k-1}\colon \|f_i - f_i^k\| < 2r_k\} \in \mathfrak{U}, \text{ for } k \ge 2.$$

Moreover, we denote $L_0 := I \setminus H_1$ and $L_k := H_k \setminus H_{k+1}$ for $k \in \mathbb{N}$. For every $i \in I$, there exists a unique $n_i \in \mathbb{N}$ so that $i \in L_{n_i}$.

Let us see that $[f_i] = [f_i^{n_i}]$. In fact, given $\varepsilon > 0$, there exists an integer $n \in \mathbb{N}$ so that $2r_k < \varepsilon$ for all $k \ge n$. Thus

$$\{i\colon \|f_i - f_i^{n_i}\| < \varepsilon\} \supset \bigcup_{k=n}^{\infty} L_k \in \mathfrak{U},$$

hence $\lim_{\mathfrak{U}} \|f_i - f_i^{n_i}\| = 0$.

Now, we claim that $\{f_i^{n_i}\}_{i\in I}$ is equiintegrable. Indeed, given $\varepsilon > 0$, we take as before $n \in \mathbb{N}$ so that $2r_k < \varepsilon$ for $k \ge n$. Observe that each i belongs either to $\bigcup_{k=1}^{n-1} L_k$ or to H_n. If $i \in \bigcup_{k=1}^{n-1} L_k$ then $|f_i^{n_i}(x)| < n$, hence

$$\int_{\{|f_i^{n_i}| > n\}} |f_i^{n_i}|\, d\mu = 0;$$

and if $i \in H_n$ then

$$\int_{\{|f_i^{n_i}| > n\}} |f_i^{n_i}|\, d\mu \le \int_{\{|f_i| > n\}} |f_i|\, d\mu = \|f_i - f_i^n\| < 2r_n < \varepsilon.$$

Thus $\{f_i^{n_i}\}_{i\in I}$ is equiintegrable; equivalently, it is relatively weakly compact. ∎

Now we study the elements of the second component of $L_1(\mu)_{\mathfrak{U}}$.

THEOREM 5: *Let* $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$. *Then* $\mathbf{f} \in L_1(\nu_{\mathfrak{U}})$ *if and only if there is a representative* $(f_i)_{i\in I}$ *of* $\mathbf{f}$ *for which* $\mu_{\mathfrak{U}}((D(f_i))^{\mathfrak{U}}) = \lim_{\mathfrak{U}} \mu(D(f_i)) = 0$.

Proof: Assume that $\mu_{\mathfrak{U}}((D(f_i))^{\mathfrak{U}}) = 0$. Then $\nu_{\mathbf{f}} \perp \mu_{\mathfrak{U}}$, and consequently, $\mathbf{f} \in L_1(\nu_{\mathfrak{U}})$.

For the converse, let $\mathbf{f} = [f_i] \in L_1(\nu_{\mathfrak{U}})$. For every $i \in I$, we consider the measurable sets $A_i := \{|f_i| < 1\}$, $B_i := \Omega \setminus A_i$, $A := (A_i)^{\mathfrak{U}}$ and $B := (B_i)^{\mathfrak{U}}$, and the functions $g_i := f_i \cdot \chi_{A_i}$, $h_i := f_i \cdot \chi_{B_i}$, $\mathbf{g} := [g_i]$, $\mathbf{h} := [h_i]$.

Since $\mathbf{f} \in L_1(\nu_{\mathfrak{U}})$, we have $\nu_{|\mathbf{f}|} \perp \mu_{\mathfrak{U}}$. Note that the measures $\nu_{|\mathbf{g}|}$ and $\nu_{|\mathbf{f}|}$ are concentrated in A and $(D(f_i))^{\mathfrak{U}}$ respectively. So $\nu_{|\mathbf{g}|} \ll \nu_{|\mathbf{f}|}$, hence $\nu_{|\mathbf{g}|} \perp \mu_{\mathfrak{U}}$, which leads to $\mathbf{g} \in L_1(\nu_{\mathfrak{U}})$. On the other hand, since $|g_i(t)| < 1$ for all t and all i, Theorem 4 gives that $\mathbf{g} \in L_1(\mu_{\mathfrak{U}})$. Thus $\mathbf{g} = 0$; or equivalently, $\mathbf{f} = \mathbf{h}$.

To finish the proof, it is enough to show that $\mu_{\mathfrak{U}}(B) = 0$. Note that $\nu_{|\mathbf{f}|}$ is concentrated in a $\mu_{\mathfrak{U}}$-null set L, and we can assume $L \subset B$. Moreover, by the definition of B, for every $C \in \Sigma_{\mathfrak{U}}$ contained in $B \setminus L$ we have $\mu_{\mathfrak{U}}(C) \leq \nu_{|\mathbf{f}|}(C) = 0$. Therefore, by Lemma 3, we have $\mu_{\mathfrak{U}}(B) = \mu_{\mathfrak{U}}(B \setminus L) = 0$. ∎

The characterizations given in Theorems 4 and 5 lead to a very pleasant representation of the projection P described in Proposition 2. For the sake of the following results, let us realize that the set $A := \{(t_i)^{\mathfrak{U}}\colon \lim_{\mathfrak{U}} f_i(t_i) = \infty\}$ is $\mu_{\mathfrak{U}}$-null, for every $[f_i] \in L_1(\mu)_{\mathfrak{U}}$. In fact, note that $A = \bigcap_{n=1}^{\infty}(\{f_i > n\})^{\mathfrak{U}(i)}$ and $\mu(\{f_i > n\}) \leq n^{-1}\|f_i\|$, so $\mu_{\mathfrak{U}}(A) = \lim_n \mu_{\mathfrak{U}}(\{f_i > n\}^{\mathfrak{U}(i)}) = 0$.

