On probabilities of large claims that are compound Poisson distributed

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Introduction and results

In this paper we shall derive exponential bounds for the probability of claims exceeding premiums that are given according to the loaded pure risk principle and to the so-called σ-loading principle. These probabilities appear to be of great importance concerning reinsurances. Especially in the first case it turns out that the enlargement of a (homogeneous) portfolio effectively reduces the considered probability.

For sums of independent random variables Theorem 1 below is well known in probability theory ("large deviations"). Since we do not know any literature on the subject in risk theory we give a short derivation of this interesting result.

In the following Q is a probability measure with $Q(0,\infty)=1$ such that the moment generating function $M(t)=\int e^{tx}\,Q(dx)$ of Q exists for some $t>0$. Let

$$t_0=\sup\{t>0:\ M(t)<\infty\}$$

and

$$K=\sup\{M'(t):\ t<t_0\}.$$

Observe that in general $K=\infty$. In the case $t_0=\infty$ this follows from

$$M'(t)=\int x\,e^{tx}\,Q(dx)\geq\int_{\varepsilon}^{\infty} x\,e^{tx}\,Q(dx)\geq\varepsilon\,e^{t\varepsilon}\,Q(\varepsilon,\infty),\qquad t>0,$$

where $\varepsilon>0$ is chosen such that $Q(\varepsilon,\infty)>0$.

Let

$$\mu=M'(0)\quad\text{and}\quad\mu_2=M''(0).$$

Theorem 1: *Assume that* X *is compound Poisson distributed with intensity* $\lambda>0$ *and individual claim size distribution* Q. *Then for* $c>1$

$$\mathbb{P}\{X>c\lambda\mu\}\leq e^{-\lambda f(r)},$$

where

$$f(t)=c\mu t-M(t)+1$$

and $r\in(0,t_0)$ *is the unique solution of the equation* $M'(t)=c\mu$, *if* $c<K/\mu$. *In the case* $c\geq K/\mu$ *we have* $r=t_0$ *with the convention* $M(t_0)=M(t_0-)$. *In both cases* $f(r)>0$ *holds true.*

Examples: (i) Assume that Q is the exponential distribution with mean $a>0$. Then we have

$$M(t)=\frac{1}{1-at},\qquad t<t_0=\frac{1}{a},\quad\text{and}\quad K=\infty.$$

Furthermore, $r=\frac{1}{a}\left(1-\frac{1}{\sqrt{c}}\right)$, which gives

$$\mathbb{P}\{X>c\lambda a\}\leq\exp[-\lambda(\sqrt{c}-1)^2],\qquad c>1.$$

(ii) If we consider a portfolio of life insurance policies with the same risk sum μ, then $Q=\delta_\mu$ and $M(t)=e^{t\mu}$, $t>0$. Here we obtain

$$\mathbb{P}\{X>c\lambda\mu\}\le\exp[-\lambda(c\log c-c+1)],\qquad c>1\,.$$

Observe that in both cases the given bound does not depend on Q.

In the following we consider the probability of claims exceeding a premium given according to the σ-loading principle, i.e.

$$\mathbb{P}\{X>\lambda\mu+\alpha\sqrt{\lambda\mu_2}\},\qquad \alpha>0\,.$$

Here, the central limit theorem implies that this probability is asymptotically constant, i.e. it cannot tend exponentially fast to zero for $\lambda\to\infty$ (if the portfolio is enlarged). Observe that we may apply Theorem 1 with

$$c=1+\alpha\mu^{-1}(\mu_2/\lambda)^{\frac{1}{2}}$$

to derive an upper bound for this probability. To obtain a feeling for what comes out consider example (i) above. Here, $c=1+x$ with $x=\alpha(2/\lambda)^{\frac{1}{2}}$. From

$$(1+x)^{\frac{1}{2}}\le 1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$$

we get

$$(\sqrt{c}-1)^2=c+1-2\sqrt{c}\ge\frac{1}{2}\alpha^2\lambda^{-1}\left(1-\frac{\sqrt{2}}{2}\alpha\lambda^{-\frac{1}{2}}\right),$$

i.e. in this case we have

$$\mathbb{P}\{X>\lambda a+\alpha(2\lambda)^{\frac{1}{2}}a\}\le\exp\left[-\frac{1}{2}\alpha^2\left(1-\frac{\sqrt{2}}{2}\alpha\lambda^{-\frac{1}{2}}\right)\right].$$

