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A GENERALIZATION OF THE BUSEMANN–PETTY PROBLEM ON SECTIONS OF CONVEX BODIES

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ABSTRACT

The 1956 Busemann–Petty problem asks whether symmetric convex bodies in $\mathbb{R}^n$ with larger central hyperplane sections must also have greater volume. The solution to the problem has recently been completed, and the answer is negative if $n \geq 5$ and affirmative when $n \leq 4$. We show a more general result, where the inequalities for the volume of central sections are replaced by similar inequalities for the derivatives of the parallel section functions at zero. The dimension of affirmative answer goes up together with the order of the derivatives. The proof is based on a version of Parseval's formula.

1. Introduction

The Busemann–Petty problem (see [BP]) asks the following question. Suppose that K and L are origin-symmetric convex bodies in $\mathbb{R}^n$ such that

$$\mathrm{vol}_{n-1}(K \cap \xi^\perp) \leq \mathrm{vol}_{n-1}(L \cap \xi^\perp)$$

for every ξ from the unit sphere S^{n-1} in $\mathbb{R}^n$, where $\xi^\perp = \{x \in \mathbb{R}^n : (x, \xi) = 0\}$ is the central hyperplane perpendicular to ξ. Does it follow that

$$\mathrm{vol}_n(K) \leq \mathrm{vol}_n(L)?$$

The answer to the problem is negative if $n \geq 5$, which was established in a series of papers by Larman and Rogers [LR] (for $n \geq 12$), Ball [Ba] ($n \geq 10$),

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Giannopoulos [Gi] and Bourgain [Bo] ($n \geq 7$), Gardner [Ga1] and Papadimitrakis [Pa] ($n \geq 5$). Gardner [Ga2] proved that the answer is affirmative for $n = 3$. The case $n = 4$ has a troubled history (see [Ko7] for details), but finally Zhang [Z3] proved that the answer in the four dimensional case is affirmative, and, a little later, a unified solution to the Busemann–Petty problem was given in [GKS].

The proof in [GKS] is based on three major ingredients. First, the crucial role belongs to the concept of an intersection body and its connection with the Busemann–Petty problem found by Lutwak [Lu]. This connection tells that the answer to the Busemann–Petty problem in $\mathbb{R}^n$ is affirmative if and only if every origin symmetric convex body in $\mathbb{R}^n$ is an intersection body.

Secondly, it was proved in [Ko1, Th. 1] that an origin symmetric star body K in $\mathbb{R}^n$ is an intersection body if and only if $\|\cdot\|_K^{-1}$ is a positive definite distribution, where $\|\cdot\|_K$ is the Minkowski functional of K. This result allows one to check whether a given body is an intersection body by means of direct computations. This was then applied to certain classes of bodies in [Ko1, Ko2, Ko3].

Finally, the following result of [GKS, Th. 1] established a connection between the Fourier transform of powers of the Minkowski functional and the derivatives of the parallel section functions.

THEOREM A: *Let K be an origin-symmetric star body in $\mathbb{R}^n$ with C^∞ boundary, and let $k \in \mathbb{N} \cup \{0\}$, $k \neq n-1$. Suppose that $\xi \in S^{n-1}$, and let $A_{K,\xi}$ be the corresponding parallel section function of K (see definition in Section 2).*

(a) *If k is even, then*

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{k/2}\pi(n-k-1)A_{K,\xi}^{(k)}(0);$$

(b) *if k is odd, then*

$$(\|x\|_K^{-n+k+1})^\wedge(\xi) = (-1)^{(k+1)/2}\,2(n-1-k)k!$$

$$\int_0^\infty \frac{A_{K,\xi}(z) - A_{K,\xi}(0) - A''_{K,\xi}(0)\frac{z^2}{2!} - \cdots - A_{K,\xi}^{(k-1)}(0)\frac{z^{k-1}}{(k-1)!}}{z^{k+1}}\,dz,$$

where $A_{K,\xi}^{(k)}(0)$ is the derivative of the order k of the parallel section function at zero, and $(\|x\|_K^{-n+k+1})^\wedge$ is the Fourier transform in the sense of distributions.

The proof of Theorem A in [GKS] is simple and based on the technique of fractional derivatives. Another short proof was recently given in [BMF].

The solution to the Busemann–Petty problem shows that starting from dimension 5 it is not enough to know that $A_{K,\xi}(0) \leq A_{L,\xi}(0)$ for every $\xi \in S^{n-1}$ to conclude that the volume of K is smaller that the volume of L. The reason is that convexity controls only the second derivatives of parallel section functions (the Brunn-Minkowski theorem tells that these second derivatives are negative at zero; put $n = 4,\ k = 2$ in Theorem A to get the affirmative answer to the Busemann–Petty problem in the dimension 4; a counterexample in the case $n = 5,\ k = 3$ requires a little extra work).

This article was motivated by a question of what one has to know about the behavior of the parallel section functions of the bodies K and L at zero to make a conclusion about the relation between the volumes of K and L in every dimension. In Section 3 we offer the following answer to this question.