LEMMA 6: *Let* $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$ *and* $c \in \mathbb{R}$. *Then we have*

$$\{g_{\mathbf{f}} < c\} \subset (\{f_i < c\})^{\mathfrak{U}} \subset (\{f_i \leq c\})^{\mathfrak{U}} \subset \{g_{\mathbf{f}} \leq c\} \quad \mu_{\mathfrak{U}}\text{-a.e.}$$

Proof: By Theorems 2 and 5, we can write $\mathbf{f} = [u_i] + [v_i]$, where $[u_i] \in L_1(\mu_{\mathfrak{U}})$, $[v_i] \in L_1(\nu_{\mathfrak{U}})$, and $\mu_{\mathfrak{U}}((D(v_i))^{\mathfrak{U}}) = 0$. The last condition implies that

$$(\{f_i < c\})^{\mathfrak{U}} = (\{u_i < c\})^{\mathfrak{U}} \text{ and } (\{f_i \leq c\})^{\mathfrak{U}} = (\{u_i \leq c\})^{\mathfrak{U}}\ \mu_{\mathfrak{U}}\text{-a.e.}$$

Since $\{g_{\mathbf{f}} < c\} = \bigcup_{n=1}^{\infty}\{g_{\mathbf{f}} < c - 1/n\}$, for the first inclusion it is enough to prove that $\{g_{\mathbf{f}} < c-1/n\} \subset (\{u_i < c\})^{\mathfrak{U}}$, $\mu_{\mathfrak{U}}$-a.e. And, by Lemma 3, we only have to show that every $B = (B_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$ contained in $\{g_{\mathbf{f}} < c - 1/n\} \setminus (\{u_i < c\})^{\mathfrak{U}}$ is $\mu_{\mathfrak{U}}$-null. On the one hand, we have $w_{\mathbf{f}}(B) = \int_B g_{\mathbf{f}}\, d\mu_{\mathfrak{U}} \leq (c - 1/n)\mu_{\mathfrak{U}}(B)$. On the other hand, observe that $\Omega^{\mathfrak{U}} \setminus (\{u_i < c\})^{\mathfrak{U}} = (\{u_i \geq c\})^{\mathfrak{U}}$. Thus

$$w_{\mathbf{f}}(B) = \lim_{\mathfrak{U}} \int_{B_i} u_i\, d\mu \geq c\mu_{\mathfrak{U}}(B),$$

and we conclude that $\mu_{\mathfrak{U}}(B) = 0$.

The proof of the third inclusion is analogous: it is enough to show that every $B = (B_i)^{\mathfrak{U}} \in \Sigma_{\mathfrak{U}}$ contained in $(\{u_i \le c\})^{\mathfrak{U}} \setminus \{g_{\mathbf{f}} \le c + 1/n\}$ is $\mu_{\mathfrak{U}}$-null. On the one hand, $w_{\mathbf{f}}(B) = \int_B g_{\mathbf{f}}\, d\mu_{\mathfrak{U}} \ge (c + 1/n)\mu_{\mathfrak{U}}(B)$. On the other hand,

$$w_{\mathbf{f}}(B) = \lim_{\mathfrak{U}} \int_{B_i} u_i \, d\mu \le c\mu_{\mathfrak{U}}(B),$$

hence $\mu_{\mathfrak{U}}(B) = 0$. ∎

The first inclusion of Lemma 6 is sometimes strict. For example, the zero function in $L_1(\mu)_{\mathfrak{U}}$ can be represented by means of strictly negative functions. The same happens for the third inclusion. However, we have the following result.

COROLLARY 7: *Let $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$. Then there is a countable subset $C \subset \mathbb{R}$ so that*

$$\{g_{\mathbf{f}} < c\} = (\{f_i < c\})^{\mathfrak{U}} \text{ and } \{g_{\mathbf{f}} \le c\} = (\{f_i \le c\})^{\mathfrak{U}} \ \mu_{\mathfrak{U}}\text{-a.e.},$$

for every $c \in \mathbb{R} \setminus C$.

Proof: The set $\{c \in \mathbb{R}\colon \mu_{\mathfrak{U}}(\{g_{\mathbf{f}} = c\}) > 0\}$ cannot be uncountable. ∎

The following Theorem is a precise representation of the projection of $\mathbf{f} \in L_1(\mu)_{\mathfrak{U}}$ on $L_1(\mu_{\mathfrak{U}})$.

