CONVOLUTION OPERATOR AND MAXIMAL FUNCTION FOR THE DUNKL TRANSFORM

By

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Abstract. For a family of weight functions h_κ invariant under a finite reflection group on $\mathbb{R}^d$, analysis related to the Dunkl transform is carried out for the weighted L^p spaces. Making use of the generalized translation operator and the weighted convolution, we study the summability of the inverse Dunkl transform, including as examples the Poisson integrals and the Bochner–Riesz means. We also define a maximal function and use it to prove the almost everywhere convergence.

1 Introduction

The classical Fourier transform, initially defined on $L^1(\mathbb{R}^d)$, extends to an isometry of $L^2(\mathbb{R}^d)$ and commutes with the rotation group. For a family of weight functions h_κ invariant under a reflection group G, there is a similar isometry of $L^2(\mathbb{R}^d, h_\kappa^2)$, called the Dunkl transform ([3]), which enjoys properties similar to those of the classical Fourier transform. This transform is defined by

$$\widehat{f}(x) = c_h \int_{\mathbb{R}^d} E(x, -iy) f(y) h_\kappa^2(y) dy,$$

where the usual character $e^{-i\langle x,y\rangle}$ is replaced by $E(x,-iy) = V_\kappa(e^{-i\langle \cdot, y\rangle})(x)$ for some positive linear operator V_κ (see the next section). If the parameter $\kappa = 0$, then $h_\kappa(x) \equiv 1$ and $V_\kappa = \mathrm{id}$, so that $\widehat{f}$ becomes the classical Fourier transform.

The basic properties of the Dunkl transform have been studied in [3, 7, 13, 15] and also in [12, 19] (see also the references therein). These studies are mostly for $L^2(\mathbb{R}^d)$ or for Schwartz class functions.

The purpose of this paper is to develop an L^p theory for the summability of the inverse Dunkl transform and to prove a maximal inequality that implies almost everywhere convergence.

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The classical Fourier transform behaves well with the translation operator $f \mapsto f(\cdot - y)$, which leaves Lebesgue measure on $\mathbb{R}^d$ invariant. However, the measure $h_\kappa^2(x)dx$ is no longer invariant under the usual translation. One ends up with a generalized translation operator, defined on the Dunkl transform side by

$$\widehat{\tau_y f}(x) = E(y, -ix)\widehat{f}(x), \qquad x \in \mathbb{R}^d.$$

An explicit formula for τ_y is unknown in general. In fact, τ_y may not even be a positive operator. Consequently, even the boundedness of τ_y in $L^p(\mathbb{R}^d; h_\kappa^2)$ becomes a challenging problem. At the moment, an explicit formula for $\tau_y f$ is known only in two cases: when f is a radial function and when $G = \mathbb{Z}_2^d$. Properties of τ_y are studied in Section 3. In particular, the boundedness of the τ_y for radial functions is established.

For f, g in $L^2(\mathbb{R}^d; h_\kappa^2)$, their convolution can be defined in terms of the translation operator as

$$(f *_\kappa g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x g^\vee(y) h_\kappa^2(y)dy.$$

Based on a sharp Paley–Wiener theorem, we are able to prove that $f *_\kappa \phi_\varepsilon$ converges to f in $L^p(\mathbb{R}^d; h_\kappa^2)$ for certain radial ϕ , where ϕ_ε is a proper dilation of ϕ. This and other results are given in Section 4.

The convolution $*_\kappa$ can be used to study the summability of the inverse Dunkl transform. We prove L^p convergence of the summability under mild conditions, including as examples Gaussian means (heat kernel transform), Abel means and the Bochner–Riesz means for the Dunkl transform in Section 5.

In Section 6, we define a maximal function and prove that it is strong type (p, p) for $1 < p \leq \infty$ and weak type $(1, 1)$. As usual, the maximal inequality implies almost everywhere convergence for the summability.

In the case $G = \mathbb{Z}_2^d$, the generalized translation operator is bounded on $L^p(\mathbb{R}^d; h_\kappa^2)$. Many of the results proved in the previous sections hold under conditions that are more relaxed in this case and the proof is more conventional. This case is discussed in Section 7.

The following section is devoted to the preliminaries and background. The basic properties of the Dunkl transform are also given there.

2 Preliminaries

Let G be a finite reflection group on $\mathbb{R}^d$ with a fixed positive root system R_+, normalized so that $\langle v, v\rangle = 2$ for all $v \in R_+$, where $\langle x, y\rangle$ denotes the usual Euclidean inner product. For a nonzero vector $v \in \mathbb{R}^d$, let σ_v denote the reflection with respect to the hyperplane perpendicular to v, $x\sigma_v := x - 2(\langle x,v\rangle/\|v\|^2)v$, $x \in \mathbb{R}^d$. Then G is a subgroup of the orthogonal group generated by the reflections $\{\sigma_v : v \in R_+\}$.

In [1], Dunkl defined a family of first order differential-difference operators $\mathcal{D}_i$, which play the role of the usual partial differentiation for the reflection group structure. Let κ be a nonnegative multiplicity function $v \mapsto \kappa_v$ defined on R_+ with the property that $\kappa_u = \kappa_v$ whenever σ_u is conjugate to σ_v in G; then $v \mapsto \kappa_v$ is a G-invariant function. Dunkl's operators are defined by

$$\mathcal{D}_i f(x) = \partial_i f(x) + \sum_{v\in R_+} k_v \frac{f(x) - f(x\sigma_v)}{\langle x,v\rangle}\langle v,\varepsilon_i\rangle, \qquad 1 \le i \le d,$$

where $\varepsilon_1, \dots, \varepsilon_d$ are the standard unit vectors of $\mathbb{R}^d$. These operators map $\mathcal{P}_n^d$ to $\mathcal{P}_{n-1}^d$, where $\mathcal{P}_n^d$ is the space of homogeneous polynomials of degree n in d variables. More importantly, these operators mutually commute; that is, $\mathcal{D}_i\mathcal{D}_j = \mathcal{D}_j\mathcal{D}_i$.

Associated with the reflection group and the function κ is the weight function h_κ defined by

$$h_\kappa(x) = \prod_{v\in R_+} |\langle x,v\rangle|^{\kappa_v}, \qquad x \in \mathbb{R}^d. \tag{2.1}$$

This is a positive homogeneous function of degree $\gamma_\kappa := \sum_{v\in R_+}\kappa_v$ and is invariant under the reflection group G. The simplest example is given by the case $G = \mathbb{Z}_2^d$, for which h_κ is just the product weight function

$$h_\kappa(x) = \prod_{i=1}^d |x_i|^{\kappa_i}, \qquad \kappa_i \ge 0.$$

The Dunkl transform is taken with respect to the measure $h_\kappa^2(x)dx$.

There is a linear isomorphism which intertwines the algebra generated by Dunkl's operators with the algebra of partial differential operators. The intertwining operator V_κ is a linear operator determined uniquely by

$$V_\kappa\mathcal{P}_n \subset \mathcal{P}_n, \quad V_\kappa 1 = 1, \quad \mathcal{D}_i V_\kappa = V_\kappa\partial_i, \quad 1 \le i \le d.$$

An explicit formula of V_κ is not known in general. For the group $G = \mathbb{Z}_2^d$, it is an integral transform

$$V_\kappa f(x) = b_\kappa \int_{[-1,1]^d} f(x_1t_1, \dots, x_dt_d)\prod_{i=1}^d (1+t_i)(1-t_i^2)^{\kappa_i - 1}dt. \tag{2.2}$$

If some $\kappa_i = 0$, then the formula holds under the limit relation

$$\lim_{\lambda\to 0} b_\lambda \int_{-1}^1 f(t)(1-t)^{\lambda-1}dt = [f(1) + f(-1)]/2.$$

It is known that V_κ is a positive operator ([13]); that is, $p \geq 0$ implies $V_\kappa p \geq 0$.

The function $E(x,y) := V_\kappa^{(x)}\left[e^{\langle x,y\rangle}\right]$, where the superscript means that V_κ is applied to the x variable, plays an important role in the development of the Dunkl transform. Some of its properties are listed below ([2]).

Proposition 2.1. *For $x, y \in \mathbb{R}^n$,*

1. $E(x,y) = E(y,x)$.
2. $|E(x,y)| \leq e^{\|x\|\cdot\|y\|}$, $x, y \in \mathbb{C}^n$.
3. *Let $\nu(z) = z_1^2 + \cdots + z_d^2$, $z_i \in \mathbb{C}$. For $z, w \in \mathbb{C}^d$,*

$$c_h \int_{\mathbb{R}^d} E(z,x)E(w,x)h_\kappa^2(x)e^{-\|x\|^2/2}dx = e^{(\nu(z)+\nu(w))/2}E(z,w),$$

where c_h is the constant defined by $c_h^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x)e^{-\|x\|^2/2}dx$.

In particular, the function

$$E(x,iy) = V_\kappa^{(x)}\left[e^{i\langle x,y\rangle}\right], \qquad x, y \in \mathbb{R}^d,$$

plays the role of $e^{i\langle x,y\rangle}$ in ordinary Fourier analysis. The Dunkl transform is defined in terms of it by

$$\hat{f}(y) = c_h \int_{\mathbb{R}^d} f(x)E(x,-iy)h_\kappa^2(x)dx. \tag{2.3}$$

If $\kappa = 0$, then $V_\kappa = id$ and the Dunkl transform coincides with the usual Fourier transform. If $d = 1$ and $G = \mathbb{Z}_2$, then the Dunkl transform is related closely to the Hankel transform on the real line. In fact, in this case,

$$E(x,-iy) = \Gamma(\kappa + 1/2)(|xy|/2)^{-\kappa+1/2}\left[J_{\kappa-1/2}(|xy|) - i\,\mathrm{sign}(xy)J_{\kappa+1/2}(|xy|)\right],$$

where J_α denotes the usual Bessel function

$$J_\alpha(t) = \left(\frac{t}{2}\right)^\alpha \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\alpha+1)}\left(\frac{t}{2}\right)^{2n}. \tag{2.4}$$

We list some of the known properties of the Dunkl transform ([3, 7]) below.

Proposition 2.2. *1. For $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, $\hat{f}$ is in $C_0(\mathbb{R}^d)$.*

2. When both f and $\hat{f}$ are in $L^1(\mathbb{R}^d; h_\kappa^2)$, we have the inversion formula

$$f(x) = \int_{\mathbb{R}^d} E(ix,y)\hat{f}(y)h_\kappa^2(y)dy.$$

3. *The Dunkl transform extends to an isometry of* $L^2(\mathbb{R}^d; h_\kappa^2)$.

4. *For Schwartz class functions* f, $\widehat{\mathcal{D}_j f}(y) = iy_j \widehat{f}(y)$.

