SOME APPLICATIONS OF THE WEIL REPRESENTATION

By

DAVID KAZHDAN

§1. The Weil representation in the local case

Let F be a field with char $F \neq 2$, $V = F^{2n}$ be a finite-dimensional vector space and Ω be a nondegenerate skew-symmetric bilinear form on V. We will denote by $\mathrm{Sp}(V, \Omega)$ or simply Sp the algebraic group of linear transformations $v \to v^g$ of V preserving Ω.

Let N be the following central extension of V (as an algebraic group):

$$N = \{(v, t), v \in V, t \in F\}$$

with the composition law:

$$(v_1, t_1) \cdot (v_2, t_2) = (v_1 + v_2, t_1 + t_2 + \Omega(v_1, v_2)).$$

We have a natural action $n \to n^g$ of Sp on N:

$$g : (v, t) \to (v^g, t).$$

Suppose now that k is a local field, and $\psi : F^+ \to C^*$ is a nontrivial additive character. Let F_1, $V_2 \subset V$ be maximal isotropic subspaces such that $V_1 \cap V_2 = \{0\}$. Then $V = V_1 \oplus V_2$. The restrictions of the natural imbedding $i : V \to N$, $i(v) = (v, 0)$ to V_1 and V_2 are homomorphisms and so we can consider V_1, V_2 as subgroups of N. It is clear that they generate N. Now we define the unitary representation ρ of N in the space $W \overset{\mathrm{def}}{=} L^2(V_1)$ by describing its restrictions to V_1, V_2:

$$(\rho(v_1, 0)f)(v_1^0) = f(v_1 + v_1^0), \ (\rho(v_2, 0)f)(v_1^0) = \psi(\rho(v_1^0, v_2))f(v_1^0),$$

$$v_1 \in V_1, \quad v_2 \in V_2.$$

JOURNAL D'ANALYSE MATHÉMATIQUE, Vol. 32 (1977)

It is not hard to check that ρ is really a representation of N in W and $\rho(0,t)=\psi(t)\mathrm{Id}$.

It is well known ([6]) that every irreducible unitary representation

$$\rho' : N \to \operatorname{Aut} W$$

such that $\rho'(0,t)=\psi(t)E$, $\forall t \in k$, is equivalent to ρ.

It follows that for every $g \in \mathrm{Sp}$ the representation $\rho^g(n) \overset{\text{def}}{=} \rho(n^g)$ is equivalent to ρ. So for every g there exists a unitary operator $A(g)$: $W \to W$ such that

$$A(g)\rho(n)=\rho(n^g)A(g).$$

ρ is irreducible. Therefore $A(g)$ is uniquely defined up to a constant and we obtain a projective representation $A:\mathrm{Sp}\to \operatorname{Aut} P(W)$. A can be realized as a usual representation only if $k=\mathbf{C}$. If $k \neq \mathbf{C}$, then ([6]) there exists an honest representation $\tau : \widetilde{\mathrm{Sp}} \to \operatorname{Aut} W$ of the two-sheeted covering $p:\widetilde{\mathrm{Sp}}\to \mathrm{Sp}$ such that

$$\tau(\tilde g)\rho(n)=\rho(n^g)\tau(\tilde g) \text{ for all } n \in N,\quad \tilde g \in \widetilde{\mathrm{Sp}},\quad g=p(\tilde g).$$

This representation τ we will call the Weil representation.

We denote by $W_0 \subset W$ the subspace of smooth vectors, i.e., in the case when F is non-archimedean then $W_0=\{w \in W \mid \operatorname{Stab} w$ is an open subgroup in $\widetilde{\mathrm{Sp}}\}$; for archimedean F, $W_0=\{w\in W \mid \forall w' \in W$, the function $\varphi_{w,w'}(g) \overset{\text{def}}{=} \langle w, \tau(g)w'\rangle$ on $\widetilde{\mathrm{Sp}}$ is smooth$\}$. In our realization $W=L^2(V_1)$, $W_0=\{$Space of Schwartz–Bruhat functions on $V_1\}$ ([6]).