THEOREM 8: *For $\mathbf{f} = [f_i] \in L_1(\mu)_{\mathfrak{U}}$ we have that $g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) = \lim_{\mathfrak{U}} f_i(t_i)$ $\mu_{\mathfrak{U}}$-a.e.*

Proof: It is enough to observe that, as a consequence of Lemma 6, we have that the sets $\{(t_i)^{\mathfrak{U}}\colon g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) < \lim_{\mathfrak{U}} f_i(t_i)\}$ and $\{(t_i)^{\mathfrak{U}}\colon g_{\mathbf{f}}((t_i)^{\mathfrak{U}}) > \lim_{\mathfrak{U}} f_i(t_i)\}$ are $\mu_{\mathfrak{U}}$-null. ∎

Note that the above Theorem works for any representative of $\mathbf{f}$. In addition, Theorem 8 yields a direct way to calculate the absolutely continuous part $w_{\mathbf{f}}$ and the singular part $m_{\mathbf{f}}$ of the measure $\nu_{\mathbf{f}}$ introduced in Proposition 2.

4. The splitting principle for $L_1(\mu)_{\mathfrak{U}}$

Rosenthal, by refining an argument of Kadec and Pełczyński, obtained the following result.

THEOREM 9 (subsequence splitting principle): *Every bounded sequence (f_n) in $L_1(\mu)$ has a subsequence (f_{n_k}) that can be decomposed into $f_{n_k} = y_k + z_k$, where (y_k) is weakly convergent and (z_k) is a disjoint sequence.*

In this section, we show that for ultrafilters $\mathfrak{V}$ on $\mathbb{N}$ which are **rare p-points** (definitions below), Theorems 4 and 5 can be notably improved. In fact, we obtain that every $\mathbf{f} \in L_1(\nu_{\mathfrak{V}})$ has a disjoint representative, and that $[f_n]$ belongs to $L_1(\mu_{\mathfrak{V}})$ if and only if (f_n) is weakly convergent following $\mathfrak{V}$. As a consequence, we obtain an ultrapower version of the subsequence splitting principle of $L_1(\mu)$. Note that the above problem only makes sense for ultrafilters on countable sets. Moreover, we show that for ultrafilters on $\mathbb{N}$ which are not p-points, these results are no longer true.

Definition 10: [1, Def. 4.6 and Th. 4.7]. A (non-trivial) ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is said to be a **p-point** if for every countable partition $\{I_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ disjoint with $\mathfrak{U}$, there are finite subsets $A_n \subset I_n$ so that $\bigcup_{n=1}^{\infty} A_n \in \mathfrak{U}$. Equivalently, $\mathfrak{U}$ is a p-point if and only if every bounded sequence (a_n) of real numbers contains a convergent subsequence (a_{n_k}) such that $\{n_k\}_{k=1}^{\infty} \in \mathfrak{U}$.

Definition 11: An ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ is said to be rare if for every increasing sequence $(m_k)_{k=1}^{\infty} \subset \mathbb{N}$, there exists $\{n_k\}_{k=1}^{\infty} \in \mathfrak{U}$ so that

$$n_1 \geq m_1 \quad \text{and} \quad n_{k+1} \geq m_{n_k} \quad \text{for } k = 1, 2, \ldots.$$

We give now the main result of this section.

THEOREM 12: *Let $\mathfrak{V}$ be a rare p-point on $\mathbb{N}$, and*

$$\mathbf{f} \in L_1(\mu)_{\mathfrak{V}} = L_1(\mu_{\mathfrak{V}}) \oplus_1 L_1(\nu_{\mathfrak{V}}).$$

(a) $\mathbf{f} \in L_1(\nu_{\mathfrak{V}})$ *if and only if* $\mathbf{f}$ *has a representative consisting of pairwise disjoint functions.*

(b) $\mathbf{f} \in L_1(\mu_{\mathfrak{V}})$ *if and only if all the representatives of* $\mathbf{f}$ *are weakly convergent following* $\mathfrak{V}$.

Proof: (a) If $\mathbf{f}$ has a representative $[f_i]$ consisting of pairwise disjoint functions, since $\mu(\Omega) < \infty$, we have $\lim_{\mathfrak{V}} D(f_i) = 0$, and by Proposition 5, $\mathbf{f} \in L_1(\nu_{\mathfrak{V}})$.

For the converse, let $0 \neq \mathbf{f} \in L_1(\nu_{\mathfrak{V}})$. By Proposition 5, $\mathbf{f}$ has a representative (f_n) such that $\lim_{\mathfrak{V}} \mu(D(f_n)) = 0$. Without loss of generality, we can assume that $\|f_n\| = 1$ for all n. Since $\mathfrak{V}$ is a p-point, there is a subsequence (f_{n_k}) verifying $\lim_k \mu(D(f_{n_k})) = 0$ and $\{n_k\}_{k=1}^{\infty} \in \mathfrak{V}$. Write $g_m := f_{n_k}$ for all $m = n_k, \ldots, n_{k+1} - 1$, and $\mu_n := \mu(D(g_n))$. Note that $\lim_n \mu_n = 0$. Now we choose inductively positive real numbers $\delta_1, \delta_2, \ldots$ so that

$$\int_A |g_n| \, d\mu < \frac{1}{n} \quad \text{if } \mu(A) < 2\delta_n$$

for $n \geq 1$, and $0 < \delta_n < 2^{-1}\delta_{n-1}$ for $n \geq 2$.