THEOREM 3: *Let K and L be $(k-1)$-smooth origin symmetric convex bodies in $\mathbb{R}^n$ such that, for every $\xi \in S^{n-1}$,*

$$(1) \qquad (-1)^{(k-1)/2} A_{K,\xi}^{(k-1)}(0) \le (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0),$$

where k is an odd integer and $1 \le k \le n-1$. Then

(i) *if $k \ge n-3$ then $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$;*
(ii) *if $k < n-3$ then it is still possible that $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.*

The condition (1) can be formulated for every $k \in (1, n)$ in terms of fractional derivatives, but we avoid this language in order to simplify our statements. Also, if k is an even integer then $A_{K,\xi}^{(k-1)}(0) = A_{L,\xi}^{(k-1)}(0) = 0$ for every $\xi \in S^{n-1}$, so the condition (1) does not make sense for even k. In this case one can replace the condition (1) by an inequality involving integrals from the part (b) of Theorem A.

If we put $n = 4$ and $k = 1$ in Theorem 3, we get an affirmative answer to the four dimensional Busemann–Petty problem. The case $n = 5,\ k = 1$ of Theorem 3 confirms the negative answer to the Busemann–Petty problem in the dimension 5. Therefore, Theorem 3 represents a generalization of the solution to the Busemann–Petty problem. We actually prove that if, in addition to the condition (1), we know that $\|x\|_K^{-k}$ is a positive definite distribution, then $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$. The property that $\|x\|_K^{-k}$ is positive definite was studied for different norms in [Ko4]. In particular, $\|x\|_K^{-k}$ is positive definite for every $k \in (0, n)$ if K is the unit ball of any n-dimensional subspace of L_p with $0 < p \le 2$. In view of this fact, we also obtain a generalization of a result from [Ko1, Th. 2] that the answer to the Busemann–Petty problem is affirmative if K is the unit ball of any finite dimensional subspace of L_p, $0 < p \le 2$, and L is any symmetric star body.

We show that Theorem 3 is a simple consequence of a version of Parseval's formula (see Lemma 3 below), the proof of which is purely Fourier analytic. Theorems 1 and 2 generalize Lutwak's results [Lu]. In Section 4 we further clarify the geometric sense of what happens in previous sections by introducing a concept of a k-intersection body.

2. A version of Parseval's formula on the sphere

Let K be a body that is starshaped with respect to the origin in $\mathbb{R}^n$. The **Minkowski functional** of K is given by

$$\|x\|_K = \min\{a > 0 : \ x \in aK\}, \quad x \in \mathbb{R}^n.$$

We call K a **star body** if $\|\cdot\|_K$ is continuous on S^{n-1} and K contains the origin in its interior.

For every $\xi \in S^{n-1}$, the **parallel section function** $A_{K,\xi}$ is defined by

$$z \to A_{K,\xi}(z) = \operatorname*{vol}_{n-1}(K \cap \{\xi^\perp + z\xi\}), \quad z \in \mathbb{R}.$$

For $k \in \mathbb{N} \cup \{0\}$, we say that a body K is **k-smooth** if the restriction of the Minkowski functional to the sphere S^{n-1} belongs to the space $C^{(k)}(S^{n-1})$ of continuously differentiable up to to the order k functions. If a similar condition holds with $k = \infty$ we say that K is **infinitely smooth** (or has C^∞-boundary.)

As usual, we denote by $\mathcal{S}$ the space of rapidly decreasing infinitely differentiable functions on $\mathbb{R}^n$ with values in $\mathbb{C}$. We use notation and results from [GS]. By $\mathcal{S}'$ we denote the space of distributions over $\mathcal{S}$. The **Fourier transform** of a distribution f is defined by $\langle \hat{f}, \hat{\varphi}\rangle = (2\pi)^n \langle f, \varphi\rangle$ for every test function φ. If a test function φ is even, we have

$$(\hat{\varphi})^\wedge = (2\pi)^n \varphi \qquad \text{and} \qquad \langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$$

for every $f \in \mathcal{S}'$.

A distribution f is called **even homogeneous** of degree $p \in \mathbb{R}$ if $\langle f(x), \phi(x/t)\rangle = |t|^{n+p}\langle f, \phi\rangle$ for every test function ϕ and every $t \in \mathbb{R},\ t \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n-p$.

A distribution f is called **positive definite** if, for every test function φ,

$$\left\langle f, \varphi * \overline{\varphi(-x)} \right\rangle \geq 0.$$

By L. Schwartz's generalization of Bochner's theorem, a distribution is positive definite if and only if its Fourier transform is a positive distribution (in the sense that $\langle f, \varphi\rangle \geq 0$ for every non-negative test function φ; see, for example, [GV, p. 152]). On the other hand, every positive distribution is a tempered measure, i.e. a Borel non-negative, locally finite measure γ on $\mathbb{R}^n$ such that, for some $\beta > 0$,

$$\int_{\mathbb{R}^n} (1 + \|x\|_2)^{-\beta}\, d\gamma(x) < \infty,$$

where $\|\cdot\|_2$ stands for the Euclidean norm (see [GV, p. 147].)

We need a few remarks on Theorem A.

Remark 1: (i) It follows from Theorem A (for integers p) and Theorem 2 from [GKS] (for non-integers p), that if an origin symmetric star body K is infinitely smooth then $(\|x\|_K^{-p})^\wedge$ is a continuous function on the sphere S^{n-1} for every $p \in (0, n)$.

(ii) The condition of Theorem A, that K has C^∞-boundary, can be weakened. It is enough to assume that the body K is k-smooth to prove the statement (a). In fact, if K is k-smooth, it is easy to see that the function $A_{K,\xi}$ is k times differentiable in a neighborhood of zero. Also, the integral (7) in [GKS] converges to $A_{K,\xi}^{(k-1)}(0)$ when $q \to k-1$ (replace n by k in that formula).

