ISRAEL JOURNAL OF MATHEMATICS 80 (1992), 97-133

CONTINUITY OF THE HAUSDORFF DIMENSION FOR PIECEWISE MONOTONIC MAPS

BY

PETER RAITH*

Institut für Mathematik, Universität Wien
Strudlhofgasse 4, A-1090 Wien, Austria

ABSTRACT

In this paper a piecewise monotonic map $T : X \to \mathbb{R}$, where X is a finite union of intervals, is considered. Define $R(T) = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$. The influence of small perturbations of T on the Hausdorff dimension $\mathrm{HD}(R(T))$ of $R(T)$ is investigated. It is shown, that $\mathrm{HD}(R(T))$ is lower semi-continuous, and an upper bound of the jumps up is given. Furthermore a similar result is shown for the topological pressure.

Introduction

Let X be a finite union of closed intervals, and consider a piecewise monotonic map $T : X \to \mathbb{R}$, that means there exists a finite partition $\mathcal{Z}$ of X into pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$, such that $T \mid Z$ is bounded, strictly monotone and continuous for all $Z \in \mathcal{Z}$. Set $R(T) := \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$, which can be considered as the set, where T^n is defined for all $n \in \mathbb{N}$. We have $R(T) = \bigcap_{n\in\mathbb{N}} X_n$, where for $n \in \mathbb{N}$ we define $X_n := \bigcap_{j=0}^{n-1} \overline{T^{-j}X}$, which can be considered as the set, where T^n is defined. We consider the dynamical system $(R(T), T)$, and we are interested in the influence of small perturbations of T on the set $R(T)$.

Such dynamical systems occur in a natural way. If $T : [0,1] \to [0,1]$ is a piecewise monotonic map, and L is a maximal topologically transitive subset of $[0,1]$ with $h_{\mathrm{top}}(L,T) > 0$, then there exists an $X \subseteq [0,1]$, which is a finite union of intervals, such that $L = \bigcap_{n=0}^{\infty} \overline{T^{-n}X}$ (cf. [2]).

* Research was supported by Projekt Nr. P8193-PHY of the Austrian Fond zur Förderung der wissenschaftlichen Forschung
Received June 12, 1991 and in revised form February 16, 1992

The "size" of $R(T)$ can be described in different ways. We consider the topological entropy $h_{\text{top}}(R(T),T)$, the topological pressure $p(R(T),T,f)$ of a piecewise continuous function $f : X \to \mathbb{R}$, and the Hausdorff dimension $\mathrm{HD}(R(T))$. To investigate the influence of perturbations of T on these quantities we have to introduce topologies for piecewise monotonic maps. A function $f : X \to \mathbb{R}$ is called piecewise continuous with respect to $\mathcal{Z}$, if $f \mid Z$ can be extended to a continuous function on $\overline{Z}$ for all $Z \in \mathcal{Z}$. We suppose that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with $Z_1 < Z_2 < \cdots < Z_K$. Let $\tilde{X}$ be a finite union of closed intervals, let $\tilde{\mathcal{Z}}$ be a finite partition of $\tilde{X}$ into disjoint open intervals with $\bigcup_{\tilde{Z}\in\tilde{\mathcal{Z}}} \overline{\tilde{Z}} = \tilde{X}$, and let $\tilde{f} : \tilde{X} \to \mathbb{R}$ be piecewise continuous with respect to $\tilde{\mathcal{Z}}$. Then $\tilde{f}$ is said to be close to f, if $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_K\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_K$, and if for all $j \in \{1, 2, \ldots, K\}$ the graph of $\tilde{f} \mid \tilde{Z}_j$ is contained in a small neighbourhood of the graph of $f \mid Z_j$, considered as a subset of $\mathbb{R}^2$ (observe that this definition depends on the partitions $\mathcal{Z}$ and $\tilde{\mathcal{Z}}$). Denote by R^0 the family of all systems $(X, T, \mathcal{Z})$, where $T : X \to \mathbb{R}$ is piecewise monotone with respect to $\mathcal{Z}$. Two systems $(X, T, \mathcal{Z}), (\tilde{X}, \tilde{T}, \tilde{\mathcal{Z}}) \in R^0$ are said to be close in the R^0-topology, if T and $\tilde{T}$ are close in the sense defined above for piecewise continuous functions. Let W^0 be the family of all systems $(X, T, f, \mathcal{Z})$, where $(X, T, \mathcal{Z}) \in R^0$ and f is piecewise continuous with respect to $\mathcal{Z}$. We say that $(X, T, f, \mathcal{Z}), (\tilde{X}, \tilde{T}, \tilde{f}, \tilde{\mathcal{Z}}) \in W^0$ are close in the W^0-topology, if they are close in the R^0-topology and f and $\tilde{f}$ are close in the sense defined above for piecewise continuous functions. Furthermore let R^1 be all $(X, T, \mathcal{Z}) \in R^0$, such that T' is piecewise continuous with respect to $\mathcal{Z}$. $(X, T, \mathcal{Z}), (\tilde{X}, \tilde{T}, \tilde{\mathcal{Z}}) \in R^1$ are said to be close in the R^1-topology, if $(X, T, T', \mathcal{Z})$ and $(\tilde{X}, \tilde{T}, \tilde{T}', \tilde{\mathcal{Z}})$ are close in the W^0-topology.

Theorem 1 of this paper says, that the pressure function $p : W^0 \to \mathbb{R}$ is lower semi-continuous in $(X, T, f, \mathcal{Z})$ with respect to the W^0-topology, if a certain condition on f, which generalizes $p(R(T), T, f) > \sup_{x\in X} f(x)$, is satisfied. Special cases of this result are already known. In [8] this result is obtained for systems (T, f), where T is a continuous piecewise monotonic map on $[0,1]$, and $f : [0,1] \to \mathbb{R}$ is continuous with $\sup_{x\in[0,1]} f(x) < p(T, f)$. Another special case is the well-known result on the lower semi-continuity of $h_{\text{top}}(R(T), T)$ in the R^0-topology (see [6]). In Theorem 2 we show that the jumps up of $p : W^0 \to \mathbb{R}$ in $(X, T, f, \mathcal{Z})$ with respect to the W^0-topology are bounded by the maximum of $p(R(T), T, f)$ and the logarithm of the spectral radius of the matrix $G(f)$ (see (2.2) for definition) associated to the graph $(\mathcal{G}, \to)$ (defined in Section 2), which is defined in terms of the orbits of the critical points under T and which is similar to the graph considered in [5]. This is a generalization of the results in [4] and [5], where upper bounds for the jumps up of $h_{top}(R(T),T)$ in the R^0-topology are given. A system $(X,T,\mathcal{Z}) \in R^1$ with the property, that there exists an $n \in \mathbb{N}$ with $\inf_{x\in X_n} |(T^n)'(x)| > 1$, is considered in Theorem 3. It is shown, that the function $(X,T,\mathcal{Z}) \mapsto \mathrm{HD}(R(T))$ is lower semi-continuous in the R^1-topology, and an upper bound for the jumps up in a fixed $(X,T,\mathcal{Z})$ in terms of the graph $(\mathcal{G},\to)$ mentioned above, is given (see Section 5).

The proofs use a graph $(\mathcal{D},\to)$, called Markov diagram, associated to $(X,T,\mathcal{Z})$. In Lemma 6 it is shown, that $(X,T,\mathcal{Z})$ and $(\tilde{X},\tilde{T},\tilde{\mathcal{Z}})$ have "similar initial parts" of their Markov diagrams, if they are close in the R^0-topology. In order to use this result to derive Theorem 1 and Theorem 2 we have to approximate a function f, which is piecewise continuous with respect to $\mathcal{Z}$, by piecewise constant functions. Lemma 6 remains true, if we replace the partition $\mathcal{Z}$ by a suitable refinement $\mathcal{Y}$. This implies Theorem 1 and Theorem 2 by Lemma 6 of [7], which says that for piecewise constant functions f the pressure $p(R(T),T,f)$ can be obtained as the logarithm of the spectral radius of a certain matrix $F(f)$ (see (2.6) for definition) associated to the Markov diagram. Then Theorem 3 follows from Theorem 2 in [7], which says that $\mathrm{HD}(R(T))$ equals the unique real number t_R with $p(R(T),T,-t_R \log|T'|) = 0$.

1. Definitions and notations

Suppose that X is a finite union of closed intervals. We say that $\mathcal{Z}$ is a **finite partition** of X, if $\mathcal{Z}$ consists of pairwise disjoint open intervals with $\bigcup_{Z\in\mathcal{Z}} \overline{Z} = X$. A function $f : X \to \mathbb{R}$ is called **piecewise continuous** with respect to the finite partition $\mathcal{Z}(f)$ of X, if $f \mid Z$ can be extended to a continuous function on the closure of Z for all $Z \in \mathcal{Z}(f)$. For every $x \in X$ at least one of the numbers $f(x^+) := \lim_{y\to x^+} f(y)$ and $f(x^-) := \lim_{y\to x^-} f(y)$ exist. We assume throughout this paper, that for every $x \in X$ we have $f(x) = f(x^+)$ or $f(x) = f(x^-)$. A function $f : X \to \mathbb{R}$ is called **piecewise constant** with respect to the finite partition $\mathcal{Z}(f)$ of X, if $f \mid Z$ is constant for all $Z \in \mathcal{Z}(f)$.

A piecewise continuous map $T : X \to \mathbb{R}$ is called **piecewise monotone**, if there exists a finite partition $\mathcal{Z}$ of X, such that $T \mid Z$ is strictly monotone and continuous for all $Z \in \mathcal{Z}$. We call $(T,\mathcal{Z})$ a **piecewise monotonic map of class R^0**. We assume throughout this paper that $\mathcal{Z} = \{Z_1, Z_2, \ldots, Z_K\}$ with

$Z_1 < Z_2 < \cdots < Z_K$. If $(T, \mathcal{Z})$ is of class R^0, then we set $E(T) := \{\inf Z, \sup Z : Z \in \mathcal{Z}\}$ and $E_1(T) := E(T) \setminus (\overline{\mathbb{R} \setminus X})$. $E(T)$ is the set of all endpoints of elements of $\mathcal{Z}$, and $E_1(T)$ is the set of all elements of $E(T)$, which are inner points of X. Furthermore we define $R(T) := \bigcap_{j=0}^{\infty} \overline{T^{-j}X}$. We want to remark that the elements of $E(T)$ need not be neither discontinuities nor turning points of T.

If $(T, \mathcal{Z})$ is a piecewise monotonic map of class R^0 and $f : X \to \mathbb{R}$ is a piecewise continuous function, then we can assume that $\mathcal{Z}(f)$ is a refinement of $\mathcal{Z}$.

Let $n \in \mathbb{N}_0 \cup \{\infty\}$. $(T, \mathcal{Z})$ is called a **piecewise monotonic map of class R^n**, if $(T, \mathcal{Z})$ is of class R^0 and T is piecewise C^n. T is piecewise C^n means, that $T^{(j)}$ is a piecewise continuous function with respect to $\mathcal{Z}$ on X for every $j \in \mathbb{N}_0$ with $0 \le j \le n$. $(T, \mathcal{Z})$ is called a **piecewise monotonic map of class E^n** for an $n \ge 1$, if $(T, \mathcal{Z})$ is of class R^n and there exists a $j \ge 1$, such that $(T^j)'$ is a piecewise continuous function on $X_j := \bigcap_{l=0}^{j-1} \overline{T^{-l}X}$ and $\inf_{x \in R(T)} |(T^j)'(x)| > 1$, where we assume that $|(T^j)'(x)| = \min\{|(T^j)'(x^-)|, |(T^j)'(x^+)|\}$ for all $x \in X_j$. If $(T, \mathcal{Z})$ is of class R^n and there exists a piecewise continuous function $f : X \to \mathbb{R}$, such that $f^{(j)}$ is a piecewise continuous function with respect to $\mathcal{Z}$ for all $j \in \mathbb{N}_0$ with $0 \le j \le n$, then $(T, f, \mathcal{Z})$ is called a **weighted piecewise monotonic map of class W^n**. If $(T, \mathcal{Z})$ is of class E^n for an $n \in \mathbb{N} \cup \{\infty\}$, then $(T, -t \log |T'|, \mathcal{Z})$ is of class W^{n-1} for all $t \in \mathbb{R}$.

In order to define topologies on R^n, W^n and E^n we define first topologies for piecewise continuous functions and for partitions. Let $\varepsilon > 0$. Two continuous functions $f : (a, b) \to \mathbb{R}$ and $\tilde{f} : (\tilde{a}, \tilde{b}) \to \mathbb{R}$ are ε–close, if

(1) $|a - \tilde{a}| < \varepsilon$ and $|b - \tilde{b}| < \varepsilon$,

(2) $|f(x) - \tilde{f}(x)| < \varepsilon$ for all $x \in (a, b) \cap (\tilde{a}, \tilde{b})$,

(3) $\sup_{x \in (a, \tilde{a})} |f(x) - \tilde{f}(\tilde{a}^+)| < \varepsilon$, if $a < \tilde{a}$, or $\sup_{x \in (\tilde{a}, a)} |\tilde{f}(x) - f(a^+)| < \varepsilon$, if otherwise $\tilde{a} \le a$,

(4) $\sup_{x \in (\tilde{b}, b)} |f(x) - \tilde{f}(\tilde{b}^-)| < \varepsilon$, if $\tilde{b} < b$, or $\sup_{x \in (b, \tilde{b})} |\tilde{f}(x) - f(b^-)| < \varepsilon$, if otherwise $b \le \tilde{b}$.

We want to remark, that $(a, b) \cap (\tilde{a}, \tilde{b}) \neq \emptyset$ by (1), if ε is small enough.

Suppose that X and $\tilde{X}$ are finite unions of closed intervals. Let $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ be a finite partition of X, where $Y_1 < Y_2 < \cdots < Y_N$, and suppose that $Y_j = (c_j, d_j)$ for $j \in \{1, 2, \ldots, N\}$. Suppose that $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\}$ is a finite partition of $\tilde{X}$, where $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$, and suppose that $\tilde{Y}_j = (\tilde{c}_j, \tilde{d}_j)$ for $j \in \{1, 2, \ldots, N\}$. Then we say the partitions $\mathcal{Y}$ and

$\tilde{\mathcal{Y}}$ are ε–close, if $|c_j - \tilde{c}_j| < \varepsilon$ and $|d_j - \tilde{d}_j| < \varepsilon$ for $j \in \{1,2,\ldots,N\}$.

Now let $f : X \to \mathbb{R}$ and $\tilde{f} : \tilde{X} \to \mathbb{R}$ be piecewise continuous functions with respect to $\mathcal{Z}(f)$, resp. $\tilde{\mathcal{Z}}(\tilde{f})$. Then f and $\tilde{f}$ are said to be ε–close, if there exists a finite partition $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ of X refining $\mathcal{Z}(f)$ with $Y_1 < Y_2 < \cdots < Y_N$ and a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\}$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}(\tilde{f})$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$, such that $f \mid Y_j$ is ε–close to $\tilde{f} \mid \tilde{Y}_j$ for $j \in \{1,2,\ldots,N\}$.

Observe that this definition implies that $\tilde{\mathcal{Y}}$ is ε–close to $\mathcal{Y}$, if $\tilde{f}$ is ε–close to f. In Section 4 we shall need the following result.

LEMMA 1: *Let $f : X \to \mathbb{R}$ be a piecewise continuous function with respect to the finite partition $\mathcal{Z}(f)$ of X, and let $\varepsilon > 0$. Suppose that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ is a finite partition of X, which refines $\mathcal{Z}(f)$, where $Y_1 < Y_2 < \cdots < Y_N$. Set*

$$s := \sup_{Y \in \mathcal{Y}} \sup_{x,y \in Y} |f(x) - f(y)|.$$

If $\tilde{f} : \tilde{X} \to \mathbb{R}$ is a piecewise continuous function, which is ε–close to f, then there exists a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\}$ of $\tilde{X}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$, which is ε–close to $\mathcal{Y}$, such that for every $j \in \{1,2,\ldots,N\}$

$$\sup_{x \in \tilde{Y}_j} \tilde{f}(x) \le \inf_{x \in Y_j} f(x) + s + \varepsilon \quad \text{and} \quad \inf_{x \in \tilde{Y}_j} \tilde{f}(x) \ge \sup_{x \in Y_j} f(x) - s - \varepsilon \, .$$

Proof: By the definition of ε–closeness there exist finite partitions $\mathcal{V} = \{V_1, V_2, \ldots, V_l\}$ of X and $\tilde{\mathcal{V}} = \{\tilde{V}_1, \tilde{V}_2, \ldots, \tilde{V}_l\}$ of $\tilde{X}$, such that $\tilde{\mathcal{V}}$ is ε–close to $\mathcal{V}$ and $\tilde{f} \mid \tilde{V}_j$ is ε–close to $f \mid V_j$ for all $j \in \{1,2,\ldots,l\}$, where we assume $V_1 < V_2 < \cdots < V_l$ and $\tilde{V}_1 < \tilde{V}_2 < \cdots < \tilde{V}_l$. We can assume, that $\mathcal{V}$ is a refinement of $\mathcal{Y}$. Hence for every $j \in \{1,2,\ldots,N\}$ there are numbers $i_j, s_j \in \{1,2,\ldots,l\}$ with $\overline{Y_j} = \bigcup_{k=i_j}^{s_j} \overline{V_k}$. Now we define $\tilde{Y}_j := (\inf \tilde{V}_{i_j}, \sup \tilde{V}_{s_j})$ and $\tilde{\mathcal{Y}} := \{\tilde{Y}_j : j = 1,2,\ldots,N\}$. Then $\tilde{\mathcal{Y}}$ is ε–close to $\mathcal{Y}$. Let $j \in \{1,2,\ldots,N\}$. By the definition of ε–closeness, by definition of s, and as $\tilde{f}$ is piecewise continuous, we get $f(x) - s - \varepsilon \le \tilde{f}(y) \le f(x) + s + \varepsilon$ for every $x \in Y_j$ and every $y \in \tilde{Y}_j$, which gives the desired result. ∎

Now we define the following topology on W^0. Let $(T, f, \mathcal{Z})$ and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ be both of class W^0, and let $\varepsilon > 0$. Suppose that $\tilde{\mathcal{Z}} = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_K\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_K$. Then $(T, f, \mathcal{Z})$ and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ are said to be ε–close in W^0, if

(1) $T \mid Z_j$ and $\tilde{T} \mid \tilde{Z}_j$ are ε–close for $j = 1,2,\ldots,K$,

(2) $f \mid Z_j$ and $\tilde{f} \mid \tilde{Z}_j$ are ε–close for $j = 1, 2, \dots, K$. Observe that this definition implies that $\tilde{\mathcal{Z}}$ is ε–close to $\mathcal{Z}$, if $(T, f, \mathcal{Z})$ is ε–close to $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$.