In the following theorem we show that the asymptotic bound $\exp[-\frac{1}{2}\alpha^2]$ can be halved. (For $\alpha=2$, e.g., this gives 0.068 instead of 0.136.)

Theorem 2: *Let X be compound Poisson distributed with intensity $\lambda>0$ and individual claim size distribution Q such that $K=\infty$. Then for $\alpha>0$,*

$$\mathbb{P}\{X>\lambda\mu+\alpha\sqrt{\lambda\mu_2}\}\le a(\lambda)\exp[-\tfrac{1}{2}\alpha^2\mu_2^2(M''(s))^{-2}]\,,$$

where $a(\lambda)\le 1$ and

$$|a(\lambda)-\tfrac{1}{2}|\le 0.8\,\lambda^{-\frac{1}{2}}M'''(s)\,\mu_2^{-\frac{3}{2}}$$

with $s=\alpha(\lambda\mu_2)^{-\frac{1}{2}}$. In particular,

$$\lim_{\lambda\to\infty}\mu_2^2(M''(s))^{-2}=1\,.$$

Proofs

Theorem 1: For $t\in(0,t_0)$

$$\mathbb{P}\{X>c\lambda\mu\}=\mathbb{P}\{tX>c\lambda\mu t\}\le e^{-c\lambda\mu t}\,E\,e^{tX}=e^{-\lambda f(t)}\,,$$

where $f(t)=c\mu t-M(t)+1$. We have

$$f(0)=0,\quad f'(0)=(c-1)\mu>0,\quad f''(t)=-M''(t)<0\,.$$

In the case $c \geq K/\mu$ (which implies $t_0 < \infty$)

$$f'(t) = c\mu - M'(t) \geq K - M'(t) > 0, \qquad t \in (0, t_0).$$

Hence, $f(t)$ strictly increases and attains its maximum (positive) value for $t = t_0$. Observe that $M(t_0-) < \infty$, as

$$M(t) = \int e^{xt} Q(dx) \leq \exp[t_0^2] + t_0^{-1} \int_{t_0}^{\infty} x e^{xt} Q(dx) \leq \exp[t_0^2] + t_0^{-1} K, \quad t < t_0.$$

In the case $1 < c < K/\mu$ the maximum (positive) value of $f(t)$ is given for $t = r$, where $f'(r) = 0$, i.e. $M'(r) = c\mu$.

For the proof of Theorem 2 we need the concept of a generalized compound Poisson distribution. Let m be a measure with $m(-\infty, 0) = 0$, $0 < m[0, \infty) < \infty$ and define

$$P(\lambda, m) = e^{-\lambda m[0,\infty)} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} m^{*k}, \qquad \lambda > 0,$$

where $m^{*0} = \delta_0$. Then the characteristic function $\varphi(t)$ of $P(\lambda, m)$ is given by

$$\varphi(t) = e^{\lambda(\psi(t) - m[0,\infty))},$$

where

$$\psi(t) = \int_0^{\infty} e^{itx} m(dx), \qquad t \in \mathbb{R}.$$

This gives

$$P(\lambda, m) * P(\tau, m) = P(\lambda + \tau, m), \qquad \lambda, \tau > 0,$$

i.e., the distributions $P(\lambda, m)$ are infinitely divisible.