There are two more results which we need. First we require the definition of h-harmonics. The h-Laplacian is defined by $\Delta_h = \mathcal{D}_1^2 + \cdots + \mathcal{D}_d^2$; it plays the role similar to that of the ordinary Laplacian. Let $\mathcal{P}_n^d$ denote the subspace of homogeneous polynomials of degree n in d variables. An h-harmonic polynomial P of degree n is a homogeneous polynomial $P \in \mathcal{P}_n^d$ such that $\Delta_h P = 0$. Furthermore, let $\mathcal{H}_n^d(h_\kappa^2)$ denote the space of h-harmonic polynomials of degree n in d variables and define

$$\langle f, g \rangle_\kappa := a_\kappa \int_{S^{d-1}} f(x) g(x) h_\kappa^2(x) d\omega(x),$$

where $a_\kappa^{-1} = \int_{S^{d-1}} h_\kappa^2(x) d\omega$. Then $\langle P, Q \rangle_\kappa = 0$ for $P \in \mathcal{H}_n^d(h_\kappa^2)$ and $Q \in \Pi_{n-1}^d$. The spherical h-harmonics are the restriction of h-harmonics to the unit sphere. Standard Hilbert space theory shows that

$$L^2(h_\kappa^2) = \sum_{n=0}^{\infty} \bigoplus \mathcal{H}_n^d(h_\kappa^2).$$

Throughout this paper, we fix the value of $\lambda := \lambda_\kappa$ as

$$\lambda := \gamma_\kappa + \frac{d-2}{2} \qquad \text{with } \gamma_\kappa = \sum_{v \in R_+} \kappa_v. \tag{2.5}$$

Using the spherical-polar coordinates $x = rx'$, where $x' \in S^{d-1}$, we have

$$\int_{\mathbb{R}^d} f(x) h_\kappa^2(x) dx = \int_0^\infty \int_{S^{d-1}} f(rx') h_\kappa^2(x') d\omega(x') r^{2\lambda_\kappa + 1} dr, \tag{2.6}$$

from which it follows that

$$c_h^{-1} = \int_{\mathbb{R}^d} h_\kappa^2(x) e^{-\|x\|^2/2} dx = 2^{\lambda_\kappa} \Gamma(\lambda_\kappa + 1) a_\kappa^{-1}.$$

The following formula is useful for computing the Dunkl transform of certain functions ([3]).

Proposition 2.3. *Let* $f \in \mathcal{H}_n^d(h_\kappa^2)$, $y \in \mathbb{R}^d$ *and* $\mu > 0$. *Then the function*

$$g(x) = a_\kappa \int_{S^{d-1}} f(\xi) E(x, -i\mu\xi) h_\kappa^2(\xi) d\omega(\xi)$$

satisfies $\Delta_h g = -\mu^2 g$ *and*

$$g(x) = (-i)^n f\left(\frac{x}{\|x\|}\right) \left(\frac{\mu \|x\|}{2}\right)^{-\lambda_\kappa} J_{n+\lambda_\kappa}(\mu \|x\|).$$

We also use the Hankel transform H_α defined on the positive reals $\mathbb{R}_+$. For $\alpha > -1/2$,

$$H_\alpha f(s) := \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(r) \frac{J_\alpha(rs)}{(rs)^\alpha} r^{2\alpha+1} dr. \tag{2.7}$$

The inverse Hankel transform is given by

$$f(r) = \frac{1}{\Gamma(\alpha+1)} \int_0^\infty H_\alpha f(s) \frac{J_\alpha(rs)}{(rs)^\alpha} s^{2\alpha+1} ds, \tag{2.8}$$

which holds under mild conditions on f, for example, if f is piecewise continuous and of bounded variation in every finite subinterval of $(0, \infty)$, and $\sqrt{r} f \in L^1(\mathbb{R}_+)$ ([20, p. 456]).

Proposition 2.4. *If* $f(x) = f_0(\|x\|)$, *then* $\widehat{f}(x) = H_{\lambda_\kappa} f_0(\|x\|)$.

Proof. This follows immediately from (2.6) and Proposition 2.3. □

3 Generalized translation

One of the important tools in the classical Fourier analysis is the convolution

$$(f * g)(x) = \int_{\mathbb{R}^d} f(y) g(x-y) dy,$$

which depends on the translation $\tau_y : f(x) \mapsto f(x-y)$. There is a generalized translation for the reflection invariant weight function, which we study in this section.

3.1 Basic properties and explicit formulas.

Taking the Fourier transform, we see that the translation $\tau_y f = f(\cdot - y)$ of $\mathbb{R}^d$ satisfies $\widehat{\tau_y f}(x) = e^{-i\langle x, y\rangle} \widehat{f}(x)$. Looking at the Fourier transform side, we can define an analogue of the translation operator for the Dunkl transform as follows.

Definition 3.1. Let $y \in \mathbb{R}^d$ be given. The generalized translation operator $f \mapsto \tau_y f$ is defined on $L^2(\mathbb{R}^d; h_\kappa^2)$ by the equation

$$\widehat{\tau_y f}(x) = E(y, -ix) \widehat{f}(x), \qquad x \in \mathbb{R}^d. \tag{3.1}$$

Note that the definition makes sense, as the Dunkl transform is an isometry of $L^2(\mathbb{R}^d; h_\kappa^2)$ onto itself and the function $E(y, -ix)$ is bounded. When the function f is in the Schwartz class, the above equation holds pointwise. Otherwise it is to be interpreted as an equation for L^2 functions. As an operator on $L^2(\mathbb{R}^d; h_\kappa^2)$, τ_y

is bounded. A priori it is not at all clear whether the translation operator can be defined for L^p functions for p different from 2. One of the important issues is to prove the L^p boundedness of the translation operator on the dense subspace of Schwartz class functions. If it can be done, then we can extend the definition to all L^p functions.

The above definition gives $\tau_y f$ as an L^2 function. It is useful to have a class of functions on which (3.1) holds pointwise. One such class is given by the subspace

$$A_\kappa(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d; h_\kappa^2) : \widehat{f} \in L^1(\mathbb{R}^d; h_\kappa^2)\}.$$

Note that $A_\kappa(\mathbb{R}^d)$ is contained in the intersection of $L^1(\mathbb{R}^d; h_\kappa^2)$ and L^∞ and hence is a subspace of $L^2(\mathbb{R}^d; h_\kappa^2)$. For $f \in A_\kappa(\mathbb{R}^d)$, we have

$$\tau_y f(x) = \int_{\mathbb{R}^d} E(ix,\xi)E(-iy,\xi)\widehat{f}(\xi)h_\kappa^2(\xi)d\xi. \tag{3.2}$$

Before stating some properties of the generalized translation operator, let us mention that there is an abstract formula for τ_y given in terms of the intertwining operator V_κ and its inverse. It takes the form [19]

$$\tau_y f(x) = V_\kappa^{(x)} \otimes V_\kappa^{(y)} \left[(V_\kappa^{-1} f)(x-y)\right] \tag{3.3}$$

for f a Schwartz class function. Note that V_κ^{-1} satisfies the formula $V_\kappa^{-1} f(x) = e^{-\langle y, \mathcal{D}\rangle} f(x)|_{y=0}$. The above formula, however, does not provide much information on $\tau_y f$. The generalized translation operator has been studied in [13, 15, 19]. In [19], the equation (3.3) is taken as the starting point.

The following proposition collects some of the elementary properties of this operator which are easy to prove when both f and g belong to $A_\kappa(\mathbb{R}^d)$.

Proposition 3.2. *Assume that $f \in A_\kappa(\mathbb{R}^d)$ and that $g \in L^1(\mathbb{R}^d; h_\kappa^2)$ is bounded. Then*

1. $\int_{\mathbb{R}^d} \tau_y f(\xi) g(\xi) h_\kappa^2(\xi) d\xi = \int_{\mathbb{R}^d} f(\xi)\tau_{-y} g(\xi) h_\kappa^2(\xi) d\xi$;
2. $\tau_y f(x) = \tau_{-x} f(-y)$.

Proof. The property (2) follows from the definition since $E(\lambda x, \xi) = E(x, \lambda\xi)$ for any $\lambda \in \mathbb{C}$. To prove (1), assume first that both f and g belong to $A_\kappa(\mathbb{R}^d)$. Then both integrals in (1) are well-defined. From the definition,

$$\begin{aligned}\int_{\mathbb{R}^d} \tau_y f(\xi) g(\xi) h_\kappa^2(\xi) d\xi &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} E(ix,\xi)E(-iy,\xi)\widehat{f}(\xi)h_\kappa^2(\xi)d\xi\right) g(x)h_\kappa^2(x)dx \\ &= \int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{g}(-\xi)E(-iy,\xi)h_\kappa^2(\xi)d\xi.\end{aligned}$$

We also have

$$\begin{aligned}\int_{\mathbb{R}^d} f(\xi)\tau_{-y}g(\xi)h_\kappa^2(\xi)d\xi &= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} E(ix,\xi)E(iy,\xi)\widehat{g}(\xi)h_\kappa^2(\xi)d\xi\right) f(x)h_\kappa^2(x)dx\\ &= \int_{\mathbb{R}^d} \widehat{f}(-\xi)\widehat{g}(\xi)E(iy,\xi)h_\kappa^2(\xi)d\xi\\ &= \int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{g}(-\xi)E(-iy,\xi)h_\kappa^2(\xi)d\xi.\end{aligned}$$

This proves (1) when both f and g belong to $A_\kappa(\mathbb{R}^d)$.

Suppose now $f \in A_\kappa(\mathbb{R}^d)$ but g is in the intersection of $L^1(\mathbb{R}^d; h_\kappa^2)$ and L^∞. Note that $g \in L^2(\mathbb{R}^d; h_\kappa^2)$, so $\tau_y g$ is defined as an L^2 function. Since f is in $L^2(\mathbb{R}^d; h_\kappa^2)$ and bounded, both integrals are finite. The equation

$$\int_{\mathbb{R}^d} f(\xi)\widehat{g}(\xi)h_\kappa^2(\xi)d\xi = \int_{\mathbb{R}^d} \widehat{f}(\xi)g(\xi)h_\kappa^2(\xi)d\xi,$$

which is true for Schwartz class functions, remains true for $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$ as well. Using this, we get

$$\begin{aligned}\int_{\mathbb{R}^d} \tau_y f(x)g(x)h_\kappa^2(x)dx &= \int_{\mathbb{R}^d} \tau_y f(-x)g(-x)h_\kappa^2(x)dx\\ &= \int_{\mathbb{R}^d} E(y,-i\xi)\widehat{f}(\xi)\widehat{g}(-\xi)h_\kappa^2(\xi)d\xi.\end{aligned}$$

By the same argument, the integral on the right hand side is also given by the same expression. Hence (1) is proved. □

We need to prove further properties of τ_y. In the classical case, the ordinary translation satisfies

$$\int_{\mathbb{R}^d} f(x-y)dx = \int_{\mathbb{R}^d} f(x)dx.$$

Such a property is true for τ_y if f is a Schwartz class function. Indeed,

$$\int_{\mathbb{R}^d} \tau_y f(x)h_\kappa^2(x)dx = \widehat{(\tau_y f)}(0) = \widehat{f}(0).$$

Here we have used the fact that τ_y takes $\mathcal{S}$ into itself. Although $\tau_y f$ is defined for $f \in A_\kappa(\mathbb{R}^d)$, we do not know if it is integrable. We now address the question whether the above property holds at least for a subclass of functions.

For this purpose, we use the following result, which gives an explicit formula for $\tau_y f$ when f is radial; see [15]. We write $x' = x/|x|$ for non-zero $x \in \mathbb{R}^d$.