If $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are both of class R^0, then $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are said to be ε–close in R^0, if (1) above is satisfied. This topology coincides with the topology considered in [5]. If $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are both of class R^n, then $(T, \mathcal{Z})$ and $(\tilde{T}, \tilde{\mathcal{Z}})$ are said to be ε–close in R^n, if $(T, T^{(j)}, \mathcal{Z})$ and $(\tilde{T}, \tilde{T}^{(j)}, \tilde{\mathcal{Z}})$ are ε–close in W^0 for $j \in \mathbb{N}_0$, $j \leq n$. If $(T, \mathcal{Z})$ is of class E^n for an $n \geq 1$, if $\varepsilon > 0$ is small enough, and if the map $(\tilde{T}, \tilde{\mathcal{Z}})$ of class R^n is ε–close to $(T, \mathcal{Z})$ in R^n, then $(\tilde{T}, \tilde{\mathcal{Z}})$ is of class E^n. Two weighted piecewise monotonic maps $(T, f, \mathcal{Z})$ and $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ of class W^n are said to be ε–close in W^n, if they are ε–close in R^n and $(T, f^{(j)}, \mathcal{Z})$ and $(\tilde{T}, \tilde{f}^{(j)}, \tilde{\mathcal{Z}})$ are ε–close in W^0 for $j \in \mathbb{N}_0$, $j \leq n$.

Next we modify (X, T) in order to get a topological dynamical system.

Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. We assume throughout this paper, that $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_N\}$ with $Y_1 < Y_2 < \dots < Y_N$. Set $x_0 := \inf TX$ and $x_1 := \sup TX$. Now define

$$E := \{\inf Y, \sup Y : Y \in \mathcal{Y}\} \quad \text{and} \quad W := \Big(\bigcup_{j=0}^{\infty} T^{-j}(E \setminus \{x_0, x_1\})\Big) \setminus \{x_0, x_1\}.$$

Set $\mathbb{R}_{\mathcal{Y}} := \mathbb{R} \setminus W \cup \{x^-, x^+ : x \in W\}$, and define $y < x^- < x^+ < z$, if $y < x < z$ holds in $\mathbb{R}$. This means, that we have doubled all endpoints of elements of $\mathcal{Y}$, and we have also doubled all inverse images of doubled points. For $x \in \mathbb{R}_{\mathcal{Y}}$ define $\pi(x) := y$, where $y \in \mathbb{R}$ satisfies either $x = y$ or $y \in W$ and $x \in \{y^-, y^+\}$. We have that $x, y \in \mathbb{R}_{\mathcal{Y}}$, $\pi(x) < \pi(y)$ implies $x < y$. Now we define a map $c : \pi^{-1}(W) \to \pi^{-1}(W)$ by $c(x^-) := x^+$ and $c(x^+) := x^-$ for $x \in W$.

For $x, y \in \mathbb{R}_{\mathcal{Y}}$ with $x < y$ let $n(x, y)$ be the minimum of all $k \in \mathbb{N}_0$, such that there exists a $z \in (\bigcup_{j=0}^{k} T^{-j}E) \setminus \{x_0, x_1\}$ with $x < z^+$ and $z^- < y$ ($n(x, y) := \infty$, if such a k does not exist). Then define $d(x, y) := |\pi(x) - \pi(y)| + \frac{1}{n(x,y)+1}$ ($d(x, y) := |\pi(x) - \pi(y)|$, if $n(x, y) = \infty$). This gives rise to a metric d on $\mathbb{R}_{\mathcal{Y}}$. The topology generated by d is exactly the order topology on $\mathbb{R}_{\mathcal{Y}}$.

Let $X_{\mathcal{Y}}$ be the closure of $X \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Observe that $X_{\mathcal{Y}}$ is compact. Now define $E_0(\mathcal{Y}) := \{x \in \mathbb{R}_{\mathcal{Y}} : \pi(x) \in E\}$ and $E_{\mathcal{Y}} := E_0(\mathcal{Y}) \cap X_{\mathcal{Y}}$. Then $E_{\mathcal{Y}} = \{a_1, a_2, \dots, a_{2N}\}$ with $a_1 < a_2 < \dots < a_{2N}$, where $N := \operatorname{card} \mathcal{Y}$. For a perfect subset A of $\mathbb{R}$ let $\hat{A}$ be the closure of $A \setminus W$ in $\mathbb{R}_{\mathcal{Y}}$. Now set $\hat{\mathcal{Y}} := \{\hat{Y} : Y \in \mathcal{Y}\}$ and $\hat{\mathcal{Z}} := \{\hat{Z} : Z \in \mathcal{Z}\}$. Then $\hat{\mathcal{Y}} = \{[a_{2j-1}, a_{2j}] : j = 1, 2, \dots, N\}$, where

$[a,b] := \{x \in \mathbb{R}_{\mathcal{Y}} : a \leq x \leq b\}$. The map $T \mid X \setminus (W \cup E)$ can be extended to a unique continuous piecewise monotonic map $T_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}_{\mathcal{Y}}$. Then $(T_{\mathcal{Y}}, \hat{\mathcal{Z}})$ is a continuous piecewise monotonic map of class R^0 on $X_{\mathcal{Y}} = \bigcup_{\hat{Y} \in \hat{\mathcal{Y}}} \hat{Y}$. If there is no confusion we shall use the notation $\mathcal{Y}$ instead of $\hat{\mathcal{Y}}$ and $\mathcal{Z}$ instead of $\hat{\mathcal{Z}}$. Define $E(T_{\mathcal{Y}}) := \{x \in X_{\mathcal{Y}} : \pi(x) \in E(T)\}$ and $E_1(T_{\mathcal{Y}}) := \{x \in X_{\mathcal{Y}} : \pi(x) \in E_1(T)\}$. The set $R_{\mathcal{Y}} := \bigcap_{j=0}^{\infty} T_{\mathcal{Y}}^{-j} X_{\mathcal{Y}}$ satisfies $R_{\mathcal{Y}} = \bigcap_{j=0}^{\infty} \overline{T_{\mathcal{Y}}^{-j} X_{\mathcal{Y}}} = \{x \in \mathbb{R}_{\mathcal{Y}} : \pi(x) \in R(T)\} \cap X_{\mathcal{Y}}$. $T_{\mathcal{Y}}$ is called the completion of T with respect to $\mathcal{Y}$.

A **topological dynamical system** (X,T) is a continuous map T of a compact metric space X into itself. Hence $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ is a topological dynamical system.

Let $(T, f, \mathcal{Z})$ be a piecewise monotonic map of class W^0, and suppose, that $\mathcal{Y}$ is a refinement of $\mathcal{Z}$. Let $T_{\mathcal{Y}}$ be the completion of T with respect to $\mathcal{Y}$. Then there exists a unique continuous function $f_{\mathcal{Y}} : X_{\mathcal{Y}} \to \mathbb{R}$ with $f_{\mathcal{Y}}(x) = f(x)$ for all $x \in X \setminus (W \cup E)$. Then $(T_{\mathcal{Y}}, f_{\mathcal{Y}}, \mathcal{Z})$ is called the completion of $(T, f, \mathcal{Z})$ with respect to $\mathcal{Y}$.

If (X,T) is a topological dynamical system, and $f : X \to \mathbb{R}$ is a continuous function, then the **topological pressure** $p(X,T,f)$ is defined by

$$p(X,T,f) := \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \sup_{E} \sum_{x \in E} \exp\Big(\sum_{j=0}^{n-1} f(T^j x)\Big) \,, \tag{1.1}$$

where the supremum is taken over all (n,ε)–separated subsets E of X. $E \subseteq X$ is called (n,ε)–separated, if for every $x \neq y \in E$ there exists a $j \in \{0,1,\ldots,n-1\}$ with $d(T^j x, T^j y) > \varepsilon$.

If $(T, f, \mathcal{Z})$ is a piecewise monotonic map of class W^0, and $\mathcal{Y}$ is a finite partition, which refines $\mathcal{Z}$, then $(R_{\mathcal{Y}}, T_{\mathcal{Y}})$ is a topological dynamical system and $f_{\mathcal{Y}} : R_{\mathcal{Y}} \to \mathbb{R}$ is a continuous function, where $(T_{\mathcal{Y}}, f_{\mathcal{Y}}, \mathcal{Z})$ is the completion of $(T, f, \mathcal{Z})$ with respect to $\mathcal{Y}$. Then we define

$$p(R(T),T,f) := p(R_{\mathcal{Y}}, T_{\mathcal{Y}}, f_{\mathcal{Y}}) \,. \tag{1.2}$$

Lemma 2 of [7] says, that this definition does not depend on the partition $\mathcal{Y}$. Furthermore we define for $n \in \mathbb{N}$

$$S_n(R(T),f) := \sup_{x \in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}{}^j x) \,. \tag{1.3}$$

Observe that this definition does not depend on the partition $\mathcal{Y}$.

Finally we define the Hausdorff dimension. For an $A \subseteq \mathbb{R}$, $A \neq \emptyset$ define $\operatorname{diam} A := \sup_{x,y \in A} |x-y|$. Let $Y \subseteq \mathbb{R}$. For $t \geq 0$ and $\varepsilon > 0$ set

$$m(Y,t,\varepsilon) := \inf\{ \sum_{A \in \mathcal{A}} (\operatorname{diam} A)^t : \mathcal{A} \text{ is an at most countable cover of } Y \text{ with } \operatorname{diam} A < \varepsilon \text{ for all } A \in \mathcal{A}\} .$$

Then define the **Hausdorff dimension** $\mathrm{HD}(Y)$ of Y by

$$\mathrm{HD}(Y) := \inf\{t \geq 0 : \lim_{\varepsilon \to 0} m(Y,t,\varepsilon) = 0\} . \tag{1.4}$$

In [7] this definition is slightly modified, which allows to define the Hausdorff dimension also on $X_{\mathcal{Y}}$ — the space, where the completion $T_{\mathcal{Y}}$ of a piecewise monotonic map T of class R^0 acts — in a way, such that $\mathrm{HD}(R_{\mathcal{Y}}) = \mathrm{HD}(R(T))$. At this point we remark, that all results of this paper hold also in the situation considered in [7], where a bit more general situation is treated.

2. Oriented graphs associated to a piecewise monotonic map

Now we define an at most countable oriented graph $(\mathcal{D}, \to)$, called Markov diagram, which describes the orbit structure of $(R(T), T)$ (cf. [2]). Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. Let $T_{\mathcal{Y}}$ be the completion of T with respect to $\mathcal{Y}$ and set $K := \operatorname{card} \mathcal{Z}$ and $N := \operatorname{card} \mathcal{Y}$. Then we can write $E_{\mathcal{Y}} = \{a_1, a_2, \ldots, a_{2N}\}$ with $a_1 < a_2 < \cdots < a_{2N}$. As $E(T_{\mathcal{Y}}) \subseteq E_{\mathcal{Y}}$ there exists an $I \subseteq \{1,2,\ldots,2N\}$ with $\operatorname{card} I = 2K$ and $E(T_{\mathcal{Y}}) = \{a_i : i \in I\}$. Then every $Y \in \hat{\mathcal{Y}}$ can be written as $Y = [a_{2j-1}, a_{2j}]$ for a $j \in \{1,2,\ldots,N\}$. Let $Y_0 \in \hat{\mathcal{Y}}$ and let D be a perfect subinterval of Y_0. A nonempty $C \subseteq X_{\mathcal{Y}}$ is called **successor** of D, if there exists a $Y \in \hat{\mathcal{Y}}$ with $C = T_{\mathcal{Y}} D \cap Y$, and we write $D \to C$. We get that every successor C of D is again a perfect subinterval of an element of $\hat{\mathcal{Y}}$. Let $\mathcal{D}$ be the smallest set with $\hat{\mathcal{Y}} \subseteq \mathcal{D}$ and such that $D \in \mathcal{D}$ and $D \to C$ imply $C \in \mathcal{D}$. Then $(\mathcal{D}, \to)$ is called the **Markov diagram** of T with respect to $\mathcal{Y}$. $\mathcal{D}$ is at most countable and its elements are perfect subintervals of elements of $\hat{\mathcal{Y}}$.

Set $\mathcal{D}_0 := \hat{\mathcal{Y}}$, and for $r \in \mathbb{N}$ set $\mathcal{D}_r := \mathcal{D}_{r-1} \cup \{D \in \mathcal{D} : \exists C \in \mathcal{D}_{r-1} \text{ with } C \to D\}$. Then we have $\mathcal{D}_0 \subseteq \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots$ and $\mathcal{D} = \bigcup_{r=0}^{\infty} \mathcal{D}_r$.

We shall need also another oriented graph $(\mathcal{G}, \to)$. To this end we introduce the following notations. Let $i \in \{1,2,\ldots,2N\}$. Define

$$j(i) := \min\{j \in \mathbb{N} : T_{\mathcal{Y}}{}^j a_i \notin X_{\mathcal{Y}}\},$$

where we set $j(i) := \infty$, if $T_{\mathcal{Y}}^j a_i \in X_{\mathcal{Y}}$ for all $j \in \mathbb{N}$. Now define

$$a_{i,j} := T_{\mathcal{Y}}^j a_i \quad \text{for } i \in \{1, 2, \ldots, 2N\} \text{ and } j \in \mathbb{N}_0,\ 0 \le j < j(i). \tag{2.1}$$

Set $\mathcal{G} := \{a_{i,j} : i \in I, j \in \mathbb{N}_0, 0 \le j < j(i)\}$. For $a, b \in \mathcal{G}$ we introduce an arrow $a \to b$, if and only if either $T_{\mathcal{Y}} a = b$ or $b \in E_1(T_{\mathcal{Y}})$ and $T_{\mathcal{Y}} a = c(b)$. Observe that $(\mathcal{G}, \to)$ does not depend on the partition $\mathcal{Y}$. The graph considered in [5] is similar to $(\mathcal{G}, \to)$.

Let $(\mathcal{H}, \to)$ be an oriented graph. For $n \in \mathbb{N}$ we call $c_0 \to c_1 \to \cdots \to c_n$ a **path of length** n **in** $\mathcal{H}$, if $c_j \in \mathcal{H}$ for $j \in \{0, 1, \ldots, n\}$ and $c_{j-1} \to c_j$ for $j \in \{1, 2, \ldots, n\}$. $c_0 \to c_1 \to c_2 \to \cdots$ is called an **infinite path in** $\mathcal{H}$, if $c_j \in \mathcal{H}$ for all $j \in \mathbb{N}_0$ and $c_{j-1} \to c_j$ for all $j \in \mathbb{N}$. A subset $\mathcal{C}$ of $\mathcal{H}$ is called **closed**, if $c \in \mathcal{C}$, $d \in \mathcal{H}$ and $c \to d$ imply $d \in \mathcal{C}$. $\mathcal{H}$ is called **irreducible**, if for every $c, d \in \mathcal{H}$ there exists a finite path $c_0 \to c_1 \to \cdots \to c_n$ in $\mathcal{H}$ with $c_0 = c$ and $c_n = d$. If $\mathcal{H}$ is irreducible and finite, then $\mathcal{H}$ is called **finite irreducible.** An irreducible subset $\mathcal{C}$ of $\mathcal{H}$ is called **maximal irreducible** in $\mathcal{H}$, if every $\mathcal{C}' \ne \mathcal{C}$ with $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{H}$ is not irreducible.

Suppose that $(T, f, \mathcal{Z})$ is a piecewise monotonic map of class W^0. Suppose that $\mathcal{Y}$ is a finite partition of X, which refines $\mathcal{Z}$. Let $(T_{\mathcal{Y}}, f_{\mathcal{Y}}, \mathcal{Z})$ be the completion of $(T, f, \mathcal{Z})$ with respect to $\mathcal{Y}$, and let $(\mathcal{G}, \to)$ be the graph introduced above. For $a, b \in \mathcal{G}$ define

$$G_{a,b}(f) := \begin{cases} e^{f_{\mathcal{Y}}(a)} & \text{if } a \to b, \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

Set $G(f) := (G_{a,b}(f))_{a,b \in \mathcal{G}}$. As card $\{b \in \mathcal{G} : a \to b\} \le 2$ for all $a \in \mathcal{G}$, we get $\sum_{b \in \mathcal{G}} G_{a,b} \le 2e^{\|f\|_\infty}$. As in [7] this implies that $u \mapsto uG(f)$ is an $\ell^1(\mathcal{G})$–operator and $v \mapsto G(f)v$ is an $\ell^\infty(\mathcal{G})$–operator. Both operators have the same norm $\|G(f)\|$ and the same spectral radius $r(G(f))$. We have

$$\|G(f)\| = \sup_{a \in \mathcal{G}} \sum_{b \in \mathcal{G}} G_{a,b}(f) , \tag{2.3}$$

$$\|G(f)^n\| = \sup_{a \in \mathcal{G}} \sum_{b_0 = a \to b_1 \to \cdots \to b_n} \prod_{j=0}^{n-1} e^{f_{\mathcal{Y}}(b_j)} \quad \text{for all } n \in \mathbb{N}, \tag{2.4}$$

where the sum is taken over all paths $b_0 \to b_1 \to \cdots \to b_n$ of length n in $\mathcal{G}$ with $b_0 = a$, and

$$r(G(f)) = \lim_{n \to \infty} \|G(f)^n\|^{\frac{1}{n}} = \inf_{n \in \mathbb{N}} \|G(f)^n\|^{\frac{1}{n}} . \tag{2.5}$$

Observe that the matrix $G(f)$ does not depend on the partition $\mathcal{Y}$. The next lemma gives an upper bound for $r(G(f))$, if there is no $x \in E_1(T\mathcal{Y})$ with $T\mathcal{Y}^n x \in E_1(T\mathcal{Y})$ for an $n \in \mathbb{N}$.

LEMMA 2: *Let $(T, f, \mathcal{Z})$ be a piecewise monotonic map of class W^0.*

(1) *Suppose that $\mathcal{G}$ contains no closed paths $b_0 \to b_1 \to \cdots \to b_n$ with $b_0 = b_n \in E_1(T\mathcal{Y})$. Then*

$$\log r(G(f)) \le \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f) \le p(R(T), T, f) .$$

(2) *If $T\mathcal{Y}^n x \notin E_1(T\mathcal{Y})$ for all $x \in E_1(T\mathcal{Y})$ and all $n \in \mathbb{N}$, then*

$$\log r(G(f)) \le \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f) \le p(R(T), T, f) .$$

Proof: (1) For $a \in \mathcal{G}$ let $n(a)$ be the smallest number $n \in \mathbb{N}$, such that there exists a path $b_0 \to b_1 \to \cdots \to b_n$ of length n in $\mathcal{G}$ with $b_0 = a$ and $b_n \in E_1(T\mathcal{Y})$, where we set $n(a) := \infty$, if there exists no such n. This gives that every path $b_0 \to b_1 \to \cdots \to b_n$ of length $n < n(a)$ in $\mathcal{G}$ with $b_0 = a$ satisfies $b_j = T\mathcal{Y}^j a$ for $j \in \{0, 1, \ldots, n\}$. Furthermore we have $T\mathcal{Y}^{n(a)} a \in E_1(T\mathcal{Y})$, if $n(a) < \infty$. The definition of $\mathcal{G}$ gives that there exists an $s \in \mathbb{N}$, such that $n(a) < \infty$ implies $n(a) \le s$.

