ON THE ZEROS OF $f(g(z))$ WHERE f AND g ARE ENTIRE FUNCTIONS

BY

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Introduction. Pólya proved in [2]:

If f and g are entire functions, then $f(g(z))$ is of infinite order unless (i) *f is of finite order and g is a polynomial or* (ii) *f is of order zero and g is of finite order.*

This result suggests the following questions.

I. *What can be said about the exponent of convergence of the zeros of $f(g(z))$ if the zeros of f have a positive exponent of convergence and $g(z)$ is entire and not a polynomial?*

II. *What is the order of the meromorphic function $F(g(z))$, if F is meromorphic and g is entire?*

As far as we can see the solution of these problems can not be deduced from Pólya's theorem. Nor does it seem possible to obtain an answer from the theorems of Schottky, Landau and Bloch (a corollary of Schottky's theorem is used in Pólya's proof of his theorem).

We shall derive a solution of these two problems from an investigation of the following question, which seems of independent interest.

III. *Let $g(z)$ be an entire function. Denote the maximum modulus of g on*

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$|z| = r$ by $M(r) = M(r, g)$. Define, for every sufficiently large complex number w, the positive number $t = t(|w|)$ by

$$|w| = M(t, g). \tag{1}$$

Is it possible to find a 'universal' function $\phi(t)$ such that the equation

$$g(z) = w \tag{2}$$

has a solution in

$$|z| < \phi(t),$$

provided that $|w|$ exceeds(3) a constant $K_0(g, \phi)$?

We have been unable to find a general solution of problem III, but for functions of finite order we can prove, rather simply,

Theorem 1. *Let $g(z)$ be an entire function of finite order which is not a polynomial. Let ξ be a positive number. For $|w| > |g(0)|$ define $t = t(|w|)$ by (1). Then (2) has a solution in*

$$|z| < t^{1+\xi} \quad , \tag{3}$$

provided $|w| > K(g, \xi)$.

This theorem contains a complete answer to questions I and II. We prove:

Corollary 1.1. *Let f and g be entire functions. Assume that the zeros of f have a positive exponent of convergence and that g is not a polynomial. Then the zeros of $f(g(z))$ do not have a finite exponent of convergence.*

Corollary 1.2. *Let F be a meromorphic function which is not of order zero and let $g(z)$ be an entire function which is not a polynomial.*

Then $F(g(z))$ is of infinite order.

Corollary 1.2 gives a new proof of Pólya's theorem, independent of Schottky's theorem(4).

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(3) Picard's theorem shows that the condition $|w| > K_0(g, \phi)$ is essential.

(4) Our proof is, in fact, completely elementary and depends only on the simplest properties of analytic functions.

Theorem 1 has two shortcomings: it is restricted to functions of finite order and the bound $\phi(t) = t^{1+\xi}$ seems unnecessarily large. The following theorem gives a much better bound and applies to functions of infinite order provided their rate of growth is subject to the rather mild regularity conditions (4) and (5). It is immediately verified that these conditions are satisfied for all functions of finite order and positive lower order.

Theorem 2. *Let $g(z)$ be an entire function of positive lower order, i.e. for some $\beta(>0)$ and $r_0(>0)$,*

$$\log M(r,g) > r^{\beta} \quad (r > r_0). \tag{4}$$

Assume that there exist positive numbers r_1, B such that

(5 $$\log M\{r(1+(\log M(r,g))^{-1/2})\} < (\log M(r))^{B} \quad (r > r_1).$$

If t is defined by (1), *then* (2) *has a root in*

$$|z| < t\{1 + 2(\log M(t,g))^{-1/2}\}$$

provided $|w|$ exceeds a bound depending only on g.

It is easy to deduce, from a well-known lemma of E. Borel [1; p. 61], that the left-hand side of (5) is less than $B \log M(r)$ (with any fixed $B > 1$), provided r avoids an exceptional set of finite logarithmic measure(5). This shows that condition (5) is a rather weak restriction, in particular, this condition is compatible with arbitrarily high rates of growth of $M(r,g)$, provided this growth does not take place too irregularly.