Proposition 3.3. *Let $f \in A_\kappa(\mathbb{R}^d)$ be radial and let $f(x) = f_0(\|x\|)$. Then*

$$\tau_y f(x) = V_\kappa\left[f_0\left(\sqrt{\|x\|^2+\|y\|^2-2\|x\|\,\|y\|\langle x',\cdot\rangle}\right)\right](y').$$

This formula is proved in [15] for all Schwartz class functions. A different proof can be given using expansions in terms of h-harmonics. For that, one needs to invert Hankel transforms of h-harmonic coefficients of f of various orders. Once we assume that $f \in A_\kappa(\mathbb{R}^d)$, it follows that all h-harmonic coefficients of f and their Hankel transforms are integrable, so that inversion is valid. A special case of the above theorem is the formula

$$\tau_y q_t(x) = e^{-t(\|x\|^2+\|y\|^2)} E(2tx, y), \tag{3.4}$$

where

$$q_t(x) = (2t)^{-(\gamma+d/2)} e^{-t\|x\|^2}$$

is the so-called heat kernel. This formula has already appeared in [12]. The other known formula for $\tau_y f$ is the case when $G = \mathbb{Z}_2^d$.

Theorem 3.4. *Let $f \in A_\kappa(\mathbb{R}^d)$ be radial and nonnegative. Then $\tau_y f \geq 0$, $\tau_y f \in L^1(\mathbb{R}^d; h_\kappa^2)$ and*

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x) dx.$$

Proof. As f is radial, the explicit formula in Proposition 3.3 shows that $\tau_y f \geq 0$ since V_κ is a positive operator. Taking $g(x) = e^{-t\|x\|^2}$ and making use of (3.4), we get

$$\int_{\mathbb{R}^d} \tau_y f(x) e^{-t\|x\|^2} h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) e^{-t(\|x\|^2+\|y\|^2)} E(\sqrt{2t}x, \sqrt{2t}y) h_\kappa^2(x) dx.$$

As $|E(x, y)| \leq e^{\|x\| \|y\|}$, we can take the limit as $t \to 0$ to get

$$\lim_{t\to 0} \int_{\mathbb{R}^d} \tau_y f(x) e^{-t\|x\|^2} h_\kappa^2(x) dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x) dx.$$

Since $\tau_y f \geq 0$, monotone convergence theorem applied to the integral on the left completes the proof. □

We would like to relax the condition on f in the above proposition. In order to do that, we introduce the notion of generalized (Dunkl) convolution.

Definition 3.5. For $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$,

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) \tau_x g^\vee(y) h_\kappa^2(y) dy,$$

where $g^\vee(y) = g(-y)$.

Note that as $\tau_x g^\vee \in L^2(\mathbb{R}^d; h_\kappa^2)$, the above convolution is well-defined. We can also write the definition as

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} \widehat{f}(\xi)\widehat{g}(\xi)E(ix,\xi)h_\kappa^2(\xi)d\xi.$$

If we assume that g is also in $L^1(\mathbb{R}^d; h_\kappa^2)$, so that $\widehat{g}$ is bounded, then by the Plancherel theorem we obtain

$$\|f *_\kappa g\|_{\kappa,2} \le \|g\|_{\kappa,1}\|f\|_{\kappa,2}.$$

We are interested in knowing under what conditions on g the operator $f \to f *_\kappa g$ defined on the Schwartz class can be extended to $L^p(\mathbb{R}^d; h_\kappa^2)$ as a bounded operator. But now we use the L^2 boundedness of the convolution to prove the following.

Theorem 3.6. *Let $g \in L^1(\mathbb{R}^d; h_\kappa^2)$ be radial, bounded and nonnegative. Then $\tau_y g \ge 0, \tau_y g \in L^1(\mathbb{R}^d; h_\kappa^2)$ and*

$$\int_{\mathbb{R}^d} \tau_y g(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} g(x)h_\kappa^2(x)dx.$$

Proof. Let q_t be the heat kernel defined earlier, so that $\widehat{q_t}(\xi) = e^{-t\|\xi\|^2}$. By the Plancherel theorem,

$$\|g *_\kappa q_t - g\|_{\kappa,2}^2 = \int_{\mathbb{R}^d} |\widehat{g}(\xi)|^2(1 - e^{-t\|\xi\|^2})^2 h_\kappa^2(\xi)d\xi,$$

which shows that $g *_\kappa q_t \to g$ in $L^2(\mathbb{R}^d; h_\kappa^2)$ as $t \to 0$. Since τ_y is bounded on $L^2(\mathbb{R}^d; h_\kappa^2)$, we have $\tau_y(g *_\kappa q_t) \to \tau_y g$ in $L^2(\mathbb{R}^d; h_\kappa^2)$ as $t \to 0$. By passing to a subsequence if necessary, we can assume that the convergence is also almost everywhere.

Now as g is radial and nonnegative, the convolution

$$g *_\kappa q_t(x) = \int_{\mathbb{R}^d} g(y)\tau_x q_t(y) h_\kappa^2(y)dy$$

is also radial and nonnegative. Also, $g *_\kappa q_t \in A_\kappa(\mathbb{R}^d)$, as g is both in $L^1(\mathbb{R}^d; h_\kappa^2)$ and $L^2(\mathbb{R}^d; h_\kappa^2)$; in fact, $g *_\kappa q_t \in L^1(\mathbb{R}^d, h_\kappa^2)$, as $q_t \in A_\kappa(\mathbb{R}^d)$ and, by the Plancherel theorem and Hölder's inequality, $\|\widehat{g *_\kappa q_t}\|_{\kappa,1} = \|\widehat{g}\cdot\widehat{q_t}\|_{\kappa,1} \le \|g\|_{\kappa,2}\|q_t\|_{\kappa,2}$. Thus, by Theorem 3.6, $\tau_y(g *_\kappa q_t)(x) \ge 0$. This gives

$$\lim_{t\to 0} \tau_y(g *_\kappa q_t)(x) = \tau_y g(x) \ge 0$$

for almost every x. Once the nonnegativity of $\tau_y g(x)$ is proved, it is easy to show that it is integrable. As before,

$$\int_{\mathbb{R}^d} \tau_y g(x)e^{-t\|x\|^2}h_\kappa^2(x)dx = \int_{\mathbb{R}^d} g(x)e^{-t(\|x\|^2+\|y\|^2)}E(\sqrt{2t}x, \sqrt{2t}y)h_\kappa^2(x)dx.$$

Taking limits as t goes to 0 and using monotone convergence theorem, we get

$$\int_{\mathbb{R}^d} \tau_y g(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} g(x) h_\kappa^2(x)dx.$$

This completes the proof. □

There is another way of proving the above result which avoids the intermediate steps. If we assume that $\int_{\mathbb{R}^d} g(x)h_\kappa^2(x)dx = 1$, our result is an immediate consequence of Proposition 6.2 in [19]. We thank the referee for pointing this out. We are now in a position to prove the following result. Let $L^p_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$ denote the space of all radial functions in $L^p(\mathbb{R}^d; h_\kappa^2)$.

Theorem 3.7. *The generalized translation operator τ_y, initially defined on the intersection of $L^1(\mathbb{R}^d; h_\kappa^2)$ and L^∞, can be extended to all radial functions in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le 2$; and $\tau_y : L^p_{\text{rad}}(\mathbb{R}^d, h_\kappa^2) \to L^p(\mathbb{R}^d; h_\kappa^2)$ is a bounded operator.*

Proof. For real valued $f \in L^1(\mathbb{R}^d; h_\kappa^2) \cap L^\infty$ which is radial, the inequality $-|f| \le f \le |f|$ together with the nonnegativity of τ_y on radial functions in $L^1(\mathbb{R}^d; h_\kappa^2) \cap L^\infty$ shows that $|\tau_y f(x)| \le \tau_y |f|(x)$. Hence

$$\int_{\mathbb{R}^d} |\tau_y f(x)| h_\kappa^2(x)dx \le \int_{\mathbb{R}^d} |f|(x) h_\kappa^2(x)dx \le \|f\|_{\kappa,1}.$$

We also have $\|\tau_y f\|_{\kappa,2} \le \|f\|_{\kappa,2}$. As L^p is the interpolation space between L^1 and L^2, we get $\|\tau_y f\|_{\kappa,p} \le \|f\|_{\kappa,p}$ for all $1 \le p \le 2$ for all $f \in L^p_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$. This proves the theorem. For the interpolation theorem used here, see [18]. □

Theorem 3.8. *For every $f \in L^1_{\text{rad}}(\mathbb{R}^d; h_\kappa^2)$,*

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x)dx.$$

Proof. Choose radial functions $f_n \in A_\kappa(\mathbb{R}^d)$ such that $f_n \to f$ and $\tau_y f_n \to \tau_y f$ in $L^1(\mathbb{R}^d; h_\kappa^2)$. Since

$$\int_{\mathbb{R}^d} \tau_y f_n(x) g(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} f_n(x) \tau_{-y} g(x) h_\kappa^2(x)dx$$

for every $g \in A_\kappa(\mathbb{R}^d)$, taking the limit as n tends to infinity, we get

$$\int_{\mathbb{R}^d} \tau_y f(x) g(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} f(x) \tau_{-y} g(x) h_\kappa^2(x)dx.$$

Now set $g(x) = e^{-t\|x\|^2}$ and take the limit as t goes to 0. Since $\tau_y f \in L^1(\mathbb{R}^d; h_\kappa^2)$, by the dominated convergence theorem we obtain

$$\int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} f(x) h_\kappa^2(x)dx$$

for $f \in L^1(\mathbb{R}^d; h_\kappa^2)$. □

It remains an open problem whether $\tau_y f$ can be defined for all $f \in L^1(\mathbb{R}^d; h_\kappa^2)$.

3.2 Positivity of τ_y. As an immediate consequence of the explicit formula for the generalized translation of radial functions, if $f(x) \in A_\kappa(\mathbb{R}^d)$ is nonnegative, then $\tau_y f(x) \ge 0$ for all $y \in \mathbb{R}^d$ ([15]).

One would naturally expect that the generalized translation defines a positive operator; that is, $\tau_y f(x) \ge 0$ whenever $f(x) \ge 0$. This, however, turns out not to be the case. For $G = \mathbb{Z}_2$, the explicit formula given in Section 7 shows that τ_y is not positive in general (signed hypergroup, see [11]). Below we give an example to show that τ_y is not positive in a case where the explicit formula is not available. It depends on a method of computing the generalized translation of simple functions. The explicit formula (3.2) can be used to define $\tau_y f$ when f is a polynomial.

