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A REMARK ON SCHRÖDINGER OPERATORS

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ABSTRACT

We study the almost everythere convergence to the initial data $f(x) = u(x,0)$ of the solution $u(x,t)$ of the two-dimensional linear Schrödinger equation $\Delta u = i\partial_t u$. The main result is that $u(x,t) \to f(x)$ almost everywhere for $t \to 0$ if $f \in H^\rho(\mathbb{R}^2)$, where ρ may be chosen $<1/2$. To get this result (improving on Vega's work, see [6]), we devise a strategy to capture certain cancellations, which we believe has other applications in related problems.

We are interested in the problem of almost everywhere convergence $u(x,t) \to u(x,0)$ for solutions u of $\Delta u = i\partial_t u$ on $\mathbb{R}_+^{d+1}$. In dimension $d = 1$, it was shown by L. Carleson [2] that the condition $u(x,0) = f(x) \in H^{1/4}(\mathbb{R})$, i.e. L^2-control on the derivative of order $\frac{1}{4}$ of f, suffices. This result is sharp as observed by Dahlberg and Kenig in [3]. In dimension ≥ 2 the correct exponents are unknown. The best result up to date for $d \geq 2$ is the condition $f(x) \in H^s(\mathbb{R}^d)$ for some $s > \frac{1}{2}$, obtained independently in [1], [5], [6]. We will be concerned here with dimension $d = 2$. The purpose of this note is to improve to $f \in H^\rho(\mathbb{R}^2)$ for some $\rho < \frac{1}{2}$. (We consider only the local problem.) The heart of the matter consists of some new estimates on certain particular integral operators. This line of investigation seems of interest by itself and the technique developed here most likely also applies to related problems, such as certain Korteweg–deVries equations. We don't intend to pursue these matters here, neither will we try to optimize the result of the method. The author is grateful to M. Ben-Artzi for discussions on the subject.

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Consider the operator

$$Tf(x) = \int_{\mathbb{R}^2} \hat{f}(\xi) e^{it(x)|\xi|^2} e^{i\langle x,\xi\rangle} d\xi \tag{1}$$

where $0 < t(x) < 1$. We consider the local problem, i.e. $x \in D(0,1)$ = unit disc. Here $t(x)$ is an arbitrary function. We will first reduce the problem to studying integral operators of the form (12) below. Our methods have a certain formal resemblance to those used in analyzing the Luzin maximal function (cf. [4]) and involve combinatorial considerations.

Consider the usual Littlewood–Paley decomposition of f, i.e.

$$f = \sum_{k\geq 0} f_k$$

where $\hat{f}_k(\xi) = \hat{f}(\xi)\varphi_k(|\xi|)$ and φ_k is a bumpfunction, and write

$$T = \sum T_k,$$

$$T_k f(x) = \int \hat{f}(\xi) e^{it(x)|\xi|^2} e^{i\langle x,\xi\rangle} \varphi_k(|\xi|) d\xi. \tag{2}$$

We evaluate the individual components T_k on L^2. Our aim is to get an estimate

$$\|T_k\| \leq 2^{\rho k} \quad \text{for some } \rho < \frac{1}{2}. \tag{3}$$

This will yield boundedness of T on H^ρ where $\rho < \frac{1}{2}$.

Dualizing T_k, one has to estimate

$$T_k^* g(\xi) = \int g(x) e^{it(x)|\xi|^2} e^{i\langle x,\xi\rangle} dx \cdot \varphi_k(|\xi|)$$

acting from $L^2(D(0,1))$ to $L^2(\mathbb{R}^2)$.

One has (repeating Carleson's argument)

$$\int |T_k^* g|^2 d\xi = \int g(x)\overline{g(y)} K_1(x,y) dx dy$$

where

$$K_1(x,y) = \int e^{i[\langle x-y,\xi\rangle + (t(x)-t(y))|\xi|^2]} \varphi_k(|\xi|)^2 d\xi.$$

Using polar coordinates $\xi = (r\cos\theta, r\sin\theta)$ and stationary phase, $K_1(x,y)$ has a main term which is captured by a kernel of the form

$$\begin{aligned} K_2(x,y) =& 2^{k/2}\varphi_k\left(-\frac{|x-y|}{2[t(x)-t(y)]}\right)\frac{e^{i\frac{|x-y|^2}{4[t(x)-t(y)]}}}{|x-y|^{1/2}|t(x)-t(y)|^{1/2}} \\ & + O\left(\frac{2^{k/2}}{|x-y|}\right). \end{aligned} \tag{4}$$