Let $n \in \mathbb{N}_0$. If $b_0 \to b_1 \to \cdots \to b_{n+Ks}$ is a path of length $n + Ks$ in $\mathcal{G}$, then our assumption on $\mathcal{G}$ gives card $\{j \in \{0, 1, \ldots, Ks\} : b_j \in E_1(T\mathcal{Y})\} \le K - 1$ and $n(b_{Ks}) = \infty$. Hence for every $a \in \mathcal{G}$ there are at most 2^{K-1} different paths $b_0 \to b_1 \to \cdots \to b_{n+Ks}$ of length $n + Ks$ with $b_0 = a$. If n is large enough, this gives

$$\sum_{b_0=a\to b_1\to\cdots\to b_{n+Ks}} \prod_{j=0}^{n+Ks-1} e^{f\mathcal{Y}(b_j)} \le 2^{K-1} e^{Ks\|f\|_\infty} \exp\Big(\sum_{j=0}^{n-1} f\mathcal{Y}(b_{j+Ks})\Big) =$$
$$2^{K-1} e^{Ks\|f\|_\infty} \exp\Big(\sum_{j=0}^{n-1} f\mathcal{Y}(T\mathcal{Y}^j b_{Ks})\Big) \le 2^{K-1} e^{Ks\|f\|_\infty} \exp(S_n(R(T), f)) .$$

Now we get by (2.4) and (2.5)

$$\log r(G(f)) = \lim_{n\to\infty} \frac{1}{n+Ks} \log \|G(f)^{n+Ks}\| =$$
$$\lim_{n\to\infty} \frac{1}{n+Ks} \log \sup_{a\in\mathcal{G}} \sum_{b_0=a\to b_1\to\cdots\to b_{n+Ks}} \prod_{j=0}^{n+Ks-1} e^{f_{\mathcal{Y}}(b_j)} \leq$$
$$\lim_{n\to\infty} \left(\frac{K-1}{n+Ks}\log 2 + \frac{Ks\|f\|_\infty}{n+Ks} + \frac{n}{n+Ks}\frac{1}{n}S_n(R(T),f)\right) =$$
$$\lim_{n\to\infty} \frac{1}{n} S_n(R(T),f)\,.$$

Using (1.2) and (1.3) this gives the desired result, since

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x\in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}^j x) \leq p(R_{\mathcal{Y}}, T_{\mathcal{Y}}, f_{\mathcal{Y}}).$$

(2) Suppose that $b_0 \to b_1 \to \cdots \to b_n$ is a path of length n in $\mathcal{G}$ with $b_0 = b_n \in E_1(T_{\mathcal{Y}})$. Set $k := \min\{l \in \mathbb{N} : b_l \in E_1(T_{\mathcal{Y}})\}$. As $b_n \in E_1(T_{\mathcal{Y}})$ we get $k \leq n$. Since $b_l \notin E_1(T_{\mathcal{Y}})$ for $l \in \{1, 2, \ldots, k-1\}$ we get $T_{\mathcal{Y}}^k b_0 \in E_1(T_{\mathcal{Y}})$, which contradicts our assumption. Now (1) gives the desired result. ∎

In the sequel we shall need also a description of the Markov diagram in a different way (which will be called variant of the Markov diagram). This will be similar to that used in [3].

Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0 and suppose that $\mathcal{Y}$ is a finite partition of X, which refines $\mathcal{Z}$. Let $(T_{\mathcal{Y}}, \mathcal{Z})$ be the completion, and $(\mathcal{D}, \to)$ the Markov diagram of T with respect to $\mathcal{Y}$. First set

$$\mathcal{M} := \{(i,j) : i \in \{1, 2, \ldots, 2N\}, j \in \mathbb{N}_0, 0 \leq j < j(i)\},$$

and for $r \in \mathbb{N}_0$ define $\mathcal{M}_r := \{(i,j) \in \mathcal{M} : j \leq r\}$. Now we define a map $A : \mathcal{M} \to \mathcal{D}$ with $A(\mathcal{M}) = \mathcal{D}$ and $A(\mathcal{M}_r) = \mathcal{D}_r$ for all $r \in \mathbb{N}_0$, such that $a_{i,j}$ is an endpoint of $A(i,j)$ for all $(i,j) \in \mathcal{M}$. This map will be surjective, but need not be injective, that means a $C \in \mathcal{D}$ can be represented by different elements of $\mathcal{M}$. Furthermore we define arrows between elements of $\mathcal{M}$, such that $c \to d$ in $\mathcal{M}$ implies $A(c) \to A(d)$ in $\mathcal{D}$, and for every $c \in \mathcal{M}$ the map A is bijective from $\{d \in \mathcal{M} : c \to d\}$ to $\{D \in \mathcal{D} : A(c) \to D\}$. Furthermore we shall have, that $c \in \mathcal{M}_r$ implies the existence of a $d \in \mathcal{M}_r$ with $A(c) \subseteq A(d)$ and either $A(c) = [a_d, a_c]$ or $A(c) = [a_c, a_d]$.

For $j \in \{1,2,\dots,N\}$ set $A(2j-1,0) := A(2j,0) := Y_j$. Hence we have $a_{i,0}$ is an endpoint of $A(i,0)$ for all $i \in \{1,2,\dots,2N\}$. Now suppose that $A \mid \mathcal{M}_r$ is constructed, and all arrows beginning in $\mathcal{M}_{r-1}$ are described for an $r \in \mathbb{N}_0$. Let $i \in \{1,2,\dots,2N\}$, and suppose that $j(i) \geq r+1$. Then $(i,r) \in \mathcal{M}_r$ and $A(i,r) \in \mathcal{D}_r$. We have that $A(i,r) \subseteq A(u,v)$ and either $A(i,r) = [a_{u,v}, a_{i,r}]$ or $A(i,r) = [a_{i,r}, a_{u,v}]$ for a $(u,v) \in \mathcal{M}_r$. First we suppose, that there exists an $s \in \{0,1,\dots,r-1\}$ with $A(i,r) = A(i,s)$. In this case we introduce an arrow $(i,r) \to d$ if and only if either $d = (i,r+1)$ or $d \neq (i,s+1)$ and $(i,s) \to d$. Furthermore we set $A(i,r+1) = A(i,s+1)$. Now we consider the case $A(i,r) \neq A(i,s)$ for all $s \in \{0,1,\dots,r-1\}$. Set $\mathcal{C} := \{C \in \mathcal{D} : A(i,r) \to C, T_{\mathcal{Y}}a_{i,r} \notin C, T_{\mathcal{Y}}a_{u,v} \notin C\}$. For every $C \in \mathcal{C}$ there exists an $i(C) \in \{1,2,\dots,2N\}$ with $A(i(C),0) = C$. We introduce an arrow $(i,r) \to (i(C),0)$. If $A(i,r)$ has a successor C with $T_{\mathcal{Y}}a_{u,v} \in C$ and $T_{\mathcal{Y}}a_{i,r} \notin C$, then we introduce an arrow $(i,r) \to (u,v+1)$. If $j(i) > r+1$, then there exists a successor D of $A(i,r)$ with $a_{i,r+1} = T_{\mathcal{Y}}a_{i,r} \in D$. We introduce an arrow $(i,r) \to (i,r+1)$ and define $A(i,r+1) := D$. We have that $a_{i,r+1}$ is an endpoint of $A(i,r+1)$. If $T_{\mathcal{Y}}a_{u,v} \in A(i,r+1)$, then $A(i,r+1) \subseteq A(u,v+1)$ and we have either $A(i,r+1) = [a_{u,v+1}, a_{i,r+1}]$ or $A(i,r+1) = [a_{i,r+1}, a_{u,v+1}]$. Otherwise there exists a $w \in \{1,2,\dots,2N\}$ with $A(i,r+1) \subseteq A(w,0)$, such that either $A(i,r+1) = [a_{w,0}, a_{i,r+1}]$ or $A(i,r+1) = [a_{i,r+1}, a_{w,0}]$. This finishes the construction of the oriented graph $(\mathcal{M},\to)$ and the function A.

$(\mathcal{A},\to)$ is called a **variant of the Markov diagram** of T with respect to $\mathcal{Y}$, if $\mathcal{A} \subseteq \mathcal{M}$ satisfies the following properties.

(1) If $i \in \{1,2,\dots,2N\}$ and $j \in \mathbb{N}_0$, then $(i,j) \in \mathcal{A}$ implies $(i,l) \in \mathcal{A}$ for $l \in \{0,1,\dots,j\}$.
(2) $c,d \in \mathcal{A}$ and $c \to d$ in $\mathcal{M}$ imply $c \to d$ in $\mathcal{A}$.
(3) $c,d \in \mathcal{A}$ and $c \to d$ in $\mathcal{A}$ imply either $c \to d$ in $\mathcal{M}$ or there exists a $d_0 \in \mathcal{M} \setminus \mathcal{A}$ with $c \to d_0$ in $\mathcal{M}$ and $A(d) = A(d_0)$.
(4) For $c \in \mathcal{A}$ the map $A : \{d \in \mathcal{A} : c \to d\} \to \{D \in \mathcal{D} : A(c) \to D\}$ is bijective.
(5) $A(\mathcal{A} \cap \mathcal{M}_r) = \mathcal{D}_r$ for all $r \in \mathbb{N}_0$.

Observe that $(\mathcal{M},\to)$ is a variant, and $(\mathcal{D},\to)$ can be considered as a variant of the Markov diagram of T with respect to $\mathcal{Y}$. For $r \in \mathbb{N}_0$ set $\mathcal{A}_r := \mathcal{A} \cap \mathcal{M}_r$.

Now suppose, that $(T,\mathcal{Z})$ is a piecewise monotonic map of class R^0, that $f : X \to \mathbb{R}$ is piecewise constant, and that $\mathcal{Y}$ is a finite partition of X, which refines both $\mathcal{Z}$ and $\mathcal{Z}(f)$. Let $(\mathcal{A},\to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$. For $c \in \mathcal{A}$ let f_c be the unique real number with $f_{\mathcal{Y}}(x) = f_c$

for all $x \in A(c)$. For $c, d \in \mathcal{A}$ define

$$F_{c,d}(f) := \begin{cases} e^{f_c} & \text{if } c \to d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.6}$$

Set $F(f) := (F_{c,d}(f))_{c,d\in\mathcal{A}}$, and for $\mathcal{C} \subseteq \mathcal{A}$ set $F_{\mathcal{C}}(f) := (F_{c,d}(f))_{c,d\in\mathcal{C}}$. It is shown in [7], that $u \mapsto uF_{\mathcal{C}}(f)$ is an $\ell^1(\mathcal{C})$–operator and $v \mapsto F_{\mathcal{C}}(f)v$ is an $\ell^\infty(\mathcal{C})$–operator, where both operators have the same norm $\|F_{\mathcal{C}}(f)\|$ and the same spectral radius $r(F_{\mathcal{C}}(f))$. We have

$$\|F_{\mathcal{C}}(f)\| = \sup_{c\in\mathcal{C}} \sum_{d\in\mathcal{C}} F_{c,d}(f) \ , \tag{2.7}$$

$$\|F_{\mathcal{C}}(f)^n\| = \sup_{c\in\mathcal{C}} \sum_{c_0=c\to c_1\to\cdots\to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}} \qquad \text{for every } n \in \mathbb{N}, \tag{2.8}$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{C}$ with $c_0 = c$, and

$$r(F_{\mathcal{C}}(f)) = \lim_{n\to\infty} \|F_{\mathcal{C}}(f)^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|F_{\mathcal{C}}(f)^n\|^{\frac{1}{n}} \ . \tag{2.9}$$

The next lemma shows, that the spectral radius of $F(f)$ does not depend on the variant $\mathcal{A}$.

LEMMA 3: *Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0, and let $f : X \to \mathbb{R}$ be a piecewise constant function. Suppose that $\mathcal{Y}$ is a finite partition of X, which refines both $\mathcal{Z}$ and $\mathcal{Z}(f)$. Let $(\mathcal{A}, \to)$ and $(\mathcal{A}', \to)$ be two variants of the Markov diagram of T with respect to $\mathcal{Y}$, and set $F_{\mathcal{A}}(f) := (F_{c,d}(f))_{c,d\in\mathcal{A}}$ and $F_{\mathcal{A}'}(f) := (F_{c,d}(f))_{c,d\in\mathcal{A}'}$, where $F_{c,d}(f)$ is defined as in (2.6). Then $\|F_{\mathcal{A}}(f)^n\| = \|F_{\mathcal{A}'}(f)^n\|$ for all $n \in \mathbb{N}$, and $r(F_{\mathcal{A}}(f)) = r(F_{\mathcal{A}'}(f))$.*

Proof: By (2.9) it suffices to show $\|F_{\mathcal{A}}(f)^n\| = \|F_{\mathcal{A}'}(f)^n\|$ for all $n \in \mathbb{N}$. For $c \in \mathcal{A}$ let f_c be the unique real number with $f_{\mathcal{Y}}(x) = f_c$ for all $x \in A(c)$, and for $d \in \mathcal{A}'$ let f_d be the unique real number with $f_{\mathcal{Y}}(x) = f_d$ for all $x \in A(d)$. Let $n \in \mathbb{N}$, and let $c \in \mathcal{A}$. Then there exists a $c' \in \mathcal{A}'$ with $A(c) = A(c')$. As for every $d \in \mathcal{A}$ the map $a \mapsto A(a)$ from $\{a \in \mathcal{A} : d \to a\}$ to $\{C \in \mathcal{D} : A(d) \to C\}$ is bijective, and as the same holds for $\mathcal{A}'$, we get by (2.8) that

$$\sum_{c_0=c\to c_1\to\cdots\to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}} = \sum_{d_0=c'\to d_1\to\cdots\to d_n} \prod_{j=0}^{n-1} e^{f_{d_j}} \le \|F_{\mathcal{A}'}(f)^n\| \ ,$$

where the first sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{A}$ with $c_0 = c$, and the second sum is taken over all paths $d_0 \to d_1 \to \cdots \to d_n$ of length n in $\mathcal{A}'$ with $d_0 = c'$. Hence (2.8) gives $\|F_{\mathcal{A}}(f)^n\| \leq \|F_{\mathcal{A}'}(f)^n\|$. Changing the rules of $\mathcal{A}$ and $\mathcal{A}'$ in this calculation gives the desired result. ∎

Remark: The above proof shows a bit more. Suppose that $\mathcal{C} \subseteq \mathcal{A}$ and $\mathcal{C}' \subseteq \mathcal{A}'$ satisfy, that for every $c \in \mathcal{C}$ there exists a $c' \in \mathcal{C}'$ with $A(c) = A(c')$. Furthermore we suppose, that if $c \in \mathcal{C}$, $c' \in \mathcal{C}'$ and $A(c) = A(c')$, then $d \in \mathcal{C}$ and $c \to d$ in $\mathcal{A}$ imply the existence of a $d' \in \mathcal{C}'$ with $c' \to d'$ in $\mathcal{A}'$ and $A(d) = A(d')$. Then $\|F_{\mathcal{C}}(f)^n\| \leq \|F_{\mathcal{C}'}(f)^n\|$ for all $n \in \mathbb{N}$ and $r(F_{\mathcal{C}}(f)) \leq r(F_{\mathcal{C}'}(f))$.

The next lemma gives a way to estimate $r(F(f))$, if only a finite part of the Markov diagram is known.

LEMMA 4: *Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0, and let $f : X \to \mathbb{R}$ be a piecewise constant function. Suppose that $\mathcal{Y}$ is a finite partition of X, which refines both $\mathcal{Z}$ and $\mathcal{Z}(f)$. Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$, and for $c \in \mathcal{A}$ denote by f_c the unique real number with $f_{\mathcal{Y}}(x) = f_c$ for all $x \in A(c)$. Set $F(f) := F_{\mathcal{A}}(f)$. Then*

(2.10)
$$\|F(f)^n\| = \|F_{\mathcal{A}_n}(f)^n\| = \sup_{c\in\mathcal{A}_0} \sum_{c_0=c\to c_1\to\cdots\to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}} \qquad \text{for all } n \in \mathbb{N},$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{A}$ with $c_0 = c$, and

$$r(F(f)) = \lim_{n\to\infty} \|F_{\mathcal{A}_n}(f)^n\|^{\frac{1}{n}} = \inf_{n\in\mathbb{N}} \|F_{\mathcal{A}_n}(f)^n\|^{\frac{1}{n}} . \tag{2.11}$$

Proof: As (2.10) implies (2.11) by (2.9) it remains to show (2.10). To this end we use (2.8). Fix $n \in \mathbb{N}$. Now fix $c \in \mathcal{A}$. Then there exists a $d \in \mathcal{A}_0$ with $A(c) \subseteq A(d)$. Let $c_0 \to c_1 \to \cdots \to c_n$ be a path of length n in $\mathcal{A}$ with $c_0 = c$. We show by induction, that there exists a path $d_0 \to d_1 \to \cdots \to d_n$ of length n in $\mathcal{A}$ with $d_0 = d$ and $A(c_j) \subseteq A(d_j)$ for all $j \in \{0, 1, \ldots, n\}$. If $d_0, d_1, \ldots, d_{k-1}$ are constructed, then $A(c_{k-1}) \subseteq A(d_{k-1})$ and there exists a $Y \in \mathcal{Y}$ with $A(c_k) \subseteq Y$. Hence $A(c_k) = T_{\mathcal{Y}}A(c_{k-1}) \cap Y \subseteq T_{\mathcal{Y}}A(d_{k-1}) \cap Y$. Therefore there exists a $d_k \in \mathcal{A}$ with $d_{k-1} \to d_k$ and $A(d_k) = T_{\mathcal{Y}}A(d_{k-1}) \cap Y$. This construction gives an injective map from the set of all paths of length n in $\mathcal{A}$ with $c_0 = c$ to those with $d_0 = d$. Hence

$$\sum_{c_0=c\to c_1\to\cdots\to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}} \leq \sum_{d_0=d\to d_1\to\cdots\to d_n} \prod_{j=0}^{n-1} e^{f_{d_j}} .$$

This gives

$$\|F_{\mathcal{A}_n}(f)^n\| \le \|F(f)^n\| = \sup_{c\in\mathcal{A}_0} \sum_{c_0=c\to c_1\to\cdots\to c_n} \prod_{j=0}^{n-1} e^{f_{c_j}}.$$

Observing that every path $c_0 \to c_1 \to \cdots \to c_n$ of length n in $\mathcal{A}$ with $c_0 \in \mathcal{A}_0$ is in $\mathcal{A}_n$ we get the desired result. ∎

Using Lemma 3 we get by the proof of Lemma 6 in [7] that

$$p(R(T),T,f) = \log r(F_{\mathcal{A}}(f)) \tag{2.12}$$

for every variant $(\mathcal{A},\to)$ of the Markov diagram of T with respect to $\mathcal{Y}$. This gives together with Lemma 4 a method to estimate $p(R(T),T,f)$ from above. We shall also need a result to estimate $p(R(T),T,f)$ from below.