We define an increasing sequence of positive integers (m_n) by

$$m_1 := \min\{k \geq 1\colon \mu_l < \delta_1 \text{ for all } l \geq k\},$$
$$m_n := \min\{k > m_{n-1}\colon \mu_l < \delta_n \text{ for all } l \geq k\}, \quad \text{for } n \geq 2.$$

As $\mathfrak{V}$ is rare, there exists $(p_n) \in \mathfrak{V}$ so that $p_1 \geq m_1$ and $p_{n+1} \geq m_{p_n}$ for $n \in \mathbb{N}$. Thus

$$\mu(D(g_{p_n})) = \mu_{p_n} < \delta_{p_{n-1}} < \frac{1}{2^2}\delta_{p_{n-3}} < \cdots < \frac{1}{2^{n-2}}\delta_{p_1}.$$

Therefore, $\mu_{p_{n+k}} \leq 2^{-k+1}\delta_{p_n}$. For every positive integer n, we denote $F_n := \bigcup_{k=n+1}^{\infty} D(g_{p_k})$ and define the element of $L_1(\mu)$ by

$$h_{p_n} := g_{p_n} - g_{p_n} \cdot \chi_{F_n}.$$

Obviously, $h_{p_n} h_{p_m} = 0$ for $n \neq m$ and

$$\mu(D(g_{p_n} \cdot \chi_{F_n})) \leq \sum_{k=1}^{\infty} \mu_{p_{n+k}} < \sum_{k=1}^{\infty} \frac{1}{2^{k-1}}\delta_{p_n} = 2\delta_{p_n}.$$

Therefore

$$\|h_{p_n} - g_{p_n}\| = \int_{F_n} |g_{p_n}|\, d\mu < \frac{1}{p_n}.$$

Thus, writing $h_k \equiv 0$ for $k \neq p_n$, since $\{p_n\}_{n=1}^{\infty} \in \mathfrak{V}$, we have that $\lim_{\mathfrak{V}} \|h_n - g_n\| = 0$, which leads to $[h_n] = [f_n]$, and the proof is finished.

(b) Given $\mathbf{f} \in L_1(\mu)_{\mathfrak{V}}$, Proposition 2, Theorem 4 and part (a) allow us to write $\mathbf{f} = [g_i] + [h_i]$, where $\{g_i\}_{i=1}^{\infty}$ is relatively weakly compact and $\{h_i\}_{i=1}^{\infty}$ is pairwise disjoint. In order to obtain the result, we just need to prove that $[h_i] = 0$ if and only if there is w-$\lim_{\mathfrak{V}} h_i$. The "only if" part is trivial.

For the "if" part, assume that $[h_i] \neq 0$. Without loss of generality, we suppose that $\|h_i\| = 1$ for all i. For every $i \in \mathbb{N}$, we define $l_i := \sum_{j=i}^{\infty} \operatorname{sgn} h_j \in L_\infty(\mu)$, where $\operatorname{sgn}(0) = 0$. Thus, the action of l_i on h_j is given by $l_i(h_j) = 1$ if $i \leq j$ and $l_i(h_j) = 0$ if $j < i$; so

$$\lim_{\mathfrak{V}(j)} l_i(h_j) = 1.$$

Moreover, given any $f \in L_1(\mu)$, there is i_0 so that $l_{i_0}(f) < 1/2$, so

$$\lim_{\mathfrak{V}(j)} l_{i_0}(h_j - f) > \frac{1}{2},$$

which proves that (h_i) is not convergent following $\mathfrak{V}$ in the weak topology. ∎

Remark: Note that if one of the representatives of $\mathbf{f} \in L_1(\mu)_{\mathfrak{V}}$ is weakly convergent following $\mathfrak{V}$ then all of its representatives are so.

From Theorem 12, we derive the following ultrapower version of the subsequence splitting principle for $L_1(\mu)$.

COROLLARY 13: *Let $\mathfrak{V}$ be a rare p-point on $\mathbb{N}$. Then for every bounded sequence $(f_n) \subset L_1(\mu)$, there exist a sequence (y_k) which is weakly convergent following $\mathfrak{V}$ and a pairwise disjoint sequence (z_k) so that $[f_k] = [y_k] + [z_k]$ in $L_1(\mu)_{\mathfrak{V}}$.*

Remark: Note that in Corollary 13 we do not pass to a subsequence, like in Theorem 9. Moreover, Theorem 9 may be obtained easily from our Theorems 4 and 12(a). Indeed, given a bounded sequence (f_n) in $L_1(\mu)$, the mentioned theorems give a relatively weakly compact sequence (g_n) and a pairwise disjoint sequence of functions (z_n) of $L_1(\mu)$ such that $\lim_{\mathfrak{V}} \|f_n - g_n - z_n\| = 0$. Therefore, passing to a subsequence, we can assume that g_n is weakly convergent and $h_n := f_n - g_n - z_n$ is norm null, and we write $f_n = (g_n + h_n) + z_n$, where $(g_n + h_n)$ is weakly convergent and (z_n) is disjoint.