The contribution of the second term in operator norm is $\leq 2^{k/2}$, hence negligible. The first term is clearly bounded by

$$2^k\frac{1}{|x-y|} \tag{5}$$

which has operator norm 2^k. This approach would lead to the condition $f \in H^{1/2}$. Our aim is to make a more delicate estimate in order to show that the operator norm

$$\left\|\varphi_k\left(\frac{|x-y|}{t(x)-t(y)}\right)\frac{e^{i\frac{|x-y|^2}{t(x)-t(y)}}}{|x-y|^{1/2}|t(x)-t(y)|^{1/2}}\right\| \leq 2^{(\frac{1}{2}-\epsilon)k}. \tag{6}$$

This estimate is local in the sense that we keep $x, y \in D(0,1)$. It should be uniform over all possible functions $0 < t(x) < 1$.

We do a first reduction. Use letters $b_1, b_2, \ldots$ for parameters to be specified later. Fix $0 < b_1 < 1$ and define for $j \geq 0$

$$D_j = \{x \in D(0,1) | jb_1 2^{-k} \leq t(x) < (j+1)b_1 2^{-k}\}. \tag{7}$$

If $K_3(x,y)$ is the kernel appearing in (6), write for $b_2 > 1$

$$\begin{aligned} K_3(x,y) =& \sum_{1\leq |j-j'|\leq b_1^{-1}b_2} K_3(x,y)\chi_{D_j}(x)\chi_{D_{j'}}(y) \\ & + O\left(|K_3(x,y)|\chi_{[|t(x)-t(y)|\leq b_1 2^{-k}]\cup[|t(x)-t(y)|>b_2 2^{-k}]}\right). \end{aligned} \tag{8}$$

The error term in (8) is bounded by

$$\frac{2^{k/2}}{|x-y|}\chi_{[|x-y|\leq b_1]} + b_2^{-1/2}2^{k/2}\frac{1}{|x-y|^{1/2}} \tag{9}$$

which operator norm on L^2 is clearly

$$\leq 2^{k/2}\left(b_1 + b_2^{-1/2}\right). \tag{10}$$

The operator norm of the first term in (8) is at most

$$2^{k/2} b_2 b_1^{-c} E_1 \tag{11}$$

where E_1 is a uniform bound on operators with kernel of the form

$$K_4(x,y) = \exp i \frac{2^k |x-y|^2}{a(x) - \bar{a}(y)} \tag{12}$$

where

$$0 \le a \le b_1 \tag{12'}$$

and

$$2b_1 \le \bar{a} \le b_2. \tag{12''}$$

Thus we have to show that the norm of such K_4 is bounded by $2^{-\varepsilon k}$, independently of $a, \bar{a}$. Observe that if $a, \bar{a}$ would be constant, K_4 would be a convolution operator of norm $\le 2^{-k} b_2$.

Fix $0 < b_3 < 1$ and consider the sets

$$\begin{aligned} U_j &= \{x \in D(0,1) | j b_3 2^{-k} \le a(x) < (j+1) b_3 2^{-k}\}, \\ \bar{U}_j &= \{x \in D(0,1) | j b_3 2^{-k} \le \bar{a}(x) < (j+1) b_3 2^{-k}\}. \end{aligned} \tag{13}$$

A direct verification shows that if a (resp. $\bar{a}$) is replaced by a function constant on each U_j (resp. $\bar{U}_j$), an error appears of at most

$$\frac{b_3}{b_1^2}. \tag{14}$$

Since K_4 is uniformly bounded, an L^2–L^2 estimate of $2^{-\varepsilon k}$ will result from a bound:

$$\left| \iint_{D(0,1) \times D(0,1)} f(x) g(y) K_4(x,y) dx dy \right| \le 2^{-\varepsilon k} \|f\|_\infty \|g\|_\infty. \tag{15}$$