(iii) Also, if K is k-smooth, then $A_{K,\xi}^{(k)}(0)$ is a continuous function of ξ on the sphere. If k is an even integer, then, by the statement (a) of Theorem A, $(\|x\|_K^{-n+k+1})^\wedge$ is a continuous function on the sphere.

(iv) Another consequence of Theorem A is that if k is even, K and K_m, $m \in \mathbb{N}$ are k-smooth star bodies such that the distance between the functions $\|\cdot\|_K$ and $\|\cdot\|_{K_m}$ in the space $C^{(k)}(S^{n-1})$ approaches zero as $m \to \infty$, then the distance between the functions $(\|x\|_K^{-n+k+1})^\wedge$ and $(\|x\|_{K_m}^{-n+k+1})^\wedge$ in the space $C(S^{n-1})$ also has limit zero. We can choose the bodies K_m to be infinitely smooth.

Let K be an origin-symmetric star body in $\mathbb{R}^n$. Define a function $\mu_K(\xi) = (\exp(-\|x\|_K^4))^\wedge(\xi)$.

LEMMA 1: *Suppose that an origin-symmetric star body K has C^∞-boundary. Then the function μ_K is continuous and integrable on $\mathbb{R}^n$.*

Proof: Let n be an even integer. Consider a function

$$F(x) = \Delta^{(n+2)/2}\exp(-\|x\|_K^4) = h(x)\exp(-\|x\|_K^4),$$

where Δ is the Laplace operator and h is a locally integrable function on $\mathbb{R}^n$ (it is a combination of homogeneous functions of degrees higher than $-n+2$) which is continuous on the sphere S^{n-1}. Hence, F is integrable on $\mathbb{R}^n$ (see Remark 2), and the Fourier transform of F is a bounded function on $\mathbb{R}^n$. Using the connection between the Fourier transform and differentiation we see that

$$\hat{F}(x) = (-1)^{(n+2)/2}\|x\|_2^{n+2}\mu_K(x).$$

Therefore, there exists a constant C so that $|\mu_K(x)| \leq C\|x\|_2^{-n-2}$ for every $x \in \mathbb{R}^n$. Since $\exp(-\|x\|_K^4)$ is integrable on $\mathbb{R}^n$, μ_K is also a continuous function and is locally integrable, so the result follows. A similar argument (with $\Delta^{(n+1)/2}$) works in the case where n is an odd integer. ∎

Remark 2: We repeatedly use the fact that if $0 < p < n$ and f is a continuous on S^{n-1} homogeneous of degree $-p$ function on $\mathbb{R}^n$, then f is locally integrable on $\mathbb{R}^n$. Moreover, if μ is a bounded integrable function on $\mathbb{R}^n$ then the integral $\int_{\mathbb{R}^n} f(x)\mu(x)\,dx$ converges absolutely. In particular, if μ_K is the function from Lemma 1 then, for every $p \in (0,n)$, we have

$$\int_{\mathbb{R}^n} \|x\|_2^{-n+p}|\mu_K(x)|\,dx = \int_{S^{n-1}}\int_0^\infty r^{p-1}|\mu(r\theta)|\,dr\,d\theta < \infty,$$

so the function $\theta \mapsto \int_0^\infty r^{p-1}|\mu(r\theta)|\,dr$ belongs to the space $L_1(S^{n-1})$.

LEMMA 2: *Let $0 < p < n$ and K be an infinitely smooth origin symmetric star body in $\mathbb{R}^n$. Then for every $\theta \in S^{n-1}$*

$$\frac{\Gamma((n-p)/4)}{4}(\|x\|_K^{-n+p})^\wedge(\theta) = \int_0^\infty t^{p-1}\mu_K(t\theta)\,dt.$$

Proof: For any test function $\phi \in \mathcal{S}(\mathbb{R}^n)$ consider the integral

$$(2) \qquad \int_{\mathbb{R}^n} \mu_K(x)\Big(\int_0^\infty t^{n-p-1}\phi(tx)\,dt\Big)\,dx.$$

This integral converges absolutely by Lemma 1 and Remark 2, since the inner integral is a continuous on S^{n-1} and homogeneous of degree $-n+p$ function of the variable x. First, let us write the integral (2) in spherical coordinates (r,θ) (we make a substitution $z = rt$ in the inner integral). We get

$$(2) = \int_{S^{n-1}}\Big(\int_0^\infty r^{p-1}\mu_K(r\theta)\,dr\Big)\Big(\int_0^\infty z^{n-p-1}\phi(z\theta)\,dz\Big)\,d\theta.$$

Now let us write the same integral in a different way. Note that, by Remark 1(i), $(\|x\|_K^{-n+p})^\wedge$ is a continuous function on S^{n-1}, homogeneous of degree $-p$, and that $(\exp(-t^4\|x\|_K^4))^\wedge(\xi) = t^{-n}\mu_K(\xi/t)$. We have

