DISTINGUISHED SUBSETS AND SUMMABILITY INVARIANTS

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1. Introduction

A property of certain summability matrices and integral transforms has been discussed and applied in various situations for over fifty years. The property has been given various names, the one of most frequent occurrence being "the mean value property", a free translation of *Mittelwertsatz*. Applications of the property have been made to the study of inclusion and equivalence theorems, convergence and summability factors, Tauberian theorems and other classical topics. A survey of the history and applications may be found in [25], §24, pp. 42–45, and §34, p. 65. I would like to mention specifically that W. Jurkat, A. Peyerimhoff, and K. Zeller have played an important role in these applications, for example in [7], [8], [11], [23], [24].

The present article is an essentially self-contained generalization (not new in every case) of the mean value property and many other properties which are equivalent to it. Specifically, we shall define subsets B, I_i, F, S, and others, of the convergence domain c_A of a matrix A. Many of these have been considered earlier, sometimes with restrictions on the matrix A. The mean value property is, simply, $B = c_A$. Various authors have shown that this is equivalent to each of the assumptions $I_i = c_A$, $F = c_A$,and one of the type $S \oplus 1 = c_A$.

Our aim is, first, to study these sets in the more general case in which they are not assumed to fill up c_A, to obtain relations between them, and thus to generalize the known results.

Secondly, we ask which of these sets are invariant, i.e. depend only on c_A and not on A, and which of the invariant sets can be given an invariant definition so that they can be defined for more general sequence spaces.

Our work has bearing on the problem of characterizing convergence domains among FK spaces . Certainly all the results given will be necessary.

No really satisfactory set of sufficient conditions is known to the author. For example, if X is an FK space such that $\dim X/c = \dim X/m < \infty$, then there exists A such that $X = c_A$. [17].

2. Standard definitions

We recall that an FK space is an F space (i.e. a locally convex linear complete metric space) which is also a sequence space with continuous coordinates; "sequence" means "sequence of complex numbers".

For example, consider c, the space of convergent sequences with $\|x\| = \sup|x_n|$. Since, for each n, $|x_n| \leqq \|x\|$ we conclude that c is an FK space, the other details, such as completeness, being familiar exercises. We shall also be interested in c_0, m, the spaces of null, bounded sequences, respectively, with the same norm as c; and l, the space of sequences x with $\|x\| = \Sigma\ |x_n| < \infty$.

FK topologies are usually placed upon sequence spaces by means of a sequence of seminorms. For example, the space s of all sequences is an FK space with $\{p_n\}$, where $p_n(x) = |x_n|$.

The elementary theory of FK spaces is given in [16], Chapter 12. We mention, in particular, the fact that the topology of an FK space is unique, i.e. a sequence space has at most one FK topology.

For $k = 1, 2, \cdots$, δ^k is the sequence $\{\delta_n^k\} = (0, 0, \cdots, 0, 1, 0, \cdots)$. By X', we mean the set of continuous linear functionals on the linear topological space X.

A sequence is called *basic* if it is a (Schauder) basis for its linear closure.

Let A be a matrix, $A = (a_{nk})$, $n, k = 1, 2, \cdots$ Let Ax be the sequence $\{(Ax)_n\}$ where $(Ax)_n = \sum_{k=1}^{\infty} a_{nk} x_k$. (This is ordinary multiplication, treating x as a column vector.)

The following facts are given in [16], Chapter 12. The space $c_A = \{x : Ax \in c\}$ is an FK space with the property that every continuous linear functional f on c_A has the form

$$f(x) = \alpha \lim\nolimits_A x + t(Ax) + \Sigma\ \beta_k x_k, \quad t \in l \tag{1}$$

where $\lim_A x = \lim Ax = \lim_{n\to\infty} (Ax)_n$, $t(Ax) = \Sigma\ t_n (Ax)_n$, (treat t as a row vector, x as a column vector,) and the series $\Sigma\ \beta_k x_k$ converges for all $x \in c_A$.

A basic fact of the theory is that if A, D are matrices with $c_A = c_D$, *A and D yield the same FK topology for c_A.*

Although much of the following does not require it, *we shall assume everywhere in the following work that the matrix A is conservative,* i.e. $c_A \supset c$. It is well known that this is true if and only if

$$\| A \| \equiv \sup_n \Sigma_k | a_{nk} | < \infty, \quad a_k \equiv \lim_{n\to\infty} a_{nk} = \lim_A \delta^k$$

exists for each k (the column limits,) and $1 \in c_A$. The matrix A is called *perfect* if c is dense in c_A.

Let $\chi(A) = \lim_A 1 - \Sigma a_k = \lim_{m\to\infty} \lim_{n\to\infty} \sum_{k=m}^{\infty} a_{nk}$. If $\chi(A) = 0$, A is called *conull,* otherwise *coregular.*

A *regular* matrix (*i.e.* $\lim_A x = \lim x$ for all $x \in c$) is coregular. A *multiplicative-χ* matrix (i.e. $\lim_A x = \chi \lim x$ for all $x \in c$) has $\chi(A) = \chi$. A is multiplicative if and only if $a_k = 0$. This follows from the general formula

$$\text{(2)} \qquad \lim_A x = \chi(A) \lim x + \Sigma a_k x_k \quad \text{for } x \in c,$$

which in turn follows easily from the representation of c', see [16], Chapter 6, § 4, Example 5.

If A is *reversible,* i.e. one to one and onto c, we may take $\beta = 0$ in (1), since c_A is congruent with c. In particular, this holds if A is a *triangle,* i.e. $a_{nk} = 0$ for $k > n$, $a_{nn} \neq 0$.

The representation (1) is not necessarily unique, (see §8) even if A is a coregular triangle, or if A is a triangle and $t = 0$. However, if f is given by (1), we have, [16], Chapter 6, §4, Problem 25,

$$\text{(3)} \qquad \chi(f) = \alpha \chi(A)$$

where $\chi(f) = f(1) - \Sigma f(\delta^k)$.

It follows that α is unique if A is coregular. If A is coregular and reversible, there is a unique representation of the form (1) with $\beta = 0$.