The element $(-\mathrm{Id})\in \mathrm{Sp}$ is in the center of Sp. The subspaces W^+, $W^- \subset W$ of even and odd functions on V_1 are eigensubspaces of $\tau(-\mathrm{Id})$, and therefore are invariant under $\rho(\widetilde{\mathrm{Sp}})$. We denote $W_0^{\pm}=W^{\pm}\cap W_0$.

Lemma 1. *The restrictions $\tau^{\pm}$ to W^+ and W^- are irreducible.*

Proof. Let us consider two subgroups $L_1, L_2 \subset \mathrm{Sp}$:

$$L_1=\{g\in \mathrm{Sp} \mid V_1^g=V_1,\ V_2^g=V_2\},$$

$$L_2=\{g \in \mathrm{Sp} \mid V_1^g=V_1,\ g \mid V_1=\mathrm{Id},\ g \mid V/V_1=\mathrm{Id}\}.$$

It is well known ([6]) that there exists a natural isomorphism

$$\alpha : L_1 \xrightarrow{\sim} \mathrm{GL}(V_1),\ \beta : L_2 \longrightarrow \{\text{quadratic forms on } V_1\}$$

and

$$(*)(\rho(l_1)f)(v_1) = \varepsilon(l_1)|\det\alpha(l_1)|^{-1/2} f(\alpha(l_1)v_1), \qquad (\rho(l_2)f)(v_1) = \psi(\beta(l_2)(v_1))f(v_1)$$

($\varepsilon(l_1)$ is a complex number, $|\varepsilon| = 1$) for all $l_1 \in L_1$, $l_2 \in L_2$, $v_1 \in V_1$, $f \in W$. The lemma follows immediately from this result.'

Suppose now that k is a non-archimedean local field, and denote by $\mathcal{O} \subset F$ the ring of integers. Let $\Lambda_1 \subset V_1$ be a compact open $\mathcal{O}$-submodule, let $\Lambda_2 \subset V_2$ be the annihilator of Λ_1, i.e.,

$$\Lambda_2 = \{v \in V_2 \mid \psi(\Omega(v, \lambda)) = 1 \quad \text{for} \quad \forall\lambda \in \Lambda_1\},$$

and let $\Lambda = \Lambda_1 \oplus \Lambda_2 \subset V$. We let $N_\Lambda \subset N$ denote the subgroup generated by $i(\Lambda_1)$ and $i(\Lambda_2)$ in N, and let $K_\Lambda \subset \mathrm{Sp}$ denote the stabilizer of $\Lambda \subset V$. It is well known that K_Λ is a maximal subgroup in Sp, i.e., for every $K_\Lambda \subset H \subset \mathrm{Sp}$, either $H = K_\Lambda$ or $H = \mathrm{Sp}$.

Lemma 2. a) *The covering* $p: \widetilde{\mathrm{Sp}} \to \mathrm{Sp}$ *splits over* K_Λ, *i.e., there exists a section* $j: K_\Lambda \to \widetilde{\mathrm{Sp}}$.

b) *The subspace* $W^{K_\Lambda} \subset W$ *of vectors which are invariant under* $\tau \circ j(K_\Lambda)$ *is one dimensional and lies in* W^+

Proof. a) Let us denote by $f_0 \in W$ the characteristic function of Λ_1. It is clear that f_0 is invariant under $\rho(N_\Lambda)$ and, moreover, $W^{\rho(N_\Lambda)} = \mathbf{C} f_0$. The subgroup K_Λ normalizes N_Λ. Therefore, $A(k)$ preserves $W^{\rho(N_\Lambda)}$ for every $k \in K_\Lambda$, and we can choose the operator $A_0(k) \subset A(k)$ such that $A_0(k)f_0 = f_0$. It is now clear that the map $k \to A_0(k)$ gives us an honest representation of K_Λ.

Let $\tilde{K}_\Lambda$ be $p^{-1}(K_\Lambda) \subset \widetilde{\mathrm{Sp}}$. We denote by $M \subset \tilde{K}$ the commutator subgroup of $\tilde{K}_\Lambda$.