Next we show that Theorem 12 fails if $\mathfrak{V}$ is replaced by an ultrafilter on $\mathbb{N}$ which is not a p-point. First we give the next Lemma in order to avoid some trivial but annoying cases. Its proof is immediate and we leave it to the reader.

LEMMA 14: *For every ultrafilter $\mathfrak{U}$ on $\mathbb{N}$ which is not a p-point, there exists a partition $\{I_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ disjoint with $\mathfrak{U}$ such that every I_n is infinite, and for every $A \in \mathfrak{U}$ there exists $n \in \mathbb{N}$ so that $A \cap I_n$ is infinite.*

We will say that this partition $\{I_n\}_{n=1}^{\infty}$ is **non-controllable by $\mathfrak{U}$**.

In the following two results we assume that μ is the Lebesgue measure on $[0,1]$, and we denote by $\mathbf{1}$ the function $\chi_{[0,1]} \in L_1(\mu)$.

PROPOSITION 15: *Let $\mathfrak{W}$ be an ultrafilter on $\mathbb{N}$ which is not a p-point. Then there exists a vector $\mathbf{f} \in L_1(\nu_{\mathfrak{W}})$ having no pairwise disjoint representative.*

Proof: Let $\{I_n\}_{n=1}^{\infty}$ be a partition of $\mathbb{N}$ non-controllable by $\mathfrak{W}$. For each $n \in \mathbb{N}$ and every $i \in I_n$, we define $f_i := n\chi_{[0,1/n]}$. Since $\{I_n\}_{n=1}^{\infty}$ is disjoint with $\mathfrak{W}$, we have that $\lim_{\mathfrak{W}} \mu(D(f_i)) = 0$. By Proposition 5, we get $\mathbf{f} \in L_1(\nu_{\mathfrak{W}})$.

Suppose that there is a disjoint sequence $(h_i) \subset L_1(\mu)$ such that $[h_i] = [f_i]$. Given $0 < \varepsilon < 1$, there exists $J \in \mathfrak{W}$ such that $\|f_i - h_i\| < \varepsilon$ for all $i \in J$. Note that $J \cap I_n$ is infinite for some $n \in \mathbb{N}$. But observe that, in order to have $\|f_i - h_i\| < \varepsilon$ for $i \in J \cap I_n$, it is necessary that $\mu(D(h_i)) > n^{-1}(1-\varepsilon)$. Otherwise, we would have that $f_i - h_i = f_i = n$ in a set of measure bigger than $n^{-1}\varepsilon$. Thus the sequence (h_i) cannot be disjoint, and we get a contradiction. ∎

PROPOSITION 16: *Let $\mathfrak{W}$ be an ultrafilter on $\mathbb{N}$ which is not a p-point. Then there is $0 \neq \mathbf{f} \in L_1(\nu_{\mathfrak{W}})$ such that the limit w-$\lim_{\mathfrak{W}} f_i$ exists for all its representatives $[f_i]$.*

Proof: Let $\{I_n\}_{n=1}^{\infty}$ be a partition of $\mathbb{N}$ non-controllable by $\mathfrak{W}$, and rename the elements of $\mathbb{N}$ as pairs of positive integers (n,k) so that $(n,k) \in I_n$. We shall build a normalized sequence $(f_{(n,k)}) \subset L_1(\mu)$ such that

(1) $\lim_{\mathfrak{W}} \mu(D(f_{(n,k)})) = 0$, and

(2) for every measurable set $A \subset [0,1]$, we have that $\lim_{\mathfrak{W}} \int_A f_{(n,k)} = \mu(A)$.

Thus, by condition (1) and Proposition 5, we obtain that $[f_{(n,k)}] \in L_1(\nu_{\mathfrak{W}})$. On the other hand, since the simple functions form a dense subset of $L_1(\mu)^*$, condition (2) implies that $(f_{(n,k)})$ is weakly convergent following $\mathfrak{W}$ to $\mathbf{1} \in L_1(\mu)$.

Let us build the functions $f_{(n,k)}$. For every $k \in \mathbb{N}$, every $1 \leq i \leq 2^k$ and every $n \in \mathbb{N}$, we define

$$I_i^k := \left[\frac{i-1}{2^k}, \frac{i}{2^k}\right],$$

$$F_{(n,k)} := \bigcup_{i=1}^{2^k} \left[\frac{i-1}{2^k}, \frac{i-1}{2^k} + \frac{1}{n2^k}\right],$$

$$f_{(n,k)} := n\chi_{F_{(n,k)}}.$$

Clearly, we have that $\|f_{(n,k)}\|_1 = 1$ and $\mu(F_{(n,k)}) = 1/n$. Fix a positive integer m. Then $\mu(F_{(n,k)}) < 1/m$ for all $(n,k) \in \bigcup_{l=m+1}^{\infty} I_l \in \mathfrak{W}$, hence $\lim_{\mathfrak{W}} \mu(D(f_{(n,k)})) = 0$ and condition (1) is proved.