Lemma 3.9. *Let $y \in \mathbb{R}^d$. For $1 \le j \le d$, $\tau_y\{x_j\} = x_j - y_j$; and for $1 \le j, k \le d$,*

$$\tau_y\{x_j x_k\} = (x_j - y_j)(x_k - y_k) - \kappa \sum_{v \in R_+} \left[V_\kappa(\langle x, y\rangle) - V_\kappa(\langle x\sigma_v, y\rangle)\right].$$

Proof. We use (3.4) and the fact that $\mathcal{D}_j \tau_y = \tau_y \mathcal{D}_j$. On the one hand, since the difference part of $\mathcal{D}_j$ becomes zero when applied to radial functions,

$$\tau_y \mathcal{D}_j e^{-t\|x\|^2} = -2t\tau_y \left(\{\cdot\}_j e^{-t\|\cdot\|^2}\right)(x).$$

On the other hand, it is easy to verify that

$$\mathcal{D}_j \tau_y e^{-t\|x\|^2} = \mathcal{D}_j \left[e^{-t(\|x\|^2+\|y\|^2)} E(2tx, y)\right] = 2te^{-t(\|x\|^2+\|y\|^2)} E(2tx, y)(y_j - x_j).$$

Together, this leads to the equation

$$\tau_y \left(x_j e^{-t\|x\|^2}\right) = 2e^{-t(\|x\|^2+\|y\|^2)} E(2tx, y)(x_j - y_j). \tag{3.5}$$

Taking the limit as $t \to 0$ gives $\tau_y\{x_j\} = x_j - y_j$.

Next we repeat the above argument, taking (3.5) as the starting point. Using the product formula for $\mathcal{D}_k$ [5, p. 156], we have after a simple computation

$$\begin{aligned}\mathcal{D}_k \tau_y \left(x_j e^{-t\|x\|^2}\right) &= \mathcal{D}_k \left[(x_j - y_j) e^{-t(\|x\|^2+\|y\|^2)}\right] \\ &= e^{-t(\|x\|^2+\|y\|^2)} \Big[-2t(x_j - y_j)(x_k - y_k) E(2tx, y) \\ &\qquad + \delta_{k,j} E(2tx, y) + 2 \sum_{v \in R_+} \kappa_v \frac{v_k v_j}{\|v\|^2} E(2tx\sigma_v, y)\Big].\end{aligned}$$

On the other hand, computing $\mathcal{D}_k(x_j e^{-t\|x\|^2})$ leads to

$$\tau_y \left(\mathcal{D}_k(x_j e^{-t\|x\|^2})\right) = -2t\tau_y(x_j x_k e^{-t\|x\|^2}) + \tau_y e^{-t\|x\|^2}\left[\delta_{k,j} + 2 \sum_{v \in R_+} \kappa_v \frac{v_k v_j}{\|v\|^2}\right].$$

Hence, by (3.4), the equation $\mathcal{D}_k \tau_y (x_j e^{-t\|x\|^2}) = \tau_y \mathcal{D}_k (x_j e^{-t\|x\|^2})$ gives

$$\tau_y \left(x_j x_k e^{-t\|x\|^2} \right) = e^{-t(\|x\|^2+\|y\|^2)} \Big[(x_j - y_j)(x_k - y_k) E(2tx, y) + \sum_{v \in R_+} \kappa_v \frac{v_k v_j}{\|v\|^2} \frac{E(2tx, y) - E(2tx\sigma_v, y)}{t} \Big].$$

Taking the limit as $t \to 0$ gives the formula of $\tau_y\{x_j x_k\}$. □

Proposition 3.10. *The generalized translation τ_y is not a positive operator for the symmetric group S_d.*

Proof. The formula $\tau_y\{x_j x_k\}$ depends on the values of $V_\kappa x_j$. For the symmetric group S_d of d objects, the formula of $V_\kappa x_j$ is given [4] by

$$V_\kappa x_j = \frac{1}{d\kappa + 1}(1 + x_j + \kappa |x|), \qquad |x| = x_1 + \cdots + x_d.$$

Let $x(j, k)$ denote the transposition of the variables x_j and x_k. It follows that

$$\begin{aligned} \tau_y\{x_j^2\} &= (x_j - y_j)^2 + \kappa \sum_{k \neq j} [V_\kappa(\langle x, y \rangle) - V_\kappa(\langle x(k, j), y \rangle)] \\ &= (x_j - y_j)^2 + \kappa \sum_{k \neq j} [(x_j - x_k) V_\kappa (y_j - y_k)] \\ &= (x_j - y_j)^2 + \frac{\kappa}{d\kappa + 1} \sum_{k \neq j} [(x_j - x_k)(y_j - y_k)] . \end{aligned}$$

Choosing $x = (1, 0, 0, \ldots, 0)$ and $y = (0, 2, 2, \ldots, 2)$, we see that $\tau_y(\{\cdot\}_1^2)(x) = -((d - 2)\kappa + 1)/(d\kappa + 1) \leq 0$. This proves the proposition. □

By (3.2), this proposition also shows that V_κ^{-1} is not a positive operator for the symmetric group. In the case of $\mathbb{Z}_2$, an explicit formula for V_κ^{-1} is known ([22]) which is not positive.

3.3 A Paley–Wiener theorem and the support of τ_y.

In this subsection, we prove a sharp Paley–Wiener theorem and study its consequences. The usual version of the Paley–Wiener theorem has been already proved by de Jeu in his thesis (Leiden, 1994). Another type of Paley–Wiener theorem has been proved in [19]. Our result is a refined version of the usual Paley–Wiener which is analogous to an intrinsic version of the Paley–Wiener theorem for the Fourier transform studied by Helgason [6]. Recently, a geometric form of the Paley–Wiener theorem has been conjectured and studied in [8].

Let us denote by $\{Y_{j,n} : 1 \leq j \leq \dim \mathcal{H}_n^d(h_\kappa^2)\}$ an orthonormal basis of $\mathcal{H}_n^d(h_\kappa^2)$. First we prove a Paley–Wiener theorem for the Dunkl transform.

Theorem 3.11. *Let $f \in S$ and B be a positive number. Then f is supported in $\{x : \|x\| \le B\}$ if and only if for every j and n, the function*

$$F_{j,n}(\rho) = \rho^{-n} \int_{S^{d-1}} \widehat{f}(\rho x) Y_{j,n}(x) h_\kappa^2(x) d\omega(x)$$

extends to an entire function of $\rho \in \mathbb{C}$ satisfying the estimate

$$|F_{j,n}(\rho)| \le c_{j,n} e^{B\|\Im \rho\|}.$$

Proof. By the definition of $\widehat{f}$ and Proposition 2.3,

$$\begin{aligned}
&\int_{S^{d-1}} \widehat{f}(\rho x) Y_{j,n}(x) h_\kappa^2(x) d\omega(x) \\
&\quad = c \int_{\mathbb{R}^d} \int_{S^{d-1}} E(y, -i\rho x) Y_{j,n}(x) h_\kappa^2(x) d\omega(x) f(y) h_\kappa^2(y) dy \\
&\quad = c \int_{\mathbb{R}^d} f(y) Y_{j,n}(y') \frac{J_{\lambda_\kappa+n}(\rho\|y\|)}{(\rho\|y\|)^{\lambda_\kappa}} h_\kappa^2(y) dy \\
&\quad = c \int_0^\infty f_{j,n}(r) \frac{J_{\lambda_\kappa+n}(r\rho)}{(r\rho)^{\lambda_\kappa}} r^{2\lambda_\kappa+n+1} dr,
\end{aligned}$$

where c is a constant and

$$f_{j,n}(r) = r^{-n} \int_{S^{d-1}} f(ry') Y_{j,n}(y') h_\kappa^2(y') d\omega(y').$$

Thus, $F_{j,n}$ is the Hankel transform of order $\lambda_\kappa + n$ of the function $f_{j,n}(r)$. The theorem then follows from the Paley–Wiener theorem for the Hankel transform (see, for example, [9]). □

Corollary 3.12. *A function $f \in S$ is supported in $\{x : \|x\| \le B\}$ if and only if $\widehat{f}$ extends to an entire function of $\zeta \in \mathbb{C}^d$ which satisfies*

$$|\widehat{f}(\zeta)| \le c\, e^{B\|\Im \zeta\|}.$$

Proof. The direct part follows from the fact that $E(x, -i\zeta)$ is entire and $|E(x, -i\zeta)| \le c\, e^{\|x\| \cdot \|\Im \zeta\|}$. For the converse, we look at

$$\int_{S^{d-1}} \widehat{f}(\rho y') Y_{j,n}(y') h_\kappa^2(y') d\omega(y'), \qquad \rho \in \mathbb{C},$$

where $d\omega$ is the surface measure on S^{d-1}. This is certainly entire and, from the proof of the previous theorem, has a zero of order n at the origin. Hence

$$\rho^{-n} \int_{S^{d-1}} \widehat{f}(\rho y') Y_{j,n}(y') h_\kappa^2(y') d\omega(y')$$

is an entire function of exponential type B. The converse now follows from the theorem. □

Proposition 3.13. *Let $f \in S$ be supported in $\{x : \|x\| \le B\}$. Then $\tau_y f$ is supported in $\{x : \|x\| \le B + \|y\|\}$.*

Proof. Let $g(x) = \tau_y f(x)$. Then $\widehat{g}(\xi) = E(y, -i\xi)\widehat{f}(\xi)$ extends to $\mathbb{C}^d$ as an entire function of type $B + \|y\|$. □

This property of τ_y has appeared in [19]. We note that the explicit formula for τ_y shows that the support set of τ_y given in Proposition 3.13 is sharp.

An important corollary in this regard is the following result.

Theorem 3.14. *If $f \in C_0^\infty(\mathbb{R}^d)$ is supported in $\|x\| \le B$, then $\|\tau_y f - f\|_p \le c\|y\|(B + \|y\|)^{N/p}$ for $1 \le p \le \infty$, where $N = d + 2\gamma_\kappa$. Consequently, $\lim_{y\to 0} \|\tau_y f - f\|_{\kappa,p} = 0$.*

Proof. From the definition, we have

$$\tau_y f(x) - f(x) = \int_{\mathbb{R}^d} (E(y, -i\xi) - 1)\, E(x, i\xi)\widehat{f}(\xi)h_\kappa^2(\xi)d\xi.$$

Using the mean value theorem and estimates on the derivatives of $E(x, i\xi)$, we have the estimate

$$\|\tau_y f - f\|_\infty \le c\|y\| \int_{\mathbb{R}^d} \|\xi\|\, |\widehat{f}(\xi)| h_\kappa^2(\xi)d\xi.$$

As $\tau_y f$ is supported in $\|x\| \le (B + \|y\|)$, we obtain

$$\|\tau_y f - f\|_p \le c\|y\|(B + \|y\|)^{N/p},$$

which goes to zero as y goes to zero. □

4 The generalized convolution

4.1 Convolution. Recall that in Section 3 we defined the convolution $f *_\kappa g$ for $f, g \in L^2(\mathbb{R}^d; h_\kappa^2)$ by

$$(f *_\kappa g)(x) = \int_{\mathbb{R}^d} f(y)\tau_x g^\vee(y)h_\kappa^2(y)dy.$$

This convolution has been considered in [12, 19]. It satisfies the following basic properties:

1. $\widehat{f *_\kappa g} = \widehat{f} \cdot \widehat{g}$;

2. $f *_\kappa g = g *_\kappa f$.

We also noted that the operator $f \to f *_\kappa g$ is bounded on $L^2(\mathbb{R}^d; h_\kappa^2)$ provided $\widehat{g}$ is bounded. We would like to know under what conditions on g the operator $f \to f *_\kappa g$ can be extended to L^p as a bounded operator. If the generalized translation operator can be extended as a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$, then the convolution will satisfy the usual Young's inequality. At present, we can only say something about convolution with radial functions.