Now, for $k = 1, 2, \cdots$, let $x = \delta^k$ in (1). This yields

$$f(\delta^k) = \alpha a_k + \sum_{r=1}^{\infty} t_r a_{rk} + \beta_k.$$

Eliminating β_k between this and (1) gives, for $x \in c_A$, $f \in c'_A$,

$$(4) \qquad f(x) = \alpha \lim_A x + t(Ax) + \sum_k \left[f(\delta^k) - \alpha a_k - \sum_r t_r a_{rk} \right] x_k.$$

3. Distinguished subsets

As always we are making the blanket assumption that A is a conservative matrix. Let

$B = \left\{ x \in c_A \text{: there exists } M \text{ with } \left| \sum_{k=1}^{m} a_{nk} x_k \right| < M \text{ for all } m, n = 1, 2, \cdots \right\}.$
M may depend on x.

$F = \{x \in c_A\text{: } \sum x_k \delta^k$ is weakly Cauchy, i.e. $\sum x_k f(\delta^k)$ is convergent for each $f \in c'_A$.$\}$

$I = \{x \in c_A\text{: } \sum x_k a_k$ is convergent.$\}$

$I_i = \cap \{I_D : c_D = c_A\}$.

$L = \{x \in c_A : (tA)x = \sum_k \sum_r t_r a_{rk} x_k$ exists for all $t \in l$.$\}$

$S = \{x \in c_A : x = \sum x_k \delta^k\}$

$W = \{x \in c_A\text{: } \sum x_k \delta^k$ is weakly convergent, i.e. $\sum x_k f(\delta^k) = f(x)$ for each $f \in c'_A$.$\}$

Sometimes, if the dependence on A is in doubt, we shall write B_A for B, and similarly for the others. This was already done in the definition of I_i.

B is the set of "cut-bounded" sequences, F is associated with "functional convergence", S with "strong convergence". W with "weak convergence", hence the initials.

F, S, W were called, respectively, the set of all x with *FAK*, *AK*, *SAK* in [23], [11]. B, S were designated A_1, A_3 in [24], §§4.1 and 4.3. L was called A''' in [4]. I was called the *inset* in [15]. We shall call I_i the ***internal inset.***

If $B = c_A$, A is said to have the *mean value property*. (See [25].) If $F = c_A$, A is said to have *FAK* (funktionale Abschnittskonvergenz) (See [23]). If $I = c_A$, A is said to have ***maximal inset***, and if $I_i = c_A$, A is said to have

PMI (propagation of maximal inset) (See [15]) If $L = c_A$, A is called *associative*. (See [3], also [12] p. 56.) There has also been investigation of matrices A for which $S = c_A$ (i.e. $\{\delta^n\}$ is a basis for c_A) or for which c_A has as basis $\{\delta^n\}$ and another sequence u, which in some general cases may be taken to be 1. (See [18]).

Many of these properties are known to be equivalent. Their equivalence, together with references, will be given as a corollary to our main theorems, Corollary 5.9.

Associated with I is the function Λ defined by $\Lambda(x) = \lim_A x - \sum a_k x_k$ for $x \in I$. Clearly $\Lambda(x) = \lim_{n\to\infty} \lim_A x^{(n)}$, where

$$x^{(n)} = (0, 0, \cdots, 0, x_{n+1}, x_{n+2}, \cdots) = x - \sum_{k=1}^{n} x_k \delta^k.$$

It follows from (2) that $\Lambda(x) = \chi(A) \lim x$ for $x \in c$. In particular $\Lambda(1) = \chi(A)$ and so A is conull if and only if $1 \in \Lambda^\perp$.

The set $\Lambda^\perp$ was called A_0 in [24].

4. Preliminary lemmas

Some acknowledgments of priority are given in §5.

In the following, A is a conservative matrix.

Lemma 4.1. *For each $x \in L$, $t \in l$, we have $(tA)x = t(Ax)$.*

This has been proved independently by A. Peyerimhoff, K. Zeller and the author. See [15], Lemma 13.

Lemma 4.2. $B = L$.

For a fixed $x \in c_A$ and m, $n = 1, 2, \cdots$, set $u_{mn} = \sum_{k=1}^{m} a_{nk} x_k$.

Let $x \in B$. If $t \in l$, the series $\sum_r t_r u_{mr}$ is uniformly convergent. It follows, taking the limit as $m \to \infty$, that $(tA)x$ is convergent. Hence $x \in L$.

Conversely, for $x \in L$ and $m = 1, 2, \cdots$, define v_m on l by $v_m(t) = \sum t_n u_{mn}$. Then $\| v_m \| = \sup_n | u_{mn} |$. Since $\lim_{m\to\infty} v_m(t)$ exists for each $t \in l$, the uniform boundedness principle yields $x \in B$.

Lemma 4.3. $F = L \cap I$.

This is Lemma 3 of [3].

Lemma 4.4. $I_t \subset L$.

Let $x \in I_t$ and $t \in l$. Let J be S. Mazur's matrix (j_{nk}) given by $j_{nn} = 1, j_{nk} = t_k$ for $k \leqq n-1$, $j_{nk} = 0$ otherwise. Let $D = JA$. Then $c_D = c_A$ (S. Mazur. See [19] p. 382, lines 9–11, [5], #3, or [15], Lemma 1). Thus $x \in I_D$ and so $\Sigma\ d_k x_k$ is convergent. But $(tA)x = \Sigma\ d_k x_k - \Sigma\ a_k x_k$ and so $(tA)x$ is convergent and $x \in L$.

Lemma 4.5. *Suppose that $\bar{c}_0 \subset B$, for example if B is closed. Then $\{\delta^n\}$ is basic and $S = W = \bar{c}_0$.*

For $x \in B$, a straightforward calculation shows that $\Sigma\ x_k \delta^k$ is bounded. In a complete space this implies that $\{\delta^n\}$ is basic; this may be applied to $\bar{c}_0$. By definition of S, it follows that S is the linear closure of $\{\delta^n\}$. Finally, $\bar{c}_0$, being a closed linear subspace is weakly closed and so $W \subset \bar{c}_0$.

Lemma 4.6. *A is coregular if and only if $1 \notin W$.*

If $1 \in W$, take $f = \lim_A$ in the definition of W. This yields $\chi(A) = 0$. Conversely, if A is conull, (3) shows that $1 \in W$.

5. Relations among the distinguished subsets

In the following, A is a conservative matrix.

Theorem 5.1. $c_0 \subset S \subset W \subset F = I_t \subset B = L \subset P$.

Theorem 5.2. $c \subset c_A \cap m \subset F = L \cap I$.