Check now condition (2). Let A be a measurable subset of $[0,1]$ and fix a positive integer n. For every k, there exists $J_k \subset \{1,\ldots,2^k\}$ so that, denoting $M_k := \bigcup_{i \in J_k} I_i^k$ and $A \triangle M_k := (A \setminus M_k) \cup (M_k \setminus A)$ the symmetric difference, we have that

$$\alpha(k) := \mu(A \triangle M_k) \xrightarrow{k\to\infty} 0.$$

Observe that $\int_{M_k} f_{(n,k)} = \int_{M_k} \mathbf{1}$, so

$$\left|\int_A f_{(n,k)} - \int_A \mathbf{1}\right| \leq \left|\int_A f_{(n,k)} - \int_{M_k} f_{(n,k)}\right| + \left|\int_{M_k} \mathbf{1} - \int_A \mathbf{1}\right| \leq (n+1)\alpha(k).$$

Take now $\varepsilon > 0$. Then there is k_0 such that $(n+1)\alpha(k) < \varepsilon$ for all $k \geq k_0$. That means that, for every fixed n, the set

$$\left\{k\colon \left|\int_A f_{(n,k)} - \int_A \mathbf{1}\right| \geq \varepsilon\right\}$$

is finite. But $\{I_n\}_{n=1}^{\infty}$ is not controllable by $\mathfrak{W}$, so

$$\left\{(n,k)\colon \left|\int_A f_{(n,k)} - \int_A \mathbf{1}\right| < \varepsilon\right\} \in \mathfrak{W},$$

which proves that condition (2) holds. ∎

5. Final remarks

In this section we discuss the existence of the rare p-points on $\mathbb{N}$ used in Theorem 12. Moreover, as an application of the representation Theorems 4 and 5, we show that every reflexive subspace of $L_1(\mu)$ is superreflexive.

We have seen in Propositions 15 and 16 that Theorem 12 is true only if the ultrafilter is a p-point. On the one hand, it is easy to show the existence of countably incomplete ultrafilters on $\mathbb{N}$ which are not p-points. For instance, let $\mathfrak{U}$ be any ultrafilter on $\mathbb{N}$. Then the product $\mathfrak{U}\times\mathfrak{U}$ is an ultrafilter on a countable set and is not a p-point. By means of the Continuum Hypothesis (henceforth CH), Rudin [12] proved that p-points on $\mathbb{N}$ do exist. Furthermore, Choquet [3] used again CH to find p-points $\mathfrak{V}$ with the following additional condition:

for every countable partition $\{F_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ into finite sets there exists $U \in \mathfrak{V}$ such that

$$|U \cap F_k| \leq 1. \tag{2}$$

Ultrafilters satisfying condition (2) were named **rare** by Choquet. Actually, our Definition 11 turns out to agree with condition (2), as we will prove later.

Moreover, Shelah (see [15]) showed that the existence of p-points on $\mathbb{N}$ is undecidable in the Zermelo–Fraenkel set theory plus the Axiom of Choice (henceforth, ZFC). So, if we wish to work within a Set Theory for which the existence of p-points is not acceptable, it follows from Propositions 15 and 16 that Theorem 12 is not valid.

For some mathematicians, CH is too demanding and they prefer a weaker statement, the Martin axiom (MA from now on), which is consistent with both ZFC + CH and ZFC + ¬ CH (for this topic, we recommend the expository paper [11]). In this context, Booth [1, Th. 4.9 and 4.14] shows that ZFC + MA yields the existence of rare p-points on $\mathbb{N}$.

In the literature, rare p-points are called *selective.* These ultrafilters have been extensively studied in [4].

Those who cherish the flexibility of MA, rather than CH, will look for the following equivalence between our Definition 11 and Choquet's definition given in (2).

PROPOSITION 17: *Given an ultrafilter $\mathfrak{U}$ on $\mathbb{N}$, the following statements are equivalent:*

(1) *for every increasing map $f\colon \mathbb{N} \longrightarrow \mathbb{N}$ there exists $A \in \mathfrak{U}$ so that for every pair $k, l \in A$ with $k < l$, we have that $f(k) \leq l$;*

(2) *for every countable partition $\{F_n\}_{n=1}^{\infty}$ of $\mathbb{N}$ into finite sets F_k there exists $A \in \mathfrak{U}$ such that $|A \cap F_k| \leq 1$.*

Proof: Assume condition (1) and let $\{F_k\}_{n=1}^{\infty}$ be a partition of $\mathbb{N}$ into finite sets. Take an increasing map $h\colon \mathbb{N} \longrightarrow \mathbb{N}$ so that $\max F_k \leq \min h(F_k)$ for all k. Let $A \in \mathfrak{U}$ such that if $\{k, l\} \subset A$ and $k < l$ then $h(k) \leq l$. Now, if $k \in F_n$, then we have $l \geq h(k) > \max F_n$, hence $l \notin F_n$. So $|A \cap F_n| \leq 1$, and condition (2) holds.

