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GROUPS WITH FEW CHARACTERS OF SMALL DEGREES

BY

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ABSTRACT

Let $d > 1$ be a proper divisor of the order of a finite group G and let $\sigma_d(G)$ be the sum of squares of degrees of those irreducible characters whose degrees are not divisible by d. It is easy to see that d divides $\sigma_d(G)$. The groups G such that $\sigma_d(G) = d$ coincide with Frobenius groups whose kernel has index d (see G. Karpilovsky, *Group Representations*, Volume 1, Part B, North-Holland, Amsterdam, 1992, Theorem 37.5.5). In this note we study the case $\sigma_d(G) = 2d$ in some detail. In particular, if G is a 2-group, it is of maximal class (Remark 3(b)).

Let $d > 1$ be a proper divisor of the order $|G|$ of a finite group G and $\mathrm{Irr}(G)$ the set of all complex irreducible characters of G. Let $X_d = X_d(G)$ be the set of all irreducible characters of G whose degrees are not divisible by d and $\sigma_d(G)$ the sum of squares of the degrees of characters in X_d. If $\mathrm{Irr}(G)$ contains a character of degree divisible by d (this is the case if and only if $|G| > \sigma_d(G)$), then d divides $\sigma_d(G)$. Set $K = K_d(G) = \bigcap_{\chi \in X_d} \ker(\chi)$. It is clear that $|G : K| = \sigma_d(G)$ if and only if $\mathrm{Irr}(G/K) = X_d$. In the last case, K is p-nilpotent for any prime divisor p of d (see [B1] or Remark 1 below) and, moreover, K is solvable (see [B2], Proposition 9 and Remark 1 following it). If G is a Frobenius group with kernel of index d, then $\sigma_d(G) = d$ (see [I], Theorem 6.34). Conversely, if $|G| > d = \sigma_d(G)$, then G is a Frobenius group with kernel of index d (see [K], Theorem 37.5.5; for another proof, see Remark 2, below), i.e., the groups G with $\sigma_d(G) = d$ are classified. In this note we will proceed to the following step and consider the case $\sigma_d(G) = 2d$. Theorem 6 yields the classification in the case when d is

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odd. The proof of that theorem is partially based on Lemma 2, which is due to the referee. In the general case, when d is arbitrary, we prove that the socle of G is abelian (Corollary 4; we may consider that result as a generalization of Thompson's Theorem on solvability of Frobenius kernels). Lemma 5, which was inspired by the referee's report, also yields essential information on the structure of G.

The following two lemmas are due to the referee. Lemma 2 is the key.

LEMMA 1: *Let X be a subset of $\mathrm{Irr}(G)$ and assume that whenever α and β lie in X, all irreducible constituents of $\alpha\beta$ also lie in X. Let $K = \bigcap_{\alpha \in X} \ker(\alpha)$. Then $X = \mathrm{Irr}(G/K)$.*

Proof: Let $\chi = \sum_{\alpha \in X} \alpha$; then χ is a faithful character of G/K. By the Burnside–Brauer Theorem (see [I], Theorem 4.3), every irreducible character of G/K is a constituent of some power of χ and thus lies in X, by assumption. ∎

LEMMA 2: *Let $d > 1$ be a divisor of the order of a group G and $\sigma_d(G) \leq 2d$. Then if $\alpha, \beta \in X_d$, all irreducible constituents of $\alpha\beta$ lie in X.*

Proof: We may assume that neither α nor β is linear. Note that $1_G \in X_d$. So if $\alpha \neq \beta$, then $2d \geq \sigma_d(G) > \alpha(1)^2 + \beta(1)^2 \geq 2\alpha(1)\beta(1)$, hence $\alpha(1)\beta(1) < d$. It follows that all irreducible constituents of $\alpha\beta$ lie in X_d.