That $F = L \cap I$ is Lemma 3 of [3]. That $c_A \cap m \subset B$ is mentioned in [24] §5.6. That $B = L$ is given in [4] in case A is regular and reversible. The proof given here (Lemma 4.2) is probably the one hinted at in [9].

$\{\delta^n\}$ is a basis for c_0 in the norm topology given by $\|x\| = \sup|x_n|$, hence in the weaker topology of c_A, i.e. $c_0 \subset S$. We also have $F \subset I$ by Lemma 4.3. Now F has an invariant definition, so that $F_A = F_D$ if $c_A = c_D$. Thus $F \subset I_t$. Also $F = I \cap L \supset I_t \cap L = I_t$ by Lemma 4.4. The rest of Theorems 5.1, 5.2 is staightforward in view of the lemmas.

The relation $F = I_t$ would be trivial if we could affirm that for every $f \in c_A'$ there exists D with $c_D = c_A$, $d_k = f(\delta^k)$ for all k. This is, however, false even for $f = 0$. See Example 11.

Lemma 5.3. *Let f be a continuous linear functional on c_A. Then for $x \in F, f(x) - \sum x_k f(\delta^k) = \alpha \Lambda(x)$, where α is given in* (1).

Since $x \in L$ and $x \in I$, by Theorem 5.1, the result follows from (4), taking account of Lemma 4.1.

Theorem 5.4. $W = F \cap \Lambda^{\perp} = L \cap \Lambda^{\perp}$. *Either $F = W$ or $F = W \oplus u$ for some sequence u. If A is coregular, $F = W \oplus 1$.*

Here $\Lambda^{\perp}$ means $\{x \in c_A : \Lambda(x) = 0\}$. It is either a maximal subspace of I or is all of I.

The result $W = B \cap \Lambda^{\perp}$ is stated in [24], §4.3. Special cases are proved in [23], Satz 3.6, and [11], Satz 5.2.

If $x \in W$, taking $f = \lim_A$ in the definition of W yields $\Lambda(x) = 0$. Thus $W \subset L \cap \Lambda^{\perp}$ by Theorem 5.1. Conversely, if $x \in L \cap \Lambda^{\perp}$, $x \in F$ by Theorem 5.2, and by Lemma 5.3, $x \in W$. Finally $F \cap \Lambda^{\perp} = L \cap I \cap \Lambda^{\perp} = L \cap \Lambda^{\perp}$, using Theorem 5.2. The second result follows since $F \subset I$ and so Λ is defined on F. The last result holds since $\Lambda(1) = \chi(A) \neq 0$.

Lemma 5.5. *If c_0 is not dense in L, $L = F \neq W$.*

There exists $f \in c_A'$ with $f = 0$ *on* c_0, $f \neq 0$ on L. Taking $x \in L$ in (4) we have, using Lemma 5.3, $f(x) = \alpha \lim_A x - \sum \alpha a_k x_k$. Since $f \neq 0$ on L, it follows that $\alpha \neq 0$ and so $x \in I$. Thus $L \subset I$, and so by Theorem 5.2, $L = F$. It also follows that $f = \alpha \Lambda$ and $\Lambda \neq 0$ on L. Thus $L \not\subset \Lambda^{\perp}$ and so $L \neq W$ by Theorem 5.4.

Lemma 5.6. *If L is closed, $L = F$. Hence if L is closed and $L \supset I$, it follows that $L = I$.*

By Lemma 5.5, we may assume that c_0 is dense in L. By Lemma 4.5 and the fact that $B = L$, it follows that $L = S$ Hence, by Theorem 5.1, $L = F$. The last sentence is true because $F \subset I$.

The next result extends the second part of Satz 3.4 of [23] which is the special case of Theorem 5.7 in which $F = \bar{c}_0$.

Theorem 5.7. *Suppose that F is closed. Then $S = W = \bar{c}_0$. Moreover F has a basis which is either $\{\delta^n\}$, in which case $F = \bar{c}_0 = S$, or $\{\delta^n\}$ and one more sequence u, in which case $F = \bar{c}_0 \oplus u = S \oplus u$. Both cases may* *occur with A conull; if A is coregular, the second case holds and u may be taken to be* 1. *Exactly the same conclusion holds if L is closed. (In which case $L = F$.)*

If A is coregular, $\Lambda(1) \neq 0$, so, by Theorem 5.4, $F = W \oplus 1$. Hence $F = \bar{c}_0 \oplus 1$ by Lemma 4.5. If A is conull, there are two cases to consider. Suppose first that $\Lambda = 0$ on I. Then by Theorems 5.2, 5.4, $F = W$. Hence $F = \bar{c}_0$ by Lemma 4.5. If, on the other hand, $\Lambda(u) \neq 0$ for some $u \in I$, the same reasoning yields $F = \bar{c}_0 \oplus u$. The last sentence follows from Lemma 5.6.

The various cases are illustrated in Examples 1, 2, 3, 4.

Theorem 5.8. *If any one of S, W, F are closed, all three are, and the conclusion of Theorem 5.7 holds. If L is closed, S, W, F are closed.*

We have $c_0 \subset S \subset W \subset \bar{c}_0$ since the strong and weak closures of c_0 are the same. Thus if S is closed, $S = W = \bar{c}_0$. If W is closed, it is a complete space hence its weak basis $\{\delta^n\}$ must be a strong basis and so $W = S$. By Theorem 5.4, and the fact that $F \subset I$ we see that either $F = W$ or $F = W \oplus u$ for some u. In either case F is closed if W is. Finally, if F is closed, S, W are, by Theorem 5.7.

Theorems 5.7, 5.8 extend [24] §8.2, as well as the first part of [23], Satz 3.4.

With one possible exception, the following corollary is known, occasionally with some extra assumptions on A. See [10], Theorems 3, 4, 5, [23], Beispiel 4.4, [18], Satz 1, [3], Lemmas 4, 6, and Theorems 2, 3, and [15] Lemma 16.

The exception is the sufficiency of iii. This improves [3], Theorem 1 which gives it under more restrictive hypotheses.