Lemma 5: *Let $(T,\mathcal{Z})$ be a piecewise monotonic map of class R^0, and let $f : X \to \mathbb{R}$ be a piecewise constant function. Suppose that $\mathcal{Y}$ is a finite partition of X, which refines both $\mathcal{Z}$ and $\mathcal{Z}(f)$, and suppose that*

$$p(R(T),T,f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T),f).$$

Then for every $\varepsilon > 0$ there exists an $r \in \mathbb{N}$, such that for every variant $(\mathcal{A},\to)$ of the Markov diagram of T with respect to $\mathcal{Y}$ there exists an irreducible $\mathcal{C} \subseteq \mathcal{A}_r$ with $\log r(F_{\mathcal{C}}(f)) > p(R(T),T,f) - \varepsilon$.

Proof: By Theorem 11 of [2] the nonwandering set $\Omega(R_{\mathcal{Y}},T_{\mathcal{Y}})$ of $(R_{\mathcal{Y}},T_{\mathcal{Y}})$ can be written as $\Omega(R_{\mathcal{Y}},T_{\mathcal{Y}}) = \bigcup_{\mathcal{E}\in\Gamma} L(\mathcal{E}) \cup L_\infty \cup P \cup W$, where the sets $L(\mathcal{E})$, L_∞, P and W have the properties described in Theorem 11 of [2]. As $h_{\text{top}}(L_\infty \cup P) = 0$ we get

$$p(L_\infty \cup P) \le \lim_{n\to\infty} \frac{1}{n} \sup_{x\in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}{}^j x) = \lim_{n\to\infty} \frac{1}{n} S_n(R(T),f)$$

by (1.3). Using this and the fact, that every $x \in W$ is not contained in the centre of $(R_{\mathcal{Y}},T_{\mathcal{Y}})$, we get by (1.2) and by Corollary 2.18 of [1]

$$p(R(T),T,f) = p(R_{\mathcal{Y}},T_{\mathcal{Y}},f_{\mathcal{Y}}) = \sup_{\mathcal{E}\in\Gamma} p(L(\mathcal{E}),T_{\mathcal{Y}},f_{\mathcal{Y}}).$$

Hence there exists an $\mathcal{E} \in \Gamma$ with $p(L(\mathcal{E}), T_{\mathcal{Y}}, f_{\mathcal{Y}}) > p(R(T), T, f) - \varepsilon$ and

$$p(L(\mathcal{E}), T_{\mathcal{Y}}, f_{\mathcal{Y}}) > \lim_{n\to\infty} \frac{1}{n} \sup_{x\in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}^j x).$$

Now the proof of Lemma 6 in [7] shows, that $\log r(F_{\mathcal{E}}(f)) = p(L(\mathcal{E}), T, f)$.

Let $(\mathcal{A}, \to)$ be a variant of the Markov diagram of T with respect to $\mathcal{Y}$. Define $\mathcal{E}_{\mathcal{M}} := \{c \in \mathcal{M} : A(c) \in \mathcal{E}\}$ and $\mathcal{E}_{\mathcal{A}} := \{c \in \mathcal{A} : A(c) \in \mathcal{E}\}$. By the proof of Lemma 3 (see the remark after Lemma 3) we get $r(F_{\mathcal{E}_{\mathcal{M}}}(f)) = r(F_{\mathcal{E}}(f))$. As

$$r(F_{\mathcal{E}_{\mathcal{M}}}(f)) > \lim_{n\to\infty} \exp \frac{1}{n} \sup_{x\in R_{\mathcal{Y}}} \sum_{j=0}^{n-1} f_{\mathcal{Y}}(T_{\mathcal{Y}}^j x)$$

we get by the proof of Lemma 6 in [7] that

$$\lim_{r\to\infty} r(F_{\mathcal{E}_{\mathcal{M}}\cap\mathcal{M}_r}(f)) = r(F_{\mathcal{E}_{\mathcal{M}}}(f)) .$$

Therefore there exists an $r \in \mathbb{N}$ with $\log r(F_{\mathcal{E}_{\mathcal{M}}\cap\mathcal{M}_r}(f)) > p(R(T), T, f) - \varepsilon$. Using the proof of Lemma 3 we get $r(F_{\mathcal{E}_{\mathcal{M}}\cap\mathcal{M}_r}(f)) \le r(F_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{A}_r}(f))$, which gives $\log r(F_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{A}_r}(f)) > p(R(T), T, f) - \varepsilon$. Since this implies $r(F_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{A}_r}(f)) > 0$, and as $F_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{A}_r}(f)$ is a finite matrix, there exists an irreducible $\mathcal{C} \subseteq \mathcal{E}_{\mathcal{A}} \cap \mathcal{A}_r$ with $r(F_{\mathcal{C}}(f)) = r(F_{\mathcal{E}_{\mathcal{A}}\cap\mathcal{A}_r}(f))$, and hence $\log r(F_{\mathcal{C}}(f)) > p(R(T), T, f) - \varepsilon$. ∎

3. Continuity of the Markov diagram

In this section let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0, and let $\mathcal{Y}$ be a finite partition of X, which refines $\mathcal{Z}$. Let $\mathcal{Z}_0$ be the set, which consists of all elements of $\mathcal{Z}$ and all maximal open subintervals of $[\inf TX, \sup TX] \setminus X$. We assume throughout this section, that $\mathcal{Z}_0 = \{Z_1, Z_2, \ldots, Z_L\}$ with $Z_1 < Z_2 < \cdots < Z_L$. Set $J := \{j \in \{1, 2, \ldots, L\} : Z_j \in \mathcal{Z}\}$. If $\delta > 0$ is small enough, if $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ–close to $(T, \mathcal{Z})$ in R^0, and if $\tilde{\mathcal{Y}}$ is a finite partition of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, which is δ–close to $\mathcal{Y}$, then we get that $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \ldots, \tilde{Y}_N\}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$, $\tilde{\mathcal{Z}}_0 = \{\tilde{Z}_1, \tilde{Z}_2, \ldots, \tilde{Z}_L\}$ with $\tilde{Z}_1 < \tilde{Z}_2 < \cdots < \tilde{Z}_L$, $\tilde{\mathcal{Z}} = \{\tilde{Z}_j : j \in J\}$, and for $j \in J$ we have $\{i : Y_i \subseteq Z_j\} = \{i : \tilde{Y}_i \subseteq \tilde{Z}_j\}$. We assume these properties throughout this section. The aim of this section is to show, that if $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ–close to $(T, \mathcal{Z})$, and $\tilde{\mathcal{Y}}$ is δ–close to $\mathcal{Y}$, then their Markov diagrams have similar initial parts.

Define the numbers $j(i)$ and the elements $a_{i,j}$ as in Section 2 for the map T, and define analoguously the numbers $\tilde{j}(i)$ and the elements $\tilde{a}_{i,j}$ for the map $\tilde{T}$. Then we have for $i_1, i_2 \in \{1, 2, \dots, 2N\}$ that $a_{i_1,0} < a_{i_2,0}$ is equivalent to $\tilde{a}_{i_1,0} < \tilde{a}_{i_2,0}$.

Denote by $(T_{\mathcal{Y}}, \mathcal{Z})$ the completion of $(T, \mathcal{Z})$ with respect to $\mathcal{Y}$, and let $(\mathcal{D}, \to)$ be the Markov diagram of T with respect to $\mathcal{Y}$. $C, D \in \mathcal{D}$ are called **$\mathcal{Y}$–close**, if there exists a $Z \in \mathcal{Z}$ with $C \cup D \subseteq Z$ and if there exists a $j \in \{1, 2, \dots, N\}$ with $C \subseteq Y_j$ and $D \subseteq Y_{j-1} \cup Y_j \cup Y_{j+1}$, where we set $Y_0 := Y_{N+1} := \emptyset$. Observe that $\mathcal{Y}$–closeness does not depend only on $\mathcal{Y}$, but also on $\mathcal{Z}$.

LEMMA 6: *Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class R^0. Suppose that $\mathcal{Y}$ is a finite partition of X, which refines $\mathcal{Z}$. Then for every $r \in \mathbb{N}$ there exists a $\delta > 0$, such that for every piecewise monotonic map $(\tilde{T}, \tilde{\mathcal{Z}})$, which is δ–close to $(T, \mathcal{Z})$ with respect to the R^0–topology and for every finite partition $\tilde{\mathcal{Y}}$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, which is δ–close to $\mathcal{Y}$, there exists a variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$ and a variant $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Y}}$ with the following properties.*

(1) *$\tilde{\mathcal{A}}_r$ can be written as a disjoint union $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$, such that $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_2$ are closed in $\tilde{\mathcal{A}}_r$ and $\tilde{\mathcal{A}}_0 \subseteq \mathcal{B}_0$ ($\mathcal{B}_1$ and $\mathcal{B}_2$ may be possibly empty).*

(2) *Every $c \in \tilde{\mathcal{A}}_r$ has at most two successors in $\mathcal{B}_1 \cup \mathcal{B}_2$.*

(3) *There exists a bijective function $\varphi : \mathcal{A}_r \to \mathcal{B}_0$, and there exists a function $\psi : \mathcal{B}_2 \to \mathcal{G}$.*

(4) *For $c, d \in \mathcal{A}_r$ the property $c \to d$ in $\mathcal{A}$ is equivalent to $\varphi(c) \to \varphi(d)$ in $\tilde{\mathcal{A}}$. For $c, d \in \mathcal{B}_2$ the property $c \to d$ in $\tilde{\mathcal{A}}$ implies $\psi(c) \to \psi(d)$ in $\mathcal{G}$.*

(5) *$A(c) = Y_j$ for a $c \in \mathcal{A}_0$ and a $j \in \{1, 2, \dots, N\}$ implies $\varphi(c) \in \tilde{\mathcal{A}}_0$ and $\tilde{A}(\varphi(c)) = \tilde{Y}_j$.*

(6) *$c \in \mathcal{A}_r$ and $A(c) \subseteq A(d)$ for a $d \in \mathcal{A}_0$ imply $\tilde{A}(\varphi(c)) \subseteq \tilde{A}(\varphi(d))$. $c \in \mathcal{B}_2$ and $\psi(c) \in A(d)$ for a $d \in \mathcal{A}_0$ imply $\tilde{A}(c)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(\varphi(d))$.*

(7) *Let $\mathcal{P}$ be the set of all paths $c_0 \to c_1 \to \cdots \to c_r$ of length r in $\tilde{\mathcal{A}}_r$ with $c_0 \in \tilde{\mathcal{A}}_0$, and set $\mathcal{N} := \{(d_0, d_1, \dots, d_r) : d_j \in \mathcal{A}_r \cup \mathcal{G} \text{ for } j \in \{0, 1, \dots, r\}\}$. Then there exists a function $\chi : \mathcal{P} \to \mathcal{N}$.*

(8) *Let $c_0 \to c_1 \to \cdots \to c_r \in \mathcal{P}$, $\chi(c_0 \to c_1 \to \cdots \to c_r) = (d_0, d_1, \dots, d_r)$ and $j \in \{0, 1, \dots, r\}$. $c_j \in \mathcal{B}_2$ is equivalent to $d_j \in \mathcal{G}$, and we have then $\psi(c_j) = d_j$. $c_j \in \mathcal{B}_0$ implies $\varphi(d_j) = c_j$. $c_j \in \mathcal{B}_0 \cup \mathcal{B}_1$ implies $\tilde{A}(c_j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(\varphi(d_j))$. Furthermore $c_j \in \mathcal{B}_0 \cup \mathcal{B}_1$ and $j \geq 1$ imply $d_{j-1} \to d_j$ in $\mathcal{A}$.*

(9) *For a fixed* $c \in \tilde{\mathcal{A}}_0$ *and for a fixed* $(d_0, d_1, \dots, d_r) \in \mathcal{N}$ *there are at most* $2r+1$ *different paths* $c_0 \to c_1 \to \cdots \to c_r \in \mathcal{P}$ *with* $c_0 = c$ *and* $\chi(c_0 \to c_1 \to \cdots \to c_r) = (d_0, d_1, \dots, d_r)$. *Furthermore for* $s \in \{0, 1, \dots, r-1\}$ *and fixed* $d_0, d_1, \dots, d_s \in \mathcal{A}_r \cup \mathcal{G}$ *there are at most* 4 *different* $b \in \mathcal{G}$, *such that there exist* $d_{s+2}, d_{s+3}, \dots, d_r \in \mathcal{A}_r \cup \mathcal{G}$ *with* $(d_0, d_1, \dots, d_s, b, d_{s+2}, \dots, d_r) \in \chi(\mathcal{P})$.

Proof: To prove this lemma it suffices to consider the completions of T and $\tilde{T}$. We use the notations $X, T, \mathcal{Y}, \dots$, resp. $\tilde{X}, \tilde{T}, \tilde{\mathcal{Y}}, \dots$ for these completions.

We show by induction that the following extended version of the lemma holds. For every $r \in \mathbb{N}_0$ and for every $\eta > 0$ there exists a $\delta > 0$, such that for every piecewise monotonic map $(\tilde{T}, \tilde{\mathcal{Z}})$, which is δ-close to $(T, \mathcal{Z})$ with respect to the R^0-topology, and for every finite partition $\tilde{\mathcal{Y}}$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, which is δ-close to $\mathcal{Y}$, there exists a variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$ and a variant $(\tilde{\mathcal{A}}, \to)$ of the Markov diagram of $\tilde{T}$ with respect to $\tilde{\mathcal{Y}}$ with the following properties.

(a) $(i,j) \in \mathcal{A}_{r+1}$ implies $(i,j) \in \tilde{\mathcal{A}}_{r+1}$. Furthermore $(i,0) \in \mathcal{A}_0$ for every $i \in \{1, 2, \dots, 2N\}$.

(b) $\varphi(i,j) = (i,j)$, whenever $(i,j) \in \mathcal{A}_r$.

(c) If $(i,j) \in \mathcal{A}_r$ and $(u,v) \in \mathcal{A}_{r+1}$, then $(i,j) \to (u,v)$ in $\mathcal{A}$ is equivalent to $(i,j) \to (u,v)$ in $\tilde{\mathcal{A}}$.

(d) If $(i,j) \in \mathcal{A}_{r+1}$, then there exists an $s \in J$ and a $t \in \{1, 2, \dots, N\}$ with $a_{i,j} \in Z_s \cap Y_t$ and $\tilde{a}_{i,j} \in \tilde{Z}_s \cap \tilde{Y}_t$.

(e) If $(i,j) \in \mathcal{A}_{r+1}$, then there exists a $(u,v) \in \mathcal{A}_{r+1}$ with $A(i,j) \subseteq A(u,v)$ and $\tilde{A}(i,j) \subseteq \tilde{A}(u,v)$, such that either $A(i,j) = [a_{u,v}, a_{i,j}]$ and $\tilde{A}(i,j) = [\tilde{a}_{u,v}, \tilde{a}_{i,j}]$ or $A(i,j) = [a_{i,j}, a_{u,v}]$ and $\tilde{A}(i,j) = [\tilde{a}_{i,j}, \tilde{a}_{u,v}]$.

(f) $\tilde{\mathcal{A}}_r$ can be written as a disjoint union $\mathcal{B}_0 \cup \mathcal{B}_1 \cup \mathcal{B}_2$, such that $\mathcal{B}_1 \cup \mathcal{B}_2$ and $\mathcal{B}_2$ are closed in $\tilde{\mathcal{A}}_r$, and $\mathcal{B}_0 = \varphi(\mathcal{A}_r)$. Furthermore we have $\tilde{\mathcal{A}}_0 \subseteq \mathcal{B}_0$, and every $(i,j) \in \tilde{\mathcal{A}}_r$ has at most two successors in $\mathcal{B}_1 \cup \mathcal{B}_2$.

(g) $|\pi(a_{i,j}) - \tilde{\pi}(\tilde{a}_{i,j})| < \eta$, whenever $(i,j) \in \mathcal{B}_0 \cup \mathcal{B}_1$.

(h) If $(i,j) \in \mathcal{A}_r$, $(i,j+1) \in \tilde{\mathcal{A}}_{r+1} \setminus \mathcal{A}_{r+1}$, then $|\tilde{\pi}(x) - \pi(Ta_{i,j})| < \eta$ for all $x \in \tilde{A}(i,j+1)$.

(i) If $(i,j) \in \mathcal{B}_1$, then we have $|\tilde{\pi}(x) - \pi(a_{i,j})| < \eta$ for all $x \in \tilde{A}(i,j)$, and $|\tilde{\pi}(x) - \pi(Ta_{i,j})| < \eta$ for all $x \in C$, if $\tilde{A}(i,j) \to C$ in $\tilde{\mathcal{D}}$.

(j) If $(i,j) \in \mathcal{B}_2$, then there exists a $p \in \{1, 2, \dots, 2N\}$ with $a_{p,0} \in E(T)$ and a $k \in \{0, 1, \dots, j-1\}$ with $k < j(p)$, such that $a_{p,l} \notin E(T)$ for $l \in$

$\{1,2,\dots,k\}$, $|\tilde{\pi}(\tilde{a}_{i,j})-\pi(a_{p,k})|<\eta$, $|\tilde{\pi}(\tilde{a}_{p,k})-\pi(a_{p,k})|<\eta$, and there exists an $s\in J$ with $a_{p,k}\in Z_s$ and $\tilde{a}_{i,j}\in\tilde{Z}_s$. We have then, that $\psi(i,j)=a_{p,k}$.

(k) If $(i,j)\in\mathcal{B}_2$, then there exists a $(u,0)\in\mathcal{A}_0$ with $a_{p,k}\in A(u,0)$. We have then, that $\tilde{A}(i,j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(u,0)$. Furthermore we have $|\tilde{\pi}(x)-\pi(a_{p,k})|<\eta$ for all $x\in\tilde{A}(i,j)$, and $|\tilde{\pi}(x)-\pi(Ta_{p,k})|<\eta$ for all $x\in C$, if $\tilde{A}(i,j)\to C$ in $\tilde{\mathcal{D}}$.

(l) $(i,j),(u,v)\in\mathcal{B}_2$ and $(i,j)\to(u,v)$ in $\tilde{\mathcal{A}}$ imply $\psi(i,j)\to\psi(u,v)$ in $\mathcal{G}$.

(m) Suppose that $s\in\mathbb{N}$, $s\le r$, and that $(q,0)\in\tilde{\mathcal{A}}_0$ and $(i,j)\in\mathcal{B}_0\cup\mathcal{B}_1$. If $c_0=(q,0)\to c_1\to\cdots\to c_s=(i,j)$ is a path of length s in $\tilde{\mathcal{A}}$, then there exists a path $d_0=(q,0)\to d_1\to\cdots\to d_s$ in $\mathcal{A}_r$, where $A(d_s)=[a_{i_1,j_1},a_{i_2,j_2}]$ with $(i_1,j_1),(i_2,j_2)\in\mathcal{A}_r$, such that either $\pi(a_{i_1,j_1})=\pi(a_{i,j})$ or $\pi(a_{i_2,j_2})=\pi(a_{i,j})$. Then we set $\chi_s(c_0\to c_1\to\cdots\to c_s):=(d_0,d_1,\dots,d_s)$. We have that $\tilde{A}(i,j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(d_s)$, and $(i,j)\in\mathcal{B}_0$ implies $d_s=(i,j)$. Furthermore we have either $d_s=(i,j)$ or $|\tilde{\pi}(x)-\pi(a_{i_1,j_1})|<\eta$ for all $x\in\tilde{A}(i,j)$ or $|\tilde{\pi}(x)-\pi(a_{i_2,j_2})|<\eta$ for all $x\in\tilde{A}(i,j)$.