Writing $K_4 = \Sigma_{j,j'} K_4(x,y) \chi_{U_j}(x) \chi_{\bar{U}_{j'}}(y)$, one gets by the preceding hypothesis on a, $\bar{a}$ the following bound on the left of (15):

$$b_2 2^{-k} \sum_{j,j'} \|f|_{U_j}\|_2 \|g|_{\bar{U}_{j'}}\|_2 \le b_2 2^{-k} \sum_{j,j' \le b_3^{-1} 2^k} |U_j|^{1/2} |\bar{U}_{j'}|^{1/2}. \tag{16}$$

Fix $b_4 > 1$ and define

$$J = \{j \,||U_j| > b_4 2^{-k}\} \quad \text{and } J' = \{j \,||\bar{U}_{j'}| > b_4 2^{-k}\}. \tag{17}$$

Thus $\# \ J$, $\# \ J' \le b_4^{-1} 2^k$ and hence by Cauchy–Schwartz

(18)

$$\sum_{j \in J \text{ or } j' \in J'} |U_j|^{1/2} |\bar{U}_{j'}|^{1/2} \le b_3^{-1/2} b_4^{-1/2} 2^k \left(\Sigma |U_j|\right)^{1/2} \left(\Sigma |\bar{U}_{j'}|\right)^{1/2} \le (b_3 b_4)^{-1/2} 2^k.$$

Substituting (18) in (16) yields a contribution

$$b_2 (b_3 b_4)^{-1/2}. \tag{19}$$

Eliminating this contribution, one gets image measures ν (resp $\bar{\nu}$) of $a : D(0,1) \to \mathbb{R}$ (resp $\bar{a}$) satisfying by construction

$$\nu(I) \le \frac{b_4}{b_3} |I| + 2^{-k} \quad \text{and} \quad \bar{\nu}(I) \le \frac{b_4}{b_3} |I| + 2^{-k}. \tag{20}$$

In fact, only the properties of a will be used.

Come back to (15). Consider subsets V_j of U_j and estimate

$$\sum_j \int\limits_{D(0,1)} \left| \int\limits_{V_j} f(x) K_4(x,y) dx \right| dy \le \left[\sum_j |V_j|^{-1} \int \left| \int_{V_j} f(x) K_4(x,y) dx \right|^2 dy \right]^{1/2}$$

$$\le \left[\sum_j |V_j|^{-1} \iint_{V_j \times V_j} \left| \int K_4(x_1,y) \overline{K_4(x_2,y)} dy \right| dx_1 dx_2 \right]^{1/2}. \tag{21}$$

Denote a_j the value of a on U_j. It follows from (12) that

$$K_4(x_1,y) \overline{K_4(x_2,y)} = \exp i 2^k \frac{|x_1|^2 - |x_2|^2 - 2\langle y, x_1 - x_2 \rangle}{a_j - \bar{a}(y)}. \tag{22}$$

We will use the following lemma:

LEMMA 23: *Let μ be a probability measure on $D(0,R) \backslash D(0,\rho)$ and $\mu = \int_0^{2\pi} \mu_\theta d\theta$ its radial disintegration. Let $N \in \mathbb{Z}_+$ and assume*

$$\int\limits_0^{2\pi} \int\limits_0^1 \int\limits_0^1 \frac{\mu_\theta(dr_1) \mu_\theta(dr_2)}{|r_1 - r_2| + \frac{1}{N}} d\theta < b_5. \tag{24}$$

Let $\xi_1, \xi_2, \dots, \xi_r \in \mathbb{R}^2$ satisfy $|\xi_s - \xi_{s'}| > 1$ for $s \neq s'$ and $|\xi_s| < N$. If for $s = 1, \dots, r$

(25)
$$|\hat{\mu}(\xi_s)| > b_6,$$

then r satisfies

(26)
$$r \leq C \frac{R^2}{\rho} \frac{b_5}{b_6^4}.$$

Proof: By passing to a proportional subset of $\{1, \dots, r\}$, one may assume for instance $\operatorname{Re} \hat{\mu}(\xi_s) > b_6$,

(27)
$$\int \left| \sum_{s=1}^{r} e^{2\pi i \langle \xi_s, x \rangle} \right| \mu(dx) > r b_6,$$

and hence also

(28)
$$\int \left| \sum_{s=1}^{r} e^{2\pi i \langle \xi_s, x \rangle} \right| \nu(dx) > \frac{1}{2} r b_6,$$

where ν is the expectation of μ w.r.t. a partition of $D(0,1)$ in squares of size $\sim 1/N$. Defining