Corollary 5.9. *The following are equivalent conditions for a conservative matrix A.*

i. *A has PMI* (i.e. $I_t = c_A$)
ii. *A has FAK* (i.e. $F = c_A$)
iii. *A is associative* (i.e. $L = c_A$)
iv. *A has the mean value property* (i.e. $B = c_A$).
v. *c_A has as basis $\{\delta^n\}$, or $\{\delta^n\}$ and one more sequence $u \in L$. (Thus $S = c_A$ or $S \oplus u = c_A$.)*

Furthermore $W = c_A$ *if and only if* $S = c_A$ *and these are true if and only if* $\Lambda = 0$ *and any one of the first five conditions holds. (In this case* c_A *has* $\{\delta^n\}$ *as basis.)*

i and ii are equivalent because $F = I_l$. iii and iv are equivalent because $L = B$. ii implies iii because $F \subset L$, and iii implies ii by Lemma 5.6. ii implies v by Theorem 5.7. Finally, if v holds, either $S = c_A$ or $S \oplus u = c_A$, either one forcing iii.

Remarks. I. If A is coregular, v may be written: c_A has $\{\delta^n\}$ and one more sequence u as basis. Indeed we may take $u = 1$. (See Theorem 5.7).

II. If in v the condition $u \in L$ were omitted, then v would no longer be sufficient to imply the other parts. (Example 12.) It is an eight year old conjecture [18], p. 262, that $u \in I$ may be substituted for $u \in L$. (See Remark I.)

III. A sufficient condition for an affirmative answer to the conjecture in II is $L \neq S$. (Compare Lemma 5.5.)

Theorem 5.10. *Suppose that A is a regular matrix, or more generally that there exists a regular matrix D such that* $c_D = c_A$. *Then* $L = F$.

The hypothesis ensures that $1 \notin \bar{c}_0$ since $\lim_D$ is continuous, vanishes on c_0 but not at 1. The result follows from Lemma 5.5 since $1 \in L$.

I do not know whether $L = F$ for every coregular matrix. In Example 9, $L \neq F$.

Example 11 shows that the converse of Theorem 5.10 is false even if A is assumed coregular.

6. Two more distinguished subsets

Let

$F^0 = \{x \in c_A\colon \Sigma\, x_k f(\delta^k)$ is convergent for each $f \in c'_A$ for which α, in (1), is 0.$\}$

$P = x \in c_A\colon (tA)x = t(Ax)$ for all $t \in l$ such that $(tA)y$ exists for all $y \in c_A\}$.

In connection with the definition of F^0, it must be noted that α is not uniquely determined by f. Any f for which a representation (1) exists with $\alpha = 0$ is eligible as a test function for determining membership in F^0. See §8 for further information on the uniqueness of α.

Theorem 6.1. $L = F^0$.

Let $x \in L$ and $f \in c'_A$ with $\alpha = 0$. From (4), $\Sigma\, x_k f(\delta^k)$ is convergent, and so $x \in F^0$. Conversely, for $t \in l$ define $f \in c'_A$ by $f(x) = t(Ax)$. For $x \in F^0$, $(tA)x = \sum_k \sum_r t_r a_{rk} x_k = \sum x_k f(\delta^k)$ is convergent, hence $x \in L$.

We next observe that P is the largest of the distinguished subsets.

Theorem 6.2. $L \subset P$.

This follows from Lemma 4.1.

Theorem 6.3. *$\bar{c} \subset P$ and P is closed. If A is coregular, $P = \bar{c}$.*

Let $t \in l$ have the property that $(tA)x$ exists for all $x \in c_A$. Define f_t by $f_t(x) = (tA)x - t(Ax)$. It is continuous, by the Banach-Steinhaus closure theorem, hence $f_t^{\perp}$ is closed. Since P is the intersection of all such sets P is closed. Since $c \subset P$, it follows that $\bar{c} \subset P$.

Next, assume that A is coregular and let $f \in c'_A$ satisfy $f = 0$ on c. By (3) and (4), $f(x) = t(Ax) - (tA)x$ for all $x \in c_A$, and so, by definition of P, $f = 0$ on P. This proves that $P \subset \bar{c}$.

We can now improve Theorem 5.8 in the coregular case.

Corollary 6.4. *If A is coregular, c is dense in all the distinguished subsets, and if* any one of S, W, F, L is closed *then $F = L = P$.*

Theorem 6.3 generalizes the following result of Coomes and Cowling. This was given (with a serious misprint) as Lemma 1 of [3].

Corollary 6.5. *A coregular matrix A is perfect if and only if $P = c_A$.*

Forms of the next result have been given in many places. Probably its first formulation (a special case of Corollary 6.6) was given in [4].

Corollary 6.6. *A coregular matrix with the mean value (or equivalent) property is perfect.*

This follows from Corollary 6.4.

For conull matrices the result is false. Example 3. Indeed a multiplicative-0 reversible matrix cannot be perfect since $\lim_A = 0$ on c but not on c_A.

The next result extends the first part of Lemma 5.5 (in view of Theorem 6.2). It has the additional advantage that its hypothesis is invariant (see §7) and so its conclusion is $L_D = F$ for all D with $c_D = c_A$.

Theorem 6.7. *If c_0 is not dense in P, $L = F$.*

(In case A is coregular, the hypothesis is, by Theorem 6.3, equivalent to the assumption that c_0 is not dense in c. This implies the hypothesis of Theorem 5.10.)

There exists $f \in c'_A$ with $f = 0$ on c_0, $f \neq 0$ on P. Now if $\alpha = 0$ for this function, we would have, from (4), that $f = 0$ on P. Thus $\alpha \neq 0$. For $x \in L$, it follows from (4) that $x \in I$. Hence $L \subset I$. By Theorem 5.2, $L = F$.

Theorem 6.8. *c_0 is dense either in L or a maximal subspace of L. If A has maximal inset, replace L by P in this sentence.*

Let $f \in c'_A, f = 0$ on c_0. Then for $x \in L$ we have $f(x) = \alpha \lim_A x - \sum \alpha a_k x_k = 0$ or $\alpha\Lambda(x)$. Thus $\bar{c}_0$ has codimension at most one.

If A has maximal inset and $f = 0$ on c_0 we have, for all x,

$$f(x) = \alpha\Lambda(x) + t(Ax) - (tA)x.$$

For $x \in P$, $f(x) = \alpha\Lambda(x)$ and the conclusion follows.

Theorem 6.9. *If A sums no bounded divergent sequences, then $L = P = F = c$, and these sets are also closed.*

This follows from Theorem 6.3 and the fact that A is coregular and c is closed, [17] Theorem 1. It also follows from §5.3 of [24] which says that if B has any unbounded sequences in it, it also has bounded divergent ones.