(n) Suppose that $s\in\mathbb{N}$, $s\le r$, and that $(q,0)\in\tilde{\mathcal{A}}_0$ and $(i,j)\in\mathcal{B}_2$. If $c_0=(q,0)\to c_1\to\cdots\to c_s=(i,j)$ is a path of length s in $\tilde{\mathcal{A}}$, then there exists a $t\in\{0,1,\dots,s-1\}$ with $t=\max\{l\in\{0,1,\dots,s\}:c_l\in\mathcal{B}_0\cup\mathcal{B}_1\}$. Then we set $\chi_s(c_0\to c_1\to\cdots\to c_s):=(d_0,d_1,\dots,d_s)$, where $(d_0,d_1,\dots,d_t):=\chi_t(c_0\to c_1\to\cdots\to c_t)$ and $d_l:=\psi(c_l)$ for $l\in\{t+1,t+2,\dots,s\}$.

(o) Suppose that $s\in\mathbb{N}$, $s\le r$, and that $(q,0)\in\tilde{\mathcal{A}}_0$ and $(i,j)\in\tilde{\mathcal{A}}_r$. If $c_0=(q,0)\to c_1\to\cdots\to c_s=(i,j)$ is a path of length s in $\tilde{\mathcal{A}}$, if (p,k) satisfies $k\le j$ and $|\tilde{\pi}(x)-\pi(a_{p,k})|<\eta$ for all $x\in\tilde{A}(i,j)$, and if $t\in J$ satisfies $a_{p,k}\in Z_t$ and $\tilde{A}(i,j)\subseteq\tilde{Z}_t$, then there exist $(u_1,v_1),(u_2,v_2)\in\tilde{\mathcal{A}}_r$ with the following properties. $x\in[\tilde{a}_{u_1,v_1},\tilde{a}_{u_2,v_2}]$ implies $x\in\tilde{Z}_t$ and $|\tilde{\pi}(x)-\pi(a_{p,k})|<\eta$. There are at most s different paths $d_0=(q,0)\to d_1\to\cdots\to d_s$ of length s in $\tilde{\mathcal{A}}_r$ with $\chi_s(d_0\to d_1\to\cdots\to d_s)=\chi_s(c_0\to c_1\to\cdots\to c_s)$ and $|\tilde{\pi}(x)-\pi(a_{p,k})|<\eta$ for all $x\in\tilde{A}(d_s)$, and each of these paths satisfies $\tilde{A}(d_s)\subseteq[\tilde{a}_{u_1,v_1},\tilde{a}_{u_2,v_2}]$, and if $d_0\to d_1\to\cdots\to d_s\ne c_0\to c_1\to\cdots\to c_s$ then $\tilde{A}(d_s)\cap\tilde{A}(c_s)=\emptyset$.

As (f) implies (1) and (2), (a), (b), (f) and (j) imply (3), (c) and (l) imply (4), (a), (b) and (d) imply (5), (a), (b), (d), (j) and (k) imply (6), (m) and (n) imply (7) and (8) (set $\chi:=\chi_r$), and (h), (i), (k), (l), (m), (n) and (o) imply (9) (η can be chosen so small, that $j,v\le r$ and $|\pi(a_{i,j})-\pi(a_{u,v})|<2\eta$ imply $\pi(a_{i,j})=\pi(a_{u,v})$), it remains to show (a)–(o). First we define $\mathcal{A}_0:=\{(i,0):i\in\{1,2,\dots,2N\}\}$ and

$\tilde{\mathcal{A}}_0 := \mathcal{A}_0$. We have that $a_{i,0} \in A(i,0)$ and $\tilde{a}_{i,0} \in \tilde{A}(i,0)$.

Now suppose that $r \in \mathbb{N}_0$. If $r = 0$ then the induction hypothesis is partially trivial, and partially shown in the following proof. If otherwise $r > 0$, then we assume that the extended lemma is shown for $r-1$. Let $\eta > 0$. Define $\alpha_1 := \min\{|\pi(a_{i_1,j_1}) - \pi(a_{i_2,j_2})| : i_1, i_2 \in \{1,2,\dots,2N\}, j_1, j_2 \in \{0,1,\dots,r+1\}, j_1 < j(i_1), j_2 < j(i_2), |\pi(a_{i_1,j_1}) - \pi(a_{i_2,j_2})| \neq 0\}$ and $\alpha_2 := \min\{|\pi(Ta_{i_1,j}) - \pi(a_{i_2,0})| : i_1, i_2 \in \{1,2,\dots,2N\}, j \in \{0,1,\dots,r\}, j < j(i), |\pi(Ta_{i_1,j}) - \pi(a_{i_2,0})| \neq 0\}$. Hence $\alpha_1 > 0$ and $\alpha_2 > 0$. By the piecewise monotonicity of T there exists an $\alpha_3 > 0$, such that $x, y \in Z_s$ for an $s \in J$ and $|\pi(x) - \pi(y)| < \alpha_3$ imply $|\pi(Tx) - \pi(Ty)| < \frac{1}{4}\min\{\eta, \alpha_1, \alpha_2\}$. Set $\eta_0 := \frac{1}{4}\min\{\eta, \alpha_1, \alpha_2, \alpha_3\}$. Then there exists a $\delta > 0$ with $\delta \leq \eta_0$, such that the extended lemma holds for r replaced by $r-1$ and η replaced by η_0 (in the case $r = 0$ set $\delta = \eta_0$). Now let $(\tilde{T}, \tilde{\mathcal{Z}})$ be δ–close to $(T, \mathcal{Z})$ with respect to the R^0–topology and let $\tilde{\mathcal{Y}}$ be a finite partition of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, which is δ–close to $\mathcal{Y}$.

We show at first

$$
\begin{aligned}
&s \in J\,,\ x \in Z_s\,,\ y \in \tilde{Z}_s\,,\ \text{then } |\pi(x) - \tilde{\pi}(y)| < \eta_0 \implies \\
&\qquad |\pi(Tx) - \tilde{\pi}(\tilde{T}y)| < \frac{1}{2}\min\{\eta, \alpha_1, \alpha_2\}\,.
\end{aligned} \tag{3.1}
$$

As $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ–close to $(T, \mathcal{Z})$, there exists an $a \in Z_s$ with $|\pi(x) - \pi(a)| \leq \eta_0 < \alpha_3$, such that $|\pi(Ta) - \tilde{\pi}(\tilde{T}y)| < \delta \leq \frac{1}{4}\min\{\eta, \alpha_1, \alpha_2\}$. Hence

$$
\begin{aligned}
&|\pi(Tx) - \tilde{\pi}(\tilde{T}y)| \leq |\pi(Tx) - \pi(Ta)| + |\pi(Ta) - \tilde{\pi}(\tilde{T}y)| < \\
&\frac{1}{4}\min\{\eta, \alpha_1, \alpha_2\} + \delta \leq \frac{1}{2}\min\{\eta, \alpha_1, \alpha_2\},
\end{aligned}
$$

which shows (3.1).

Next we show for $i, i_1, i_2 \in \{1,2,\dots,2N\}$ and $j \in \mathbb{N}_0$, $0 \leq j < \min\{j(i), r+2\}$

$$
\begin{aligned}
&\pi(a_{i_1,0}) < \pi(a_{i,j}) < \pi(a_{i_2,0})\,,\ x \in \tilde{X}\,, \\
&|\tilde{\pi}(x) - \pi(a_{i,j})| < \frac{3}{4}\min\{\eta, \alpha_1, \alpha_2\} \implies \tilde{\pi}(\tilde{a}_{i_1,0}) < \tilde{\pi}(x) < \tilde{\pi}(\tilde{a}_{i_2,0})\,,
\end{aligned} \tag{3.2}
$$

and for $i, i_1, i_2 \in \{1,2,\dots,2N\}$ and $j \in \mathbb{N}_0$, $0 \leq j < \min\{j(i), r+1\}$

$$
\begin{aligned}
&\pi(a_{i_1,0}) < \pi(Ta_{i,j}) < \pi(a_{i_2,0})\,,\ x \in \tilde{X}\,, \\
&|\tilde{\pi}(x) - \pi(Ta_{i,j})| < \frac{3}{4}\min\{\eta, \alpha_1, \alpha_2\} \implies \tilde{\pi}(\tilde{a}_{i_1,0}) < \tilde{\pi}(x) < \tilde{\pi}(\tilde{a}_{i_2,0})\,.
\end{aligned} \tag{3.3}
$$

By the definition of α_1 we get $\pi(a_{i_1,0}) + \alpha_1 \leq \pi(a_{i,j}) \leq \pi(a_{i_2,0}) - \alpha_1$. As $\tilde{\mathcal{Y}}$ is δ–close to $\mathcal{Y}$ and $\delta \leq \eta_0 \leq \frac{\alpha_1}{4}$ this gives

$$\tilde{\pi}(\tilde{a}_{i_1,0}) < \pi(a_{i_1,0}) + \frac{\alpha_1}{4} \leq \pi(a_{i,j}) - \frac{3\alpha_1}{4} < \tilde{\pi}(x) < \pi(a_{i,j}) + \frac{3\alpha_1}{4} \leq \pi(a_{i_2,0}) - \frac{\alpha_1}{4} < \tilde{\pi}(\tilde{a}_{i_2,0}),$$

which shows (3.2). An analoguous calculation shows (3.3). From (3.2) and (3.3) it follows easily, that for $i \in \{1,2,\ldots,2N\}$ and $j \in \mathbb{N}_0$, $0 \leq j < \min\{j(i), r+2\}$

(3.4)
$$a_{i,j} \notin E_0\,,\ a_{i,j} \in Y_t \text{ for a } t \in \{1,2,\ldots,N\}\,,\ x \in \tilde{X}\,,\quad |\tilde{\pi}(x) - \pi(a_{i,j})| < \frac{3}{4}\min\{\eta,\alpha_1,\alpha_2\} \implies x \notin \tilde{E}_0\,,\ x \in \tilde{Y}_t\,,$$

and for $i \in \{1,2,\ldots,2N\}$ and $j \in \mathbb{N}_0$, $0 \leq j < \min\{j(i), r+1\}$

(3.5)
$$Ta_{i,j} \notin E_0\,,\ Ta_{i,j} \in Z_s \text{ for an } s \in \{1,2,\ldots,L\}\,,\ x \in \tilde{X}\,,\quad |\tilde{\pi}(x) - \pi(Ta_{i,j})| < \frac{3}{4}\min\{\eta,\alpha_1,\alpha_2\} \implies x \notin \tilde{E}_0\,,\ x \in \tilde{Z}_s\,.$$

Suppose that $(i,r) \in \tilde{\mathcal{A}}_r$. Throughout this proof we assume, that there exists a $(u,v) \in \tilde{\mathcal{A}}_r$ with $\tilde{A}(i,r) \subseteq \tilde{A}(u,v)$ and $\tilde{A}(i,r) = [\tilde{a}_{u,v}, \tilde{a}_{i,r}]$, and that $\tilde{T} \mid \tilde{A}(i,r)$ is strictly increasing (analoguous considerations show the desired properties, if $\tilde{A}(i,r) = [\tilde{a}_{i,r}, \tilde{a}_{u,v}]$, or if $\tilde{T} \mid \tilde{A}(i,r)$ is strictly decreasing).

At first we suppose that $(i,r) \in \mathcal{A}_r$. Then set $\varphi(i,r) := (i,r)$. We have $|\pi(a_{i,r}) - \tilde{\pi}(\tilde{a}_{i,r})| < \eta_0 < \eta$ and there exist $s_1 \in J$, $t_1 \in \{1,2,\ldots,N\}$ with $a_{i,r} \in Z_{s_1} \cap Y_{t_1}$ and $\tilde{a}_{i,r} \in \tilde{Z}_{s_1} \cap \tilde{Y}_{t_1}$. Furthermore there exists a $(u,v) \in \mathcal{A}_r$ with $A(i,r) = [a_{u,v}, a_{i,r}] \subseteq A(u,v)$ and $\tilde{A}(i,r) = [\tilde{a}_{u,v}, \tilde{a}_{i,r}] \subseteq \tilde{A}(u,v)$. $T \mid Z_s$ is strictly increasing. As $|\pi(a_{u,v}) - \tilde{\pi}(\tilde{a}_{u,v})| < \eta_0 < \eta$ we get by (3.1) that $|\pi(Ta_{i,r}) - \tilde{\pi}(\tilde{T}\tilde{a}_{i,r})| < \frac{1}{2}\min\{\eta,\alpha_1,\alpha_2\} < \eta$ and $|\pi(Ta_{u,v}) - \tilde{\pi}(\tilde{T}\tilde{a}_{u,v})| < \frac{1}{2}\min\{\eta,\alpha_1,\alpha_2\} < \eta$. Set $\mathcal{C} := \{C \in \mathcal{D} : A(i,r) \to C, Ta_{i,r} \notin C\}$, $\mathcal{C}_1 := \{C \in \mathcal{C} : Ta_{u,v} \notin C\}$. If $\mathcal{C}_1 \neq \emptyset$, then there exist $j_1, j_2 \in \{1,2,\ldots,N\}$ with $\mathcal{C}_1 = \{Y_j : j_1 \leq j \leq j_2\}$ and $Ta_{u,v} < \inf Y_{j_1} < \sup Y_{j_2} < Ta_{i,r}$. Now (3.3) gives $\tilde{T}\tilde{a}_{u,v} < \inf \tilde{Y}_{j_1} < \sup \tilde{Y}_{j_2} < \tilde{T}\tilde{a}_{i,r}$, which shows $\tilde{A}(i,r) \to \tilde{Y}_j$ for $j_1 \leq j \leq j_2$. For every $C \in \mathcal{C}_1$ there exists an $i(C) \in \{1,2,\ldots,2N\}$ with $C = A(i(C),0)$ and the arrow $(i,r) \to (i(C),0)$ is allowed in $\mathcal{A}$ and $\tilde{\mathcal{A}}$. Furthermore we have $C = Y_j$ for a $j \in \{j_1, j_1+1, \ldots, j_2\}$ and $\tilde{Y}_j = \tilde{A}(i(C),0)$. Hence we make the arrows $(i,r) \to (i(C),0)$ in $\mathcal{A}$ and $\tilde{\mathcal{A}}$. The construction below shows, that if $\mathcal{C}\backslash\mathcal{C}_1 \neq \emptyset$, then there exists a $(\tilde{u},\tilde{v}) \in \mathcal{A}_{r+1}$ with

$(i,r) \to (\tilde{u},\tilde{v})$ in $\mathcal{A}$ and $\tilde{\mathcal{A}}$, $A(\tilde{u},\tilde{v}) \subseteq Y_{j_1-1}$ and $\tilde{A}(\tilde{u},\tilde{v}) \subseteq \tilde{Y}_{j_1-1}$. Besides these arrows there exists at most one $(u_0,v_0) \in \tilde{\mathcal{A}}_{r+1} \setminus \mathcal{A}_{r+1}$ with $(i,r) \to (u_0,v_0)$ in $\tilde{\mathcal{A}}$ and $\tilde{T}\tilde{a}_{u,v} \in \tilde{A}(u_0,v_0)$ (in this case $(\tilde{u},\tilde{v}) \in \mathcal{A}_0$, $A(\tilde{u},\tilde{v}) = Y_{j_1-1}$ and $\tilde{A}(\tilde{u},\tilde{v}) = \tilde{Y}_{j_1-1}$). This shows (c) for the arrows $(i,r) \to d$ with $Ta_{i,r} \notin A(d)$.

Now we describe all other arrows in $\mathcal{A}$ and $\tilde{\mathcal{A}}$ beginning in (i,r). Suppose, that $Ta_{i,r} \in X$. Then there exists an $l \in \{1,2,\ldots,2N\}$ with $a_{i,r+1} = Ta_{i,r} \in A(l,0)$ and $a_{i,r+1} \le a_{l,0}$. Hence either $a_{i,r+1} = a_{l,0}$ or $a_{l,0} \notin TA(i,r)$. Set $D := TA(i,r) \cap A(l,0)$, which is a successor of $A(i,r)$, and set $\tilde{D} := \tilde{T}\tilde{A}(i,r) \cap \tilde{A}(l,0)$. Using $Ta_{u,v} < a_{l,0}$ (3.3) gives $\tilde{T}\tilde{a}_{u,v} < \tilde{a}_{l,0}$, which implies that $\tilde{D}$ is a successor of $\tilde{A}(i,r)$. If $\tilde{T}\tilde{a}_{u,v} \in \tilde{D}$, then $\tilde{a}_{u,v+1} = \tilde{T}\tilde{a}_{u,v} \in \tilde{A}(l,0)$, and (3.3) gives $a_{u,v+1} = Ta_{u,v} \in A(l,0)$. In this case set $u_1 := u$ and $v_1 := v+1$, which implies $v_1 \le r+1$. If otherwise $\tilde{T}\tilde{a}_{u,v} \notin \tilde{D}$, then there exists a $u_1 \in \{1,2,\ldots,2N\}$ with $u_1 \ne l$ and $\tilde{a}_{u_1,0} \in \tilde{D}$. (3.3) gives $a_{u_1,0} \in D$. Set $v_1 := 0$ in this case, which gives $v_1 \le r+1$. In both cases we have $D = [a_{u_1,v_1}, a_{i,r+1}] \subseteq A(u_1,v_1)$ and $\tilde{D} = [\tilde{a}_{u_1,v_1}, \sup\tilde{D}] \subseteq \tilde{A}(u_1,v_1)$ with $\tilde{a}_{u_1,v_1} < \sup\tilde{D} = \min\{\tilde{a}_{l,0}, \tilde{T}\tilde{a}_{i,r}\}$.