Theorem 4.1. *Let g be a bounded radial function in $L^1(\mathbb{R}^d; h_\kappa^2)$. Then*

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y)\tau_x g^\vee(y) h_\kappa^2(y)dy,$$

initially defined on the intersection of $L^1(\mathbb{R}^d; h_\kappa^2)$ and $L^2(\mathbb{R}^d; h_\kappa^2)$, extends to all $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p \le \infty$, as a bounded operator. In particular,

$$\|f *_\kappa g\|_{\kappa,p} \le \|g\|_{\kappa,1}\|f\|_{\kappa,p}. \tag{4.1}$$

Proof. For $g \in L^1(\mathbb{R}^d; h_\kappa^2)$ which is bounded and radial, we have $|\tau_y g| \le \tau_y|g|$, which shows that

$$\int_{\mathbb{R}^d} |\tau_y g(x)| h_\kappa^2(x)dx \le \int_{\mathbb{R}^d} |g(x)| h_\kappa^2(x)dx.$$

Therefore,

$$\int_{\mathbb{R}^d} |f *_\kappa g(x)| h_\kappa^2(x)dx \le \|f\|_{\kappa,1}\|g\|_{\kappa,1}.$$

We also have $\|f *_\kappa g\|_\infty \le \|f\|_\infty \|g\|_{\kappa,1}$. By interpolation, we obtain $\|f *_\kappa g\|_{\kappa,p} \le \|g\|_{\kappa,1}\|f\|_{\kappa,p}$. □

For $\phi \in L^1(\mathbb{R}^d; h_\kappa^2)$ and $\varepsilon > 0$, we define the dilation ϕ_ε by

$$\phi_\varepsilon(x) = \varepsilon^{-(2\gamma_\kappa+d)}\phi(x/\varepsilon). \tag{4.2}$$

A change of variables shows that

$$\int_{\mathbb{R}^d} \phi_\varepsilon(x) h_\kappa^2(x)dx = \int_{\mathbb{R}^d} \phi(x) h_\kappa^2(x)dx, \qquad \text{for all } \varepsilon > 0.$$

Theorem 4.2. *Let $\phi \in L^1(\mathbb{R}^d; h_\kappa^2)$ be a bounded radial function and assume that $c_h \int_{\mathbb{R}^d} \phi(x) h_\kappa^2(x)dx = 1$. Then for $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$, and $f \in C_0(\mathbb{R}^d)$, $p = \infty$,*

$$\lim_{\varepsilon \to 0} \|f *_\kappa \phi_\varepsilon - f\|_{\kappa,p} = 0.$$

Proof. For a given $\eta > 0$, we choose $g \in C_0^\infty$ such that $\|g - f\|_{\kappa,p} < \eta/3$. The triangle inequality and (4.1) lead to

$$\|f *_\kappa \phi_\varepsilon - f\|_{\kappa,p} \le \frac{2}{3}\eta + \|g *_\kappa \phi_\varepsilon - g\|_{\kappa,p},$$

where we have used $\|g - f\|_{\kappa,p} < \eta/3$. Since ϕ is radial, we can choose a radial function $\psi \in C_0^\infty$ such that

$$\|\phi - \psi\|_{\kappa,1} \le (12\|g\|_{\kappa,p})^{-1}\eta.$$

If we let $a = c_h \int_{\mathbb{R}^d} \psi(y)h_\kappa^2(y)dy$, then by the triangle inequality, (4.1) and (4.2),

$$\begin{aligned}\|g *_\kappa \phi_\varepsilon - g\|_{\kappa,p} &\le \|g\|_{\kappa,p}\|\phi - \psi\|_{\kappa,1} + \|g *_\kappa \psi_\varepsilon - ag\|_{\kappa,p} + |a-1|\|g\|_{\kappa,p} \\ &\le \eta/6 + \|g *_\kappa \psi_\varepsilon - ag\|_{\kappa,p}\end{aligned}$$

since $\|g\|_{\kappa,p}\|\phi - \psi\|_{\kappa,1} \le \eta/12$ and

$$|a-1| = \left| c_h \int_{\mathbb{R}^d} (\phi_\varepsilon(x) - \psi_\varepsilon(x))\, h_\kappa^2(x)dx \right| \le (12\|g\|_{\kappa,p})^{-1}\eta.$$

Thus

$$\|f *_\kappa \phi_\varepsilon - f\|_{\kappa,p} \le \frac{5}{6}\eta + \|g *_\kappa \psi_\varepsilon - ag\|_{\kappa,p}.$$

Hence it suffices to show that $\|g *_\kappa \psi_\varepsilon - ag\|_{\kappa,p} \le \eta/6$.

But now $g \in A_\kappa(\mathbb{R}^d)$, and so

$$g *_\kappa \phi_\varepsilon(x) = \int_{\mathbb{R}^d} g(y)\tau_x\phi_\varepsilon^\vee(y)h_\kappa^2(y)dy = \int_{\mathbb{R}^d} \tau_{-x}g(y)\phi_\varepsilon(-y)h_\kappa^2(y)dy.$$

We also know that $\tau_{-x}g(y) = \tau_{-y}g(x)$ as $g \in C_0^\infty$. Therefore,

$$g *_\kappa \phi_\varepsilon(x) = \int_{\mathbb{R}^d} \tau_y g(x)\phi_\varepsilon(y)h_\kappa^2(y)dy.$$

In view of this,

$$g *_\kappa \psi_\varepsilon(x) - ag(x) = \int_{\mathbb{R}^d} (\tau_y g(x) - g(x))\, \psi_\varepsilon(y)h_\kappa^2(y)dy,$$

which gives by Minkowski's integral inequality,

$$\|g *_\kappa \psi_\varepsilon - ag\|_{\kappa,p} \le \int_{\mathbb{R}^d} \|\tau_y g - g\|_{\kappa,p}|\psi_\varepsilon(y)|h_\kappa^2(y)dy.$$

If g is supported in $\|x\| \le B$, then the estimate in Theorem 3.14 gives

$$\begin{aligned}\|g *_\kappa \psi_\varepsilon - ag\|_{\kappa,p} &\le c\int_{\mathbb{R}^d} \|y\|\,(B + \|y\|)^{N/p}\,|\psi_\varepsilon(y)|h_\kappa^2(y)dy \\ &\le c\varepsilon \int_{\mathbb{R}^d} \|y\|\,(B + \|\varepsilon y\|)^{N/p}\,|\psi(y)|h_\kappa^2(y)dy,\end{aligned}$$

which can be made smaller than $\eta/6$ by choosing ε small. This completes the proof of the theorem. □

The explicit formula in the case of $G = \mathbb{Z}_2^d$ allows us to prove an analogous result without the assumption that ϕ is radial; see Section 7.

5 Summability of the inverse Dunkl transform

Let $\Phi \in L^1(\mathbb{R}^d; h_\kappa^2)$ be continuous at 0 and assume $\Phi(0) = 1$. For $f \in \mathcal{S}$ and $\varepsilon > 0$, define

$$T_\varepsilon f(x) := \int_{\mathbb{R}^d} \widehat{f}(y) E(ix, y) \Phi(-\varepsilon y) h_\kappa^2(y) dy.$$

It follows from Plancherel's theorem that T_ε extends to the whole of L^2 as a bounded operator. We study the convergence of $T_\varepsilon f$ as $\varepsilon \to 0$. Note that $T_0 f = f$, by the inversion formula for the Dunkl transform. If $T_\varepsilon f$ can be extended to all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and if $T_\varepsilon f \to f$ in $L^p(\mathbb{R}^d; h_\kappa^2)$, we say that the inverse Dunkl transform is Φ-summable.

Proposition 5.1. *Let Φ and $\phi = \widehat{\Phi}$ both belong to $L^1(\mathbb{R}^d; h_\kappa^2)$. If Φ is radial, then*

$$T_\varepsilon f(x) = (f *_\kappa \phi_\varepsilon)(x)$$

for all $f \in L^2(\mathbb{R}^d; h_\kappa^2)$ and $\varepsilon > 0$.

Proof. Under the hypothesis on Φ, both T_ε and the operator taking f into $(f *_\kappa \phi_\varepsilon)$ extend to $L^2(\mathbb{R}^d; h_\kappa^2)$ as bounded operators. So it is enough to verify $T_\varepsilon f(x) = (f *_\kappa \phi_\varepsilon)(x)$ for all f in the Schwartz class. By the definition of the Dunkl transform,

$$\begin{aligned} T_\varepsilon f(x) &= \int_{\mathbb{R}^d} \widehat{\tau_{-x} f}(y) \Phi(-\varepsilon y) h_\kappa^2(y) dy \\ &= \int_{\mathbb{R}^d} \tau_{-x} f(\xi) c_h \int_{\mathbb{R}^d} \Phi(-\varepsilon y) E(y, -i\xi) h_\kappa^2(y) dy h_\kappa^2(\xi) d\xi \\ &= \varepsilon^{-(d+2\gamma_\kappa)} \int_{\mathbb{R}^d} \tau_{-x} f(\xi) \widehat{\Phi}(-\varepsilon^{-1}\xi) h_\kappa^2(\xi) d\xi \\ &= (f *_\kappa \phi_\varepsilon)(x), \end{aligned}$$

where we have changed variable $\xi \mapsto -\xi$ and used the fact that $\tau_{-x} f(-\xi) = \tau_\xi f(x)$. □

If the radial function ϕ satisfies the conditions of Theorem 4.2, we obtain the following result.

Theorem 5.2. *Let $\Phi(x) \in L^1(\mathbb{R}^d; h_\kappa^2)$ be radial and assume that $\widehat{\Phi} \in L^1(\mathbb{R}^d; h_\kappa^2)$ is bounded and $\Phi(0) = 1$. For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$, $T_\varepsilon f$ converges to f in $L^p(\mathbb{R}^d; h_\kappa^2)$ as $\varepsilon \to 0$,.*

The following remarks on the above theorem are in order. In general, the convolution $f * g$ of an L^p function f with an L^1 function g is not defined, as the translation operator is not defined for general L^p functions even when $p = 1$. However, when g satisfies the conditions of Theorem 4.2, we can define the convolution $f * g$ by integrating f against $\tau_y g$, which makes sense (see Definition 3.5). It is in this sense that the above convolution $f * \varphi_\epsilon$ is to be understood. Then as $f * \varphi_\varepsilon$ agrees with $T_\varepsilon f$ on Schwartz functions, and as the convolution operator extends to L^p as a bounded operator, our theorem is proved.

We consider several examples. In our first example, we take Φ to be the Gaussian function, $\Phi(x) = e^{-\|x\|^2/2}$. By (3) of Proposition 2.1 with $z = iy$ and $w = 0$, $\widehat{\Phi}(x) = e^{-\|x\|^2/2}$. We choose $\varepsilon = 1/\sqrt{2t}$ and define

$$q_t(x) = \Phi_\varepsilon(x) = (2t)^{-(\gamma_\kappa+d/2)}e^{-\|x\|^2/4t}.$$

Then $q_t(x)$ satisfies the heat equation for the h-Laplacian,

$$\Delta_h u(x,t) = \partial_t u(x,t),$$

where Δ_h is applied to x variables. For this Φ, our summability method is just $f *_\kappa q_t$. By (3.4), the generalized translation of q_t is given explicitly by

$$\tau_y q_t(x) = (2t)^{-(\gamma_\kappa+d/2)}e^{-(\|x\|^2+\|y\|^2)/4t}E(x/\sqrt{2t}, y/\sqrt{2t}),$$

which is the heat kernel for the solution of the heat equation for h-Laplacian. Then a corollary of Theorem 5.2 gives the following result in [14].