Now assume that α^2 has an irreducible constituent ψ of degree divisible by d. Since $\alpha(1)^2 < 2d$, it follows that ψ is the unique irreducible constituent of α^2 of degree divisible by d; moreover, $\langle \alpha^2, \psi \rangle = 1$. Then $\langle \alpha, \bar{\alpha}\psi \rangle = 1$. Since $\bar{\alpha}(1)\psi(1)$ is divisible by d and $\bar{\alpha}(1)\psi(1) - \alpha(1)$ is not, the character $\bar{\alpha}\psi - \alpha$ has an irreducible constituent γ that is a member of X_d. Since the multiplicity of α in $\bar{\alpha}\psi$ is 1, we get $\gamma \neq \alpha$. Now $\langle \alpha\gamma, \psi \rangle = \langle \gamma, \bar{\alpha}\psi \rangle > 0$, contrary to the result of the previous paragraph. ∎

In what follows we will assume that $\sigma_d(G) \leq 2d$ and $|G| > \sigma_d(G)$ (in that case, X_d is a proper subset of $\mathrm{Irr}(G)$). Set $K = \bigcup_{\chi \in X_d} \ker(\chi)$. By Lemma 2, $X = X_d$ satisfies the hypothesis of Lemma 1. Hence we get

Remark 1: For a prime p, let $G(p')$ denote the intersection of kernels of nonlinear irreducible characters whose degrees are not divisible by p (if G has no such characters, set $G(p') = G$). By [B1] or [K], Corollary 27.4.4, $G(p')$ is p-nilpotent (= has a normal p-complement). By [B2], Proposition 9 and Remark 1 following it, $G(p')$ is solvable so K is (this assertion depends on the classification). Note that the solvability of K also follows from the Odd Order Theorem. Indeed, this is the case if d is even (see Corollary 3). If d is odd, K is nilpotent by Theorem 6.

COROLLARY 3: *If $\sigma_d(G) \leq 2d$, then $|G : K| = \sigma_d(G)$, every member of $\mathrm{Irr}(G \mid K) = \mathrm{Irr}(G) - \mathrm{Irr}(G/K)$ has degree divisible by d and K has a normal p-complement for every prime divisor p of d.*

Proof: By Lemma 2, $\mathrm{Irr}(G/K) = X_d$ so $|G : K| = \sigma_d(G)$. By assumption, d divides $\chi(1)$ for all $\chi \in \mathrm{Irr}(G \mid K)$. Therefore, $K \leq G(p')$ for any prime divisor p of d so K is p-nilpotent for that p (see Remark 1). ∎

Remark 2: Assume that $|G| > d > 1$ and $\sigma_d(G) = d$. We will prove that then G is a Frobenius group with kernel of index d. In the above notation, $|G : K| = d$ and d divides the degrees of all characters in $\mathrm{Irr}(G \mid K)$ so, by [K], Lemma 37.5.2, K is a Hall subgroup of G. If $\chi \in \mathrm{Irr}(G \mid K)$, then χ_K is a sum of d distinct irreducible characters of K. Let A be a subgroup of order d in G (Schur–Zassenhaus). Then every nonidentity element of A induces a fixed-point-free automorphism of K, by the Brauer Permutation Lemma (see [BZ], Lemma 10.4(c)). This proves the main part of [K], Theorem 37.5.3.

COROLLARY 4: *Let $\sigma_d(G) = 2d < |G|$. Assume that L is a subgroup of G that centralizes K and intersects K trivially. Then $|L| \leq 2$.*

Proof: By Remark 1, K is solvable. Since $K > \{1\}$, it has a nonprincipal linear character θ. By assumption, $KL = K \times L$, so $\psi = \theta \times 1_L$ is a linear character of KL and $K \not\leq \ker(\psi)$. Since $\psi^G(1)$ is a multiple of d, by Corollary 3, it follows that d divides $|G : KL|$, and thus $|L| \leq 2$ since $|G : K| = 2d$. ∎

It follows from Corollary 4 that $|\,\mathrm{Sc}(G)K : K| \leq 2$, where $\mathrm{Sc}(G)$ is the socle of G and, since K is solvable (Remark 1), $\mathrm{Sc}(G)$ is abelian.