Suppose at first, that $\tilde{T}\tilde{a}_{i,r} \notin \tilde{X}$. Then there exists an $s_2 \in \{1,2,\ldots,L\} \setminus J$ with $\tilde{T}\tilde{a}_{i,r} \in \tilde{Z}_{s_2}$. By (3.5) we have $Ta_{i,r} \in Z_{s_2}$ or $a_{i,r+1} = Ta_{i,r} \in E$. In the first case we have $j(i) = \tilde{j}(i) = r+1$ and the arrows described above are all arrows in $\mathcal{A}$ and $\tilde{\mathcal{A}}$ beginning in (i,r). We have $(i,r+1) \notin \mathcal{A}_{r+1} \cup \tilde{\mathcal{A}}_{r+1}$. In the second case the above shows, that we can choose the variants $\mathcal{A}$ and $\tilde{\mathcal{A}}$ such that $(u_1,v_1) \in \mathcal{A}_{r+1} \cap \tilde{\mathcal{A}}_{r+1}$, $D \subseteq A(u_1,v_1)$ and $\tilde{D} = \tilde{A}(u_1,v_1)$. $a_{i,r+1} \in E$ gives $a_{i,r+1} = a_{l,0}$, which implies $D = A(u_1,v_1)$. We choose $\mathcal{A}$ such, that $(i,r+1) \notin \mathcal{A}_{r+1}$ and $(i,r) \to (u_1,v_1)$ is an arrow in $\mathcal{A}$. As $\tilde{j}(i) = r+1$ we have $(i,r+1) \notin \tilde{\mathcal{A}}_{r+1}$. We make the arrow $(i,r) \to (u_1,v_1)$ in $\tilde{\mathcal{A}}$, which shows (c) in this case.

It remains to consider the case $\tilde{T}\tilde{a}_{i,r} \in \tilde{X}$. In this case $\tilde{a}_{i,r+1} = \tilde{T}\tilde{a}_{i,r}$ and there exist $s_2 \in J$, $t_2 \in \{1,2,\ldots,N\}$ with $\tilde{a}_{i,r+1} \in \tilde{Z}_{s_2} \cap \tilde{Y}_{t_2}$. By (3.4) and (3.5) we have $a_{i,r+1} = Ta_{i,r} \in Z_{s_2} \cap Y_{t_2}$ or $Ta_{i,r} \in E_0$. First we consider the case $a_{i,r+1} \in Z_{s_2} \cap Y_{t_2}$. We have $\tilde{A}(i,r+1) = \tilde{D}$. In this case we choose the variants such, that $(i,r+1) \in \mathcal{A}_{r+1} \cap \tilde{\mathcal{A}}_{r+1}$ and $(i,r) \to (i,r+1)$ is an arrow in $\mathcal{A}$ and in $\tilde{\mathcal{A}}$. This shows (a), (c), (d), (e) and (g) in this case.

Now we consider the case $Ta_{i,r} \in E_0$. At first we consider the case $a_{i,r+1} = Ta_{i,r} \in X$. We have then $a_{i,r+1} = a_{l,0}$, $D = A(u_1,v_1)$, $\tilde{D} = \tilde{A}(u_1,v_1)$ and $\tilde{A}(i,r+1) = [\tilde{a}_{p,0}, \tilde{a}_{i,r+1}]$, where $a_{p,0} = c(a_{l,0})$. We choose $\mathcal{A}$ such, that $(i,r+1) \notin \mathcal{A}_{r+1}$ and $(i,r) \to (u_1,v_1)$ is an arrow in $\mathcal{A}$. Furthermore we choose $\tilde{\mathcal{A}}$ such, that $(i,r+1) \in \tilde{\mathcal{A}}_{r+1}$ and $(i,r) \to (u_1,v_1)$ and $(i,r) \to (i,r+1)$ are arrows in $\tilde{\mathcal{A}}$.

This shows (c) and (h) in this case. If otherwise $Ta_{i,r} \notin X$, then $j(i) = r+1$, $(i, r+1) \notin \mathcal{A}$, and the arrows described above are all arrows in $\mathcal{A}$ beginning in (i,r). We have in this case $\tilde{A}(i,r+1) = [\tilde{a}_{p,0}, \tilde{a}_{i,r+1}]$, where $a_{p,0} = c(Ta_{i,r})$. Then we choose $\tilde{\mathcal{A}}$ such, that $(i,r+1) \in \tilde{\mathcal{A}}_{r+1}$ and $(i,r) \to (i,r+1)$ is an arrow in $\tilde{\mathcal{A}}$. Hence (h) is satisfied in this case.

Therefore we have shown (a)–(e) and (h). Furthermore the proof gives, that every $(i,j) \in \mathcal{A}_r \subseteq \tilde{\mathcal{A}}_r$ has at most two successors in $\tilde{\mathcal{A}}$, which are not in $\tilde{\mathcal{A}}_0 = \mathcal{A}_0$. Hence every $(i,j) \in \mathcal{B}_0 := \varphi(\mathcal{A}_r) = \mathcal{A}_r$ has at most two successors in $\mathcal{B}_1 \cup \mathcal{B}_2$.

Now we suppose that $(i,r) \in \tilde{\mathcal{A}}_r \setminus \mathcal{A}_r$. Let $s \in J$ be such, that $\tilde{A}(i,r) \subseteq \tilde{Z}_s$. If there exists a $p \in \{1,2,\ldots,2N\}$ and a $k \in \{0,1,\ldots,r-1\}$ with $a_{p,0} \in E(T)$ and $|\tilde{\pi}(\tilde{a}_{i,r}) - \pi(a_{p,k})| < \eta_0$, then we say $(i,r) \in \mathcal{B}_2$, otherwise we say $(i,r) \in \mathcal{B}_1$.

Suppose that $(i,r) \in \mathcal{B}_2$. By (3.4) and (3.5) we can assume, that $a_{p,l} \notin E(T)$ for $l \in \{1,2,\ldots,k\}$, $a_{p,k} \in Z_s$, $\tilde{a}_{p,k} \in \tilde{Z}_s$ and $|\tilde{\pi}(\tilde{a}_{p,k}) - \pi(a_{p,k})| < \eta_0$. Set $\psi(i,r) := a_{p,k}$. By the definition of α_1 and α_2, and as $\eta_0 \le \frac{1}{4}\min\{\alpha_1,\alpha_2\}$, we get $|\tilde{\pi}(x) - \pi(a_{p,k})| < \eta_0 < \eta$ for all $x \in \tilde{A}(i,r)$, which shows (j). Furthermore (3.1) gives $|\tilde{\pi}(\tilde{T}x) - \pi(Ta_{p,k})| < \frac{1}{2}\min\{\eta,\alpha_1,\alpha_2\} < \eta$ for all $x \in \tilde{A}(i,r)$. By (3.3) this property shows, that every $(i,j) \in \mathcal{B}_2$ has at most two successors in $\tilde{\mathcal{A}}$, and that $(i,j) \in \mathcal{B}_2$, $(u,v) \in \tilde{\mathcal{A}}_r$ and $(i,j) \to (u,v)$ imply $(u,v) \in \mathcal{B}_2$ and $\psi(i,j) \to \psi(u,v)$. Hence (l) is shown. Let $(u,0) \in \mathcal{A}_0$ satisfy $a_{p,k} \in A(u,0)$. Then $A(u,0) \subseteq Z_s$, which implies $\tilde{A}(u,0) \subseteq \tilde{Z}_s$. Furthermore if $t \in \{1,2,\ldots,N\}$ satisfies $A(u,0) = Y_t$, then (3.2) gives $\tilde{\pi}(\inf \tilde{Y}_{t-1}) < \tilde{\pi}(x) < \tilde{\pi}(\sup \tilde{Y}_{t+1})$ for all $x \in \tilde{A}(i,r)$ (we set $\inf \tilde{Y}_0 := -\infty$ and $\sup \tilde{Y}_{N+1} := \infty$). Hence $\tilde{A}(i,r)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(u,0)$, and this shows (k).

Now suppose that $(i,r) \in \mathcal{B}_1$. Then we have either $(i,r-1) \in \mathcal{A}_{r-1}$ or $(i,r-1) \in \mathcal{B}_1$. Hence we have $|\tilde{\pi}(x) - \pi(Ta_{i,r-1})| < \eta_0 < \eta$ for all $x \in \tilde{A}(i,r)$. Hence (3.3) gives $a_{i,r} = Ta_{i,r-1} \in X$. This gives (g), and (3.1) gives $|\tilde{\pi}(\tilde{T}x) - \pi(Ta_{i,r})| < \frac{1}{2}\min\{\eta,\alpha_1,\alpha_2\} < \eta$ for all $x \in \tilde{A}(i,r)$, which shows (i). Using (3.3), the definition of α_1 and α_2 and the fact $\eta_0 \le \frac{1}{4}\min\{\alpha_1,\alpha_2\}$ this shows, that every $(i,j) \in \mathcal{B}_1$ has at most two successors in $\tilde{\mathcal{A}}$, and $(i,j) \in \mathcal{B}_1$, $(u,v) \in \tilde{\mathcal{A}}_r$ and $(i,j) \to (u,v)$ imply $(u,v) \in \mathcal{B}_1 \cup \mathcal{B}_2$. Now we have shown (a)–(l).

As every path $c_0 \to c_1 \to \cdots \to c_s$ of length $s < r$ in $\tilde{\mathcal{A}}$ with $c_0 \in \tilde{\mathcal{A}}_0$ is in $\tilde{\mathcal{A}}_{r-1}$, it remains to show (m)–(o) for $s = r$. Suppose that $(q,0) \in \tilde{\mathcal{A}}_0$, $(i,j) \in \tilde{\mathcal{A}}_r$, and that $c_0 = (q,0) \to c_1 \to \cdots \to c_r = (i,j)$ is a path of length r in $\tilde{\mathcal{A}}_r$. Then $c_0 = (q,0) \to c_1 \to \cdots \to c_{r-1}$ is a path of length $r-1$ in $\tilde{\mathcal{A}}_{r-1}$. Suppose that $c_{r-1} = (i_0, j_0)$.

Suppose at first $(i,j) \in \mathcal{B}_0 \cup \mathcal{B}_1$. We have then $(i_0, j_0) \in \mathcal{B}_0 \cup \mathcal{B}_1$ and there exists a path $d_0 = (q,0) \to d_1 \to \cdots \to d_{r-1}$ in $\mathcal{A}_{r-1}$, where $A(d_{r-1}) = [a_{k_1,l_1}, a_{k_2,l_2}]$ with $(k_1,l_1),(k_2,l_2) \in \mathcal{A}_{r-1}$, such that either $\pi(a_{k_1,l_1}) = \pi(a_{i_0,j_0})$ or $\pi(a_{k_2,l_2}) = \pi(a_{i_0,j_0})$. Furthermore $(i_0,j_0) \in \mathcal{B}_0$ implies $d_{r-1} = (i_0,j_0)$, and $(i_0,j_0) \in \mathcal{B}_1$ implies $|\tilde{\pi}(x) - \pi(a_{i_0,j_0})| < \eta_0$ for all $x \in \tilde{A}(i_0,j_0)$. If $(i,j) \in \mathcal{B}_0$, then we have $(i_0,j_0) \in \mathcal{B}_0$ and $(i_0,j_0) \to (i,j)$ in $\tilde{\mathcal{A}}$. Hence (c) implies $(i_0,j_0) \to (i,j)$ in $\mathcal{A}$, which shows (m) in this case. If otherwise $(i,j) \in \mathcal{B}_1$, then (h) and (i) give, that we can assume $|\tilde{\pi}(x) - \pi(Ta_{k_2,l_2})| < \eta_0$ for all $x \in \tilde{A}(i,j)$. As $(i,j) \in \mathcal{B}_1$ (3.3) gives $a_{k_2,l_2+1} = Ta_{k_2,l_2} \in X$, and $\tilde{A}(i,j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(b)$, where $b \in \mathcal{A}_0$ satisfies $a_{k_2,l_2+1} \in A(b)$. Hence there exists a $d_r \in \mathcal{A}_r$ with $d_{r-1} \to d_r$ and $a_{k_2,l_2+1} \in A(d_r)$. Therefore $\tilde{A}(i,j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(d_r)$. Now (i), the definition of α_1 and the fact $\eta_0 \le \frac{\alpha_1}{4}$ give $\pi(a_{k_2,l_2+1}) = \pi(a_{i,j})$. This shows (m).

Now suppose that (p,k) satisfies $k \le j$ and $|\tilde{\pi}(x) - \pi(a_{p,k})| < \eta_0$ for all $x \in \tilde{A}(i,j)$, and there exists a $t \in J$ with $a_{p,k} \in Z_t$ and $\tilde{A}(i,j) \subseteq \tilde{Z}_t$. Then there exists a (p_0,k_0) with $k_0 \le j_0$ and $\pi(Ta_{p_0,k_0}) = \pi(a_{p,k})$, and there exists a $t_0 \in J$ with $a_{p_0,k_0} \in Z_{t_0}$ and $\tilde{A}(i_0,j_0) \subseteq \tilde{Z}_{t_0}$, such that either $|\tilde{\pi}(x) - \pi(a_{p_0,k_0})| < \eta_0$ or $(i_0,j_0) \in \mathcal{B}_0$ and $a_{p_0,k_0} \in \{\inf A(i_0,j_0), \sup A(i_0,j_0)\}$. Set $\mathcal{Q} := \{d_0 = (q,0) \to d_1 \to \cdots \to d_{r-1}$ is a path of length $r-1$ in $\tilde{\mathcal{A}}_{r-1}$ with $\chi_{r-1}(d_0 \to d_1 \to \cdots \to d_{r-1}) = \chi_{r-1}(c_0 \to c_1 \to \cdots \to c_{r-1}) : |\tilde{\pi}(x) - \pi(a_{p_0,k_0})| < \eta_0$ for all $x \in \tilde{A}(d_{r-1})$, or $d_{r-1} \in \mathcal{B}_0$ and $a_{p_0,k_0} \in \{\inf A(d_{r-1}), \sup A(d_{r-1})\}\}$ and $\mathcal{Q}_0 := \{d_0 = (q,0) \to d_1 \to \cdots \to d_{r-1} \in \mathcal{Q} : |\tilde{\pi}(x) - \pi(a_{p_0,k_0})| < \eta_0$ for all $x \in \tilde{A}(d_{r-1})\}$. There exist $(u_1,v_1),(u_2,v_2)$, which satisfy (o) for $s = r-1$ and (p,k) replaced by (p_0,k_0) (observe that the property $[\tilde{a}_{u_1,v_1}, \tilde{a}_{u_2,v_2}] \subseteq \tilde{Z}_{t_0}$ remains true in the case $(i_0,j_0) \in \mathcal{B}_0$). We have that the elements of $\{\tilde{A}(d_{r-1}) : d_0 = (q,0) \to d_1 \to \cdots \to d_{r-1} \in \mathcal{Q}\}$ are pairwise disjoint. Hence the elements of $\{\tilde{T}\tilde{A}(d_{r-1}) : d_0 = (q,0) \to d_1 \to \cdots \to d_{r-1} \in \mathcal{Q}\}$ are also pairwise disjoint. By (3.3) we get, that at most one element of $\{d_{r-1} : d_0 \to d_1 \to \cdots \to d_{r-1} \in \mathcal{Q}_0\}$ has more than one successor in $\tilde{\mathcal{A}}$. If $\mathcal{Q} \backslash \mathcal{Q}_0 \ne \emptyset$, then it contains a unique element $d_0 \to d_1 \to\to d_{r-1}$, and if $d_r \in \tilde{\mathcal{A}}_r$ satisfies $d_{r-1} \to d_r$ and $|\tilde{\pi}(x) - \pi(a_{p,k})| < \eta_0$ for all $x \in \tilde{A}(d_r)$, then d_r is uniquely determined by these properties, and every element of $\{\tilde{d}_{r-1} : \tilde{d}_0 \to \tilde{d}_1 \to \cdots \to \tilde{d}_{r-1} \in \mathcal{Q}_0\}$ has at most one successor in $\tilde{\mathcal{A}}$. As every $d_0 = (q,0) \to d_1 \to \cdots \to d_r$ with $\chi_r(d_0 \to d_1 \to \cdots \to d_r) = \chi_r(c_0 \to c_1 \to \cdots \to c_r)$ and $|\tilde{\pi}(x) - \pi(a_{p,k})| < \eta_0$ for all $x \in \tilde{A}(d_r)$ satisfies $\tilde{A}(d_r) \subseteq \tilde{T}[\tilde{a}_{u_1,v_1}, \tilde{a}_{u_2,v_2}] \cap \tilde{Z}_t$ or $d_{r-1} \in \mathcal{B}_0$, (3.1) gives (o). This finishes the proof of this lemma. ∎

4. Continuity of the pressure

In this section we shall use the results of Section 3 to prove continuity results about the pressure. We consider a weighted piecewise monotonic map $(T, f, \mathcal{Z})$ of class W^0.

THEOREM 1: *Let $(T, f, \mathcal{Z})$ be a piecewise monotonic map of class W^0, and suppose that*

$$p(R(T), T, f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f).$$

Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ is δ–close to $(T, f, \mathcal{Z})$ with respect to the W^0–topology implies

$$p(R(T), T, f) - \varepsilon < p(R(\tilde{T}), \tilde{T}, \tilde{f}) .$$

Proof: Let $\varepsilon > 0$. We can assume, that ε is small enough to ensure

$$p(R(T), T, f) > \frac{\varepsilon}{3} + \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f) . \tag{4.1}$$

By the piecewise continuity of f there exists a finite partition $\mathcal{Y} = \{Y_1, Y_2, \dots, Y_N\}$ with $Y_1 < Y_2 < \dots < Y_N$ of X refining $\mathcal{Z}$, such that

$$\sup_{Y\in\mathcal{Y}} \sup_{x,y\in Y} |f(x) - f(y)| < \frac{\varepsilon}{3} . \tag{4.2}$$

If $x \in Y$ for a $Y \in \mathcal{Y}$, then define

$$f_1(x) := \sup_{y\in Y} f(y).$$

Then f_1 is (more exactly: can be extended to) a piecewise constant function $f_1 : X \to \mathbb{R}$. By the definition of f_1 we get by (1.2) $p(R(T), T, f) \le p(R(T), T, f_1)$. Let $(T_{\mathcal{Y}}, \hat{f}_1, \hat{\mathcal{Y}})$ be the completion of $(T, f_1, \mathcal{Y})$ with respect to $\mathcal{Y}$ (for simplicity we shall use the notation $(T_{\mathcal{Y}}, f_1, \mathcal{Y})$ for this completion). Now (4.1) gives

$$p(R(T), T, f_1) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T), f_1).$$

Hence Lemma 5 gives the existence of an $r \in \mathbb{N}$, such that for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$ there exists an irreducible $\mathcal{C} \subseteq \mathcal{A}_r$ with

$$\log r(F_{\mathcal{C}}(f_1)) > p(R(T), T, f_1) - \frac{\varepsilon}{3} \ge p(R(T), T, f) - \frac{\varepsilon}{3} . \tag{4.3}$$

Fix this r for the rest of this proof. By Lemma 6 there exists a $\delta \in (0, \frac{\varepsilon}{3})$, such that the conclusions of Lemma 6 are true with respect to this r, if $(\tilde{T}, \tilde{\mathcal{Z}})$ is a piecewise monotonic map, which is δ–close to $(T, \mathcal{Z})$ in the R^0–topology.