(29)
$$Y = \left[\frac{dv}{dx} > \frac{1}{100 R^2} r b_6^2 \right]$$

one gets by Hölder's inequality

(30)
$$\int_{D \setminus Y} \left| \sum_{1}^{r} e^{2\pi i \langle \xi_s, x \rangle} \right| \nu(dx) < \frac{1}{10} r b_6$$

and hence

(31)
$$\nu(Y) > \frac{b_6}{3}.$$

The set Y is a union of $(\frac{1}{N} \times \frac{1}{N})$-squares $(I_\alpha)_{\alpha \leq \beta}$, where from (29) and (31)

(32)
$$\beta < \frac{100 R^2}{b_6^2 r} N^2$$

and

$$\Sigma\mu(I_\alpha) > \frac{b_6}{3}. \tag{33}$$

It follows from (24) that

$$N\int_0^{2\pi} \Sigma_\alpha \mu_\theta(I_\alpha^\theta)^2 < 2b_5 \tag{34}$$

where $I_\alpha^\theta = I_\alpha \cap \mathbb{R}\cdot(\cos\theta, \sin\theta)$. Also, since $\operatorname{dist}(0, I_\alpha) > \rho$

$$\mu(I_\alpha) = \int \mu_\theta(I_\alpha^\theta)d\theta < \frac{1}{\sqrt{N}\sqrt{\rho}}\left(\int \mu_\theta(I_\alpha^\theta)^2 d\theta\right)^{1/2}$$

and thus, by (33), (34) and Cauchy–Schwartz' inequality,

$$b_5 \geq N^2\rho\Sigma\mu(I_\alpha)^2 \geq \rho\frac{N^2}{\beta}b_6^2. \tag{35}$$

Substituting (32) yields the bound (26) on r.

Coming back to (21) and (22), we fix a parameter

$$\rho = b_7 \tag{36}$$

and consider the image measure μ_j of 2-dimensional Lebesgue measure on $D(0,1)\setminus D(0,\rho b_2)$ under the map

$$\phi_j : y \mapsto \frac{y}{\bar{a}(y) - a_j}. \tag{37}$$

Observe that this map preserves the rays. One gets the bound

$$R < b_1^{-1} \tag{38}$$

from the hypothesis on $a, \bar{a}$.

Since $\mu_{j,\theta}$ is the image measure of $r\cdot dr$ under the map

$$\phi_{j,\theta}(r) = \frac{r}{\bar{a}(re_\theta) - a_j}, \quad e_\theta = (\cos\theta, \sin\theta),$$

the left member of (24) equals

$$\int_0^{2\pi}\int_0^1\int_0^1 \frac{r_1 r_2 dr_1 dr_2 d\theta}{\left|\frac{r_1}{\bar{a}(r_1e_\theta) - a_j} - \frac{r_2}{\bar{a}(r_2e_\theta) - a_j}\right| + \frac{1}{N}} \tag{39}$$

which is bounded by

$$\int_0^{2\pi}\int_0^1\int_0^1 \frac{dr_1 dr_2 d\theta}{|a_j(r_1-r_2) - r_1\bar{a}(r_2 e_\theta) + r_2\bar{a}(r_1 e_\theta)| + N^{-1}b_1^2} \leq$$

$$b_1^{-2}\int_0^{2\pi}\int\int \frac{dr_1 dr_2 d\theta}{(|r_1-r_2| + N^{-9})\left(\left|a_j - \frac{r_1\bar{a}(r_2e_\theta) - r_2\bar{a}(r_1e_\theta)}{r_1-r_2}\right| + N^{-1}\right)}$$

(40) $+N^{-8}b_1^{-2}$.

Here we let $N = 2^k$. It follows from (13) that

$$\left|a_j - \frac{jb_3}{N}\right| < \frac{b_3}{N}.$$

Hence, summing (40) over all j yields the inequality

$$\sum_j \int_0^{2\pi}\int\int \frac{\mu_{j,\theta}(dr_1)\mu_{j,\theta}(dr_2)}{|r_1-r_2| + 2^{-k}} d\theta \leq 2^k b_3^{-1} b_1^{-2} k^2. \tag{41}$$

(No properties of $\bar{a}$ were used, except $(12'')$.)