$$\begin{aligned}
(2) &= \int_0^\infty t^{n-p-1}\Big(\int_{\mathbb{R}^n} \phi(tx)\mu_K(x)\,dx\Big)\,dt \\
&= \int_0^\infty t^{n-p-1}\Big(\int_{\mathbb{R}^n} \phi(\xi)\mu_K(\xi/t)t^{-n}\,d\xi\Big)\,dt \\
&= \int_0^\infty t^{n-p-1}\Big(\int_{\mathbb{R}^n} \hat{\phi}(y)\exp(-t^4\|y\|_K^4)\,dy\Big)\,dt \\
&= \int_{\mathbb{R}^n} \hat{\phi}(y)\Big(\int_0^\infty t^{n-p-1}\exp(-t^4\|y\|_K^4)\,dt\Big)\,dy \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{\mathbb{R}^n} \|y\|_K^{-n+p}\hat{\phi}(y)\,dy = \frac{\Gamma((n-p)/4)}{4}\langle \|y\|_K^{-n+p}, \hat{\phi}\rangle \\
&= \frac{\Gamma((n-p)/4)}{4}\langle (\|y\|_K^{-n+p})^\wedge, \phi\rangle = \frac{\Gamma((n-p)/4)}{4}\int_{\mathbb{R}^n} (\|x\|_K^{-n+p})^\wedge(\xi)\phi(\xi)\,d\xi \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\theta)\int_0^\infty r^{n-p-1}\phi(r\theta)\,dr\,d\theta.
\end{aligned}$$

Now if we put $\phi(x) = u(r)v(\theta)$ in both expressions for the integral (2), where v is any infinitely differentiable function on the sphere and u is a non-negative test function on $\mathbb{R}$ with compact support, we get

$$\int_{S^{n-1}} \Big(\int_0^\infty r^{p-1}\mu_K(r\theta)\,dr\Big)v(\theta)\,d\theta = \frac{\Gamma((n-p)/4)}{4}\int_{S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\theta)v(\theta)\,d\theta$$

for every $v \in C^\infty(S^{n-1})$. ∎

LEMMA 3: *Let K and D be origin symmetric star bodies with C^∞-boundaries in $\mathbb{R}^n$ and $0 < p < n$. Then*

$$\int_{S^{n-1}} (\|x\|_K^{-p})^\wedge(\theta)(\|x\|_D^{-n+p})^\wedge(\theta)\,d\theta = (2\pi)^n\int_{S^{n-1}} \|\theta\|_K^{-p}\|\theta\|_D^{-n+p}\,d\theta.$$

Proof: Passing to spherical coordinates we get

$$\begin{aligned}
\int_{\mathbb{R}^n} \|x\|_K^{-p}\exp(-\|x\|_D^4)\,dx &= \int_{S^{n-1}} \|\theta\|_K^{-p}\Big(\int_0^\infty r^{n-p-1}\exp(-r^4\|\theta\|_D^4)\,dr\Big)\,d\theta \\
&= \frac{\Gamma((n-p)/4)}{4}\int_{S^{n-1}} \|\theta\|_K^{-p}\|\theta\|_D^{-n+p}\,d\theta. \qquad (3)
\end{aligned}$$

Let γ_ϵ be the standard Gaussian density with variance ϵ; then the convolution $\exp(-\|x\|_D^4) * \gamma_\epsilon$ is a test function. We now use the fact that $(\|x\|_K^{-p})^\wedge$ is a continuous on S^{n-1} homogeneous of degree $-n+p$ function (Remark 1(i)) and the dominated convergence theorem to get a different expression for the integral in the left-hand side of (3):

$$\begin{aligned}\int_{\mathbb{R}^n} \|x\|_K^{-p}\exp(-\|x\|_D^4)\, dx &= \lim_{\epsilon\to 0}\langle \|x\|_K^{-p}, \exp(-\|x\|_D^4) * \gamma_\epsilon\rangle \\ &= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^n}\langle (\|x\|_K^{-p})^\wedge(\xi), \mu_D(\xi)\exp(-\epsilon\|\xi\|_2^2)\rangle \\ &= \lim_{\epsilon\to 0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\|x\|_K^{-p})^\wedge(\xi)\mu_D(\xi)\exp(-\epsilon\|\xi\|_2^2)\, d\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}(\|x\|_K^{-p})^\wedge(\xi)\mu_D(\xi)\, d\xi \\ &= \frac{1}{(2\pi)^n}\int_{S^{n-1}}(\|x\|_K^{-p})^\wedge(\theta)\Big(\int_0^\infty r^{p-1}\mu_D(r\theta)\, dr\Big)\, d\theta.\end{aligned}$$

The result follows from Lemma 2. ∎

Instead of the argument with Gaussains in the proof of Lemma 3, we can apply Lemma 1 from [GK] and note that the convolution of $\|x\|_K^{-p}$ and $\exp(-\|x\|_D^4)$ is a continuous function, since it is the Fourier transform of an integrable on $\mathbb{R}^n$ function $(\|x\|_K^{-p})^\wedge(\xi)\mu_D(\xi)$.

Remark 3: Suppose that K is an origin symmetric star body in $\mathbb{R}^n$ and $1 \le k \le n-1$, $k \in \mathbb{N}$ so that the distribution $\|x\|_K^{-k}$ is positive definite. Then the Fourier transform of $\|x\|_K^{-k}$ is a tempered measure γ on $\mathbb{R}^n$ which is also a homogeneous distribution of degree $-n+k$. Writing this measure in the spherical coordinates (see, for example, [Ko4, Lemma 1]) we can find a measure γ_0 on S^{n-1} so that for every even test function ϕ,

$$\langle (\|x\|_K^{-k})^\wedge, \phi\rangle = \langle \gamma, \phi\rangle = \int_{S^{n-1}} d\gamma_0(\theta)\int_0^\infty r^{k-1}\phi(r\theta)\, dr.$$