Sublemma 2. *The projection* p *induces an isomorphism* $p_0: M \to K_\Lambda$ *and* $\tau(m) = A_0(p(m))$ *for all* $m \in M$.

Proof of sublemma. It is well known that the commutator subgroup of K_Λ is all of K_Λ, and therefore p is surjective. On the other hand, we proved that for every $\tilde{K} \in \tilde{K}_\Lambda$, $\tau(\tilde{k})$ preserves $W^{\rho(N_\Lambda)}$ and acts as a scalar on it. So, for every $m \in M$, $\tau(m)$ acts trivially on $W^{\rho(N_\Lambda)}$ and therefore is equal to $A_0(p(m))$. So p is injective. The sublemma is proved.

Part a) of the lemma is proved: we can simply write $j = p_0^{-1}$. To prove Part b) we shall need the following two obvious facts.

' If k is a non-archimedean field $W_0^\pm$ are algebraically irreducible.

Let $K_1 = L_1 \cap K_\Lambda$, $K_2 = L_2 \cap K_\Lambda$. Then

(i) $W^{\rho(K_2)} = \{f \in W \mid \operatorname{Supp} f \subset \Lambda_1\}$. (See (*).)

(ii) If $\operatorname{Supp} f \subset \pi\Lambda_1$ then $\exists K_2^1 \subset L_2$, $K_2^1 \supsetneq K_2$ such that $f \in W^{K_2^1}$.

Now let f be some element in W^{K_Λ}. Then, $f \in W^{K_2}$ (and so $\operatorname{supp} f \subset \Lambda_1$) and $f \in W^{K_1}$ (and so f as a function on Λ_1 is invariant under $\mathcal{O}$-linear automorphisms of Λ_1). Therefore there exists $c \in \mathbf{C}$ such that $\operatorname{supp}(f - cf_0) \subset \pi\Lambda_1$. By construction $(f - cf_0) \in W^{K_\Lambda}$ and, by (ii), $(f - cf_0) \in W^{K_2^1}$. So the stabilizer H of $\mathbf{C}(f - cf_0)$ in Sp is strictly larger than K_Λ, and therefore is equal to all of Sp. But this is possible only when $f = cf_0$. It is clear that $f_0 \in W^+$. So Lemma 2 is proved.

§2. The global case

Suppose now that k is a global field, Char $k \neq 2$. For every valuation p on k we denote by k_p the corresponding local field (archimedean valuations will often be denoted ∞), and we let $\mathbf{A}$ denote the ring of adeles over k. For every group G over k we will let $G_{\mathbf{A}}$ denote its group of adelic points.

Now let (V, Ω) be a $2n$-dimension vector space over k with a nondegenerate bilinear skew-form Ω. As before, we shall consider the groups N and Sp and isotropic subspaces $V_1, V_2 \subset V$. Let $W_{\mathbf{A}}$ be $L^2(V_{1_{\mathbf{A}}})$, let $W_{0_{\mathbf{A}}}$ be the subspace of Schwartz–Bruhat functions. We can define as before the unitary representation $\rho_{\mathbf{A}}: N_{\mathbf{A}} \to \operatorname{Aut} W_{\mathbf{A}}$ and the projective representation $A: \mathrm{Sp}_{\mathbf{A}} \to \operatorname{Aut} P(W_{\mathbf{A}})$. It is known that A induces a representation $\tau: \widetilde{\mathrm{Sp}}_{\mathbf{A}} \to \operatorname{Aut} W_{\mathbf{A}}$ of the two-sheeted coverings of $\mathrm{Sp}_{\mathbf{A}}$. This representation preserves $W_{0_{\mathbf{A}}}$.