Define

$$\mathcal{V}_j = \left\{(x_1,x_2) \in V_j \times V_j \middle| \left| \int K_4(x_1,y)\overline{K_4(x_2,y)}dy\right| > b_8\right\} \tag{42}$$

and assume

$$\int_0^{2\pi}\int\int \frac{\mu_{j,\theta}(dr_1)\mu_{j,\theta}(dr_2)}{|r_1-r_2| + 2^{-k}} d\theta < b_5, \tag{43}$$

$$\text{mes }(\mathcal{V}_j) > b_9 4^{-k}. \tag{44}$$

We assume j fixed and drop the j-subscript.

For $\delta \in [-1,1]$, define

$$\mathcal{V}_\delta = \{(x_1,x_2) \in \mathcal{V} \,|\, |x_1|^2 - |x_2|^2 = \delta\} \tag{45}$$

and

$$V_\delta = \{x_1 - x_2 | (x_1, x_2) \in \mathcal{V}_\delta\}. \tag{46}$$

Assume $\bar{\xi}_1, \dots, \bar{\xi}_r$ a 2^{-k}-separated set in V_δ and let $\xi_s = 2^k \bar{\xi}_s$. Defining

$$\varphi(y) = \exp i 2^k \frac{\delta}{a - \bar{a}(y)}, \tag{47}$$

one has

$$\sum_1^r \left| \int \varphi(y) e^{i\langle y, \xi_s \rangle / (\bar{a}(y) - a)} dy \right| > r b_8$$

and hence

$$r^2 \rho^2 b_2^2 + \sum_{s, s' \le r} \left| \hat{\mu}(\xi_s - \xi_{s'}) \right| > r^2 b_8^2 \tag{48}$$

where μ is the image measure under the map

$$y \mapsto \frac{y}{\bar{a}(y) - a}, \quad |y| > \rho b_2.$$

Since (43) holds, Lemma 23 gives the following upper bound on (48) (by (38)):

$$r \cdot \frac{b_5}{\rho^2 b_1^2 b_6^4} + r^2 b_6 + r^2 \rho^2 b_2^2, \tag{49}$$

and letting $b_7 \equiv \rho \sim b_2^{-1} b_8, b_6 \sim b_8^2$ it follows that

$$r \le \frac{b_2^2 b_5}{b_1^2 b_8^{12}}. \tag{50}$$

Clearly this bound on the 2^{-k}-entropy of V_δ implies the measure estimate

$$|V_\delta| \le b_1^{-2} b_2^2 b_5 b_8^{-12} 4^{-k}. \tag{51}$$

Denote $x_1 = (x_{1,1}, x_{1,2}), x_2 = (x_{2,1}, x_{2,2})$ and consider the coordinate transformation $F(x_1, x_2) = (\delta, y, \delta')$ given by

$$(x_1, x_2) \overset{F}{\mapsto} \left(|x_1|^2 - |x_2|^2, x_1 - x_2, \frac{x_{11} x_{22} - x_{12} x_{21}}{|x_1 - x_2|} \right) \tag{52}$$

for which the Jacobian

$$J(F) \sim |x_1 - x_2| > \delta. \tag{53}$$

Since $\mathrm{Proj}_y[F(\mathcal{V})_\delta] = V_\delta$, one has by (44) and (51)

$$\begin{aligned} b_9 4^{-k} &< \int_{F(\mathcal{V})} \frac{d\delta dy d\delta'}{\delta + 8^{-k}} \\ &\le \int_{\mathrm{Proj}_{(\delta,y)} F(\mathcal{V})} \frac{d\delta dy}{\delta + 8^{-k}} \\ &\le \int \frac{\mathrm{mes}(V_\delta)}{\delta + 8^{-k}} d\delta \\ &\le b_1^{-2} b_2^2 b_5 b_8^{-1} k \cdot 4^{-k}. \end{aligned} \tag{54}$$

These inequalities imply the existence of δ and y such that

$$\mathrm{mes}\, F(\mathcal{V})_{\delta,y} > b_1^2 b_2^{-2} b_5^{-1} b_8^{12} b_9 k^{-1}. \tag{55}$$