COROLLARY 1: *Let $k \in \mathbb{N}$, $1 \le k \le n-1$ and suppose that K and D are origin symmetric star bodies in $\mathbb{R}^n$ such that D is $(k-1)$-smooth and $\|\cdot\|_K^{-k}$ is a positive definite distribution. Suppose that γ_0 is the measure on S^{n-1} defined in Remark 3. Then*

$$\int_{S^{n-1}} (\|x\|_D^{-n+k})^\wedge(\theta)\, d\gamma_0(\theta) = (2\pi)^n\int_{S^{n-1}} \|\theta\|_K^{-k}\|\theta\|_D^{-n+k}\, d\theta. \tag{4}$$

Proof: Let D_m be a sequence of infinitely smooth symmetric star bodies approximating D in the sense of Remark 1(iv). Also consider any sequence of infinitely smooth symmetric star bodies $K_j \subset K$, $j \in \mathbb{N}$ approximating K in the Hausdorff metric. For every even test function ϕ the sequence of integrable (by Remark 2) functions $\|x\|_{K_j}^{-k}|\hat{\phi}(x)|$ is majorated by an integrable function $\|x\|_K^{-k}|\hat{\phi}(x)|$. By the dominated convergence theorem and Remark 3,

$$\int_{\mathbb{R}^n} (\|x\|_{K_j}^{-k})^\wedge(y)\phi(y)\,dy = \int_{\mathbb{R}^n} \|x\|_{K_j}^{-k}\hat{\phi}(x)\,dx$$

$$\to \int_{\mathbb{R}^n} \|x\|_K^{-k}\hat{\phi}(x)\,dx = \int_{S^{n-1}} \gamma_0(\theta)\int_0^\infty r^{k-1}\phi(r\theta)\,dr\,d\theta, \tag{5}$$

as $j \to \infty$. Since the bodies D_m are infinitely smooth,

$$\phi_m(x) = u(r)(\|x\|_{D_m}^{-n+k})^\wedge(\theta)$$

is a test function, where u is any non-negative test function on $\mathbb{R}$ with compact support. Substituting ϕ_m in (5) and passing to spherical coordinates, we get

$$\lim_{j\to\infty}\int_{S^{n-1}} (\|x\|_{K_j}^{-k})^\wedge(\theta)(\|x\|_{D_m}^{-n+k})^\wedge(\theta)\,d\theta = \int_{S^{n-1}} (\|x\|_{D_m}^{-n+k})^\wedge(\theta)\,d\gamma_0(\theta). \tag{6}$$

By Remark 1, the functions $(\|x\|_{D_m}^{-n+k})^\wedge(\theta)$ converge in $C(S^{n-1})$ to $(\|x\|_D^{-n+k})^\wedge(\theta)$, as $m \to \infty$. Hence, the integral in the right-hand side of (6) converges to the integral in the left-hand side of (4). Now we get the result if we apply Lemma 3 to the left-hand side of (6) and let $m \to \infty$. ∎

We need the following fact that was proved in [Ko5] (see also [GKS, Lemma 5]).

LEMMA 4: *For every even test function ϕ, $\xi \in S^{n-1}$, and $-1 < q < 0$ we have*

$$\int_{\mathbb{R}^n} |(\xi, x)|^{-q-1}\phi(x)\,dx = \frac{-1}{2\Gamma(1+q)\sin\frac{q\pi}{2}}\int_{-\infty}^{\infty} |t|^q\hat{\phi}(t\xi)\,dt.$$

Our next lemma shows that the Fourier transform of a homogeneous function of degree $-p$, whose restriction to to the sphere is infinitely differentiable is a homogeneous function of degree $-n+p$, whose restriction to the sphere is also infinitely differentiable. We give a proof based on the same technique as that of Theorem A in [GKS] and leading to a precise expression for the Fourier transform.

LEMMA 5:. *Let $f \in C^\infty(S^{n-1})$ be an even function and $0 < p < n$. Then there exists an even function $g_p \in C^\infty(S^{n-1})$ such that the Fourier transform of the function $f(\theta)r^{-p}$ is equal to $g_p(\theta)r^{-n+p}$. Here the function $f(\theta)r^{-p}$ is defined on $\mathbb{R}^n$ in the spherical coordinates $x = r\theta,\ x \in \mathbb{R}^n,\ r \in [0,\infty),\ \theta \in S^{n-1}$.*

Proof: For every $\xi \in S^{n-1}$ define a function $A_{f,\xi}$ on $\mathbb{R}$ by

$$A_{f,\xi}(t) = \int_{\{\theta\in S^{n-1}:(\theta,\xi)=t\}} f(\theta)\,d\theta.$$

This function is infinitely differentiable in a neighborhood of zero, so, for every $q > -1$, we can define its fractional derivative $A^{(q)}_{f,\xi}(0)$ of the order q at zero in the same way as it was done in [GKS; Section 3]. Let first $q \in (-1,0)$. Then

$$A^{(q)}_{f,\xi}(0) = \frac{1}{2\Gamma(-q)} \int_{\mathbb{R}} |t|^{-1-q} A_{f,\xi}(t)\, dt = \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} |(\theta,\xi)|^{-1-q} f(\theta)\, d\theta.$$