Lemma 3. *The covering $p_{\mathbf{A}}: \widetilde{\mathrm{Sp}}_{\mathbf{A}} \to \mathrm{Sp}_{\mathbf{A}}$ splits over Sp_k, and there exists a non-zero functional λ_0 on $W_{0_{\mathbf{A}}}$ which is invariant under $\tau(\mathrm{Sp}_k)$.*

Proof. It is easy to see that every functional $\lambda \in (W_{0_{\mathbf{A}}})'$ invariant under $\rho(N_k)$ is proportional to $\lambda_0: f \to \Sigma_{v \in V_k} f(v)$. The group Sp_k normalizes N_k, and therefore $A(\gamma)$ preserves $\mathbf{C}\lambda_0$ for all $\gamma \in \mathrm{Sp}_k$. So we can choose the operator $A_0(\gamma) \subset A(\gamma)$ such that $A_0(\gamma)\lambda_0 = \lambda_0$. Now we can finish with the same arguments as in Lemma 2.

Let Π be the set of all valuations p on k, and let B be the set of all sequences $\boldsymbol{b} = (b_p)$, $p \in \Pi$ of ± 1 in which there is only a finite number of -1. For every $\boldsymbol{b} \in B$ we denote by $W^{\boldsymbol{b}} \subset W_{\mathbf{A}}$ the subspace of functions f on $V_{1_{\mathbf{A}}}$ such that $f((-1)_p v) = b_p f(v)$ for all $v \in V_{1_{\mathbf{A}}}$. Here $(-1)_p$ is the idele which is the image of -1 under the natural imbedding $k_p^* \hookrightarrow \mathbf{A}^*$.

The following fact is an easy consequence of Lemmas 1 and 2.

Lemma 4. *For every $\boldsymbol{b} \in B$ the subspace $W^b \subset W_{\mathbf{A}}$ is invariant under $\tau_{\mathbf{A}}(\widetilde{\mathrm{Sp}}_{\mathbf{A}})$. Moreover, the restriction $\tau_{\boldsymbol{b}}$ of $\tau_{\mathbf{A}}$ to $W^{\boldsymbol{b}}$ is an irreducible representation of $\widetilde{\mathrm{Sp}}_{\mathbf{A}}$ which is equivalent to the restricted tensor product $\bigotimes_{p \in \Pi} W_p^{b_p}$, where $W_p^{b_p}$ is the corresponding representations of $\widetilde{\mathrm{Sp}}_{k_p}$.*

Let $W_0^{\boldsymbol{b}} \overset{\text{def}}{=} W^{\boldsymbol{b}} \cap W_{0_{\mathbf{A}}}$.

Lemma 5. *The restriction of λ_0 to $W^{\boldsymbol{b}}$ is trivial iff $\boldsymbol{b}$ has an odd number of -1.*

Proof. Let $Z = \text{Center}\,\widetilde{\text{Sp}}_{\mathbf{A}} = \Pi_{p\in\Pi}\mathbf{Z}/2\mathbf{Z}$. We have a natural imbedding $Z \subset \mathbf{A}^*$ and therefore a natural action of Z on V_1, W_0 and $(W_0)'$. Every element $\boldsymbol{b} \subset B$ can be considered as a character of Z.

We have to prove that

$$\int_Z \boldsymbol{b}(z)(z\lambda_0) = 0 \Leftrightarrow \boldsymbol{b} \text{ has an odd number of } -1.$$

But this is obvious.

Let $p_1, \cdots, p_l$ be a set of valuations on k, τ_{p_i} the Weil representations $\tau_{p_i}: \widetilde{\text{Sp}}_{k_{p_i}} \to \text{Aut}\, W_{p_i}$, $N_{p_i} \subset (W_{p_i})_0$ non-zero subspaces. Let

$$N = N_{p_1} \otimes \cdots \otimes N_{p_l} \bigotimes_{\Pi - \{p_i\}} (W_p)_0 \subset (W_{\mathbf{A}})_0.$$

Proposition 1. *The restriction of λ_0 on N is not zero.*

Proof. For simplicity we suppose that for every i, $1 \leqq i \leqq l$ there exists a number $b_{p_i}^0 = \pm 1$ such that the intersection $N_{p_i} \cap W$ is not zero. We will apply the proposition only in this case. Fix now $\boldsymbol{b} = (b_p) \in B$ with even numbers of (-1) and such that $b_{p_i} = b_{p_i}^0$. Let $N^{\boldsymbol{b}}$ be the intersection $N \cap W_0^{\boldsymbol{b}}$.