There is nothing special about the exponent 1/2 in (5); it could be replaced by any other number between zero and one, both in (5) and in the conclusion of the theorem.

The growth condition (5) may be omitted entirely if g satisfies some other condition which makes it possible to apply Schottky's theorem.

We prove:

(5) This may be proved by (i) applying the usual form of Borel's lemma to the auxiliary function $\phi(x) = \exp(\sqrt{\log M(e^x)})$; (ii) using the substitution $x = \log r$ and the inequality $1 + t < e^t$ $(t > 0)$.

Theorem 3. *Let $g(z)$ be an odd entire function, which does not reduce to a polynomial.*

Assume that t is determined by (1) and that $\xi(>0)$ is fixed.

Then, if $|w|$ is sufficiently large, (2) has a solution in the disc

$$|z| \leqq (1+\xi)t.$$

It is possible to modify Theorem 3 by dropping the condition that $g(z)$ be an odd function and assuming instead that $g(z)$ omits some finite value a.

1. A fundamental lemma.

Lemma 1. *Assume that the function*

$$g(z) = \sum_{k=0}^{\infty} c_k z^k$$

is regular in $|z| \leqq R$ and that

$$g(z) \neq w \qquad (|z| \leqq R).$$

Then, for all r such that $r < R$, and

$$\max\{2|g(0)|, 4\} < 2M(r,g) \leqq |w| \tag{1.1}$$

we have

$$|c_n| r^n < 6|w| \left(\frac{r}{R}\right)^n \log M(R,g) + \frac{M^2(r,g)}{|w|} \quad (n = 1,2,3,\cdots). \tag{1.2}$$

Proof. Under the assumptions of the lemma,

$$h(z) = \log\{1 - \frac{g(z)}{w}\} = \sum_{n=0}^{\infty} b_n z^n$$

is regular in $|z| \leqq R$ and

$$A(R) = \sup_{|z| \leqq R} \operatorname{Re}\{h(z)\} \leqq \log\left\{1 + \frac{M(R,g)}{|w|}\right\}.$$

By a well known result [3; p. 86], we have

$$|b_n| R^n \leqq 4\log \left(1+\frac{M(R,g)}{|w|}\right) - 2\log \left|1-\frac{g(0)}{w}\right| \qquad (n=1,2,3,\cdots).$$

By (1.1)

$$-\log \left| 1-\frac{g(0)}{w} \right| < \log 2 < \log M(R,g),$$

$$\log \left\{ 1+\frac{M(R,g)}{|w|} \right\} < \log \left\{ \frac{M(R,g)}{2} + \frac{M(R,g)}{4} \right\} < \log M(R,g),$$

so that

$$(1.3) \qquad |b_n| < \frac{6\log M(R,g)}{R^n} \qquad (n=1,2,3,\cdots).$$

In $|z| \leqq r$, we have

$$\left|\frac{g(z)}{w}\right| \leqq \frac{1}{2},$$

and hence

$$(1.4) \quad h(z) = \log\left\{ 1-\frac{g(z)}{w} \right\} = -\{g(z)/w\} - \frac{1}{2}\{g(z)/w\}^2 - \frac{1}{3}\{g(z)/w\}^3 - \cdots$$

$$= -\{g(z)/w\} - \omega(z),$$

where

$$|\omega(z)| \leqq M(r,\omega) \leqq \frac{1}{2}\sum_{k=2}^{\infty} \frac{M^k(r,g)}{|w|^k} \leqq \frac{M^2(r,g)}{|w|^2}.$$

Putting

$$\omega(z) = \sum_{n=0}^{\infty} d_n z^n ,$$

we find

$$(1.5) \qquad |d_n| r^n \leqq M(r, \omega) \leqq \frac{M^2(r,g)}{|w|^2} \; .$$

By (1.4)

$$b_n = -\frac{c_n}{w} - d_n \; ,$$

$$\frac{|c_n| r^n}{|w|} \leqq |b_n| r^n + |d_n| r^n \; .$$

Combined with (1.5) and (1.3), this yields (1.2).