Another result of the same type is [24], §6.2 which says that if $W \not\subset c$, then $W \not\subset m$.

7. Invariance

Two problems arise in connection with any property or object attached to a matrix A.

I. *Is the property invariant* ? i.e. *is it a property of c_A rather than of A?* For example "coregular" is invariant. If A is coregular and $c_D = c_A$, then D is coregular. This follows from (3). The topology of c_A is invariant.

II. *If a property is invariant, find an invariant expression for it,* i.e. *one not mentioning A,* so that it can be associated *with an arbitrary FK space.*

For example F was essentially given an invariant definition. For any linear topological space X including $\{\delta^n\}$ we can define F to be $\{x \in X\colon \sum x_k f(\delta^k)$

converges for all $f \in X'\}$. We are using here the very important fact that the FK topology of c_A is invariant.

It was pointed out by A. K. Snyder, [13], that a solution of Problem II for "coregular" is given by Lemma 4.6, above. A similar observation is made in [21].

S, W, F are all invariant and have invariant definitions.

If A is coregular P is invariant and has the invariant expression $P = \bar{c}$.

I_t is invariant and has the invariant expression $I_t = F$.

We shall now see that L is invariant in a wide variety of cases. I do not know whether L is always invariant. It is interesting that even though I is definitely not invariant (Example 6) and the invariance of L is in doubt, still $L \cap I$ is invariant by Lemma 4.3.

Theorem 7.1. *Any one of the following conditions is sufficient that L be invariant.*

i. *A is coregular*

ii. *c_0 is not dense in P*

iii. *$P \neq \bar{c}$*

iv. *L_D is closed for all D with $c_D = c_A$.*

v. *A has mean value (or equivalent) property.*

In addition ii, iii, iv, v *each imply that $L = F$.* (i.e. $L_D = F$ *for all D with* $c_D = c_A$.)

It should be pointed out that i does not imply ii, indeed c_0 may be dense in c_A. Example 11.

Suppose that A is coregular and that $c_D = c_A$. Let $t \in l$ and define $f \in c'_D = c'_A$ by $f(x) = t(Dx)$. In (1) with D instead of A, $\alpha = \alpha_D(f) = 0$. By (3), $\chi(f) = 0$. Again by (3), $\alpha_A(f) = 0$. By Theorem 6.1, for any $x \in L_A$, $\Sigma\, x_k f(\delta^k)$ is convergent. But $\Sigma\, x_k f(\delta^k)$ is $(tD)x$, thus $x \in L_D$. This shows that $L_A \subset L_D$ and by symmetry, L is invariant.

The rest of Theorem 7.1 follows, respectively, from Theorem 6.7, Theorem 6.3 (with Theorem 6.7), Lemma 5.6, and Corollary 5.9.

Corollary 7.2. *If L is not invariant, A is conull, $P = \bar{c}_0$, L_D is not closed for some D with $c_D = c_A$, and $L_D \neq c_D$ for all D with $c_D = c_A$.*

A special type of invariance is contained in the next result.

Theorem 7.3. *If J, A are conservative matrices and $D = JA$, then $L_D \supset L_A$. In particular if A, D are conservative triangles with $c_D = c_A$, then $L_D = L_A$.*

$$\left| \sum_{k=1}^{m} d_{nk} x_k \right| \leqq \| J \| \sup_{m,n} \left| \sum_{k=1}^{m} a_{nk} x_k \right|, \text{ proving } B_A \subset B_D.$$

Since $B = L$, the result follows.

The first sentence in the statement of Theorem 7.3 does not imply that $c_D \supset c_A$. (Agnew, Szasz, et al. See [12], pp. 56, 57.) Of course $c_D \supset L_A$ is clear since for $x \in L_A$, $Dx = J(Ax)$ by Lemma 4.1.

Unfortunately Theorem 7.3 does not prove the invariance of L since it is possible to have A, D with $c_A = c_D$ and no J. An easy example is D = identity matrix, $A = D$ with d_{11} replaced by 0.

Of the various equivalent conditions listed in Corollary 5.9, only *PMI* and *FAK* are obiously invariant. Of course it follows immediately that all five conditions are invariant.

Corollary 7.4. *PMI, Mean value, Associativity, FAK and the basis property of Corollary 5.9, part* v *are all invariant.*

Theorem 7.5. *Continuity of Λ on c is invariant. More specifically, $\Lambda = 0$ on c if and only if A is conull. If A is coregular, Λ is continuous on c if there exists a regular matrix D with $c_D = c_A$.*

Thus Λ is continuous on c in all but a pathological case (which may happen! Example 11).

The first statement follows from (2). If a regular matrix D exists with $c_D = c_A$, then, using (2) for $x \in c$, $\Lambda_A(x) = \chi(A) \lim x = \chi(A) \lim_D x$, hence Λ_A is continuous on c. The converse is given in [15], Theorem 15. (Replace (I) by c.)

Part ii of the following theorem is well known.

Theorem 7.6. *The following are invariant properties of A.*

i. $\sum_k a_{nk}$ *is uniformly convergent.*

ii. $\sum_k |a_{nk}|$ *is uniformly convergent.*

Indeed i *is equivalent to* $1 \in S$, ii *is equivalent to* $c_A \supset m$.

It is a straightforward check, which we shall omit, that i holds if and only if $p(1 - \sum_{k=1}^{n} \delta^k) \to 0$ for each of the seminorms p which define the FK topology of c_A.

Clearly ii implies i but not conversely, even for conservative triangles.

Theorem 7.7. *Let A be conull. Then if F is closed, $\sum_k a_{nk}$ is uniformly convergent.*

By Theorem 5.7, $S = W$. By Lemma 4.6, $1 \in S$, and the result follows from Theorem 7.6.

8. Uniqueness of α and t.

We mean, by saying that α is unique, that in (1), α is uniquely determined by f. If A has the property that $\Lambda = 0$, as in Example 4, then clearly α is not unique for A.

Theorem 8.1. *If α is not unique, $L = W$, in particular $L \subset \Lambda^{\perp}$, and A is conull.*

If α is not unique there is an expression for 0 of the form (4) with $\alpha = 1$, i.e. $\lim_A x + t(Ax) - \sum_k (a_k + \sum_r t_r a_{rk}) x_k = 0$. For $x \in L$, this says $\Lambda(x) = 0$. Thus $L \subset \Lambda^{\perp}$ and so $L = W$ by Theorem 5.4. A is conull by Lemma 4.6.