We know that the restriction of λ_0 to $W_0^{\boldsymbol{b}}$ is non-zero. We shall show that the restriction of λ_0 to $N^{\boldsymbol{b}}$ is non-zero. As $N^{\boldsymbol{b}}$ is $\widetilde{\text{Sp}}_{k_p}$ invariant for every $p \in \Pi - \{p_i\}$ this result follows immediately from:

Lemma 6. *Let p be a valuation on k and M be a non-zero subspace of $W_0^{\boldsymbol{b}}$ invariant under $\tau(\widetilde{\text{Sp}}_{k_p})$. Then the restriction of λ_0 to M is non-zero.*†

Proof of lemma. Let us consider the map from $W_0^{\boldsymbol{b}}$ to the space L of smooth functions on $\text{Sp}_k \backslash \widetilde{\text{Sp}}_{\mathbf{A}}$

$$\varphi_0(w)(g) = \lambda_0(\tau(g)w).$$

$\widetilde{\text{Sp}}_{\mathbf{A}}$ naturally acts on L by right shifts, and φ_0 is a morphism of representations. We know that $\varphi_0 \neq 0$, and that the representation of $\widetilde{\text{Sp}}_{\mathbf{A}}$ in $W_0^{\boldsymbol{b}}$ is irreducible.

† We consider $\widetilde{\text{Sp}}_{k_p}$ as a subgroup of $\widetilde{\text{Sp}}_{\mathbf{A}}$.

Hence, φ_0 is an imbedding, and $\varphi_0(M) \neq 0$. M is $\tau(\widetilde{\mathrm{Sp}}_{k_p})$ invariant, and so $L_0 = \varphi_0(M)$ is a non-zero subspace of L invariant under right shifts by $(\widetilde{\mathrm{Sp}}_{k_p})$. If the restriction λ_0 to V were zero, then for every function $\varphi \in L_0$ we would have $\varphi(\bar{e}) = 0$, where $\bar{e}$ is the image of the the unit in $\widetilde{\mathrm{Sp}}_{\mathbf{A}}$. L_0 is $(\widetilde{\mathrm{Sp}}(k_p))$ invariant, so the restriction of any $\varphi \in L_0$ to the orbit $\Omega = \bar{e}(\widetilde{\mathrm{Sp}}_{k_p})$ is zero. But it follows from the strong approximation theorem [2] that Ω is dense in $\mathrm{Sp}_k \backslash \widetilde{\mathrm{Sp}}_{\mathbf{A}}$. So $\varphi \equiv 0$. This contradiction proves the lemma and the proposition.

§3. The unitary subgroup

Let k be a field (char $k \neq 2$), let K be a quadratic extension, V be a finite dimensional K-vector space, and let Q be a nondegenerate Hermitian form on V. Let $i \neq 0$ be some element in K such that $\mathrm{Tr}_{K/k} i = 0$. If we consider V as a k-vector space (by restriction of scalars), then we can write Q in the form $Q = \mathrm{Re}\, Q + i \,\mathrm{Im}\, Q$, where $\mathrm{Re}\, Q$ and $\mathrm{Im}\, Q$ are bilinear forms on V. $\mathrm{Re}\, Q$ will be a symmetric and $\mathrm{Im}\, Q$ a symplectic nondegenerate form. Let us denote by U_Q (or simply U) the unitary group of K-automorphisms of V preserving Q, and by Sp_Q (or simply Sp) the group of k-automorphisms of V preserving $\mathrm{Im}\, Q$. By definition, we can consider U as a subgroup in Sp.

Let U_1 denote the center of U. It consists of the scalar transformations $v \to \lambda v$, for all $\lambda \in K^*$ with $N_{K/k}(\lambda) = 1$. We shall also consider the subgroup $SU \subset U$ of unimodular transformations in U.

Let k now be a local field.