2. Preliminaries about the central index and the maximum term of power series.

Theorems 1 and 2 will be proved by showing that (1.2) can be contradicted for suitable n, r, R, if the conclusions of these theorems do not hold. To find these contradictions we need a few simple facts about the maximum term and the central index of power series. The relatively complicated theory of Wiman-Valiron will not be required. We assume from now on that

$$(2.1) \qquad g(z) = \sum_{k=0}^{\infty} c_k z^k$$

is an entire function but not a polynomial. The function

$$\mu(r) = \sup_k \{ |c_k| r^k \}$$

is called the *maximum term* of (2.1). The largest integer n for which

$$|c_n| r^n = \mu(r)$$

is called the *central index* and is denoted by $\nu(r)$. Both $\mu(r)$ and $\nu(r)$ are non-decreasing functions of r tending to ∞ as $r \to \infty$ (unless $g(z)$ is a polynomial, a case that we exclude from consideration). By Cauchy's formula for the coefficients of a power series

(2.2) $$\mu(r) \leqq M(r) = M(r, g).$$

By the definition of the maximum term, if

$$0 < \rho < R \text{ and } \nu = \nu(\rho),$$

(2.3) $$(R/\rho)^{\nu} = |c_{\nu}| R^{\nu}/(|c_{\nu}| \rho^{\nu}) \leqq \mu(R)/\mu(\rho).$$

Also, for $0 < r < \rho$, $\nu = \nu(\rho)$,

(2.4) $$M(r) = M(r, g) \leqq \sum_{k=0}^{\infty} |c_k| r^k \leqq \nu\mu(r) + \frac{\mu(r)}{|c_{\nu}| r^{\nu}} \sum_{k=\nu}^{\infty} |c_k| r^k$$

$$\leqq \nu\mu(r) + \mu(r) \sum_{k=\nu}^{\infty} \frac{|c_k| \rho^k}{|c_{\nu}| \rho^{\nu}} \left(\frac{r}{\rho}\right)^{k-\nu}$$

$$\leqq \mu(r) \left\{ \nu(\rho) + \frac{\rho}{\rho - r} \right\}.$$

If ρ is so large that

(2.5) $$\mu(\rho) > 1$$

and if

$$R - \rho = \rho - r > 0,$$

then

(2.6) $$\frac{\rho}{\rho - r} < \frac{R}{\rho - r} = \frac{R}{R - \rho} = \frac{2R}{R - r},$$

and by (2.3), (2.5) and (2.2)

$$\nu(\rho) \frac{R - \rho}{R} \leqq \nu(\rho) \log(R/\rho) \leqq \log \mu(R) \leqq \log M(R).$$

Hence, substituting in (2.4) and using (2.6),

(2.7) $$M(r) \leqq \mu(r) \{\log M(R) + 1\} \frac{2R}{R - r} \quad (1 < \mu(r), 0 < r < R).$$

Finally we note that, as soon as r exceeds a suitable bound $r_2(>1)$,

$$|c_{\nu(r)}| < 1,$$

and hence

$$\nu(r) > \frac{\log \mu(r)}{\log r} \qquad (r > r_2). \tag{2.8}$$

3. A consequence of Lemma 1 and inequality (2.7). The function

$$\frac{2\,M^2(r, g)}{\mu(r)} \qquad (r > 0),$$

is a continuous function of r which tends to ∞ as $r \to \infty$.

For all sufficiently large $|w|$, it is therefore possible to find some $r = r(|w|)$ such that

$$2M(r) = 2M(r, g) \leqq \frac{2M^2(r)}{\mu(r)} = |w| \tag{3.1}$$

[if the quantity $r(|w|)$ is not unique, we consider any one of the acceptable values of r and denote it by $r(|w|)$].

In view of the definition (1), we see that (3.1) implies

$$r(|w|) < t(|w|), \tag{3.2}$$

provided $|w|$ is large enough.