On the other hand t is never unique in (4) since we may add to it any sequence s such that $s(Ax) = (sA)x$ for all $x \in c_A$; any finite sequence would do for example. In (1), t may be unique once the sequence $\{\beta_k\}$ is specified; for example, this is true if A is reversible, in which case we may take $\beta_k = 0$ for all k.

9. Replaceability

We shall call a matrix A *replaceable* if there exists a multiplicative matrix D with $c_D = c_A$. Thus a coregular matrix is replaceable if and only if there exists a regular matrix D with $c_D = c_A$, while if A is conull any multiplicative D with $c_D = c_A$ must be multiplicative – 0. These facts follow from the nvariance of coregularity.

Replaceability is invariant.

Theorem 9.1. *If c_0 is not dense in P, A is replaceable.*

Note that a replaceable matrix need not be coregular.

In the proof of Theorem 6.7 it is shown that there exists $f \in c'_A$ with $f = 0$ on c_0, $\alpha \neq 0$. The result follows from [22], Satz 5.3 b.

Note. Under the hypotheses of Theorem 9.1, the quoted construction shows that if A is a triangle, the equipotent multiplicative matrix can be made a triangle. This is not true of all replaceable triangles ([5], § 6,) but is, if the triangle is coregular. Indeed if A is a coregular matrix and there is a triangle anywhere among the class of matrices D with $c_D = c_A$, then any matrix D with $c_D = c_A$ has an equivalent triangle. ([15], Lemma 7.)

Theorem 9.2. *Let A be coregular. Then A is replaceable if and only if c_0 is not dense in P.*

Since $P = \bar{c}$ this follows easily from [15], Lemma 11.

Theorem 9.2 is false if coregular is replaced by conull, i.e. the converse of Theorem 9.1 is false. See Example 4.

Theorem 9.2 shows that if A is coregular and c_0 is not dense in P, then c_0 is not even dense in F, indeed $1 \notin \bar{c}_0$ since A is replaceable.

Lemma 9.3. *If $\sum a_k x_k$ is bounded for all $x \in \bar{I}$, I is closed. If A is perfect and $\sum a_k x_k$ is bounded for all $x \in c_A$, A has maximal inset.*

Let $u_n(x) = \sum_{k=1}^{n} a_k x_k$. Then $\{u_n\}$ is equicontinuous on $\bar{I}$ and so $I = \{x\colon \lim u_n(x) \text{ exists}\}$ is closed. The second sentence follows since $c \subset I$.

Example 9 shows that "perfect" cannot be dropped in the second sentence and that $\sum a_k x_k$ may be bounded and divergent.

Lemma 9.4. *If $\bar{I} \subset L$, I, F are closed, $F = I$ and $S = W = \Lambda^\perp$.*

Since $L = B$, we see that $\sum a_k x_k$ is bounded for all $x \in \bar{I}$. By Lemma 9.3, I is closed. By hypothesis, then, $I \subset L$, hence, by Theorem 5.2, $I = F$. Thus also F is closed and the rest follows by Theorems 5.8 and 5.4.

Lemma 9.5. *If A is coregular and I is closed, A is replaceable.*

Λ is continuous on I. The result follows from Theorem 7.5.

Theorem 9.6. *If A is coregular and $\sum a_k x_k$ is bounded for all $x \in \bar{l}$, A is replaceable.*

This follows from Lemmas 9.3 and 9.5.

Theorem 9.7. *If A is coregular and $\bar{l} \subset L$, A is replaceable.*

This follows from Lemmas 9.4 and 9.5.

10. Perfectness and type M.

We see from Theorem 6.3 that for a perfect matrix, $P = c_A$. For coregu ar matrices there are many sufficient conditions for perfectness, namely *PMI*, *FAK* and the others. (Corollary 6.4) A well known necessary and sufficient condition in the case of coregular reversible matrices is *type M*, i.e. $t \in l$, $tA = 0$ implies $t = 0$. [1], pp. 90, 91. For more general matrices, perfect and type M are no longer equivalent, indeed, in some situations they are incompatible! (For example if $c_A \neq c$ and A has a two sided conservative inverse, A *cannot* be perfect and *must* be of type M.) However both properties continue to play important roles in summability.

We can extend the equivalence between type M and a kind of perfectness a little further so as to include some conull matrices.

Theorem 10.1. *Let A be reversible and multiplicative. Then A is of type M if and only if $\bar{c}_0$ is a maximal linear subspace of c_A.*

Let A be of type M and let $f \in c_A'$ satisfy $f = 0$ on c_0. By (1), $f(x) = \alpha \lim_A x + t(Ax)$, each β_k being zero since A is reversible. Thus $0 = f(\delta^k) = \sum t_r a_{rk}$ and so $t = 0$. This proves that $f = \alpha \lim_A$ and so either $\bar{c}_0 = c_A$ or $\bar{c}_0$ is a maximal linear subspace of c_A. The former cannot happen, for $\lim_A$ vanishes on c_0 but not on c_A since A is reversible.

Conversely, suppose that A is not of type M. Let $t \in l$, $t \neq 0$, $tA = 0$, and set $f(x) = t(Ax)$. Since both f and $\lim_A$ vanish on c_0 it will follow that $\bar{c}_0$ is not a maximal linear subspace when we show that f and $\lim_A$ are linearly independent. Since A is reversible, this is the same as the trivial fact that lim and $\sum t_k x_k$ are linearly independent on c.

We deduce from Theorem 10.1 the very well known result quoted earlier and an extension to multiplicative $-\,0$ matrices.

Corollary 10.2. *Let A be reversible* and *multiplicative. Then A is of type M if and only if the codimension of $\bar{c}$ is at most one.* (*It is* 0 *if A is coregular,* 1 *if A is conull.*)

The second statement in parentheses is true because $\bar{c} = \bar{c}_0$.

Of course a reversible, multiplicative-0 matrix cannot have $\bar{c} = c_A$ since $\lim_A = 0$ on $\bar{c}$. Examples 11 and 4 show reversible coregular and conull matrices (triangles!) with $\bar{c}_0 = \bar{c} = c_A$.

An immediate corollary of Theorem 10.1 is the following result of J. Copping ([2], Theorem 11.) The phrasing is extended slightly so as to include the coregular case.