Remark 1 allows us to propose some nontrivial assertions on the structure of G/K. To this end, we may assume that K is an elementary abelian p-subgroup for some prime p (see Remark 1).

LEMMA 5: *Let $d > 1$ be a proper divisor of the order of the group G, $|G| > \sigma_d(G) = wd$. Write $K = \bigcap_{\chi \in X_d} \ker(\chi)$. Then K has a normal p-complement for any prime divisor p of d and K is solvable. Let, in addition, $|G : K| = wd$.*

- (a) *Suppose that K is an elementary abelian p-subgroup, $(w, pd) = 1$. Then K is a Sylow p-subgroup of G. Let q be a prime divisor of d and $Q \in \mathrm{Syl}_q(G)$. Then QK is a Frobenius group.*
- (b) *Suppose that $w = 2$ and K is an elementary abelian 2-subgroup. If 4 divides G/K (i.e., d is even), then G is nonabelian of order 8. If q is an*

odd prime divisor of d and $Q \in \mathrm{Syl}_q(G)$, then QK is a Frobenius group.

(c) *Let $w = 2$. If $(|K|, d) > 1$, then $(|K|, d) = 2$ and $|G/K_{2'}| = 8$, where $K_{p'}$ denote a normal p-complement of K.*

(d) *Suppose that $(w, |K|d) = 1$. If G/K contains a subgroup H/K of index w, then H is a Frobenius group with kernel K.*

Proof: The assertions preceding (a) have already been established. By assumption, $K > \{1\}$.

(a) Assume that G/K has a subgroup Z/K of order p. Then p divides d, since $(w, pd) = 1$, and $|Z| \geq p^2$, so that $Z' < K$. It follows that Z has a linear character ϕ whose kernel does not contain K. Since d does not divide $\phi^G(1) = w \cdot (d/p)$, there exists an irreducible constituent χ of ϕ^G such that d does not divide $\chi(1)$, a contradiction, since $K \not\leq \ker(\chi)$. Thus Z does not exist, so K is a p-Sylow subgroup of G.

Let q be a prime divisor of d, $Q \in \mathrm{Syl}_q(G)$. To prove that QK is a Frobenius group, it is enough to show that CK is a Frobenius group for any subgroup C of order q in Q. Assume that this is false. Then C centralizes some nonidentity element of K. By Fitting's Lemma (see, for example, [BZ], Lemma 1.18), $(CK)' < K$. So CK has a linear character ϕ such that $o(\phi) = pq$ (in particular, K is not contained in $\ker(\phi)$). Then $\phi^G(1) = |G : CK| = w \cdot (d/q)$ is not divisible by d, and again, as in the previous paragraph, we obtain a contradiction. Thus QK is a Frobenius group for $Q \in \mathrm{Syl}_q(G)$.

(b) Suppose that $w = 2$. Let $p = 2$ and let T/K be a subgroup of G/K of order 4. If $T' < K$, T has a linear character ψ whose kernel does not contain K. Then $\phi^G(1) = d/2$, so all irreducible constituents have degrees not divisible by d. On the other hand, all these constituents are contained in $\mathrm{Irr}(G \mid K)$, by reciprocity, and we get a contradiction. Thus $T' = K$. Then T is of maximal class (i.e., generalized quaternion, dihedral or semi-dihedral), by Taussky's Theorem (see [H], Satz 3.11.9(a)); in particular, T/T' is the four-group. Since K is an elementary abelian normal subgroup of a 2-group T of maximal class and $|T/K| > 2$, it follows that $|K| = 2$. In particular, K is a central subgroup of G. By Corollary 4, G is a 2-group. Assume that $|G| > 8$. Then G contains an abelian subgroup A of order 8. If $K < T \leq AK$ and $|T| = 8$, then, by the above, T is not abelian, a contradiction. Thus $|G| = 8$. The last assertion now follows from (a).