Theorem 2: (i) With $c = 1 + \alpha\mu^{-1}(\mu_2/\lambda)^{\frac{1}{2}}$ we have

$$\mathbb{P}\{X > \lambda\mu + \alpha\sqrt{\lambda\mu_2}\} = \mathbb{P}\{X > c\lambda\mu\} \leq e^{-\lambda f(r)} E e^{rX} 1_{\{X > c\lambda\mu\}}/E e^{rX},$$

where $f(t)$ and r are given in the proof of Theorem 1, i.e. $r > 0$ is the unique solution of the equation

$$M'(t) = c\mu = \mu + \alpha(\mu_2/\lambda)^{\frac{1}{2}}.$$

Let D_λ be the distribution defined by

$$D_\lambda(A) = E e^{rX} 1_A(X)/E e^{rX}, \qquad A \subset [0, \infty).$$

Then we have shown that

$$\mathbb{P}\{X > \lambda\mu + \alpha\sqrt{\lambda\mu_2}\} \leq a(\lambda) e^{-\lambda f(r)},$$

where $a(\lambda) = D_\lambda(c\lambda\mu, \infty)$.

(ii) In this part we derive a lower bound for $f(r) = c\mu r - M(r) + 1$. From

$$\mu c = M'(r) = M'(0) + r M''(\xi) = \mu + r M''(\xi)$$

and $\mu_2 \leq M''(\xi) \leq M''(r)$ we obtain

$$\alpha(M''(r))^{-1}(\mu_2/\lambda)^{\frac{1}{2}} \leq r \leq \alpha(\lambda\mu_2)^{-\frac{1}{2}} = s.$$

Furthermore,

$$\begin{aligned} f(r) &= c\mu r + M(0) - M(r) \\ &= c\mu r - rM'(r) + \tfrac{1}{2} r^2 M''(\eta) = \tfrac{1}{2} r^2 M''(\eta) \\ &\geq \tfrac{1}{2} r^2 \mu_2 \geq \tfrac{1}{2} \alpha^2 \mu_2^2 (M''(r))^{-2} \lambda^{-1} \geq \tfrac{1}{2} \alpha^2 \mu_2^2 (M''(s))^{-2} \lambda^{-1}. \end{aligned}$$

(iii) It remains to prove that $a(\lambda) = \frac{1}{2} + O(\lambda^{-\frac{1}{2}})$.
Since X is distributed according to $P(\lambda, Q)$ we infer that

$$D_\lambda = P(\lambda, m) \quad \text{with} \quad m(A) = \int e^{rx} 1_A(x)\, Q(dx).$$

Let $Y, Y_1, \ldots, Y_n$, $n \in \mathbb{N}$, be i.i.d. with distribution $P\left(\frac{\lambda}{n}, m\right)$. Then, by the lemma below,

$$EY = \frac{\lambda}{n} \int x\, m(dx) = \frac{\lambda}{n} M'(r) = \frac{\lambda}{n} c\mu.$$

As the distributions $P(\lambda, m)$ are infinitely divisible, this implies

$$a(\lambda) = D_\lambda(c\lambda\mu, \infty) = P(\lambda, m)(c\lambda\mu, \infty) = \mathbb{P}\left\{\sum_{i=1}^{n} (Y_i - EY) > 0\right\}.$$

By the Berry-Esseen Theorem in connection with the estimate of the constant that has been given in [1] we obtain

$$\begin{aligned} |a(\lambda) - \tfrac{1}{2}| &\leq 0.8\, n^{-\frac{1}{2}} [E(Y - EY)^2]^{-\frac{3}{2}} E|Y - EY|^3 \\ &= 0.8\, \lambda^{-\frac{3}{2}} [M''(r)]^{-\frac{3}{2}} n E|Y - EY|^3 \\ &\leq 0.8\, \lambda^{-\frac{3}{2}} \mu_2^{-\frac{3}{2}} n E|Y - EY|^3, \end{aligned}$$

where the equality follows from the lemma and

$$\int x^2 m(dx) = \int x^2 e^{rx} Q(dx) = M''(r).$$

Furthermore,

$$\begin{aligned} E|Y - EY|^3 &\leq E(Y + EY)^3 = E(Y - EY + 2EY)^3 \\ &= \frac{\lambda}{n} M'''(r) + 6\left(\frac{\lambda}{n}\right)^2 M''(r) M'(r) + 8\left(\frac{\lambda}{n}\right)^3 [M'(r)]^3. \end{aligned}$$

Together with the estimate above this yields the result (observe that $r \leq s$).