Suppose that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ is a piecewise monotonic map of class W^0, which is δ–close to $(T, f, \mathcal{Z})$ in the W^0–topology. By Lemma 1 and (4.2) there exists a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_N\}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, such that $\tilde{\mathcal{Y}}$ is δ–close to $\mathcal{Y}$, and

$$\inf_{x \in \tilde{Y}_j} \tilde{f}(x) \geq f_1(y) - \frac{2\varepsilon}{3},$$

where $y \in Y_j$, for all $j \in \{1, 2, \dots, N\}$. Let $(\mathcal{A}, \to)$ and $(\tilde{\mathcal{A}}, \to)$ be the variants of the Markov diagram of T, resp. $\tilde{T}$, with respect to $\mathcal{Y}$, resp. $\tilde{\mathcal{Y}}$, occurring in the conclusion of Lemma 6. If $x \in \tilde{Y}$ for a $\tilde{Y} \in \tilde{\mathcal{Y}}$, then we define

$$f_2(x) := \inf_{y \in \tilde{Y}} \tilde{f}(y).$$

Then $f_2 : \tilde{X} \to \mathbb{R}$ is a piecewise constant function, which satisfies $p(R(\tilde{T}), \tilde{T}, f_2) \leq p(R(\tilde{T}), \tilde{T}, \tilde{f})$ and $f_2(y) \geq f_1(x) - \frac{2\varepsilon}{3}$, if $x \in Y_j$, $y \in \tilde{Y}_j$ for a $j \in \{1, 2, \dots, N\}$. Denote by $(\tilde{T}_{\tilde{\mathcal{Y}}}, f_2, \tilde{\mathcal{Y}})$ the completion of $(\tilde{T}, f_2, \tilde{\mathcal{Y}})$ with respect to $\tilde{\mathcal{Y}}$.

By (4.3) there exists an irreducible $\mathcal{C} \subseteq \mathcal{A}_r$ with

$$\log r(F_{\mathcal{C}}(f_1)) > p(R(T), T, f) - \frac{\varepsilon}{3}.$$

Now consider the matrix $\tilde{F}_{\varphi(\mathcal{C})}(f_2)$, where $\varphi : \mathcal{A}_r \to \tilde{\mathcal{A}}_r$ is the function described in Lemma 6. For $c, d \in \mathcal{C}$ we get by (4), (5) and (6) of Lemma 6, that

$$\tilde{F}_{\varphi(c),\varphi(d)}(f_2) \geq e^{-\frac{2\varepsilon}{3}} F_{c,d}(f_1).$$

Using (3) of Lemma 6 and (2.8) and (2.9) this gives

$$r(\tilde{F}(f_2)) \geq r(\tilde{F}_{\varphi(\mathcal{C})}(f_2)) \geq e^{-\frac{2\varepsilon}{3}} r(F_{\mathcal{C}}(f_1)).$$

Now (2.12) and (4.3) give

$$p(R(\tilde{T}), \tilde{T}, \tilde{f}) \geq p(R(\tilde{T}), \tilde{T}, f_2) = \log r(\tilde{F}(f_2)) \geq$$
$$\log r(F_{\mathcal{C}}(f_1)) - \frac{2\varepsilon}{3} > p(R(T), T, f) - \varepsilon. \quad \blacksquare$$

This theorem shows, that the topological pressure is lower semi–continuous, if

$$p(R(T),T,f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T),f).$$

It generalizes Theorem 9 of [8], where the lower semi–continuity of

$$T \mapsto p([0,1],T,f)$$

with respect to the C^0–topology is shown for continuous piecewise monotonic maps $T : [0,1] \to [0,1]$ and a fixed function $f : [0,1] \to \mathbb{R}$ with $p([0,1],T,f) > \sup f$. Furthermore the lower semi–continuity of the topological entropy with respect to the R^0–topology follows from Theorem 1, if $h_{\text{top}}(R(T),T) > 0$. If otherwise $h_{\text{top}}(R(T),T) = 0$, then the lower semi–continuity is trivial. Hence Theorem 1 implies the well known result (Theorem 5 of [6]) on the lower semi-continuity of the topological entropy. Our proof is similar to the proofs in [6] and [8], where an approximation by "horseshoes" is used instead of our approximation by finite subsets of the Markov diagram. Now we shall give an example, where the pressure is not lower semi–continuous.

Define $\mathcal{Z} := \{(0,\frac{1}{6}),(\frac{1}{6},\frac{1}{3}),(\frac{1}{3},\frac{2}{3}),(\frac{2}{3},1)\}$, define

$$Tx := \begin{cases} 2x & \text{for } x \in [0,\frac{1}{6}], \\ \frac{2}{3} - 2x & \text{for } x \in [\frac{1}{6},\frac{1}{3}], \\ 2x - \frac{2}{3} & \text{for } x \in [\frac{1}{3},\frac{2}{3}], \\ 2 - 2x & \text{for } x \in [\frac{2}{3},1], \end{cases} \tag{4.4}$$

and define

$$f(x) := \begin{cases} 0 & \text{for } x \in [0,\frac{1}{3}], \\ 30x - 10 & \text{for } x \in [\frac{1}{3},\frac{2}{3}], \\ 30 - 30x & \text{for } x \in [\frac{2}{3},1]. \end{cases} \tag{4.5}$$

Then $(T,\mathcal{Z})$ is in E^∞ and $(T,f,\mathcal{Z})$ is in W^∞. We get $R(T) = [0,1]$ and $p([0,1],T,f) = 10$. Observe that the nonwandering set of T is $[0,\frac{1}{3}] \cup \{\frac{2}{3}\}$. The function f is so large at the isolated fixed point $\frac{2}{3}$, such that it dominates the pressure on the rest of the nonwandering set. As we shall see below this fixed point can be destroyed by an arbitrarily small perturbation. The condition

$$p(R(T),T,f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T),f)$$

excludes such a phenomenon. For $\varepsilon \in (0,1)$ we define

$$(4.6) \qquad T_\varepsilon x := \begin{cases} 2x & \text{for } x \in [0, \frac{1}{6}], \\ \frac{2}{3} - 2x & \text{for } x \in [\frac{1}{6}, \frac{1}{3}], \\ (2-\varepsilon)x - \frac{2}{3} + \frac{\varepsilon}{3} & \text{for } x \in [\frac{1}{3}, \frac{2}{3}], \\ 2 - \varepsilon - (2-\varepsilon)x & \text{for } x \in [\frac{2}{3}, 1]. \end{cases}$$

Then $(T_\varepsilon, \mathcal{Z})$ is ε–close to $(T, \mathcal{Z})$ in R^∞, and $(T_\varepsilon, f, \mathcal{Z})$ is ε–close to $(T, f, \mathcal{Z})$ in W^∞. Furthermore we have $R(T_\varepsilon) = [0,1]$, the nonwandering set of T_ε is $[0, \frac{1}{3}]$, and $p([0,1], T_\varepsilon, f) = \log 2$, which shows, that the topological pressure is not lower semi–continuous in this case.

Now we show a result on upper semi–continuity properties of the pressure.

THEOREM 2: *Let $(T, f, \mathcal{Z})$ be a piecewise monotonic map of class W^0. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ is δ–close to $(T, f, \mathcal{Z})$ with respect to the W^0–topology implies*

$$p(R(\tilde{T}), \tilde{T}, \tilde{f}) < \max\{p(R(T), T, f), \log r(G(f))\} + \varepsilon \, .$$

Proof: Let $\varepsilon > 0$. By the piecewise continuity of f there exists a finite partition $\mathcal{Y}$ of X, which refines $\mathcal{Z}$, such that

$$\sup_{Y \in \mathcal{Y}} \sup_{x,y \in Y} |f(x) - f(y)| < \frac{\varepsilon}{4}.$$

Again we suppose, that $\mathcal{Y} = \{Y_1, Y_2, \ldots, Y_N\}$ with $Y_1 < Y_2 < \cdots < Y_N$. If $x \in Y$ for a $Y \in \mathcal{Y}$, then define

$$f_1(x) := \inf_{y \in Y} f(y).$$

Then $f_1 : X \to \mathbb{R}$ is a piecewise constant function, and we have for $j \in \{1, 2, \ldots, N-1\}$

$$(4.7) \qquad |f_1(x) - f_1(y)| < \frac{\varepsilon}{4} \quad \text{for } x \in Y_j, y \in Y_{j+1} \, ,$$

if there exists a $Z \in \mathcal{Z}$ with $Y_j \cup Y_{j+1} \subseteq Z$. We have

$$p(R(T), T, f_1) \leq p(R(T), T, f)$$

and $r(G(f_1)) \leq r(G(f))$. Denote by $(T_\mathcal{Y}, f_1, \mathcal{Y})$ the completion of $(T, f_1, \mathcal{Y})$ with respect to $\mathcal{Y}$. If $(\mathcal{A}, \to)$ is a variant of the Markov diagram of T with respect to

$\mathcal{Y}$, and if $d \in \mathcal{A}$, then let f_d be the unique real number with $f_1(x) = f_d$ for all $x \in A(d)$.

Define

$$R_0 := \exp(\max\{p(R(T), T, f_1), \log r(G(f_1))\} + \varepsilon) . \tag{4.8}$$

By (2.12) we get

$$r(F_{\mathcal{A}}(f_1)) = e^{p(R(T),T,f_1)} \tag{4.9}$$

for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$. As $R_0 > e^{\frac{3\varepsilon}{4}} \max\{e^{p(R(T),T,f_1)}, r(G(f_1))\}$ we can choose an

$$R \in (e^{\frac{3\varepsilon}{4}} \max\{e^{p(R(T),T,f_1)}, r(G(f_1))\}, R_0).$$

As $e^{-\frac{3\varepsilon}{4}} R > r(F_{\mathcal{A}}(f_1))$ and $e^{-\frac{3\varepsilon}{4}} R > r(G(f_1))$ we get by (2.5) and (2.9), that there exists a $C \in \mathbb{R}$, such that

$$\sup_{s \in \mathbb{N}} e^{\frac{3\varepsilon}{4} s} R^{-s} \|F_{\mathcal{A}}(f_1)^s\| \le C \tag{4.10}$$

for every variant $(\mathcal{A}, \to)$ of the Markov diagram of T with respect to $\mathcal{Y}$, and

$$\sup_{s \in \mathbb{N}} e^{\frac{3\varepsilon}{4} s} R^{-s} \|G(f_1)^s\| \le C . \tag{4.11}$$

We can assume that

$$C \ge \max\{2, 8 e^{\frac{3\varepsilon}{4}} R^{-1} \sup_{x \in X_{\mathcal{Y}}} e^{f_1(x)}\}.$$

Fix an $r \in \mathbb{N}$ with

$$\sqrt[r]{(r+1)^2 C^3}\, R < R_0 . \tag{4.12}$$

By Lemma 6 there exists a $\delta \in (0, \frac{\varepsilon}{4})$, such that the conclusions of Lemma 6 are true for every $(\tilde{T}, \tilde{\mathcal{Z}})$, which is δ–close to $(T, \mathcal{Z})$ in the R^0–topology.

Let $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ be a piecewise monotonic map of class W^0, which is δ–close to $(T, f, \mathcal{Z})$ in the W^0–topology. By the choice of $\mathcal{Y}$ Lemma 1 gives the existence of a finite partition $\tilde{\mathcal{Y}} = \{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_N\}$ with $\tilde{Y}_1 < \tilde{Y}_2 < \cdots < \tilde{Y}_N$ of $\tilde{X}$ refining $\tilde{\mathcal{Z}}$, such that $\tilde{\mathcal{Y}}$ is δ–close to $\mathcal{Y}$, and

$$\sup_{x \in \tilde{Y}_j} \tilde{f}(x) < f_1(y) + \frac{\varepsilon}{2},$$

where $y \in Y_j$, for all $j \in \{1,2,\dots,N\}$. If $x \in \tilde{Y}$ for a $\tilde{Y} \in \tilde{\mathcal{Y}}$, then define

$$f_2(x) := \sup_{y \in \tilde{Y}} \tilde{f}(y).$$

Then $f_2 : \tilde{X} \to \mathbb{R}$ is a piecewise constant function. We have $p(R(\tilde{T}),\tilde{T},\tilde{f}) \leq p(R(\tilde{T}),\tilde{T},f_2)$, and $f_2(y) \leq f_1(x) + \frac{\varepsilon}{2}$ holds for $x \in Y_j$, $y \in \tilde{Y}_j$ for a $j \in \{1,2,\dots,N\}$. Denote by $(\tilde{T}_{\tilde{\mathcal{Y}}}, f_2, \tilde{\mathcal{Y}})$ the completion of $(\tilde{T}, f_2, \tilde{\mathcal{Y}})$ with respect to $\tilde{\mathcal{Y}}$. Let $(\mathcal{A},\to)$ and $(\tilde{\mathcal{A}},\to)$ be the variants of the Markov diagram of T, resp. $\tilde{T}$, with respect to $\mathcal{Y}$, resp. $\tilde{\mathcal{Y}}$, occurring in the conclusion of Lemma 6. For $d \in \tilde{\mathcal{A}}$ let $\tilde{f}_d$ be the unique real number with $f_2(x) = \tilde{f}_d$ for all $x \in \tilde{A}(d)$. Set $F(f_1) := (F_{c,d}(f_1))_{c,d\in\mathcal{A}}$ and $\tilde{F}(f_2) := (F_{c,d}(f_2))_{c,d\in\tilde{\mathcal{A}}}$. By (4.7) and by (5) and (6) of Lemma 6 we get for $c \in \tilde{\mathcal{A}}_r$ and $d \in \mathcal{A}_r$

$$\tilde{f}_c < f_d + \frac{3\varepsilon}{4} \quad \text{, if } \tilde{A}(c) \text{ is } \tilde{\mathcal{Y}}\text{–close to } \tilde{A}(\varphi(d))\,. \tag{4.13}$$

By (2.9) we have $r(\tilde{F}(f_2)) \leq \|\tilde{F}(f_2)^r\|^{\frac{1}{r}}$. Lemma 4 (formula (2.10)) gives

$$\begin{aligned} r(\tilde{F}(f_2))^r \leq \|\tilde{F}(f_2)^r\| = \|\tilde{F}_{\tilde{\mathcal{A}}_r}(f_2)^r\| = \\ \sup_{c\in\tilde{\mathcal{A}}_0} \sum_{c_0=c\to c_1\to\cdots\to c_r} \prod_{j=0}^{r-1} e^{\tilde{f}_{c_j}}\,, \end{aligned} \tag{4.14}$$

where the sum is taken over all paths $c_0 \to c_1 \to \cdots \to c_r$ of length r in $\tilde{\mathcal{A}}_r$ with $c_0 = c$. Fix $c \in \tilde{\mathcal{A}}_0$. Then by (1) and (3) of Lemma 6 there exists a unique $d \in \mathcal{A}_0$ with $\varphi(d) = c$.

Let $\mathcal{P}_c$ be the set of all paths $c_0 \to c_1 \to \cdots \to c_r$ of length r in $\tilde{\mathcal{A}}_r$ with $c_0 = c$, and for $s \in \{0,1,\dots,r\}$ let $\mathcal{P}_c(s)$ be the set of all $c_0 \to c_1 \to \cdots \to c_r \in \mathcal{P}_c$ with $s = \max\{j \in \{0,1,\dots,r\} : c_j \in \mathcal{B}_0 \cup \mathcal{B}_1\}$. Hence $\mathcal{P}_c = \bigcup_{s=0}^r \mathcal{P}_c(s)$. Define for $s \in \{0,1,\dots,r\}$

$$H_c(s) := \sum_{c_0\to c_1\to\cdots\to c_r\in\mathcal{P}_c(s)} \prod_{j=0}^{r-1} e^{\tilde{f}_{c_j}}\,. \tag{4.15}$$

Then we have

$$H_c := \sum_{c_0\to c_1\to\cdots\to c_r\in\mathcal{P}_c} \prod_{j=0}^{r-1} e^{\tilde{f}_{c_j}} = \sum_{s=0}^{r} H_c(s)\,. \tag{4.16}$$

Let $s \in \{0,1,\dots,r\}$. If $c_0 \to c_1 \to \cdots \to c_r \in \mathcal{P}_c(s)$ and if $(d_0,d_1,\dots,d_r) = \chi(c_0 \to c_1 \to \cdots \to c_r)$, then (1) and (8) of Lemma 6 give $d_j \in \mathcal{A}_r$ for all $j \le s$, $d_0 = d$, and $d_j \in \mathcal{G}$ for all $j > s$. Furthermore we get by (8) of Lemma 6, that $\tilde{A}(c_j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(\varphi(d_j))$ for all $j \in \{0,1,\dots,s\}$, and if $s \ge 1$, then $d_0 \to d_1 \to \cdots \to d_s$ is a path of length s in $\mathcal{A}_r$ with $d_0 = d$. Hence (4.13) gives $e^{\tilde{f}_{c_s}} \le e^{\frac{3\varepsilon}{4}} e^{f_{d_s}}$ and if $s \ge 1$, then

$$\prod_{j=0}^{s-1} e^{\tilde{f}_{c_j}} \le e^{\frac{3\varepsilon}{4}s} \prod_{j=0}^{s-1} e^{f_{d_j}} .$$

If $s \le r-2$, then we get by (4), (6) and (8) of Lemma 6, that $d_{s+1} \to d_{s+2} \to \cdots \to d_r$ is a path of length $r-s-1$ in $\mathcal{G}$, and $\tilde{A}(c_j)$ is $\tilde{\mathcal{Y}}$–close to $\tilde{A}(\varphi(\tilde{d}_j))$ for all $j \in \{s+1, s+2, \dots, r\}$, where $\tilde{d}_j \in \mathcal{A}_0$ satisfies $d_j \in A(\tilde{d}_j)$. Therefore (4.13) gives

$$\prod_{j=s+1}^{r-1} e^{\tilde{f}_{c_j}} \le e^{\frac{3\varepsilon}{4}(r-s-1)} \prod_{j=s+1}^{r-1} e^{f_1(d_j)}, \quad \text{if } s \le r-2.$$

Hence using (4.15) we get by (9) of Lemma 6 for $1 \le s \le r-2$

$$\begin{aligned} H_c(s) \le (2r+1)e^{\frac{3\varepsilon}{4}r} & \sum_{(d_0,d_1,\dots,d_r)\in\chi(\mathcal{P}_c(s))} e^{f_{d_s}} \Big(\prod_{j=0}^{s-1} e^{f_{d_j}}\Big)\Big(\prod_{j=s+1}^{r-1} e^{f_1(d_j)}\Big) \le \\ & 8e^{\frac{3\varepsilon}{4}r} \sup_{x\in X_{\mathcal{Y}}} e^{f_1(x)} (r+1)\Big(\sum_{d_0=d\to d_1\to\cdots\to d_s} \prod_{j=0}^{s-1} e^{f_{d_j}}\Big) \\ & \Big(\sup_{a\in\mathcal{G}} \sum_{b_0=a\to b_1\to\cdots\to b_{r-s-1}} \prod_{j=0}^{r-s-2} e^{f_1(b_j)}\Big) . \end{aligned}$$