Theorem 5.3. *Suppose $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$ or $f \in C_0(\mathbb{R}^d)$, $p = \infty$.*

1. *The heat transform*

$$H_t f(x) := (f *_\kappa q_t)(x) = c_h \int_{\mathbb{R}^d} f(y)\tau_y q_t(x) h_\kappa^2(y)dy, \qquad t > 0,$$

converges to f in $L^p(\mathbb{R}^d; h_\kappa^2)$ as $t \to 0$.

2. *Define $H_0 f(x) = f(x)$. Then the function $H_t f(x)$ solves the initial value problem*

$$\Delta_h u(x,t) = \partial_t u(x,t), \qquad u(x,0) = f(x), \qquad (x,t) \in \mathbb{R}^d \times [0,\infty).$$

Our second example is the analogue of Poisson summability, where we take $\Phi(x) = e^{-\|x\|}$. This case has been studied in [16]. In this case, one can compute the Dunkl transform $\widehat{\Phi}$ just as in the case of the ordinary Fourier transform, namely, using

$$e^{-t} = \frac{1}{\sqrt{\pi}}\int_0^\infty \frac{e^{-u}}{\sqrt{u}} e^{-t^2/4u} du, \tag{5.1}$$

and using the fact that the transform of the Gaussian is itself (see [18, p. 6]). The result is

$$\widehat{e^{-\|x\|}} = c_{d,\kappa}\frac{1}{(1+\|x\|^2)^{\gamma_\kappa+(d+1)/2}}, \qquad c_{d,\kappa} = 2^{\gamma_\kappa+d/2}\frac{\Gamma(\gamma_\kappa+\frac{d+1}{2})}{\sqrt{\pi}}.$$

In this case, we define the Poisson kernel as the dilation of $\widehat{\Phi}$,

$$P_\varepsilon(x) := c_{d,\kappa}\frac{\varepsilon}{(\varepsilon^2+\|x\|^2)^{\gamma_\kappa+(d+1)/2}}. \tag{5.2}$$

Since $\Phi(0) = 1$, it is easy to see that $\int P(x,\varepsilon)h_\kappa^2(x)dx = 1$. We have

Theorem 5.4. *Suppose $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$, or $f \in C_0(\mathbb{R}^d)$, $p = \infty$. Then the Poisson integral $f *_\kappa P_\varepsilon$ converges to f in $L^p(\mathbb{R}^d; h_\kappa^2)$.*

Again, the proof is a corollary of Theorem 5.2. For $\kappa = 0$, it becomes Poisson summability for the classical Fourier transform on $\mathbb{R}^d$. We remark that this theorem is already proved in M. Rosler's habilitation thesis by using a different method. We thank the referee for pointing this out.

Next, we consider the analogue of the Bochner–Riesz means, for which

$$\Phi(x) = \begin{cases} (1-\|x\|^2)^\delta, & \|x\| \le 1, \\ 0, & \text{otherwise}, \end{cases}$$

where $\delta > 0$. As in the case of the ordinary Fourier transform, we take $\varepsilon = 1/R$, where $R > 0$. Then the Bochner–Riesz means are defined by

$$S_R^\delta f(x) = c_h \int_{\|y\|\le R} \left(1 - \frac{\|y\|^2}{R^2}\right)^\delta \widehat{f}(y)E(ix,y)h_\kappa^2(y)dy.$$

Recall that $\lambda_\kappa = (d-2)/2 + \gamma_\kappa$ and $N = d + 2\gamma_\kappa$.

Theorem 5.5. *If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$, or $f \in C_0(\mathbb{R}^d)$, $p = \infty$, and $\delta > (N-1)/2$, then*

$$\|S_R^\delta f - f\|_{\kappa,p} \to 0, \qquad \text{as } R \to \infty.$$

Proof. The proof follows as in the case of an ordinary Fourier transform [18, p. 171]. From Proposition 2.4 and the properties of the Bessel function, we have

$$\widehat{\Phi}(x) = 2^{\lambda_\kappa}\|x\|^{-\lambda_\kappa-\delta-1}J_{\lambda_\kappa+\delta+1}(\|x\|).$$

Hence, by $J_\alpha(r) = O(r^{-1/2})$, $\widehat{\Phi} \in L^1(\mathbb{R}^d; h_\kappa^2)$ under the condition $\delta > \lambda_\kappa + 1/2 = (N-1)/2$. □

Note that $\lambda_\kappa = (d-2)/2+\gamma_\kappa$ where γ_κ is the sum of all (nonnegative) parameters in the weight function. If all parameters are zero, then $h_\kappa(x) \equiv 1$ and we are back to the classical Fourier transform, for which the index $(d-1)/2$ is the critical index for the Bochner–Riesz means. We do not know if the index $(N-1)/2$ is the critical index for the Bochner–Riesz means of the Dunkl transforms.

6 Maximal function and almost everywhere summability

For $f \in L^2(\mathbb{R}^d; h_\kappa^2)$, we define the maximal function $M_\kappa f$ by

$$M_\kappa f(x) = \sup_{r>0} \frac{1}{d_\kappa r^{d+2\gamma_\kappa}} |f *_\kappa \chi_{B_r}(x)|,$$

where χ_{B_r} is the characteristic function of the ball B_r of radius r centered at 0 and $d_\kappa = a_\kappa/(d+2\gamma_\kappa)$. Using (2.6), we have $\int_{B_r} h_\kappa^2(y)dy = (a_\kappa/(d+2\gamma_\kappa))r^{d+2\gamma_\kappa}$. Therefore, we can also write $M_\kappa f(x)$ as

$$M_\kappa f(x) = \sup_{r>0} \frac{\left|\int_{\mathbb{R}^d} f(y)\tau_x\chi_{B_r}(y)h_\kappa^2(y)dy\right|}{\int_{B_r} h_\kappa^2(y)dy}.$$

If $\varphi \in C_0^\infty(\mathbb{R}^d)$ is a radial function such that $\chi_{B_r}(x) \leq \varphi(x)$, then from Theorem 3.6 it follows that $\tau_y\chi_{B_r}(x) \leq \tau_y\varphi(x)$. But $\tau_y\varphi$ is bounded; hence $\tau_y\chi_{B_r}$ is bounded and compactly supported, so that it belongs to $L^p(\mathbb{R}^d; h_\kappa^2)$. This means that the maximal function $M_\kappa f$ is defined for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$. We also note that as $\tau_y\chi_{B_r} \geq 0$, we have $M_\kappa f(x) \leq M_\kappa|f|(x)$.

Theorem 6.1. *The maximal function is bounded on $L^p(\mathbb{R}^d; h_\kappa^2)$ for $1 < p \leq \infty$; moreover, it is of weak type $(1,1)$, that is, for $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ and $a > 0$,*

$$\int_{E(a)} h_\kappa^2(x)dx \leq \frac{c}{a}\|f\|_{\kappa,1},$$

where $E(a) = \{x : M_\kappa f(x) > a\}$ and c is a constant independent of a and f.

Proof. Without loss of generality, we can assume that $f \geq 0$. Let $\sigma = d+2\gamma_\kappa+1$ and define for $j \geq 0$, $B_{r,j} = \{x : 2^{-j-1}r \leq \|x\| \leq 2^{-j}r\}$. Then

$$\begin{aligned}\chi_{B_{r,j}}(y) &= (2^{-j}r)^\sigma(2^{-j}r)^{-\sigma}\chi_{B_{r,j}}(y) \\ &\leq C(2^{-j}r)^{\sigma-1}\frac{2^{-j}r}{((2^{-j}r)^2+\|y\|^2)^{\sigma/2}}\chi_{B_{r,j}}(y) \\ &\leq C(2^{-j}r)^{\sigma-1}P_{2^{-j}r}(y),\end{aligned}$$

where P_ε is the Poisson kernel defined in (5.2) and C is a constant independent of r and j. Since χ_{B_r} and P_ε are both bounded integrable radial functions, it follows from Theorem 3.6 that

$$\tau_x \chi_{B_{r,j}}(y) \le C(2^{-j}r)^{\sigma-1}\tau_x P_{2^{-j}r}(y).$$

This shows that for any positive integer m,

$$\begin{aligned}\int_{\mathbb{R}^d} f(y)\sum_{j=0}^{m}\tau_x\chi_{B_{r,j}}(y)h_\kappa^2(y)dy &\le C\sum_{j=0}^{\infty}(2^{-j}r)^{\sigma-1}\int_{\mathbb{R}^d} f(y)\tau_x P_{2^{-j}r}(y)h_\kappa^2(y)dy\\ &\le C\, r^{d+2\gamma_\kappa}\sup_{t>0} f *_\kappa P_t(x).\end{aligned}$$

As $\sum_{j=0}^m \chi_{B_{r,j}}(y)$ converges to $\chi_{B_r}(y)$ in $L^1(\mathbb{R}^d;h_\kappa^2)$, the boundedness of τ_x on $L^1_{\rm rad}(\mathbb{R}^d;h_\kappa^2)$ shows that $\sum_{j=0}^m \tau_x\chi_{B_{r,j}}(y)$ converges to $\tau_x\chi_{B_r}(y)$ in $L^1(\mathbb{R}^d;h_\kappa^2)$. By passing to a subsequence, if necessary, we can assume that $\sum_{j=0}^m \tau_x\chi_{B_{r,j}}(y)$ converges to $\tau_x\chi_{B_r}(y)$ for almost every y. Thus all the functions involved are uniformly bounded by $\tau_x\chi_{B_r}(y)$. This shows that $\sum_{j=0}^m \tau_x\chi_{B_{r,j}}(y)$ converges to $\tau_x\chi_{B_r}(y)$ in $L^{p'}(\mathbb{R}^d;h_\kappa^2)$, and hence

$$\lim_{m\to\infty}\int_{\mathbb{R}^d} f(y)\sum_{j=0}^{m}\tau_x\chi_{B_{r,j}}(y)h_\kappa^2(y)dy = \int_{\mathbb{R}^d} f(y)\tau_x\chi_{B_r}(y)h_\kappa^2(y)dy.$$

Thus we have proved that

$$f *_\kappa \chi_{B_r}(x) \le Cr^{d+2\gamma_\kappa}\sup_{t>0} f *_\kappa P_t(x),$$

which gives the inequality $M_\kappa f(x) \le CP^*f(x)$, where $P^*f(x) = \sup_{t>0} f *_\kappa P_t(x)$ is the maximal function associated to the Poisson semigroup.