We now consider $A^{(q)}_{f,\xi}(0)$ as a function of $\xi \in \mathbb{R}^n \setminus \{0\}$ so that it is homogeneous of degree $-1-q$. For every even test function ϕ, using Lemma 4 we get

$$\begin{aligned} \langle A^{(q)}_{f,\xi}(0), \phi(\xi)\rangle &= \frac{1}{2\Gamma(-q)} \int_{S^{n-1}} f(\theta) \int_{\mathbb{R}^n} |\langle \theta,\xi\rangle|^{-q-1} \varphi(\xi)\, d\xi\, d\theta \\ &= \frac{-1}{4\Gamma(-q)\Gamma(q+1)\sin\frac{q\pi}{2}} \int_{S^{n-1}} f(\theta) \int_{-\infty}^{\infty} |t|^q \widehat{\varphi}(t\theta)\, dt\, d\theta \\ &= \frac{\cos\frac{q\pi}{2}}{\pi} \left\langle (f(\theta) r^{-n+q+1})^\wedge(\xi), \varphi(\xi)\right\rangle, \end{aligned} \tag{7}$$

where the last equation follows from the property $\Gamma(-q)\Gamma(q+1) = -\pi/\sin(q\pi)$ of the Γ-function and the simple calculation

$$\langle (f(\theta) r^{-n+q+1})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} f(\theta) r^{-n+q+1} \hat{\phi}(x)\, dx$$

$$= \int_{S^{n-1}} f(\theta)\, d\theta \int_0^\infty r^q \hat{\phi}(r\theta)\, dr = \frac{1}{2} \int_{S^{n-1}} f(\theta)\, d\theta \int_{\mathbb{R}} |r|^q \hat{\phi}(r\theta)\, dr.$$

It follows from (7) that, when $p \in (n-1, n)$, the function

$$g_p(\theta) = (\pi/\cos(\pi(n-p-1)/2)) A^{(n-p-1)}_{f,\theta}(0) \tag{8}$$

provides the desired result. A standard analytic extension argument (see, for example, [GKS; Section 3]) shows that the same function g_p works for every $p \in (0,n)$ such that $n-p-1$ is not an odd integer. In the case where $n-p-1 = m$ is an odd integer, we compute the limit, as $p \to n-m-1$, in the right-hand side of (8) in the same way as it was done in [GKS, Th. 1]). Then use again the analyticity of the Fourier transform of $f(\theta)r^{-p}$ in the domain $0 < \mathrm{Re}(p) < n$ to see that the following function satisfies the condition of the lemma:

$$g_{n-m-1}(\theta) = c_m \int_0^\infty \frac{A_{f,\theta}(z) - A_{f,\theta}(0) - A''_{f,\theta}(0)\frac{z^2}{2} - \cdots - A^{(m-1)}_{f,\theta}(0)\frac{z^{m-1}}{(m-1)!}}{z^{m+1}}\, dz,$$

where $c_m = (-1)^{(m+1)/2}\, 2m!$. ∎

3. A generalization of the Busemann–Petty problem

The following two theorems generalize Lutwak's (see [Lu]) connections between intersection bodies and the Busemann–Petty problem mentioned in the Introduction. In fact, if we put $k = 1$ in Theorem 1, then, in view of [Ko1, Th. 1], the condition that $\|x\|_K^{-1}$ is positive definite means that K is an intersection body, and the condition that $(\|x\|_L^{-n+1} - \|x\|_K^{-n+1})^\wedge$ is positive means that central sections of K are smaller than those of L (this was shown in [Ko6, Th. 1]). Therefore, the result of Theorem 1 in the case $k = 1$ means that the answer to the Busemann–Petty problem is affirmative if the body with smaller sections is an intersection body, which is the result of Lutwak [Lu]. Theorem 2 with $k = 1$ generalizes the other direction of Lutwak's equivalence.

THEOREM 1: *Let k be an odd integer, $1 \le k \le n-1$, and let K and L be origin symmetric $(k-1)$-smooth star bodies in $\mathbb{R}^n$. Suppose that the distributions $\|x\|_K^{-k}$ and $\|x\|_L^{-n+k} - \|x\|_K^{-n+k}$ are positive definite. Then $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$.*

Proof: Let γ_0 be the measure defined in Remark 3. Since, by Remark 1(iii), $(\|x\|_K^{-n+k})^\wedge$ and $(\|x\|_L^{-n+k})^\wedge$ are continuous functions on S^{n-1}, and $(\|x\|_L^{-n+k} - \|x\|_K^{-n+k})^\wedge \ge 0$, we have

$$\int_{S^{n-1}} (\|x\|_L^{-n+k})^\wedge(\theta)\, d\gamma_0(\theta) \ge \int_{S^{n-1}} (\|x\|_K^{-n+p})^\wedge(\theta)\, d\gamma_0(\theta).$$

By Corollary 1,

$$\int_{S^{n-1}} \|\theta\|_K^{-k}\|\theta\|_L^{-n+k}\, d\theta \ge \int_{S^{n-1}} \|\theta\|_K^{-n}\, d\theta.$$

By Holder's inequality and since $(1/n)\int_{S^{n-1}} \|\theta\|_K^{-n}\, d\theta = \mathrm{vol}_n(K)$, we have

$$(\mathrm{vol}_n(K))^{k/n}(\mathrm{vol}_n(L))^{(n-k)/n} \ge \mathrm{vol}_n(K). \qquad \blacksquare$$

It was proved in [Ko4, Th. 1] that if K is the unit ball of an n-dimensional subspace of L_p with $0 < p \le 2$, then $\|x\|_K^{-k}$ is a positive definite distribution for every $k \in (0, n)$. Therefore, by Theorems 1 and A,

COROLLARY 2: *If k is an odd integer, $1 \le k \le n-1$, K, L are $(k-1)$-smooth origin symmetric star bodies in $\mathbb{R}^n$, K is the unit ball of a subspace of L_p with $0 < p \le 2$, and for every $\xi \in S^{n-1}$,*

$$(-1)^{(k-1)/2} A_{K,\xi}^{(k-1)}(0) \le (-1)^{(k-1)/2} A_{L,\xi}^{(k-1)}(0),$$

then $\mathrm{vol}_n(K) \le \mathrm{vol}_n(L)$.