Corollary 10.3. *Let A be reversible and multiplicative. Then A is of type M if and only if every multiplicative matrix D with $c_D \supset c_A$ has $\lim_D = \mu \lim_A$ for some constant μ, depending only on A, D. The same is true if $c_D = c_A$ is substituted for $c_D \supset c_A$. If A, D are regular, $\mu = 1$.*

Let A be of type M. Then $\lim_D = \lim_A$ on the maximal subspace $\bar{c}_0$, hence $\lim_D = \mu \lim_A$.

Conversely, let A be not of type M. Let $t \in l$, $t \neq 0$, $tA = 0$, and set $f(x) = \lim_A x + t(Ax)$. Let $D = JA$ where J is given in the proof of Lemma 4.4. Then, as pointed out there, $c_D = c_A$. Also $\lim_D = f$. Thus if $t_r \neq 0$ for a certain r, let u satisfy $Au = \delta^r$. Then $f(u) - \lim_A u = t_r \neq 0$.

(The proof of the converse is essentially that of Copping.)

The following result generalizes Mazur's well known sufficient condition for type M. See for example [10], Theorem 2, [17], Lemma 1, p. 505. A similar condition was given by Darevsky [4], (with B_A instead of L_A) under the assumption that A is regular and reversible.

Theorem 10.4. *If A has a right inverse whose columns belong to L_A, A is of type M. The same result holds if all but finitely many of the columns belong to L_A.*

Let $t \in l$, $tA = 0$. Then, if u is the n^{th} column of the right inverse, $0 = (tA)u = t(Au) = t_n$ by Lemma 4.1. Thus $t = 0$. In the latter case we get only that $t_n = 0$ for almost all n. But then $(tA)u = t(Au)$ surely holds, so that, finally, $t = 0$.

Corollary 10.5. *If a reversible matrix has the mean value property, it is of type M.*

Each reversible matrix has a right inverse ([14], Lemma 3) whose columns clearly belong to c_A. The result follows by Theorem 10.4.

This result does not contain and is not contained in Corollary 6.6. They overlap when applied to coregular reversible matrices.

11. Examples

Example 1. Let $A = 0$. Then $c_A = s$ which has $\{\delta^n\}$ as basis. Thus $S = c_A$. *A has the mean value property.*

Example 2. Let A be the identity matrix. A is coregular. $c_A = c$. $S = W = c_0$. $F = c_A$. *A has the mean value property.*

Example 3. Let $a_{nn} = 1/n$, $a_{nk} = 0$ otherwise. Then A is conull. c_A has basis $\{u, \delta^1, \delta^2, \delta^3, \cdots\}$ where $u = \{n\}$.

$$S = W = \bar{c}_0 = \bar{c}.\ F = c_A = S \oplus u = \bar{c}_0 \oplus u.$$

A has the mean value property.

Example 4. Let $\{a_k\}$ be a sequence satisfying $\Sigma\ |a_k| < \infty$, $a_k \neq 0$ for all k. Let

$$A = \begin{matrix} a_1 & 0 & 0 & 0 \cdots \\ a_1 & a_2 & 0 & 0 \cdots \\ a_1 & a_2 & a_3 & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

A is a conull triangle. A is replaceable. ([15], §5). *A has the mean value property*, and $S = c_A$.

The next result is essentially known.

Theorem 11.1. *If A is a triangle and $x \in S$, then $a_{nn}x_n \to 0$.*

Let $x \in S$, then $x^{(n)} = x - \sum_{k=1}^{n} x_k \delta^k \to 0$. But $\| x^{(n-1)} \| \geqq |a_{nn}x_n|$.

Corollary 11.2. *If A is a triangle and $\{1/a_{nn}\}$ is bounded, then $S = c_0$.*

Example 5. Let

$$A = \begin{matrix} 1 & 0 & 0 & 0 \cdots \\ -1 & 1 & 0 & 0 \cdots \\ 0 & -1 & 1 & 0 \cdots \\ 0 & 0 & -1 & 1 \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

A is a multiplicative -0 triangle. $1 \in W \sim X$. By Theorem 5.8 (or 7.7), S is not closed and $\{\delta^n\}$ is not basic. Hence, by Theorem 5.7, F, L, W are not closed. $S = c_0$ by Corollary 11.2. $I = c_A$, $L = F = c_A \cap m = (c_A \cap m \cap \bar{c}) \oplus u$ where $u = \{(-1)^n\}$, since

$$(5) \qquad \left| \sum_{k=1}^{n} \sum_{r=1}^{\infty} t_r a_{rk} x_k - \sum_{r=1}^{n} t_r (Ax)_r \right| = \left| t_{n+1} x_n \right|,$$

and since $f = 0$ on c implies $f = \alpha \lim_A$ on $c_A \cap m$.

Since $I_t = F \neq I$, I is not invariant. Thus A *does not have the mean value property.*

We shall prove that $P = c_A$.

We first observe that $t \in l$ has the property that $(tA)x$ exists for all $x \in c_A$ if and only if the matrix

$$\begin{matrix} t_2 & 0 & 0 & 0 \cdots \\ t_3 & t_3 & 0 & 0 \cdots \\ t_4 & t_4 & t_4 & 0 \cdots \\ \cdots & \cdots & \cdots & \cdots \end{matrix}$$

is conservative. (This may be seen by substituting $x = A^{-1}y$, $y \in c$ in the right hand side of (5).) If such a matrix is conservative it must be multiplicative -0, thus for all such t and $x \in c_A$, the right hand side of (5) tends to 0.

Thus $P = c_A$. $P \neq \bar{c}$ since a reversible, multiplicative-0 matrix cannot be perfect because $\lim_A = 0$ on c but not on c_A.