Therefore, it is enough to prove the boundedness of P^*f. Here we follow a general procedure used in [17]. By looking at the Dunkl transforms of the Poisson kernel and the heat kernel, we infer that

$$f *_\kappa P_t(x) = \frac{t}{\sqrt{2\pi}}\int_0^\infty (f *_\kappa q_s)(x)e^{-t^2/2s}s^{-3/2}ds,$$

which implies, as in [17, p. 49], that

$$P^*f(x) \le C\sup_{t>0}\frac{1}{t}\int_0^t Q_s f(x)ds,$$

where $Q_s f(x) = f *_\kappa q_s(x)$ is the heat semigroup. Hence using the Hopf–Dunford–Schwartz ergodic theorem as in [17, p. 48], we get the boundedness of P^*f on $L^p(\mathbb{R}^d;h_\kappa^2)$ for $1 < p \le \infty$ and the weak type (1,1). □

The Hardy–Littlewood maximal function $M_\kappa f$ can be used to study almost everywhere convergence of $f *_\kappa \phi_\epsilon$ under certain conditions on ϕ. Recall that $N = d + 2\gamma_\kappa$.

Theorem 6.2. *Let $\phi \in A_\kappa(\mathbb{R}^d)$ be a real valued radial function which satisfies $|\phi(x)| \le c(1 + \|x\|)^{-N-1}$. Then*

$$\sup_{\varepsilon>0} |f *_\kappa \phi_\varepsilon(x)| \le cM_\kappa f(x).$$

Consequently, $f *_\kappa \phi_\varepsilon(x) \to f(x)$ for almost every x as ε goes to 0 for all f in $L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$.

Proof. We can assume that both f and ϕ are nonnegative. Writing

$$\phi_\varepsilon(y) = \sum_{j=-\infty}^{\infty} \phi_\varepsilon(y)\chi_{\varepsilon 2^j \le \|y\| \le \varepsilon 2^{j+1}}(y),$$

we have

$$\sum_{j=-m}^{m} \phi_\varepsilon(y)\tau_x\chi_{\varepsilon 2^j \le \|y\| \le \varepsilon 2^{j+1}}(y) \le c \sum_{j=-m}^{m} (1 + \varepsilon 2^j)^{-N-1}\tau_x\chi_{\varepsilon 2^j \le \|y\| \le \varepsilon 2^{j+1}}(y).$$

This shows that

$$\begin{aligned}\int_{\mathbb{R}^d} f(y)\phi_\varepsilon(y) \sum_{j=-m}^{m} \chi_{\varepsilon 2^j \le \|y\| \le \varepsilon 2^{j+1}}(y)h_\kappa^2(y)dy \\ &\le c \sum_{j=-m}^{m} (1 + \varepsilon 2^j)^{-N-1}(\varepsilon 2^j)^N M_\kappa f(x) \\ &\le cM_\kappa f(x).\end{aligned}$$

Since $\phi(y) \le c(1 + \|y\|)^{-N-1} \le cP_1(y)$, it follows that $\tau_x\phi(y) \le c\tau_x P_1(y)$ is bounded. Arguing as in the previous theorem, we can show that the left hand side of the above inequality converges to $f *_\kappa \phi_\varepsilon(x)$. Thus we obtain

$$\sup_{\varepsilon>0} |f *_\kappa \phi_\varepsilon(x)| \le cM_\kappa f(x),$$

from which the proof of almost everywhere convergence follows from the standard argument. □

The above two theorems show that the maximal functions $M_\kappa f$ and $P^* f$ are comparable. As a corollary, we obtain almost everywhere convergence of Bochner–Riesz means.

Corollary 6.3. *When $\delta \geq (N+1)/2$, the Bochner–Riesz means $S_R^\delta f(x)$ converge to $f(x)$ for almost every x for all $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leq p < \infty$.*

We expect the corollary to be true for all $\delta > (N-1)/2$, as in the case of the Fourier transform. This could be proved if in the above theorem the hypothesis on ϕ could be relaxed to $|\phi(x)| \leq c(1+\|x\|)^{-N-\epsilon}$ for some $\epsilon > 0$. Since we do not know that $\tau_y((1+\|x\|)^{-N-1})$ is bounded, we cannot repeat the proof of the above theorem.

7 Product weight function invariant under $\mathbb{Z}_2^d$

Recall that in the case $G = \mathbb{Z}_2^d$, the weight function h_κ is a product function

$$h_\kappa(x) = \prod_{i=1}^{d} |x_i|^{\kappa_i}, \qquad \kappa_i \geq 0. \tag{7.1}$$

In this case, the explicit formula of the intertwining operator V_κ is known (see (2.2)); and there is an explicit formula for τ_y. The following formula is contained in [11], where it is studied in the context of signed hypergroups.

Theorem 7.1. *For $G = \mathbb{Z}_2^d$ and h_κ in* (7.1),

$$\tau_y f(x) = \tau_{y_1} \cdots \tau_{y_d} f(x), \qquad y = (y_1, \ldots, y_d) \in \mathbb{R}^d;$$

here for $G = \mathbb{Z}_2$ and $h_\kappa(t) = |t|^\kappa$ on $\mathbb{R}$,

$$\begin{aligned} \tau_s f(t) = & \frac{1}{2}\int_{-1}^{1} f\left(\sqrt{t^2+s^2-2stu}\right)\left(1+\frac{t-s}{\sqrt{t^2+s^2-2stu}}\right)\Phi_\kappa(u)du \\ & + \frac{1}{2}\int_{-1}^{1} f\left(-\sqrt{t^2+s^2-2stu}\right)\left(1-\frac{t-s}{\sqrt{t^2+s^2-2stu}}\right)\Phi_\kappa(u)du, \end{aligned} \tag{7.2}$$

where $\Phi_\kappa(u) = b_\kappa(1+u)(1-u^2)^{\kappa-1}$. Consequently, for each $y \in \mathbb{R}^d$, the generalized translation operator τ_y for $\mathbb{Z}_2^d$ extends to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$. More precisely, $\|\tau_y f\|_{\kappa,p} \leq 3\|f\|_{\kappa,p}$ for $1 \leq p \leq \infty$.

Since the generalized translation operator τ_y extends to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$, many results stated in the previous sections can be improved and the proofs can be carried out more conveniently, as in classical Fourier analysis. In particular, the properties of τ_y given in Proposition 3.2, Theorem 3.6 and Theorem 3.8 all hold under the more relaxed condition of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$.

The standard proof [23] can now be used to show that the generalized convolution satisfies the following analogue of Young's inequality.

Proposition 7.2. *Let $G = \mathbb{Z}_2^d$. Let $p, q, r \ge 1$ and $p^{-1} = q^{-1} + r^{-1} - 1$. Assume $f \in L^q(\mathbb{R}^d, h_\kappa^2)$ and $g \in L^r(\mathbb{R}^d, h_\kappa^2)$, respectively. Then*

$$\|f *_\kappa g\|_{\kappa,p} \le c\|f\|_{\kappa,q}\|g\|_{\kappa,r}.$$

In the following, we give several results which improve the corresponding results in the previous sections significantly. We start with an improved version of Theorem 4.2. The boundedness of τ_y allows us to remove the assumption that ϕ is radial.

Theorem 7.3. *Let $\phi \in L^1(\mathbb{R}^d, h_\kappa^2)$ and assume $\int_{\mathbb{R}^d} \phi(x)h_\kappa^2(x)dx = 1$. Then for $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \le p < \infty$, or $f \in C_0(\mathbb{R}^d)$ if $p = \infty$,*

$$\lim_{\varepsilon \to 0} \|f *_\kappa \phi_\varepsilon - f\|_{\kappa,p} = 0, \qquad 1 \le p \le \infty.$$

Proof. First we assume that $f \in C_0^\infty(\mathbb{R}^d)$. By Theorem 3.14, $\|\tau_y f(x) - f(x)\|_{\kappa,p} \to 0$ as $y \to 0$ for $1 \le p \le \infty$. In general, for $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, we write $f = f_1 + f_2$, where f_1 is continuous with compact support and $\|f_2\|_{\kappa,p} \le \delta$. Then the first term of the inequality

$$\|\tau_y f(x) - f(x)\|_{\kappa,p} \le \|\tau_y f_1(x) - f_1(x)\|_{\kappa,p} + \|\tau_y f_2(x) - f_2(x)\|_{\kappa,p}$$

goes to zero as $\varepsilon \mapsto 0$, and the second term is bounded by $(1 + c)\delta$, as $\|\tau_y f_2\|_{\kappa,p} \le c\|f\|_{\kappa,p}$. This proves that $\|\tau_y f(x) - f(x)\|_{\kappa,p} \to 0$ as $y \to 0$. We then have

$$\begin{aligned}
c_h &\int_{\mathbb{R}^d} |f *_\kappa g_\varepsilon(x) - f(x)|^p h_\kappa^2(x)dx \\
&= c_h \int_{\mathbb{R}^d} \left| c_h \int_{\mathbb{R}^d} (\tau_y f(x) - f(x)) g_\varepsilon(y) h_\kappa^2(y) dy \right|^p h_\kappa^2(x)dx \\
&\le c_h \int_{\mathbb{R}^d} \|\tau_y f - f\|_{\kappa,p}^p |g_\varepsilon(x)| h_\kappa^2(x)dx \\
&= c_h \int_{\mathbb{R}^d} \|\tau_{\varepsilon y} f - f\|_{\kappa,p}^p |g(x)| h_\kappa^2(x)dx,
\end{aligned}$$

which goes to zero as $\varepsilon \to 0$. □

Our next result is about the boundedness of the spherical means operator. As in [10], we define the spherical mean operator on $A_\kappa(\mathbb{R}^d)$ by

$$S_r f(x) := a_\kappa \int_{S^{d-1}} \tau_{ry} f(x) h_\kappa^2(y) d\omega(y).$$

The generalized convolution of f with a radial function can be expressed in terms of the spherical means $S_r f$. In fact, if $f \in A_\kappa(\mathbb{R}^d)$ and $g(x) = g_0(\|x\|)$ is an integrable radial function, then using the spherical-polar coordinates, we have

$$\begin{aligned}(f *_\kappa g)(x) &= c_h \int_{\mathbb{R}^d} \tau_y f(x) g(y) h_\kappa^2(y) dy \\ &= c_h \int_0^\infty r^{2\lambda_\kappa+1} g_0(r) \int_{S^{d-1}} \tau_{ry'} f(x) h_\kappa^2(y') dy' dr \\ &= \frac{c_h}{a_\kappa} \int_0^\infty S_r f(x) g_0(r) r^{2\lambda_\kappa+1} dr.\end{aligned}$$

We make use of this later in this section. Regarding boundedness, we have

Theorem 7.4. *Let $G = \mathbb{Z}_2^d$. For $f \in L^p(\mathbb{R}^d, h_\kappa^2)$,*

$$\|S_r f\|_{\kappa,p} \le c\|f\|_{\kappa,p}, \qquad 1 \le p \le \infty.$$

Furthermore, $\|S_r f - f\|_{\kappa,p} \mapsto 0$ as $r \to 0+$.