(c) Suppose that $w = 2$ and a prime p divides $(d, |K|)$. If $p = 2$, then 4 divides $|G/K|$ and $|G/K_{2'}| = 8$, by (b). If $p > 2$, then $K/K_{p'} \in \mathrm{Syl}_p(G/K_{p'})$, by (a). This means that $(|K|, d) = 2$, as desired.

(d) Let H/K be a subgroup of index w in G/K and $(w, |K|d) = 1$ (we do not assume that H is normal in G). Let a prime p divide d. Then, by Corollary 3, K has the normal p-complement $K_{p'}$. Assuming that $K_{p'} < K$, we see by (a) that $K/K_{p'}$ is a Sylow p-subgroup of $G/K_{p'}$, which is not the case. It follows that $K = K_{p'}$, i.e., p does not divide $|K|$. Since p is an arbitrary prime divisor of d, we get $(|K|, d) = 1$. Assume that H is not a Frobenius group with kernel K (however, H can be a Frobenius group with another kernel). Then there exists $\psi \in \mathrm{Irr}(H \mid K)$ such that d does not divide $\psi(1)$. Since $(w, d) = 1$, it follows that d does not divide $\psi^G(1)$. Therefore, ψ^G has an irreducible constituent χ of degree not divisible by d, a contradiction, since by reciprocity K is not contained in $\ker(\chi)$. ∎

Let G have a normal subgroup H; H is a Frobenius group with kernel K. Then all characters in $\mathrm{Irr}(G \mid K)$ have degrees divisible by $|H : K|$. This follows from [I], Theorem 6.34 and transitivity of inducing. However, in the case considered, $\mathrm{Irr}(G/K)$ is not necessarily equal to X_d.

The case when d is odd may be investigated completely. Namely, the following theorem holds.

THEOREM 6: *Let d be odd. Then $\sigma_d(G) = 2d < |G|$ if and only if G has a Frobenius subgroup H with kernel K such that $|G : H| = 2$.*

Proof: The 'only if' part follows from the remark preceding the theorem and the fact that G/K has no irreducible character of degree divisible by d.

Now suppose that $|G/K| = 2d$. Since d is odd, G/K has a subgroup H/K of index 2, and H is a Frobenius group with kernel K, by Lemma 5(d). ∎

Remark 3: (a) Let us consider, in Theorem 6, the case when $d = 2$ and K is not necessarily elementary abelian. In that case, by Corollary 3, $K = G'$ has index 4 in G. If $P \in \mathrm{Syl}_2(G)$, then $\mathrm{N}_G(P) = P$ since all nonlinear irreducible characters of G have even degrees; see [B2], Proposition 9.

(b) Let G be a group of Lemma 5(b). Suppose that G is not a 2-group. It follows from Lemma 5(b) that all Sylow subgroups of G/K are cyclic and a Sylow 2-subgroup of G/K is of order 2. In that case, d is odd, and so, by Theorem 6, G/K contains a subgroup H/K of index 2 such that H is a Frobenius group with kernel K.

(c) Let G be a nonabelian p-group of order p^n, $d = p^s$, $w \le s$, $\sigma_d(G) = p^{w+s}$, $n > w + s$. Then $p^n \equiv p^{w+s} \pmod{p^{2s}}$ so $w = s$; if N is a normal subgroup of G of index p^{2s}, then $\mathrm{Irr}(G/N) \subseteq X_d$, and so $N = K = K_d$ is the unique normal subgroup of index p^{2s} in G. In particular, if $s = 1$ and $p = 2$, then G is a 2-group of maximal class.