If $F(x_1, x_2) = (\delta, y, \delta')$, one has

$$\begin{cases} \langle x_2, \bar{y} \rangle &= \frac{\delta - |y|^2}{2|y|}, \\ \langle x_2, \bar{\bar{y}} \rangle &= \delta', \end{cases} \tag{56}$$

where $\bar{y} = y/|y|$ and $\bar{\bar{y}} = (-y_2, y_1)/|y|$ the orthogonal vector. Consequently, there is a line L such that

$$|L \cap V| > b_1^2 b_2^{-2} b_5^{-1} b_8^{12} b_9 k^{-1} \equiv b_{11}. \tag{57}$$

Hence, assuming no line L satisfies (57) with respect to V_j, (43) and (44) are contradictory. Hence, one of the following properties holds

$$\int\int\int \frac{\mu_{j,\theta}(dr_1)\mu_{j\theta}(dr_2)d\theta}{|r_1 - r_2| + 2^{-k}} > b_5, \tag{58}$$

$$\mathrm{mes}\ V_j < b_{10} 2^{-k}, \tag{59}$$

$$\mathrm{mes}\ \mathcal{V}_j < \frac{b_9}{b_{10}} 2^{-k} \mathrm{mes}\ V_j. \tag{60}$$

From (41), the number of j's satisfying (58) is at most $2^k b_3^{-1} b_1^{-2} k^2 b_5^{-1}$, which contribution in the (21)-sum is bounded by $b_3^{-1} b_1^{-2} k^2 b_5^{-1} b_4$, since $|V_j| \le |U_j| \le b_4 2^{-k}$.

The contribution of (59) is bounded by $b_{10}b_3^{-1}$. Finally, the contribution of the (60)-terms is at most

$$\sum_j |V_j|^{-1}\Big\{ \iint_{V_j\times V_j\setminus\mathcal{V}_j} \cdots \Big\} + b_3^{-1}b_{10}^{-1}b_9 < b_8 + b_3^{-1}b_{10}^{-1}b_9.$$

Collecting previous estimates, one gets the following bound on (21):

$$\{b_3^{-1}b_1^{-2}k^2b_5^{-1}b_4 + b_{10}b_3^{-1} + b_8 + b_3^{-1}b_{10}^{-1}b_9\}^{1/2}. \tag{61}$$

Given U_j, there is a decomposition in disjoint sets,

$$U_j = V_j \cup U_\alpha W_\alpha, \tag{62}$$

where V_j satisfies the previous condition, i.e. no line L intersects V_j in a set of (1-dimensional) measure $> b_{11}$ and to each W_α one may associate a rectangle R_α such that

$$W_\alpha \subset R_\alpha; \quad |W_\alpha| \geq b_{11}|R_\alpha|. \tag{63}$$

We assume U_j in a union of τ-squares, where $\tau > 0$ is chosen sufficiently small. One then constructs the W_α by induction and the R_α have τ-width. This construction is standard and we omit details. Observe that by (63)

$$\Sigma|R_\alpha| \leq b_{11}^{-1}|U_j|. \tag{64}$$

Assume j satisfies (43). One has to estimate

$$\int_{D(0,1)}\left|\int_{W_\alpha} f(x)K_4(x,y)dx\right|dy. \tag{65}$$

One bounds (65) by

$$\tau\cdot\sigma \tag{66}$$

where σ is a (uniform) estimate on

$$\int\left|\int_0^1 f(u)K_4(a+\xi u,y)du\right|dy. \tag{67}$$

Here $a \in D(0,1), |\xi| = 1$ and $|f| \le 1$. Squaring again, $(67)^2$ is bounded by

$$\int_0^1 \int_0^1 \left| \int_{D(0,1)} K_4(a+\xi u_1, y)\overline{K_4(a+\xi u_2, y)}dy \right| du_1 du_2. \tag{68}$$

From (22), one gets for the integrand the expression

$$\exp i2^k \frac{2\langle a,\xi\rangle(u_1-u_2)+u_1^2-u_2^2-2\langle y,\xi\rangle(u_1-u_2)}{a_j-\bar{a}(y)}. \tag{69}$$

The transformation $G(u_1,u_2) = (u_1,u_2)$ defined by

$$\begin{cases} v_1 = 2\langle a,\xi\rangle(u_1-u_2)+u_1^2-u_2^2 \\ v_2 = u_1-u_2 \end{cases}$$

has Jacobian $J(G) = 2|u_1-u_2|$.