The following lemma is standard and we state it only for easier reference. A proot involving the existence of the moment generating function of a probability measure m is given in chapter 3.1.8 of [2].

Lemma: *Assume that* Y *is distributed according to a generalized compound Poisson distribution* $P(\lambda, m)$ *such that* $\int x^k m(dx) < \infty$ *for some* $k \in \mathbb{N}$. *Then* $E(Y - \lambda\mu) = 0$, *where* $\mu = \int x\, m(dx)$, *and for* $k \geq 2$,

$$E(Y - \lambda\mu)^k = \lambda \sum_{i=1}^{k-1} \binom{k-1}{i} \left(\int x^{i+1} m(dx)\right) E(Y - \lambda\mu)^{k-1-i}.$$

In particular,

$$E(Y - EY)^2 = \lambda \int x^2 m(dx) \quad \textit{and} \quad E(Y - EY)^3 = \lambda \int x^3 m(dx).$$

Proof: (i) At first we prove that

$$\mathrm{EY}\,f(Y) = \lambda \int y\,[\mathrm{E}\,f(Y+y)]\,m(dy),$$

provided the right hand side is finite.
With $M = m[0, \infty)$ we have

$$\mathrm{EY}\,f(Y) = e^{-\lambda M} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \int y\,f(y)\,m^{*k}(dy)$$

$$= e^{-\lambda M} \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \sum_{i=1}^{k} \int y_i\, f\left(\sum_{j=1}^{k} y_j\right) m^k(dy_1, \ldots, dy_k)$$

$$= e^{-\lambda M} \sum_{k=2}^{\infty} \frac{\lambda^k}{k!}\, k \int y \left[\int f\left(y + \sum_{j=1}^{k-1} y_j\right) m^{k-1}(dy_1, \ldots, dy_{k-1})\right] m(dy)$$

$$+ \lambda\, e^{-\lambda M} \int y\,f(y)\,m(dy).$$

This gives our assertion.

(ii) With $f=1$ we obtain from (i) that $\mathrm{EY} = \lambda \int y\,m(dy)$. Furthermore,

$$\mathrm{E}(Y - \lambda\mu)^k = \mathrm{EY}(Y - \lambda\mu)^{k-1} - \lambda\mu\,\mathrm{E}(Y - \lambda\mu)^{k-1}$$

and

$$\mathrm{EY}(Y - \lambda\mu)^{k-1} = \lambda \int y\,[\mathrm{E}(Y - \lambda\mu + y)^{k-1}]\,m(dy),$$

hence the result.

REFERENCES

[1] *Beek, P. van* (1972): An application of Fourier methods to the problem of sharpening the Berry-Esseen inequality. Z. Wahrscheinlichkeitstheorie verw. Gebiete *23*, 187–196

[2] *Heilmann, W.-R.* (1988): Fundamentals of Risk Theory. Verlag Versicherungswirtschaft, Karlsruhe.

Zusammenfassung

Über Wahrscheinlichkeiten hoher Schäden im Fall der zusammengesetzten Poisson-Verteilung

Unter der Voraussetzung, daß das betrachtete Risiko nach einer zusammengesetzten Poissonverteilung verteilt ist, geben wir quantitative (exponentielle) Schranken für die Wahrscheinlichkeit an, daß es die nach dem Erwartungswert- bzw. Standardabweichungsprinzip berechneten Prämien übersteigt.

Summary

On probabilities of large claims that are compound Poisson distributed

Under the assumption that the considered risk is distributed according to a compound Poisson distribution we give quantitative (exponential) bounds for the probability of claims exceeding premiums that are given according to the loaded pure risk principle and to the σ-loading principle.