As $C \ge 8e^{\frac{3\varepsilon}{4}} R^{-1} \sup_{x\in X_{\mathcal{Y}}} e^{f_1(x)}$ we get by (2.4), (2.8), (4.10) and (4.11)

$$H_c(s) \le CR(r+1)CR^s CR^{r-s-1} = (r+1)C^3R^r .$$

Analogously we get using $C \ge 8e^{\frac{3\varepsilon}{4}} R^{-1} \sup_{x\in X_{\mathcal{Y}}} e^{f_1(x)}$, $C \ge 2$, (2.4), (2.8), (4.10) and (4.11)

$$\begin{aligned} H_c(0) \le& 8e^{\frac{3\varepsilon}{4}r} \sup_{x\in X_y} e^{f_1(x)}(r+1)(\sup_{a\in \mathcal{G}} \sum_{b_0=a\to b_1\to\cdots\to b_{r-1}} \prod_{j=0}^{r-2} e^{f_1(b_j)}) \le \\ & CR(r+1)CR^{r-1} \le (r+1)C^3R^r , \\ H_c(r-1) \le& 8e^{\frac{3\varepsilon}{4}r} \sup_{x\in X_y} e^{f_1(x)}(r+1)(\sum_{d_0=d\to d_1\to\cdots\to d_{r-1}} \prod_{j=0}^{r-2} e^{f_{d_j}}) \le \\ & CR(r+1)CR^{r-1} \le (r+1)C^3R^r , \\ H_c(r) \le& 2e^{\frac{3\varepsilon}{4}r}(r+1)(\sum_{d_0=d\to d_1\to\cdots\to d_r} \prod_{j=0}^{r-1} e^{f_{d_j}}) \le \\ & C\,(r+1)\,CR^r \le (r+1)C^3R^r . \end{aligned}$$

Hence (4.16) gives $H_c \le (r+1)^2C^3R^r$ for all $c \in \tilde{\mathcal{A}}_0$, and by (4.14) we get $r(\tilde{F}(f_2))^r \le (r+1)^2C^3R^r$. Now (4.12) implies

$$r(\tilde{F}(f_2)) \le \sqrt[r]{(r+1)^2C^3}\,R < R_0 .$$

By (2.12) and (4.8) this gives

$$\begin{aligned} & p(R(\tilde{T}),\tilde{T},\tilde{f}) \le p(R(\tilde{T}),\tilde{T},f_2) = \log r(\tilde{F}(f_2)) < \\ & \log R_0 = \max\{p(R(T),T,f_1),\log r(G(f_1))\} + \varepsilon \\ & \le \max\{p(R(T),T,f),\log r(G(f))\} + \varepsilon. \end{aligned}$$

∎

Corollary 2.1: *Let $(T,f,\mathcal{Z})$ be a piecewise monotonic map of class W^0, such that one of the assumptions of Lemma 2 holds, and suppose that*

$$p(R(T),T,f) > \lim_{n\to\infty} \frac{1}{n} S_n(R(T),f).$$

Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T},\tilde{f},\tilde{\mathcal{Z}})$ is δ–close to $(T,f,\mathcal{Z})$ with respect to the W^0–topology implies

$$|p(R(\tilde{T}),\tilde{T},\tilde{f}) - p(R(T),T,f)| < \varepsilon .$$

Proof: This is an easy consequence of Lemma 2, Theorem 1 and Theorem 2. ∎

Now we want to give an example. Define $\mathcal{Z} := \{(\frac{1}{3}, \frac{2}{3}), (\frac{2}{3}, 1)\}$, define

$$(4.17) \qquad Tx := \begin{cases} 2x - \frac{2}{3} & \text{for } x \in [\frac{1}{3}, \frac{2}{3}], \\ 2 - 2x & \text{for } x \in [\frac{2}{3}, 1]. \end{cases}$$

Then $(T, \mathcal{Z})$ is a piecewise monotonic map of class E^∞ and $(T, 0, \mathcal{Z})$ is of class W^∞. We have $R(T) = \{\frac{2}{3}\}$, $p(R(T), T, 0) = h_{\text{top}}(R(T), T) = 0$, $\mathcal{G} = \{\frac{1}{3}, \frac{2}{3}^-, \frac{2}{3}^+, 1\}$ with $a \to b$ if and only if $a, b \in \{\frac{2}{3}^-, \frac{2}{3}^+\}$, and $\log r(G(0)) = \log 2$. For $\varepsilon \in (0, \frac{1}{3}]$ set

$$(4.18) \qquad T_\varepsilon x := Tx + \varepsilon \ .$$

Then $(T_\varepsilon, \mathcal{Z})$ is ε–close to $(T, \mathcal{Z})$ in R^∞, and $(T_\varepsilon, 0, \mathcal{Z})$ is ε–close to $(T, 0, \mathcal{Z})$ in W^∞. We have $R(T_\varepsilon) = [\frac{2}{3} - \varepsilon, \frac{2}{3} + \varepsilon]$ and $p(R(T_\varepsilon), T_\varepsilon, 0) = h_{\text{top}}(R(T_\varepsilon), T_\varepsilon) = \log 2 = \log r(G(0))$, which shows, that the topological pressure is not upper semi-continuous in this case.

Theorem 2 generalizes Theorem 2 of [5], where a similar result is shown for the topological entropy. Also Theorem 1 of [4], which gives a similar result for the topological entropy in the case of a continuous piecewise monotonic map T, can be easily deduced from Theorem 2. To calculate the upper bound of the topological entropy given in [4] we apply Theorem 2 to $(T, f, \mathcal{Z})$, where T and f are continuous. Define

$$(4.19) \qquad \begin{aligned} G_T(f) := \max\{ &\frac{\text{card}\,\{j \in \{0, 1, \ldots, n-1\} : T^j x \in E_1(T)\}}{n} \log 2 + \\ &\frac{1}{n} \sum_{j=0}^{n-1} f(T^j x) : x \in E_1(T) \text{ is a point of period } n\} \ , \end{aligned}$$

where we set $G_T(f) := -\infty$, if $E_1(T)$ contains no periodic points.

Corollary 2.2: *Let $(T, f, \mathcal{Z})$ be a piecewise monotonic map of class W^0, which satisfies that T and f are continuous on X. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T}, \tilde{f}, \tilde{\mathcal{Z}})$ is δ–close to $(T, f, \mathcal{Z})$ with respect to the W^0–topology implies*

$$p(R(\tilde{T}), \tilde{T}, \tilde{f}) < \max\{p(R(T), T, f), G_T(f)\} + \varepsilon \ .$$

Proof: By Theorem 2 it suffices to show $\log r(G(f)) = G_T(f)$, if $\log r(G(f)) > p(R(T), T, f)$. The definition of $(\mathcal{G}, \to)$ and the continuity of T and f imply that

$T\pi(a) = \pi(b)$ and $f(\pi(a)) = f_{\mathcal{Z}}(a)$, if $a, b \in \mathcal{G}$ and $a \to b$. Hence (2.4) gives for $n \in \mathbb{N}$

$$\|G(f)^n\| = \sup_{a\in\mathcal{G}} \sum_{b_0=a\to b_1\to\cdots\to b_n} \prod_{j=0}^{n-1} e^{f_{\mathcal{Z}}(b_j)} =$$

$$\sup_{a\in\mathcal{G}} \sum_{b_0=a\to b_1\to\cdots\to b_n} \exp(\sum_{j=0}^{n-1} f(T^j\pi(a))) =$$

$$\sup_{a\in\mathcal{G}} (\exp(\sum_{j=0}^{n-1} f(T^j\pi(a))))(\text{card}\,\{b_0 = a \to b_1 \to \cdots \to b_n\}) =$$

$$\sup_{a\in\mathcal{G}} (\exp(\sum_{j=0}^{n-1} f(T^j\pi(a))))(2^{\text{card}\,\{j\in\{1,2,\ldots,n\}: T^j\pi(a)\in E_1(T)\}}) .$$

Using (2.5) this implies $\log r(G(f)) = G_T(f)$. ∎

Remark: The proof shows, that $\log r(G(f)) = G_T(f)$, if $\log r(G(f)) > p(R(T), T, f)$ or $G_T(f) > p(R(T), T, f)$.

5. Continuity of the Hausdorff dimension

In this section we shall use the results of the preceeding sections and of [7] to prove continuity results about the Hausdorff dimension of $R(T)$. Throughout this section let $(T, \mathcal{Z})$ be a piecewise monotonic map of class E^1. Denote by $(T_{\mathcal{Z}}, T'_{\mathcal{Z}}, \mathcal{Z})$ the completion of $(T, T', \mathcal{Z})$ with respect to $\mathcal{Z}$. If we set

$$\alpha := \limsup_{n\to\infty} \frac{1}{n} \log \sup_{b_0\to b_1\to\cdots\to b_n} \prod_{j=0}^{n-1} |T'_{\mathcal{Z}}|^{-1}(b_j) , \tag{5.1}$$

where the supremum is taken over all paths $b_0 \to b_1 \to \cdots \to b_n$ of length n in $\mathcal{G}$, then using (2.4) and (2.5) we get for $0 \le t_1 \le t_2$

$$t_2\alpha \le \log r(G(-t_2 \log|T'|)) \le \log r(G(-t_1 \log|T'|)) + (t_2 - t_1)\alpha . \tag{5.2}$$

This gives, that either $\log r(G(-t \log|T'|)) \ge 0$ for all $t \ge 0$ (we set $t_0 := \infty$ in this case) or there exists a $t_0 \ge 0$ with $\log r(G(-t_0 \log|T'|)) = 0$ and $\log r(G(-t \log|T'|)) < 0$ for all $t > t_0$. Now set

$$d_T := \min\{t_0, 1\} . \tag{5.3}$$

Define for $t \in \mathbb{R}$

$$b_T(t) := p(R(T), T, -t \log|T'|) . \tag{5.4}$$

LEMMA 7: *Let $(T,\mathcal{Z})$ be a piecewise monotonic map of class E^1. Then b_T is continuous and strictly decreasing, has a unique zero t_R, and $\mathrm{HD}(R(T)) = t_R$.*

Proof: By Lemma 9 of [7] $R(T)$ satisfies the requirements of Theorem 2 of [7]. Lemma 3 of [7] gives, that b_T is continuous and strictly decreasing, and has a unique zero t_R. Now Theorem 2 of [7] shows $\mathrm{HD}(R(T)) = t_R$. ∎

THEOREM 3: *Let $(T,\mathcal{Z})$ be a piecewise monotonic map of class E^1. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T},\tilde{\mathcal{Z}})$ is δ–close to $(T,\mathcal{Z})$ with respect to the R^1–topology implies*

$$\mathrm{HD}(R(T)) - \varepsilon < \mathrm{HD}(R(\tilde{T})) < \max\{\mathrm{HD}(R(T)), d_T\} + \varepsilon\,.$$

Proof: We show at first, that $\mathrm{HD}(R(T))-\varepsilon < \mathrm{HD}(R(\tilde{T}))$, if $(\tilde{T},\tilde{\mathcal{Z}})$ is sufficiently close to $(T,\mathcal{Z})$. If $\mathrm{HD}(R(T)) = 0$, this is trivial.

Suppose $\mathrm{HD}(R(T)) > 0$. We can assume, that $\varepsilon < \mathrm{HD}(R(T))$. Lemma 7 gives $b_T(t_R - \varepsilon) > 0$. By (1.3) we get, that

$$\lim_{n\to\infty} \frac{1}{n} S_n(R(T), -t\log|T'|) < 0 \quad \text{for all } t > 0,$$

as there exists a q with $(T^q)'$ is piecewise continuous and $\inf_{x\in R(T)} |(T^q)'|(x) > 1$. Hence

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{n} S_n(R(T), -(t_R-\varepsilon)\log|T'|) < 0\\ &< b_T(t_R-\varepsilon) = p(R(T), T, -(t_R-\varepsilon)\log|T'|).\end{aligned}$$

By Theorem 1 there exists a $\delta_1 > 0$, such that $(\tilde{T},\tilde{\mathcal{Z}})$ is δ_1–close to $(T,\mathcal{Z})$ in R^1 implies $b_{\tilde{T}}(t_R-\varepsilon) > 0$. Now Lemma 7 gives $\mathrm{HD}(R(T)) - \varepsilon < \mathrm{HD}(R(\tilde{T}))$.

Now we show $\mathrm{HD}(R(\tilde{T})) < \max\{\mathrm{HD}(R(T)), d_T\} + \varepsilon$, if $(\tilde{T},\tilde{\mathcal{Z}})$ is sufficiently close to $(T,\mathcal{Z})$. Set $t_1 := \max\{\mathrm{HD}(R(T)), d_T\}$. If $t_1 = 1$ the result is trivially satisfied.

Suppose $t_1 < 1$. We can suppose, that $\varepsilon < 1 - t_1$. By Lemma 7 we get $b_T(t_1+\varepsilon) < 0$, and by (5.2) and (5.3) we get $\log r(G(-(t_1+\varepsilon)\log|T'|)) < 0$. Hence Theorem 2 gives, that there exists a $\delta_2 > 0$, such that $(\tilde{T},\tilde{\mathcal{Z}})$ is δ_2–close to $(T,\mathcal{Z})$ in R^1 implies $b_{\tilde{T}}(t_1+\varepsilon) < 0$. Now we get by Lemma 7, that $\mathrm{HD}(R(\tilde{T})) < \max\{\mathrm{HD}(R(T)), d_T\} + \varepsilon$.

If we set $\delta := \min\{\delta_1, \delta_2\}$, this gives the desired result. ∎

COROLLARY 3.1: *Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class E^1, such that one of the assumptions of Lemma 2 holds. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ with respect to the R^1-topology implies*

$$|\mathrm{HD}(R(\tilde{T})) - \mathrm{HD}(R(T))| < \varepsilon .$$

Proof: Using (5.3), Lemma 2 and Lemma 7 we get $d_T \leq \mathrm{HD}(R(T))$. Hence the result follows from Theorem 3. ∎

Now we give an example. Let $(T, \mathcal{Z})$ be defined as in (4.17). Then we have $\mathrm{HD}(R(T)) = 0$. As $|T'(x)| = 2$ for all $x \in (\frac{1}{3}, \frac{2}{3}) \cup (\frac{2}{3}, 1)$, we get for $t \in \mathbb{R}$ that $\log r(G(-t \log |T'|)) = (1-t)\log 2$. This implies $d_T = 1$ by (5.3) (we have $t_0 = 1$, where t_0 is the quantity introduced before (5.3)). For $\varepsilon \in (0, \frac{1}{3}]$ let $(T_\varepsilon, \mathcal{Z})$ be defined as in (4.18). Hence $(T_\varepsilon, \mathcal{Z})$ is ε-close to $(T, \mathcal{Z})$ in R^∞. Furthermore we have that $\mathrm{HD}(R(T_\varepsilon)) = 1$, which shows that the Hausdorff dimension is not upper semi-continuous in this case.

Finally we consider the case, where T is continuous and $|T'|$ can be extended to a continuous function on X. Set

$$D_T := \max\{ \frac{\mathrm{card}\,\{j \in \{0, 1, \ldots, n-1\} : T^j x \in E_1(T)\}}{\log |(T^n)'|(x)} \log 2 : x \in E_1(T) \text{ is a point of period } n\} , \tag{5.5}$$

where we set $D_T := 0$, if $E_1(T)$ contains no periodic points. Observe that if T is continuous and if $|T'|$ can be extended to a continuous map on X, then $\log |(T^n)'|(x) = \sum_{j=0}^{n-1} \log |T'|(T^j x)$ exists for every periodic point x and for every $n \in \mathbb{N}$. Furthermore the property $\inf_{x \in X_q} |(T^q)'|(x) > 1$ for a $q \in \mathbb{N}$ implies $\log |(T^n)'|(x) > 0$ for every x with period n.

COROLLARY 3.2: *Let $(T, \mathcal{Z})$ be a piecewise monotonic map of class E^1, which satisfies that T is continuous on X, and $|T'|$ can be extended to a continuous function on X. Then for every $\varepsilon > 0$ there exists a $\delta > 0$, such that $(\tilde{T}, \tilde{\mathcal{Z}})$ is δ-close to $(T, \mathcal{Z})$ with respect to the R^1-topology implies*

$$\mathrm{HD}(R(T)) - \varepsilon < \mathrm{HD}(R(\tilde{T})) < \min\{\max\{\mathrm{HD}(R(T)), D_T\}, 1\} + \varepsilon .$$

Proof: By Theorem 3 it suffices to show $d_T \leq D_T$, if $d_T > \mathrm{HD}(R(T))$. By the proof of Corollary 2.2 we have for $t \in (\mathrm{HD}(R(T)), t_0)$ that

$$\log r(G(-t\log|T'|)) = \max\{\frac{k}{n}\log 2 - \frac{t}{n}\sum_{j=0}^{n-1}\log|T'|(T^j x) : x \in P\} =$$

$$\max\{\frac{k}{n}\log 2 - \frac{t}{n}\log|(T^n)'|(x) : x \in P\},$$

where P is the set of all periodic points, which are contained in $E_1(T)$, n is the period of x, and $k := \mathrm{card}\,\{j \in \{0,1,\ldots,n-1\} : T^j x \in E_1(T)\}$. As there exists a $q \in \mathbb{N}$ with $\inf_{x\in R(T)} |(T^q)'|(x) > 1$, we get $\log|(T^n)'|(x) > 0$ for every $x \in P$. For a fixed $x \in P$ this gives that

$$\frac{k}{n}\log 2 - \frac{t}{n}\log|(T^n)'|(x) < 0 \quad \text{for all } t > \frac{k}{\log|(T^n)'|(x)}\log 2.$$

By (5.5) this implies $\log r(G(-t\log|T'|)) < 0$ for all

$$t > \max\{\frac{k}{\log|(T^n)'|(x)} : x \in P\} = D_T$$

and $\log r(G(-D_T\log|T'|)) = 0$. Hence (5.3) gives $d_T \le t_0 = D_T$. ∎

Remark: The proof shows, that $t_0 = D_T$, if $t_0 > \mathrm{HD}(R(T))$ or $D_T > \mathrm{HD}(R(T))$.

References

[1] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Lecture notes in Math. 470, Springer–Verlag, Berlin, 1975.

[2] F. Hofbauer, *Piecewise invertible dynamical systems*, Probab. Th. Rel. Fields **72** (1986), 359–386.

[3] F. Hofbauer and P. Raith, *Topologically transitive subsets of piecewise monotonic maps, which contain no periodic points*, Monatsh. Math. **107** (1989), 217–239.

[4] M. Misiurewicz, *Jumps of entropy in one dimension*, Fund. Math. **132** (1989), 215–226.

[5] M. Misiurewicz and S. V. Shlyachkov, *Entropy of piecewise continuous interval maps*, European Conference on Iteration Theory (ECIT 89), pp. 239–245, World Scientific, Singapore, 1991.

[6] M. Misiurewicz and W. Szlenk, *Entropy of piecewise monotone mappings*, Studia Math. **67** (1980), 45–63.

[7] P. Raith, *Hausdorff dimension for piecewise monotonic maps*, Studia Math. **94** (1989), 17–33.

[8] M. Urbański, *Invariant subsets of expanding mappings of the circle*, Ergod. Th. and Dynam. Sys. **7** (1987), 627–645.