Assume now that

$$g(z) \neq w$$

in the disc

$$|z| \leqq R \qquad (R > r(|w|)).$$

We apply Lemma 1 with

$$r = r(|w|), \qquad n = \nu(r), \qquad |w| > K_0(g)$$

and select K_0 so large that the conditions (1.1), (3.2) and the inequalities $\nu(r) > 1$,

$$\mu(r) > 1, \tag{3.3}$$

are satisfied simultaneously. Then, by (1.2) and (3.1),

$$\mu(r) = |c_n| r^n \leqq 12 \frac{M^2(r)}{\mu(r)} \left\{\frac{r}{R}\right\}^n \log M(R) + \frac{1}{2}\mu(r),$$

$$1 \leqq 24 M^2(r) r^n \log M(R)/\mu^2(r) R^n. \tag{3.4}$$

Now (2.7) holds because r satisfies (3.3). Hence we may use (2.7) in (3.4); this yields

$$1 \leqq \frac{96 R^2 r^n (1 + \log M(R))^2 \log M(R)}{(R-r)^2 R^n} \quad (|w| > K_0,\ r = r(|w|) < R). \tag{3.5}$$

4. Proof of Theorem 1. Let $\xi(>0)$ be given. If Theorem 1 were false, we could find numbers w, of arbitrarily large modulus, such that $g(z) \neq w$ in

$$|z| < \{t(|w|)\}^{1+\xi}.$$

In view of (3.2) there would also exist arbitrarily large values of $r(|w|)$ such that $g(z) \neq w$ for

$$|z| \leqq R = \{r(|w|)\}^{1+\xi}.$$

Since $g(z)$ is now of finite order, there exist positive bounds λ and K_1 such that

$$\log M(R) < R^\lambda = r^{\lambda + \xi\lambda} \qquad (R > K_1) \quad .$$

Hence (3.5) implies that, for sufficiently large r (i.e. for sufficiently large $|w|$),

$$1 \leqq A r^{3\lambda + 3\xi\lambda - n\xi} \qquad (A = \text{absolute constant}). \tag{4.1}$$

Now as $r \to \infty$, $n = \nu(r) \to \infty$, so that (4.1) can not hold for arbitrarily large r. This contradiction proves Theorem 1.

5. Proof of Theorem 2. We shall use the abbreviation

$$L = \log M(r),$$

where, for $|w|$ large enough, $r = r(|w|)$ is defined by (3.1).

Choosing

$$R = r(1 + L^{-1/2}), \tag{5.1}$$

we show first that (3.5) cannot hold for sufficiently large values of $|w|$.

By (5) and (5.1),

$$\log M(R) < L^B. \tag{5.2}$$

Taking logarithms in (3.5) and using (5.2), we obtain

$$0 \leqq -n\log(1 + L^{-1/2}) + 3B\log L + \log L + A,$$

$$0 \leqq -(1/2)nL^{-1/2} + (3B+1)\log L + A \quad (L > L_0), \tag{5.3}$$

where A is an absolute constant and L_0 is a sufficiently large bound. For sufficiently large r, (2.8) and (4) yield

$$n = \nu(r) > \frac{\log \mu(r)}{\log r} > \frac{\beta \log \mu(r)}{\log L}. \tag{5.4}$$

By (2.2), (2.7), (5.1) and (5.2)

$$\log \mu(r) = L + O(\log L) \quad (L \to \infty). \tag{5.5}$$

Using (5.4) and (5.5) in (5.3), we see that (3.5) can not hold for large L unless

$$0 \leqq -\frac{1}{2}\beta L^{1/2}(\log L)^{-1} + O(\log L).$$

Since this inequality is false for all large L, it follows that the choice (5.1) of R is not permissible. Hence $g(z) = w$ has a root in

$$|z| \leqq r(1 + L^{-1/2}) \qquad (r = r(|w|)).$$

Also, by (3.1) and (5.5),

$$\log 2 + 2L - L + O(\log L) = \log|w|,$$

$$L = \log|w| + O(\log L),$$

(5.6) $$L > \frac{1}{4}\log|w|$$

for all large L. By (5.6), (3.2) and (1),

$$r(1 + L^{-1/2}) < t(1 + 2(\log|w|)^{-1/2}) = t(1 + 2(\log M(t))^{-1/2}).$$

This completes the proof of Theorem 2.