Example 6. Let

$$A = \begin{matrix} 1/2 & 0 & 0 & 0 & \cdots \\ 1/2 & 1/2 & 0 & 0 & \cdots \\ 0 & 1/2 & 1/2 & 0 & \cdots \\ 0 & 0 & 1/2 & 1/2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

A is a regular triangle. $S = c_0$ by Corollary 11.2. $I = c_A$, $L = F = c_A \cap m$ since (5) holds here also. Since $F \neq c_A$, we have $I_i \neq I$ and so I is not invariant. $P = \bar{c} = c_A$ since A is of type M. Since $c \subset F \neq c_A$, F is not closed. $\{1, \delta^1, \delta^2, \cdots\}$ is not a basis for c_A but is a $(C, 1)$ basis for c_A. ([20], Lemma 1.) *A does not have the mean value property.*

Example 7. (See also [12], Remark 7, [11], pp. 52, 53.) Let

$$A = \begin{matrix} -1 & 0 & 0 & 0 & \cdots \\ -2 & 1 & 0 & 0 & \cdots \\ 0 & -2 & 1 & 0 & \cdots \\ 0 & 0 & -2 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

A is a coregular triangle. $c_A = c \oplus u$ where $u = \{2^n\}$. (See for example [16], Chapter 1, §2, Problem 21. The rest of the statements in this example are true for any matrix A with $c_A, = c \oplus$ one sequence.) $S = W = c_0$, these are closed. $L = F = c$. Thus *L is a closed proper subspace of c_A*. For every D with $c_D = c_A$, either $I_D = c$ or $I_D = c_D$. Since $I_i \neq c_A$, there exists at least one D with $I_D = c$. For this D, *I_D is a closed proper subspace of c_A*.

Example 8. Let

$$A = \begin{matrix} u_1v_1 & 0 & 0 & 0 & \cdots \\ u_2v_1 & u_2v_2 & 0 & 0 & \cdots \\ u_3v_1 & u_3v_2 & u_3v_3 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

where $u_n \neq 0$, $v_n \neq 0$, and assume that A is regular. The $(C, 1)$ matrix is a special case .Then A is a regular triangle and $F = \bar{c} = c_A$. ([15] Theorem 2.) $W = S = \bar{c}_0 \neq c_A$. *$A$ has the mean value property.*

Example 9. Let

$$A = \begin{matrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 1 & \varepsilon_2 & 0 & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & 0 & \cdots \\ 1 & 1 & \frac{1}{2} & \frac{1}{2} & \varepsilon_3 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

where the $(2n)^{\text{th}}$ row is $1, 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \cdots \frac{1}{2^{n-1}}, \frac{1}{2^{n-1}}, 0, 0, \cdots$ and the $(2n-1)^{\text{th}}$ row is $1, 1, \frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2^{n-2}}, \varepsilon_n, 0, 0, \cdots$ where $\{\varepsilon_n\}$ is a sequence such that $2^n \varepsilon_n \to 0$. (If the odd-numbered rows are omitted the matrix is simpler, though not a triangle, and has the property given below.) Let

$$x = (1, -1, 2, -2, 4, -4, \cdots).$$

A is a conull triangle. $\Sigma\, a_k x_k$ is bounded and divergent; a simple check shows that $x \in B$. Thus $x \in L \sim I$, $x \in L \sim F, L \neq F$. By Theorems 5.7 and 6.7, $P = \bar{c}_0$, L is not closed.

Example 10. Consider a matrix A such that the codimension of c in c_A is finite. (See [17], Theorems 1 and 3 for the example and the following facts.) Then c is closed, A is coregular, and so $P = L = c$.

Example 11. (K. Zeller) Let $\Sigma\, |a_k| < \infty, a_{2k} \neq 0,\ a_{2k-1} = 0$ for all k. Let

$$A = \begin{matrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & a_2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & 1 & 0 & 0 & 0 & \cdots \\ 0 & a_2 & 1 & a_4 & 0 & 0 & \cdots \\ 0 & a_2 & 0 & a_4 & 1 & 0 & \cdots \\ 0 & a_2 & 0 & a_4 & 1 & a_6 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{matrix}$$

A is a *coregular perfect non-replaceable triangle.* (See [15], p. 657. A different example is given in [6].) An easy computation yields $I = B = \{x \in c_A\colon \lim x_{2n-1} \text{ exists}\}$. Thus $L = F = I$. Since A is not replaceable, $\bar{c}_0 = \bar{c} = P = c_A$. Thus $W \neq \bar{c}_0$ since $1 \notin W$ by Lemma 4.6 and so L, F, I are not closed. (This also follows immediately from Lemma 9.5.) Since $I = F = I_i$, any D with $c_D = c_A$ has $I_D \supset I_A$, i.e. A has the smallest inset in its class. There is no D with $c_D = c_A$ with maximal inset or even closed inset, by Lemma 9.5. Of course I_A is dense.

To give functions $f \in c_A'$ such that $d_k = f(\delta^k)$ for no D with $c_D = c_A$ is easy. Since D cannot have maximal inset, we choose f so that $\Sigma f(\delta^k)x_k$ converges for all $x \in c_A$. For example, $f = 0$ would do, or $f(x) = x_1$.

Example 12. (K. Zeller) Let D, E be the matrices of Examples 4, 7 respectively and $A = ED$. Then A is a conull triangle. $\{\delta^n\}$ is a basis for c_D in the D topology, hence in the (weaker) A topology. $c_A = c_D \oplus u$ where $u = D^{-1}(\{2^n\})$ and $(u, \delta^1, \delta^2, \cdots)$ is a basis for c_A. However A does not have the mean value property by Corollary 10.5 because it is not of type M since E is not.

Note that A is coercive, i.e. $c_A \supset m$, but A does not have maximal inset (See Theorem 7.6, ii).

12. Questions

I. Must $L = F$ if A is coregular? See 5.6, 5.10.

II. If F is closed, must L be? See 5.6, 5.7. The answer is yes if A is coregular, by Corollary 6.4.

III. Is L invariant? See 7.1, 7.2, 7.3, and the following remarks. Theorem 6.8 gives no information since, possibly, $\bar{c}_0 \not\subset L$.

IV. Must there exist D with $c_A = c_A$, $I_D = F_A$? Certainly (Theorem 5.1) $F_A = \cap \{I_D : c_D = c_A\}$.

V. If A has maximal inset, must A be replaceable ? The answer is yes f A is coregular .

VI. Is the equation $\Lambda = 0$ invariant? (It implies $P = W = \bar{c}_0$.)

VII. Can $\Sigma\, a_k x_k$ be bounded and divergent if A is coregular? (If not, the answer to I is "yes".) See Example 9.

VIII. Is P invariant?

IX. Must the "external inset" $I_e = \cup(I_D : c_D = c_A)$ be equal to c_A? (This is not answered by Example 11 in which $I_D \neq c_A$ for all D with $c_D = c_A$.)

X. I conjecture that if I is invariant for a certain A, then A has the mean value property.

XI. Must I be closed if Λ is continuous?

XII. Must a conull matrix be replaceable?

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(Received August 12, 1963)