The case $k = 1$ of Corollary 2 confirms an affirmative answer to Meyer's conjecture, which was given in [Kol, Th. 4].

THEOREM 2: *Let $0 < p < n$ and let L be an origin symmetric convex body in $\mathbb{R}^n$ with C^∞-boundary and positive curvature so that the distribution $\|x\|_L^{-p}$ is not positive definite. Then there exists an origin symmetric convex body K in $\mathbb{R}^n$ with C^∞-boundary such that the distribution $\|x\|_L^{-n+p} - \|x\|_K^{-n+p}$ is positive definite but $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$.*

Proof: Since $(\|x\|_L^{-p})^\wedge$ is a continuous sign-changing function on S^{n-1}, there exists an open subset Ω in S^{n-1} on which $(\|x\|_L^{-p})^\wedge$ is negative. Let $f \in C^\infty(S^{n-1})$ be a non-negative (and not identically zero) function supported in Ω. Let g_p be the function corresponding to $f(\theta)r^{-p}$ in Lemma 5.

Define a body K by

$$\|x\|_K^{-n+p} = \|x\|_L^{-n+p} - \frac{\epsilon}{(2\pi)^n} g_p(x),$$

where $\epsilon > 0$ is small enough so that the body K is convex (a standard perturbation argument is that, given an infinitely differentiable function on S^{n-1}, one can choose a small enough ϵ so that the differential properties of the norm $\|\cdot\|_L^{-n+p}$ equivalent to convexity of L are preserved after adding an ϵ-multiple of the $(-n+p)$-homogeneous extension of this function).

By Lemma 5 we have

$$(\|x\|_L^{-n+p})^\wedge - (\|x\|_K^{-n+p})^\wedge = \epsilon f(\theta) r^{-p}, \tag{9}$$

so the distribution $\|x\|_L^{-n+p} - \|x\|_K^{-n+p}$ is positive definite.

On the other hand, by (9) and Lemma 3,

$$\int_{S^{n-1}} (\|x\|_L^{-p})^\wedge(\theta) f(\theta)\, d\theta = \frac{(2\pi)^n}{\epsilon n}\Big(\mathrm{vol}_n(L) - \int_{S^{n-1}} \|\theta\|_L^{-p} \|\theta\|_K^{-n+p}\Big).$$

Since the quantity in the left-hand side of the latter formula is negative, we use Holder's inequality (as in Theorem 1) to see that $\mathrm{vol}_n(K) > \mathrm{vol}_n(L)$. ∎

We need the following fact that is an immediate consequence of Theorem A.

COROLLARY 3: *For every origin symmetric convex body K in $\mathbb{R}^n$, the distribution $\|x\|_K^{-p}$ is positive definite for each $p = n-3, n-2, n-1$.*

Proof: If K has C^∞-boundary the result follows from Theorem A (with $k = n-p-1$) and Brunn's Theorem, which states that the parallel section function of a convex body in each direction has maximum at zero (so $A''_{K,\xi}(0) < 0$). Now let K be an arbitrary convex body and approximate K in the Hausdorff metric by origin symmetric convex bodies K_m with C^∞-boundaries so that, for every $x \in S^{n-1}$, the sequence $\|x\|_{K_m}$, $m \in \mathbb{N}$ is decreasing and converges to $\|x\|_K$. Then, by the dominated convergence theorem (like it was done in Corollary 1), for every test function ϕ,

$$\langle (\|x\|_{K_m}^{-k})^\wedge, \phi\rangle = \int_{\mathbb{R}^n} \|x\|_{K_m}^{-k} \hat{\phi}(x)\, dx \to \int_{\mathbb{R}^n} \|x\|_K^{-k}\hat{\phi}(x)\, dx = \langle (\|x\|_K^{-k})^\wedge, \phi\rangle,$$

as $m \to \infty$, and the result follows from the C^∞ case. ∎

We are ready to prove Theorem 3 formulated in the Introduction. Note that it was proved in [Ko2, Th. 2] that if $\|\cdot\|_q$ is the norm of the space ℓ_q^n, $2 < q \le \infty$, and $0 < p < n$, then the distribution $\|\cdot\|_q^{-p}$ is positive definite if and only if $p \in [n-3, n)$. We use this fact in the proof of Theorem 3.

Proof of Theorem 3: Part (i) immediately follows from Theorems 1, A and Corollary 3. To show (ii), let L be the unit ball of the space with the norm $\|x\|_L = \|x\|_4 + \epsilon\|x\|_2$, where $\epsilon > 0$. By the result mentioned before the proof, the distribution $\|x\|_4^{-k}$ is not positive definite, therefore $\|x\|_L^{-k}$ is not positive definite for small enough ϵ (one can use an argument similar to that in the end of the proof of Corollary 3). Using this value of ϵ in the definition of L (the perturbation of the ℓ_4^n-norm was made to ensure that L has positive curvature) and putting $p = k$ in Theorem 2 we get a body K giving the desired example (again use Theorem A to connect the Fourier transform with the derivatives of parallel section functions). ∎

4. k-intersection bodies

In this section we generalize Lutwak's definition of an intersection body of a star body and prove a connection with positive definite distributions similar to that from [Ko1, Th. 1]. Note that different generalizations of the concept of an intersection body were introduced by Zhang in [Z1, Z2].