Assume now condition (2). Let $h\colon \mathbb{N} \longrightarrow \mathbb{N}$ be an increasing map. There is no loss of generality if we suppose that $h(1) > 1$. Define a sequence of intervals of $\mathbb{N}$ by $F_1 = [1, h(1))$ and $F_{k+1} = [h^k(1), h^{k+1}(1))$. Notice that $\{F_k\}_{k=1}^{\infty}$ is a partition of $\mathbb{N}$. Let $B \in \mathfrak{U}$ with $|B \cap F_k| \leq 1$. If $B = (u_j)_{j=1}^{\infty}$, set $A_0 = (u_{2j})_{j=1}^{\infty}$ and $A_1 = (u_{2j-1})_{j=1}^{\infty} \in \mathfrak{U}$. Only one of these two disjoint subsets of B belongs to $\mathfrak{U}$; let us call it A. For every $k \in \mathbb{N}$ let $v(k)$ be the unique index n such that $k \in F_n$: the map $k \longrightarrow v(k)$ is non-decreasing, and its restriction to B is injective. If $\{k, l\} \subset A$ with $k < l$ there exists $m \in B$ with $k < m < l$; then $v(k) < v(m) < v(l)$. Since $h(k) \in F_{v(k)+1}$ and $l \in F_{v(l)}$, the relation $v(k) + 1 < v(l)$ implies $h(k) < l$. ∎

Those who are unlikely to delve into Set Theory and feel comfortable with CH will probably appreciate the following straightforward evidence of the existence of the rare p-points on $\mathbb{N}$ used in our Theorem 12.

PROPOSITION 18: *Assume* CH. *Then there exists an ultrafilter $\mathfrak{V}$ on $\mathbb{N}$ that satisfies the following properties:*

(i) *$\mathfrak{V}$ is a p-point;*

(ii) *$\mathfrak{V}$ is rare; equivalently, for every increasing sequence $(m_k) \subset \mathbb{N}$, there exists $\{n_k\} \in \mathfrak{V}$ so that*

$$n_1 \geq m_1 \quad \text{and} \quad n_{k+1} \geq m_{n_k} \quad \textit{for } k = 1, 2, \ldots.$$

Proof: Let us adopt some conventions. Given two increasing sequences (m_k) and (n_k) in $\mathbb{N}$, which verify the condition (ii) of the statement, we write $(n_k) \trianglerighteq (m_k)$. We denote by $\wp_\infty(\mathbb{N})$ the class of all infinite subsets of $\mathbb{N}$ as well as the class of all increasing sequences of positive integers. Given $(n_k) \in \wp_\infty(\mathbb{N})$, we say that (a_n) is convergent in (n_k) if the subsequence $(a_{n_k})_k$ is convergent.

As usual, we denote the first uncountable ordinal by ω_1. Recall that $\omega_1 \equiv [0,\omega_1)$, and since we assume the Continuum Hypothesis, we have $\omega_1 \equiv \text{card}(\omega_1) = \text{card}(\ell_\infty) = \text{card}\,\wp_\infty(\mathbb{N})$, where ℓ_∞ stands for the space of all bounded real sequences. Therefore, there exist bijective maps $\varphi_1\colon \omega_1 \longrightarrow \wp_\infty(\mathbb{N})$ and $\varphi_2\colon \omega_1 \longrightarrow \ell_\infty$.

For every $\alpha < \omega_1$ we are going to select $\psi(\alpha) \in \wp_\infty(\mathbb{N})$ so that

(a) $\psi(\alpha) \trianglerighteq \varphi_1(\alpha)$;

(b) for all finite subsets $I \subset [0,\alpha]$, we have that $\bigcap_{\gamma\in I}\psi(\gamma) \in \wp_\infty(\mathbb{N})$;

(c) $\varphi_2(\alpha)$ is convergent in $\psi(\alpha)$.

First we select $\psi(0) \in \wp_\infty(\mathbb{N})$ such that $\psi(0) \trianglerighteq \varphi_1(0)$ and $\varphi_2(0)$ is convergent in $\psi(0)$. Then we fix $\eta < \omega_1$ and assume we have selected $\psi(\beta)$ for all $\beta < \eta$.

If η is finite, then $\bigcap_{\gamma<\eta}\psi(\gamma)$ is infinite, and we can select an increasing sequence (c_j) in $\bigcap_{\gamma<\eta}\psi(\gamma)$ so that $\varphi_2(\eta)$ converges in (c_j). We write

$$\varphi_1(\eta) = \{n_1 < n_2 < \cdots\},$$

and we construct an increasing sequence $\psi(\eta) = \{l_i\}_{i=1}^\infty$ in $\mathbb{N}$ as follows: l_1 is the smallest c_i which is greater than n_1. Now, if $l_1,\ldots,l_{m-1}$ have been chosen, we take l_m as the smallest c_i which is greater than $n_{l_{m-1}}$.