Proof. By Hölder's inequality,

$$|S_r f(x)|^p \le a_\kappa \int_{S^{d-1}} |\tau_{ry} f(x)|^p h_\kappa^2(y) d\omega(y).$$

Hence, a simple computation shows that

$$\begin{aligned}c_h \int_{\mathbb{R}^d} |S_r f(x)|^p h_\kappa^2(x) dx &\le c_h \int_{\mathbb{R}^d} a_\kappa \int_{S^{d-1}} |\tau_{ry} f(x)|^p h_\kappa^2(y) d\omega(y) h_\kappa^2(x) dx \\ &= a_\kappa \int_{S^{d-1}} \|\tau_{ry} f\|_{\kappa,p}^p h_\kappa^2(y) d\omega(y) \\ &\le c\|f\|_{\kappa,p}.\end{aligned}$$

Furthermore, we have

$$\|S_r f - f\|_{\kappa,p}^p \le a_\kappa \int_{S^{d-1}} \|\tau_{ry} f - f\|_{\kappa,p}^p h_\kappa^2(y) d\omega(y),$$

which goes to zero as $r \to 0$, since $\|\tau_{ry} f - f\|_{\kappa,p} \to 0$. □

We remark that the spherical mean value operator is bounded on L^p for any finite reflection group, not just for $G = \mathbb{Z}_2^d$. To see this, we can make use of a positive integral representation of the spherical mean operator, proved in [15]. In fact, it follows easily that S_r is actually a contraction on L^p spaces.

The boundedness of $\tau_y f$ in $L^p(\mathbb{R}^d; h_\kappa^2)$ also allows us to relax the condition of Theorem 6.2.

Theorem 7.5. *Set $G = \mathbb{Z}_2^d$. Let $\phi(x) = \phi_0(\|x\|) \in L^1(\mathbb{R}^d, h_\kappa^2)$ be a radial function. Assume that ϕ_0 is differentiable, $\lim_{r\to\infty} \phi_0(r) = 0$ and $\int_0^\infty r^{2\lambda_\kappa+2}|\phi_0(r)| dr < \infty$. Then*

$$|(f *_\kappa \phi)(x)| \le c M_\kappa f(x).$$

In particular, if $\phi \in L^1(\mathbb{R}^d, h_\kappa^2)$ and $c_h \int_{\mathbb{R}^d} \phi(x) h_\kappa^2(x) dx = 1$, then

1. *for* $1 \le p \le \infty$, $f *_\kappa \phi_\varepsilon$ *converges to* f *as* $\varepsilon \to 0$ *in* $L^p(\mathbb{R}^d; h_\kappa^2)$*;*
2. *for* $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, $(f *_\kappa \phi_\varepsilon)(x)$ *converges to* $f(x)$ *as* $\varepsilon \to 0$ *for almost all* $x \in \mathbb{R}^d$.

Proof. By definition of the spherical means $S_t f$, we can also write

$$M_\kappa f(x) = \sup_{r>0} \frac{\left|\int_0^r t^{2\lambda_\kappa+1} S_t f(x) dt\right|}{\int_0^r t^{2\lambda_\kappa+1} dt}.$$

Since $|M_\kappa f(x)| \le c M_\kappa |f|(x)$, we can assume $f(x) \ge 0$. The assumption on ϕ_0 shows that

$$\begin{aligned}\lim_{r\to\infty} \phi_0(r) \int_0^r S_t f(x) t^{2\lambda_\kappa+1} dt &= \lim_{r\to\infty} \phi_0(r) \int_{\mathbb{R}^d} \tau_y f(x) h_\kappa^2(y) dy \\ &= \lim_{r\to\infty} \phi_0(r) \int_{\mathbb{R}^d} f(y) h_\kappa^2(y) dy = 0.\end{aligned}$$

Hence, using spherical-polar coordinates and integrating by parts, we get

$$\begin{aligned}(f *_\kappa \phi)(x) &= \int_0^\infty \phi_0(r) r^{2\lambda_\kappa+1} S_r f(x) dr \\ &= -\int_0^\infty \int_0^r S_t f(x) t^{2\lambda_\kappa+1} dt \phi'(r) dr,\end{aligned}$$

which implies that

$$|(f *_\kappa \phi)(x)| \le c M_\kappa f(x) \int_0^\infty r^{2\lambda_\kappa+2} |\phi_0'(r)| dr.$$

The boundedness of the last integral proves the maximal inequality. □

As an immediate consequence of this theorem, the Bochner–Riesz means converge almost everywhere if $\delta > (N-1)/2$ for $G = \mathbb{Z}_2^d$; this closes the gap left open in Corollary 6.3.

We can further enhance Theorem 7.5 by removing the assumption that ϕ is radial. For this purpose, we make the following simple observation about the maximal function.

Lemma 7.6. *If* $f \in L^1(\mathbb{R}^d, h_\kappa^2)$ *is a nonnegative function, then*

$$M_\kappa f(x) = \sup_{r>0} \frac{\int_{B_r} \tau_y f(x) h_\kappa^2(y) dy}{\int_{B_r} h_\kappa^2(y) dy}.$$

In particular, if f *and* g *are two nonnegative functions, then*

$$M_\kappa f + M_\kappa g = M_\kappa (f + g).$$

Proof. Since $\tau_y \chi_{B_r}(x)$ is nonnegative,

$$(f *_\kappa \chi_{B_r})(x) = \int_{\mathbb{R}^d} f(y)\tau_y \chi_{B_r}(x) h_\kappa^2(y)dy$$

is nonnegative if f is nonnegative. Hence we can drop the absolute value sign in the definition of $M_\kappa f$. □

Theorem 7.7. *Set $G = \mathbb{Z}_2^d$. Let $\phi \in L^1(\mathbb{R}^d, h_\kappa^2)$ and let $\psi(x) = \psi_0(\|x\|) \in L^1(\mathbb{R}^d, h_\kappa^2)$ be a nonnegative radial function such that $|\phi(x)| \le \psi(x)$. Assume that ψ_0 is differentiable, $\lim_{r\to\infty} \psi_0(r) = 0$ and $\int_0^\infty r^{2\lambda_\kappa+2}|\psi_0(r)|dr < \infty$. Then $\sup_{\epsilon>0} |f *_\kappa \phi_\epsilon(x)|$ is of weak type $(1,1)$. In particular, if $\phi \in L^1(\mathbb{R}^d, h_\kappa^2)$ and $c_h \int_{\mathbb{R}^d} \phi(x)h_\kappa^2(x)dx = 1$, then for $f \in L^1(\mathbb{R}^d, h_\kappa^2)$, $(f *_\kappa \phi_\varepsilon)(x)$ converges to $f(x)$ as $\varepsilon \to 0$ for almost all $x \in \mathbb{R}^d$.*

Proof. Since $M_\kappa f(x) \le M_\kappa|f|(x)$, we can assume that $f(x) \ge 0$. The proof uses the explicit formula for $\tau_y f$. Let us first consider the case of $d = 1$. Since ψ is an even function, by (7.2) $\tau_y \psi$ is given by the formula

$$\tau_y f(x) = \int_{-1}^1 f\left(\sqrt{x^2+y^2-2xyt}\right)\Phi_\kappa(t)dt.$$

Since $(x-y)(1+t) = (x-yt)-(y-xt)$, we have

$$\frac{|x-y|}{\sqrt{x^2+y^2-2xyt}}(1+t) \le 2.$$

Consequently, by the explicit formula (7.2) for $\tau_y f$, the inequality $|\phi(x)| \le \psi(x)$ implies that

$$|\tau_y \phi(x)| \le \tau_y \psi(x) + 2\widetilde{\tau}_y \psi(x),$$

where

$$\widetilde{\tau}_y \psi(x) = b_\kappa \int_{-1}^1 f\left(\sqrt{x^2+y^2-2xyt}\right)(1-t^2)^{\kappa-1}dt.$$

Note that $\widetilde{\tau}_y \psi$ differs from $\tau_y \psi$ by a factor of $(1+t)$ in the weight function. Changing variables $t \mapsto -t$ and $y \mapsto -y$ in the integrals shows that

$$\int_{\mathbb{R}} f(y)\widetilde{\tau}_y \psi(x) h_\kappa^2(y)dy = \int_{\mathbb{R}} F(y)\tau_y \psi(x) h_\kappa^2(y)dy,$$

where $F(y) = (f(y)+f(-y))/2$. Hence, it follows that

$$|(f *_\kappa \phi)(x)| = \left|\int_{\mathbb{R}} f(y)\tau_y \phi(x) h_\kappa^2(y)dy\right| \le (f *_\kappa \psi)(x) + 2(F *_\kappa \psi)(x).$$

The same consideration can be extended to the case of $\mathbb{Z}_2^d$ for $d > 1$. Let $\{e_1, \dots, e_d\}$ be the standard Euclidean basis. For $\delta_j = \pm 1$, define $x\delta_j = x - (1 + \delta_j)x_j e_j$ (that is, multiplying the j-th component of x by δ_j gives $x\delta_j$). For $1 \le j \le d$, we define

$$F_{j_1,\dots,j_k} = 2^{-k} \sum_{(\delta_{j_1},\dots,\delta_{j_k}) \in \mathbb{Z}_2^k\}} f(x\delta_{j_1} \cdots \delta_{j_k}).$$

In particular, $F_j(x) = (F(x) + F(x\delta_j))/2$, $F_{j_1,j_2}(x) = (F(x) + F(x\delta_{j_1}) + F(x\delta_{j_2}) + F(x\delta_{j_1}\delta_{j_2}))/4$, and the last sum is over $\mathbb{Z}_2^d$, $F_{1,\dots,d}(x) = 2^{-d}\sum_{\sigma \in \mathbb{Z}_2^d} f(x\sigma)$. Following the proof in the case of $d = 1$, we see that

$$\begin{aligned} |(f *_\kappa \phi)(x)| \le (f *_\kappa \psi)(x) + 2\sum_{j=1}^{d}(F_j *_\kappa \psi)(x) + 4\sum_{j_1 \ne j_2}(F_{j_1,j_2} *_\kappa \psi)(x) \\ + \cdots + 2^d(F_{1,\dots,d} *_\kappa \psi)(x). \end{aligned}$$

For $G = \mathbb{Z}_2^d$, the explicit formula of τ_y shows that $M_\kappa f(x)$ is even in each of its variables. Hence, applying the result of the previous theorem on each of the above terms, we get

$$\begin{aligned} |(f *_\kappa \phi)(x)| \le M_\kappa f(x) + 2\sum_{j=1}^{d} M_\kappa F_j(x) + 4\sum_{j_1 \ne j_2} M_\kappa F_{j_1,j_2}(x) \\ + \cdots + 2^d M_\kappa F_{1,\dots,d}(x). \end{aligned}$$

Since all F_j are clearly nonnegative, by Lemma 7.6, the last expression can be written as $M_\kappa H$, where H is the sum of all functions involved. Consequently, since $\|F_{j_1,\dots,j_d}\|_{\kappa,1} \le \|f\|_{\kappa,1}$, it follows that

$$\int_{\{x:(f *_\kappa \phi)(x) \ge a\}} h_\kappa^2(y)dy \le c\frac{\|H\|_{\kappa,1}}{a} \le c_d \frac{\|f\|_{\kappa,1}}{a}.$$

Hence, $f *_\kappa \phi$ is of weak type $(1, 1)$, from which the almost everywhere convergence follows as usual. □

We do not know whether the inequality $|(f *_\kappa \phi)(x)| \le cM_\kappa f(x)$ holds in this case, since we only know $M_\kappa(R(\delta)f)(x) = R(\delta)M_\kappa f(x) = M_\kappa f(x\delta)$, where $R(\delta)f(x) = f(x\delta)$ for $\delta \in G$, from which we cannot deduce that $M_\kappa F_{j_1,\dots,j_k}(x) \le cM_\kappa f(x)$.

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