In the case when $d = \chi(1)$ for some $\chi \in \mathrm{Irr}(G)$ and $\sigma_d(G) \le 2d$, $|G| = \sigma_d(G) + d^2$, we can obtain the complete classification. Note that Theorem 7 coincides with Exercise 10.18′ in [BZ].

Let $G = A \cdot B$ be a Frobenius group with kernel B and complement A. Then G is 2-transitive on cosets of A if and only if $|A| = |B| - 1$. In that case, $B^{\#}$ is a conjugacy class of G. Conversely, if $B^{\#}$ is a conjugacy class of G, then G is 2-transitive on cosets of A. Note that there exist nonsolvable 2-transitive Frobenius groups.

Suppose that all nonprincipal irreducible characters of $G > \{1\}$ have the same degree. Since $|G| - 1 = \sum_{\chi \in \mathrm{Irr}(G), \chi \ne 1_G} \chi(1)^2$ and $\chi(1)$ divides $|G|$ for all $\chi \in \mathrm{Irr}(G)$, it follows that G is abelian.

THEOREM 7: *Let $\chi \in \mathrm{Irr}(G)$ have degree $d > 1$ and let $|G| = d^2 + wd$, where $w \le 2$. If $w = 1$, then G is a 2-transitive Frobenius group and, if $w = 2$, then $|G| = 8$.*

Proof: We retain the above notation. Since $|G| \le d^2 + 2d \le 2d^2$, χ is the unique irreducible character of G of degree divisible by d and χ is faithful. It follows by reciprocity that all nonprincipal irreducible characters of K are constituents of χ_K. By Clifford's Theorem, all nonprincipal irreducible characters of K are G-conjugate. It follows from the remark preceding the theorem that K is an abelian minimal normal subgroup of G; write $|K| = p^e$.

If $w = 1$, then $|G : K| = d = \chi(1)$, by Lemma 2, and G is a Frobenius group with kernel K, by Remark 2. It is now immediate that G is a 2-transitive Frobenius group.

Now assume that $w = 2$; then $|G : K| = 2d$, by Lemma 2. We have $d(d+2) = |G| = 2d|K| = 2dp^e$, or $d = 2(p^e - 1)$. Thus d is even. If $p = 2$, then $|G| = 8$, by Lemma 5(b). Let $p > 2$. Then $K \in \mathrm{Syl}_p(G)$, by Lemma 5(a). Let μ be a nonprincipal linear character of K. Then $\mu^G = 2\chi$. Let T be the inertia subgroup of μ in G. Then $|T : K| = 2$, contrary to Clifford theory. ∎

QUESTION: Let G have a nonlinear irreducible character of degree d. Describe the structure of G if $\sigma_d(G) = 3d$.

Let H^G be the normal closure of a subgroup H in G.

PROPOSITION 8: *Let $H > \{1\}$ be a subgroup of index wd in G, where $d > 1$ is a proper divisor of $|G|$, $w \le d + 1$, $w \in \mathbb{N}$. Suppose that, for every nonprincipal*

$\lambda \in \mathrm{Irr}(H)$, the degrees of all irreducible constituents of λ^G are divisible by d. Then H^G is solvable, and either G is a Frobenius group with kernel of index $d = \sigma_d(G)$, H is normal in G and G/H is a 2-transitive Frobenius group, or $|G : H^G| = \sigma_d(G)$ and $X_d = \mathrm{Irr}(G/H^G)$. If, in addition, $w = p$, a prime, then either H is normal in G or G is a Frobenius group with kernel H^G of index d.

Proof: By assumption and reciprocity, all characters in X_d are irreducible constituents of 1_H^G (here 1_H is the principal character of H). Let $(1_H)^G = e_1\chi^1 + \cdots + e_s\chi^s$, where $\chi^1, \dots, \chi^s$ are distinct irreducible characters of G. Let $X_d = \{\chi^1, \dots, \chi^t\}$, where $t \leq s$. By assumption and reciprocity, $\chi^i_H = \chi^i(1) \cdot 1_H$ for $i = 1, \dots, t$. Set $\tau = \chi^1 + \cdots + \chi^t$; then $\ker(\tau) = K \geq H$ and $|G : K| = w_0 d$, where w_0 divides w, by Lagrange's Theorem. It is clear that $X_d \subseteq \mathrm{Irr}(G/K)$, and so $|G : K| \geq \sigma_d(G)$.