6. Proof of Corollary 1.1. Consider first the case that $g(z)$ is of infinite order. By a fundamental result of R. Nevanlinna [1;p.72] the solutions of the equation $g(z) = w$ have an infinite exponent of convergence for all w, with at most one exception. It is therefore enough to assume that $f(z)$ has at least two distinct zeros ,w_1 and w_2. For the solutions of $g(z) = w_1$and of $g(z) = w_2$ are then zeros of $f(g(z))$.

Consider now the case that $g(z)$ is of finite order. We shall apply Theorem 1 with $\xi = 1$.

Let $f(w)$ have $q(t)$ zeros in

$$K(g,1) = K^* < |w| \leqq M(t,g).$$

Since the zeros of $f(w)$ have a positive exponent of convergence, there is a $\tau > 0$ such that for some arbitrarily large values t' of t

(6.1) $$q(t') > (M(t',g))^{\tau}.$$

Since $g(z)$ is not a polynomial,

(6.2) $$M(t,g)t^{-k} \to \infty \qquad (t \to \infty)$$

for every constant k.

By Theorem 1, $g(z)$ takes on , in $|z| \leqq t^2$, every value w such that

$$K^* < |w| \leqq M(t, g).$$

Hence $f(g(z))$ has at least $q(t)$ zeros in $|z| \leqq t^2$. By (6.1) and (6.2) this number is larger than any given power of t, for some arbitrarily large values of t. This shows that the zeros of $f(g(z))$ have an infinite exponent of convergence and completes the proof of the corollary.

7. Proof of Corollary 1.2. By assumption $F(w)$ is not of zero order. Hence, if a is not one of two possible exceptional values [1; *p.* 72], we may write

$$F(w) = a + (f(w)/h(w)),$$

where f and h are entire functions without common zeros and where the zeros of f have a positive exponent of convergence. Therefore, by Corollary 1.1 the a-points of $F(g(z))$ have an infinite exponent of convergence. Hence $F(g(z))$ is of infinite order.

8. Proof of Theorem 3. Assume that $g(z) \neq w$ in

(8.1) $$|z| \leqq (1 + \xi)\, t(|w|) = (1 + \xi)t,$$

where $\xi(>0)$ is fixed and $t(|w|)$ is defined by (1).

Since $g(z)$ is an odd function, we also have

(8.2) $$g(0) = 0, \qquad g(z) \neq -w.$$

Consider the auxiliary function

$$f(s) = \frac{g((1 + \xi)st)}{2w} + \frac{1}{2} \quad .$$

Then by (8.2), we have, in $|s| \leqq 1$,

$$f(s) \neq 1, \quad f(s) \neq 0, \quad f(0) = \tfrac{1}{2}.$$

Hence by Schottky's theorem [3; p. 280],

$$\left| f\left(\frac{1 + \frac{1}{2}\xi}{1 + \xi} e^{i\theta} \right) \right| < C \quad (0 \leqq \theta < 2\pi),$$

where C depends only on ξ.

In terms of g this implies

$$M((1 + \tfrac{1}{2}\xi)t, g) < (2C + 1)\, M(t, g). \tag{8.3}$$

Consider the set $\mathscr{E}$ of all values w such that, in the disc (8.1), $g(z) \neq w$. If $\mathscr{E}$ were unbounded, there would exist arbitrarily large values of t for which (8.3) holds. Since $\log M(t)$ is a convex function of $\log t$, this would imply that $\log M(t)/\log t$ is bounded and hence that $g(z)$ is a polynomial. As this contradicts one of our assumptions, we see that $\mathscr{E}$ must be bounded. The proof of Theorem 3 is now complete.

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