Repeating the considerations leading to (50) shows that, if v_1 is fixed, then

$$\operatorname{mes}\left\{ v_2 \left| \left| \int \exp i2^k \frac{v_1 - 2\langle y,\xi\rangle v_2}{a_j-\bar{a}(y)}dy \right| > b_{12} \right.\right\} \le b_1^{-2}b_2^2 b_5 b_{12}^{-12}2^{-k}. \tag{70}$$

Hence, for $\kappa > 0$

$$\begin{aligned}(68) \le & \iint_{|u_1-u_2|<\kappa} \cdots du_1 du_2 \\ & + \int\int \frac{1}{|v_2|+\kappa} \left| \int \exp i2^k \frac{v_1-2\langle y,\xi\rangle v_2}{a_j-\bar{a}(y)} dy \right| dv_1 dv_2 \\ & < \kappa + b_{12}\log\frac{1}{\kappa} + \frac{1}{\kappa} b_1^{-2}b_2^2 b_5 b_{12}^{-12} 2^{-k}. \end{aligned} \tag{71}$$

Choosing κ appropriately, this yields the following estimate of σ:

$$\sigma \lesssim b_{12}^{1/2} + b_1^{-1/2} b_2^{1/2} b_5^{1/4} b_{12}^{-3} 2^{-k/4}. \tag{72}$$

Hence (65) is bounded by $|R_\alpha|$. (72) and summation over α yields, by (64) and (72),

$$\int_{D(0,1)} \left| \int_{\cup W_\alpha} f(x)K_4(x,y)dx \right| dy \le b_{11}^{-1} b_1^{-1/14} b_2^{1/14} b_5^{1/28} 2^{-k/28} |U_j| \tag{73}$$

for an appropriate choice of b_{12}.

From (61), (62) and (73), it follows that

$$\sum_j \int_{D(0,1)} \left| \int_{U_j} f(x) K_4(x,y)dx \right| dy \leq \tag{74}$$
$$b_1^{-1} b_3^{-1/2} b_4^{1/2} b_5^{-1/2} k + b_3^{-1/2} b_{10}^{1/2} + b_8^{1/2}$$
$$+ b_3^{-1/2} b_{10}^{-1/2} b_9^{1/2} + b_{11}^{-1} b_1^{-1/14} b_2^{1/14} b_5^{1/28} 2^{-k/28}.$$

By (19), (74) and (57), there is the following bound on E_1:

$$b_1^{-2} b_3 + b_2 b_3^{-1/2} b_4^{-1/2} + b_1^{-1} b_3^{-1/2} b_4^{1/2} b_5^{1/2} k + b_3^{-1/2} b_{10}^{1/2} \tag{75}$$
$$+ b_8^{1/2} + b_3^{-1/2} b_{10}^{-1/2} b_9^{1/2} + b_1^{-3} b_2^3 b_5^2 b_8^{-12} b_9^{-1} k 2^{-k/28}.$$

Thus (6) is bounded by

$$2^{k/2}\{b_1 + b_2^{-1/2} + b_1^{-c} b_2 \cdot (75)\} \tag{76}$$

from (10) and (11). Here c is some constant.

Choosing in order $b_1, b_2, b_3, b_4, b_5, b_8, b_{10}, b_9$, a gain of $2^{-\varepsilon k}$ is gotten, where $\varepsilon > 0$ is some constant. This yields inequality (6) and hence (3).

Appendix

There is a different approach to the estimate proved in the paper, using the recent work of the author on Fourier restriction results in $\mathbb{R}^3$. This approach will mainly combine the results from [B1] and [B2] listed at the end of this appendix and give some application of the non-L^2 restriction phenomena (in dual form) obtained in [B1]. It would be possible to derive from the argument presented below some explicit exponent, but for the sake of simplicity we won't attempt here to make things more precise and optimal. Thus we look for an inequality of the form

$$\left\| \sup_{0<t<1} \left| \int_{|\xi|<N} \hat{f}(\xi) e^{i(\langle x,\xi\rangle + t|\xi|^2)} d\xi \right| \right\|_{L^2(B(0,1))} \leq N^\rho \left(\int |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \tag{A.1}$$

for some $\rho < \frac{1}{2}$.