Lemma 7. *The restriction on the covering $p: \widetilde{\mathrm{Sp}}(2n, k) \to \mathrm{Sp}(2n, k)$ splits over SU.*

Proof. If $k = \mathbf{C}$ there is nothing to prove. If $k = \mathbf{R}$ and $K = \mathbf{C}$ we can apply topological arguments. It is enough to prove that the image of the fundamental group $\pi_1(SU)$ under the embedding $\alpha: SU \to \mathrm{Sp}(2n, \mathbf{R})$ is zero. Let the Hermitian form Q have type (s, t) $(s + t = n)$. Then the group SU is contractible on $U(s) \times U(t)$, $\mathrm{Sp}(2n, \mathbf{R})$ on $U(s + t)$ and the embedding α corresponds to the natural embedding $\bar{\alpha}: U(s) \times U(t) \to U(s + t)$. It is clear that $\bar{\alpha}(S(U(s) \times U(t))) \subset SU(n)$. As $SU(n)$ is simply connected the lemma is proved in this case.

Now suppose that k is a non-archimedean field. The following remarks were told me by Han Sah.

1) Suppose that $V = V_1 \oplus V_2$ and $Q = Q_1 \oplus Q_2$. If the lemma is true for (V, Q) it is true also for (V_1, Q_1). Indeed, we know [6] that the metaplectic covering $\widetilde{\mathrm{Sp}}(V_1) \to \mathrm{Sp}(V_1)$ can be obtained by the restriction of the covering $\widetilde{\mathrm{Sp}}(V) \to \mathrm{Sp}(V)$. So if the restriction of the latter to $SU_Q(V)$ splits then the

restriction of the former one to $SU_{Q_1}(V_1)$ also splits. So, it is enough to prove the lemma in the case when Q is quasi-split.

2) The lemma is true when $\dim_K V = 2$ ([5]).

Consider now the case when Q is quasi-split. Denote by $\tilde{p}: \widetilde{SU} \to SU$ the restriction of p to SU_Q. $\widetilde{SU}$ is the continuous central extension of the semi-simple group. For every simple root λ of SU we denote by $\mathrm{SL}(2, k)_\lambda$ the subgroup of SU generated by the root subgroups E_λ and $E_{-\lambda}$. It follows from 2) that the restriction of $\tilde{p}$ to $\mathrm{SL}(2, k)_\lambda$ splits. So $\tilde{p}$ also splits. The lemma is proved.

As SU is the commutator subgroup of U it follows from Lemma 7 that the commutator subgroup of $p^{-1}(U)$ is isomorphic to SU (the projection p induces the isomorphism). So we will consider SU as the subgroup in $\widetilde{\mathrm{SP}}(2n, k)$.

Now let $k = \mathbf{R}$ and Q have type (s, t).

As p splits over $SU(s, t)$, there exists a unique section $j: SU(s, t) \hookrightarrow \widetilde{\mathrm{Sp}}(2n, \mathbf{R})$, and we shall henceforth consider $SU(s, t)$ as a subgroup in $\widetilde{\mathrm{Sp}}(2n, \mathbf{R})$. Let us denote by C the preimage $p^{-1}(U_1) \subset \widetilde{\mathrm{Sp}}(2n, \mathbf{R})$. Let $\alpha \in C$ be a nontrivial element in $p^{-1}(e)$, we know that C is connected for even n and isomorphic to $U_1 \times \{1, \alpha\}$ for odd n. For simplicity, we fix the additive character ψ on $\mathbf{R}$ (which we need in order to define τ) as $\psi(x) = \exp(ix)$.

We let $\hat{C}^-$ denote the set of characters χ on C such that $\chi(\alpha) = -1$.