If η is infinite, then $[0,\eta)$ is countable, and there is a bijection $\phi\colon \mathbb{N} \longrightarrow [0,\eta)$. Since for every n the set $\bigcap_{i=1}^n \psi\circ\phi(i)$ is infinite, we can select an increasing sequence of positive integers (c_j) such that $c_j \in \bigcap_{i=1}^j \psi\circ\phi(i)$. Taking a subsequence if necessary, we can assume that $\varphi_2(\eta)$ converges in (c_j), and we construct the sequence $\psi(\eta) = \{l_i\}_{i=1}^\infty$ as in the case when η finite.

Clearly, $\psi(\eta)$ satisfies the conditions (a), (b) and (c). In particular, the family $\{\psi(\alpha)\colon \alpha \in [0,\omega_1)\}$ has the finite intersection property, and therefore is contained in some ultrafilter $\mathfrak{V}$ on $\mathbb{N}$ that is a p-point. Now a moment of reflection is enough to see that $\mathfrak{V}$ is the wanted ultrafilter. ∎

From a result of Rosenthal [10], it follows that all reflexive subspaces of $L_1(\mu)$ are superreflexive. Here we prove this fact by applying the representation Theorems of the ultrapower of $L_1(\mu)$ given before. Recall that a Banach space X is **superreflexive** if any Banach space finitely representable in X is reflexive; equivalently, if any ultrapower $X_\mathfrak{U}$ is reflexive [8].

PROPOSITION 19: *Let E be a subspace of $L_1(\mu)$. Then E is reflexive if and only if $E_\mathfrak{U}$ is contained in $L_1(\mu_\mathfrak{U})$.*

Proof: It is well known that a reflexive subspace E of $L_1(\mu)$ is reflexive if and only if B_E is equiintegrable. So, the direct implication follows directly from Theorem 4.

For the converse, assume that E is not reflexive. Then B_E is not equiintegrable, so there is a normalized sequence $(f_n) \subset E$, $\varepsilon > 0$ and subsets $A_n \subset D(f_n)$ such that $\lim_n \mu(A_n) = 0$ and $\int_{A_n} |f_n|\, d\mu > \varepsilon$ for all n. Let $\{I_n\}_{n=1}^{\infty}$ be a partition of I disjoint with $\mathfrak{U}$, and define $y_i := f_n$, $h_i := f_n \chi_{A_n}$ and $g_i := y_i - h_i$ for every n and every $i \in I_n$. Theorem 5 shows that $0 \neq [h_i] \in L_1(\nu_{\mathfrak{U}})$. As $[y_i] = [g_i] + [h_i] \in E_{\mathfrak{U}}$ and $g_i \cdot h_i = 0$ for all i, we obtain that $E_{\mathfrak{U}} \not\subset L_1(\mu_{\mathfrak{U}})$. ∎

It follows from Lemma 1 that, given a Banach space X and two ultrafilters $\mathfrak{U}$ and $\mathfrak{V}$ on I and J respectively, the map that takes $[x_{ij}]_{ij} \in X_{\mathfrak{U}\times\mathfrak{V}}$ to $[[x_{ij}]_j]_i \in (X_{\mathfrak{V}})_{\mathfrak{U}}$ is an isometry from $X_{\mathfrak{U}\times\mathfrak{V}}$ onto $(X_{\mathfrak{V}})_{\mathfrak{U}}$. We need this fact in the next result.

PROPOSITION 20: *Every reflexive subspace of $L_1(\mu)$ is superreflexive.*

Proof: Let G be the natural isometry from $L_1(\mu)_{\mathfrak{U}\times\mathfrak{U}}$ onto $(L_1(\mu)_{\mathfrak{U}})_{\mathfrak{U}}$. Since by Lemma 3 the completions $\overline{\mu_{\mathfrak{U}\times\mathfrak{U}}}$ and $\overline{(\mu_{\mathfrak{U}})_{\mathfrak{U}}}$ coincide, there is a canonical onto isometry F: $L_1(\mu_{\mathfrak{U}\times\mathfrak{U}}) \longrightarrow L_1((\mu_{\mathfrak{U}})_{\mathfrak{U}})$ that makes the next diagram commutative:

$$\begin{array}{ccc} L_1(\mu_{\mathfrak{U}\times\mathfrak{U}}) & \xrightarrow{F} & L_1((\mu_{\mathfrak{U}})_{\mathfrak{U}}) \\ {\scriptstyle J_1}\big\downarrow & & {\scriptstyle J_2}\big\downarrow \\ L_1(\mu)_{\mathfrak{U}\times\mathfrak{U}} & \xrightarrow[G]{} & (L_1(\mu)_{\mathfrak{U}})_{\mathfrak{U}} \end{array}$$

Here J_1 and J_2 are the natural embeddings introduced in Proposition 2.

Let E be a reflexive subspace of $L_1(\mu)$. By Proposition 19 we have that $E_{\mathfrak{U}\times\mathfrak{U}} \subset L_1(\mu_{\mathfrak{U}\times\mathfrak{U}})$. The above commutative diagram shows that $(E_{\mathfrak{U}})_{\mathfrak{U}} \subset L_1((\mu_{\mathfrak{U}})_{\mathfrak{U}})$. A new application of Proposition 19 proves that $E_{\mathfrak{U}}$ is reflexive, hence E is superreflexive. ∎

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