Suppose that $w = 1$. Then $\sigma_d(G) = d$, so G is a Frobenius group with kernel H (see [K], Corollary 37.5.4).

Suppose that $\sigma_d(G) = d$. Then G is a Frobenius group with kernel L of index d, so $(d, |L|) = 1$. It follows that $H \leq K \leq L$ (see [I], Theorem 6.34). Suppose that $H^G < L$. Then $wd \geq |G : H^G| \geq (d+1)d$ since G/H^G is a Frobenius group with kernel L/H^G. Since $d + 1 \geq w$ we get $wd = |G : H^G| = (d+1)d$ and $H^G = H$, i.e., H is normal in G and G/H is a doubly transitive Frobenius group.

Suppose that $\sigma_d(G) > d$. In that case, $w > 1$. Assume that $\chi \in \mathrm{Irr}(G/K)$ is of degree divisible by d. Then $|G/K| \geq \sigma_d(G) + \chi(1|)^2 \geq 2d + d^2 > wd = |G : H|$, which is not the case. It follows that $X_d = \mathrm{Irr}(G/K)$. Since $H^G \leq K$ it follows that $|G : K| = \sigma_d(G) \geq 2d$. Assuming that $\mathrm{Irr}(G/H^G)$ has a character χ of degree divisible by d, we get, as above, $|G : H^G| \geq \chi(1)^2 + \sigma_d(G) \geq d^2 + 2d > wd = |G : H|$, a contradiction. Thus $H^G = K$ and $|G : H^G| = \sigma_d(G)$, as desired. The solvability of H^G follows from Remark 1.

Let, in addition, $w = p$, a prime. Suppose that H is not normal in G. Then by the result of the previous paragraph, $\sigma_d(G) = |G : H^G|$ is a proper divisor of $|G : H| = pd$. Since d divides $\sigma_d(G)$, it follows that $\sigma_d(G) = d$, and G is a Frobenius group with kernel H^G, by [K], Theorem 37.5.5. ∎

Remark 4: Let H be a nontrivial subgroup of G. Suppose that, for every nonlinear $\mu \in \mathrm{Irr}(H)$, the degrees of all irreducible constituents of μ^G are divisible by $d > 1$. We claim that H is solvable. Let $\chi \in X_d$. By assumption and reciprocity, all irreducible constituents of χ_H are linear. Hence, $H' \leq K = \bigcap_{\chi \in X_d} \ker(\chi)$. Therefore, as in Remark 1, $H' \leq G(r')$, where r is a prime divisor of d, and so H' is solvable. In that case, H is also solvable, as claimed.

Let G contain an elementary abelian subgroup A of order 2^4 such that $G/A \cong \mathrm{PSL}(2,5)$ and the class number of G is 9. Let H be a subgroup of index 2 in A and $d = 15$; then $|G:H| = 8d$. If μ is a nonprincipal irreducible character of H, then $\mu^G = 2(\chi_1 + \cdots + \chi_4)$, where $\mathrm{Irr}(\mu^G) = \{\chi_1, \dots, \chi_4\}$ and $\chi_i(1) = 15 = d$ for $i = 1, \dots, 4$. In the case considered, $H^G = A$, $|G : H^G| = |G : A| = 60 = 4d = \sigma_{15}(G)$ and $\mathrm{Irr}(G/H^G) = X_{15}$, since the degrees of irreducible characters of G are $1, 3, 3, 4, 5, 15, 15, 15, 15$.

See also in [BZ], §10.4 for related results.

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