For every $\chi \in \hat{C}^-$ let $W_\chi \subset W$ be the subspace of vectors $w \in W$ such that $\tau(c)w = \chi(c)w$ for all $c \in C$. It is clear that W_χ is invariant under the restriction of τ to $SU(s, t)$. The corresponding representation of $SU(s, t)$ in W_χ will be denoted by ρ_χ. For every $l \in \mathbf{Z}$ we denote by χ_l the following character on C: $\chi_l(u) = u^{[\frac{1}{2}(s-t)+l]}$, and we will write W_l instead of W_{χ_l}. We consider U_1 as a subgroup in $\mathbf{C}^*$. If n is even, then χ_l is a character of the connected component of l in C, and we extend it to all of C by $\chi_l(\alpha) = -1$. If n is odd, then χ_l is a correctly defined character on C and $\chi_l(\alpha) = -1$.

Proposition 2. a) $W = \bigoplus_{l \in \mathbf{Z}} W_{\chi_l}$.

b) *If* $(s, t) = (n, 0)$ *then* $W_{\chi_l} \neq 0$ *iff* $l \geqq 0$ *with* $l \in \mathbf{Z}$ *non-negative, for* $u \in U_1$. *The representation* τ_l *of* $SU(n)$ *in* W_{χ_l} *is irreducible and isomorphic to the* l*-th symmetric power of the standard representation of* SU.

c) *If* $(s, t) = (0, n)$ *then* $W_\chi \neq 0$ *iff* $\chi = \chi_{-l}$ *for some non-negative* $l \in \mathbf{Z}$. *The representation of* $SU(n)$ *in* W_{-l} *is dual to the representation in* W_l.

d) *If* $s \cdot t \neq 0$, *then* $W_l \neq 0$ *for all* $l \in \mathbf{Z}$, *and the representation* τ_l *of* $SU(s, t)$ *in* W_l *is irreducible.*

e) *If* $(s, t) = (n-1, 1)$, *then the representation of* $SU(n-1, 1)$ *in* W_1 *has the following two properties*:

I) *The restriction of* τ_1 *to the maximal compact subgroup* $U(n-1) \hookrightarrow SU(n-1, 1)$ *contains the representation* $\mathrm{St} \otimes \det$, *where* St *is the standard* $(n-1)$*-dimensional representation of* $U(n-1)$, *and* det *is the one-dimensional representation* $u \mapsto \det u$.

II) *Let* $(\tau_1)_*$ *be the corresponding representation of the universal enveloping algebra* A *of the Lie algebra* $SU(n-1,1)$, *and let* $\Delta \in A$ *be the Casimir operator. Then* $(\tau_1)_*(\Delta)=0$.

f) *In the same case* τ_l *is tempered iff* $l \leqq 1-n$. *In this case it is in the discrete series.*

g) *If* $s, t > 1$ *then* τ_χ *is not in the discrete series for any* χ.

Proof. To prove the proposition we take the Bargmann–Fock realization of the Weil representation ([4]). As our field is real we shall consider the corresponding representation of the Lie algebra $\mathfrak{g}$ of $\mathrm{Sp}(2n,\mathbf{R})$.

Let Θ be the space of all holomorphic $f:\mathbf{C}^m \to \mathbf{C}$ such that

$$\int_{\mathbf{C}^m} |f(z)|^2 l^{-|z|^2} dz < \infty$$

where $|z|^2 = \Sigma_1^n |z_j|^2$ and dz is Lebesgue measure on $\mathbf{C}^m$. Then Θ is a Hilbert space with inner product

$$(f,g) = \pi^{\ m} \int f(z)\overline{g(z)} l^{-|z|^2} dz.$$

We can consider $\mathfrak{g}$ as a space of quadratic forms on $p_1,\cdots,p_n,q_1,\cdots,q_n$ where the commutator is Poisson brackets. Then we can write the representation τ_* by

$$\tau_*(p_ap_l) = -\frac{1}{2}\left(\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} - z_b\frac{\partial}{\partial z_a} + z_az_b - \delta_{ab}\right),$$

$$\tau_*(p_aq_b) = \frac{1}{2}\left(\frac{\partial^2}{\partial z_a \partial z_b} - z_a\frac{\partial}{\partial z_b} + z_b\frac{\partial}{\partial z_a} - z_az_b\right),$$

$$\tau_*(q_aq_b) = -\frac{i}{2}\left(\frac{\partial^2}{\partial z_a \partial z_b} + z_a\frac{\partial}{\partial z_b} + z_b\frac{\partial}{\partial z_a} + z_az_b + \delta_{ab}\right) \quad \text{(see [4])}.$$