Choose some $q > 2$ and estimate the left member of (A.1) by appropriate change of variables (rescaling),

$$N^{2-2/q} \left\| \sup_{0<t<N^2} \left| \int_{|\xi|<1} \hat{f}(N\xi) e^{i(\langle x,\xi\rangle + t|\xi|^2)} d\xi \right| \right\|_{L^q(dx)}. \tag{A.2}$$

Using standard considerations, one may then estimate it further by

$$\text{(A.3)} \qquad N^{2(1-1/q)} \left\| \int_{|\xi|<1} \hat{f}(N\xi) e^{i(\langle x,\xi\rangle + t|\xi|^2)} d\xi \right\|_{L^q(dx\,dt)}.$$

Consider the surface $(\xi, |\xi|^2)$ in $\mathbb{R}^3$, restricting ξ to the unit disc. Since there is obviously curvature and smoothness, the restriction and extension theory applies equally well as for the sphere. Call (p,q) an admissible exponent pair provided

$$\text{(A.4)} \qquad \|\hat{\mu}\|_q \le C \left\| \frac{d\mu}{d\sigma} \right\|_p$$

with μ a measure carried by the 2-sphere S_2 or the restricted paraboloid $\mathcal{P}$ considered above. The classical L^2 - restriction theorem states then that (2,4) is admissible and in [B1] admissible pairs were obtained with $q < 4$.

Applying the (p,q) pair in (A.3), one gets a bound

$$\text{(A.5)} \qquad N^{2(1-1/q)} \left(\int |\hat{f}(N\xi)|^p d\xi \right)^{1/p} = N^{2(1-\frac{1}{p}-\frac{1}{q})} \left(\int |\hat{f}(\xi)|^p d\xi \right)^{1/p}.$$

For $p = 2, q = 4$, that is inequality (A.1) with $\rho = \frac{1}{2}$.

It was shown in [B2] (immediate consequence of Lemma 3.23) that this (2,4)-estimate may be improved, unless the density "corresponds" to the indicator function of a cap (as a rough statement). More precisely, if we assume $\hat{f}$ to be of the form

$$\text{(A.6)} \qquad \hat{f}(\xi) = \frac{\chi_\Omega}{|\Omega|^{1/2}} \qquad (\Omega \subset B(0,N))$$

then an improvement will be obtained, unless for some square Q one has

$$\text{(A.7)} \qquad |\Omega| \sim |Q| \sim |\Omega \cap Q|.$$

Remark: $\hat{f}$ may always be broken up in level sets. The meaning of $\sim$ actually will allow factors of the form N^δ for some specific δ and so is the meaning of an "improvement". That this is the result of the reasoning below is left to the reader to check.

So assume (7) holds. If $|Q| \sim N^2$, apply estimate (A.5) for an admissible pair (p,q) with $q < 4$, gotten from [B1]. Since here

$$\left(\int |\hat{f}(\xi)|^p d\xi\right)^{1/p} \sim N^{2/p} N^{-1} \tag{A.8}$$

we get an estimate of the form

$$N^{1-2/q} \tag{A.9}$$

with $1 - 2/q < \frac{1}{2}$.

If $|Q| = N_1^2$, with $N_1 = N^{1-\epsilon}$, thus $Q \subset \xi_0 + B(0, N_1)$, we proceed as follows. Write in (A.2)

$$|\xi|^2 = |\xi_0'|^2 + 2\langle \xi_0', \eta\rangle + |\eta|^2 \tag{A.10}$$

where $\xi_0' = \xi_0/N, |\eta| < N_1/N$.

It is clear from (A.10) that the parameter values needed to recapture the supremum for $t \in [0, N^2]$ may be taken in a N/N_1-net and hence the passage to the t-integral and (A.3) gives a saving of a factor $(N_1/N)^{1/9}$. Then continue with the (2,4)-extension theorem to conclude also that case. Going through this argument a bit more explicitly, this easily leads to (A.1) with an exponent $\rho < \frac{1}{2}$.

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