Definition 1: Let $n, k \in \mathbb{N}$, $1 \le k < n$, and let K, L be origin symmetric star bodies in $\mathbb{R}^n$. We say that K is a k-intersection body of L if, for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$, we have

$$\mathrm{vol}_k(K \cap H^\perp) = \mathrm{vol}_{n-k}(L \cap H),$$

where $H^\perp$ is the subspace orthogonal to H.

The latter equality can be written in the following form:

$$\int_{S^{n-1}\cap H^\perp} \|\theta\|_K^{-k}\, d\theta = \frac{k}{n-k}\int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k}\, d\theta.$$

If $k=1$, our definition means that K is an intersection body of L in the sense of Lutwak [Lu] (one has to divide by 2 the left-hand side of the latter equality).

Clearly, if K is a k-intersection body of L, then L is a $(n-k)$-intersection body of K. We say that a star body K is a **k-intersection body of a star body** if there exists a star body L satisfying the conditions of Definition 1.

LEMMA 6: *Let $1 \le k < n$, ϕ be an even integrable function on $\mathbb{R}^n$, and H be an $(n-k)$-dimensional subspace of $\mathbb{R}^n$. Suppose that ϕ is integrable on H and that the Fourier transform $\hat{\phi}$ is integrable on $H^\perp$. Then*

$$(2\pi)^k \int_H \phi(x)\, dx = \int_{H^\perp} \hat{\phi}(x)\, dx.$$

Proof: Let $\xi_1, ..., \xi_k$ be an orthonormal basis in $H^\perp$. For every $t \in \mathbb{R}^k$ we have

$$\hat{\phi}(t_1\xi_1 + \cdots + t_k\xi_k) = \int_{\mathbb{R}^n} \phi(x)\exp(-i(x, t_1\xi_1 + \cdots + t_k\xi_k))\, dx$$

$$= \int_{\mathbb{R}^k} \exp(-i(t_1u_1 + \cdots + t_ku_k))\Big(\int_{H+u_1\xi_1+\cdots+u_k\xi_k} \phi(x)\, dx\Big)\, du.$$

This means that the function $f(t) = \hat{\phi}(t_1\xi_1 + \cdots + t_k\xi_k)$ is the Fourier transform of the function $g(u) = \int_{H+u_1\xi_1+\cdots+u_k\xi_k} \phi(x)\, dx$. The function f is integrable and even on $\mathbb{R}^k$, therefore

$$\int_{\mathbb{R}^k} f(t)\, dt = (\hat{g})^\wedge(0) = (2\pi)^k g(0),$$

which immediately implies the desired formula. ∎

LEMMA 7: *Let L be an origin symmetric star body with C^∞-boundary in $\mathbb{R}^n$. Then for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$ we have*

$$(2\pi)^k \int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k}\, d\theta = \int_{S^{n-1}\cap H^\perp} (\|x\|_L^{-n+k})^\wedge(\theta)\, d\theta.$$

Proof: By Lemma 2,

$$\int_{S^{n-1}\cap H^\perp} (\|x\|_L^{-n+k})^\wedge(\theta)\, d\theta = (4/\Gamma((n-k)/4)) \int_{S^{n-1}\cap H^\perp} d\theta \int_0^\infty t^{k-1}\mu_L(t\theta)\, dt$$
$$= (4/\Gamma((n-k)/4)) \int_{H^\perp} \mu_L(x)\, dx.$$

By Lemma 6 applied to $\phi(x) = \exp(-\|x\|_L^4)$, the latter integral is equal to

$$(4(2\pi)^k/\Gamma((n-k)/4)) \int_H \exp(-\|x\|_L^4)\ dx = (2\pi)^k \int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k}\ d\theta. \qquad \blacksquare$$

THEOREM 4: *Let K be an origin symmetric star body with C^∞-boundary in $\mathbb{R}^n$, and $1 \le k \le n-1$. The body K is a k-intersection body of a star body if and only if the distribution $\|x\|_K^{-k}$ is positive definite.*

Proof: By Lemma 7, for every $(n-k)$-dimensional subspace H of $\mathbb{R}^n$,

$$\int_{S^{n-1}\cap H^\perp} \|\theta\|_K^{-k} = \frac{1}{(2\pi)^{n-k}} \int_{S^{n-1}\cap H} (\|x\|_K^{-k})^\wedge(\theta)\ d\theta.$$

It follows that if K is a k-intersection body of L, then for every $(n-k)$-dimensional subspace H

$$\int_{S^{n-1}\cap H} (\|x\|_K^{-k})^\wedge(\theta)\ d\theta = \frac{(2\pi)^{n-k}k}{n-k} \int_{S^{n-1}\cap H} \|\theta\|_L^{-n+k}\ d\theta,$$

and, by [Ga3, Th. 7.2.3], for every $\theta \in S^{n-1}$,

$$(\|x\|_K^{-k})^\wedge(\theta) = \frac{(2\pi)^{n-k}k}{n-k} \|\theta\|_L^{-n+k}, \tag{10}$$

which means, in particular, that $\|x\|_K^{-k}$ is positive definite.

Conversely, if the distribution $\|x\|_K^{-k}$ is positive definite we can define the body L by (10), since, by Remark 1(i), $(\|x\|_K^{-k})^\wedge$ is a continuous function on S^{n-1}. The result follows from Lemma 7. $\blacksquare$

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