Let us consider the subspace $U_{s,t} \subset \mathfrak{g}$ $(s+t=n)$ generated by

$$\xi_{a,b} \overset{\text{def}}{=} p_ap_b + q_aq_b \quad \text{for} \quad a \leqq b \leqq s \quad \text{and} \quad s < a \leqq b$$

and

$$\eta_{a,b} = p_ap_b - q_aq_b \quad \text{for} \quad a \leqq s < b.$$

It is clear that $U_{s,t}$ is the subalgebra of $\mathfrak{g}$ corresponding to the imbedding $U(s,t) \hookrightarrow \mathrm{Sp}(2n,\mathbf{R})$.

Now it is easy to verify the propositions a)–e). The statements f), g) are proved in [4].

Next let k be any local non-archimedean field and K its quadratic extension, Q a nondegenerate Hermitian form on V and let U_Q be the corresponding unitary group. We can also define the subgroups U_1, $SU \subset U_Q$ and the corresponding subgroups C, $SU \subset \tilde{U} = p^{-1}(U) \subset \widetilde{\mathrm{Sp}}_k$.

As before, for every character $x \in \hat{C}$ we consider the subspace $W_x \subset W$ of $w \in W$ such that $\tau(c)w = \chi(c)w$, $c \in C$. W_x is invariant under $\tau(U)$.

Now let k be an absolutely real number field, K its purely imaginary quadratic extension, Q a Hermitian form on a finite dimension K-vector space V. For every valuation p on k we denote by SU_{k_p} the corresponding unitary group (if K splits in p then $SU_{k_p} \simeq \mathrm{GL}(n, k_p)$).

Let i be the natural imbedding $i: U_Q \hookrightarrow \mathrm{Sp}(2n, k)$ and $i_{\mathbf{A}}$ the corresponding imbedding $i_{\mathbf{A}}$; $U_Q(\mathbf{A}) \to \mathrm{Sp}(2n, \mathbf{A})$. Let $\tilde{U}_Q(\mathbf{A})$ (or simply $\tilde{U}_{\mathbf{A}}$) be the preimage $p^{-1}(U_Q(\mathbf{A}))$ in the two-sheeted covering $p: \widetilde{\mathrm{Sp}}_{\mathbf{A}} \to \mathrm{Sp}_{\mathbf{A}}$ (we will write $\mathrm{Sp}_{\mathbf{A}}$ and Sp_k instead of $\mathrm{Sp}(2n, \mathbf{A})$ and $\mathrm{Sp}(2n, k)$). We consider $U_Q(k)$ (or simply U_k) as a subgroup of $\tilde{U}_{\mathbf{A}}$. As we proved p splits over $SU_{\mathbf{A}}$ and we can consider $SU_{\mathbf{A}}$ as a subgroup in $\widetilde{\mathrm{Sp}}_{\mathbf{A}}$.

The restriction of the Weil representation τ to $\tilde{U}_{\mathbf{A}}$ (and to all subgroups) will also be denoted by τ.

Let $\infty_1, \cdots, \infty_l$ be the set of all infinite valuations of k. For every i, $1 \leqq i \leqq r$ let l_i be a number such that $W_{l_i} \neq \{0\}$. (We consider W_{l_i} as the subspace in W_{∞_i}.) (See Proposition 2.) We denote by L the space of smooth functions on $SU_k \backslash SU_{\mathbf{A}}$.

Proposition 3. *There exists an $SU_{\mathbf{A}}$-invariant subspace $L_0 \subset L$, $L \neq \{0\}$ such that the corresponding representation of $\prod_{i=1}^{l} SU_{\infty_i}$ (by right shifts) in L_0 is a multiple of $\otimes W_{l_i}$.*

Let us denote by $\varphi_1: (W_{\mathbf{A}})_0 \to L$ the map $\varphi_1(w)(u) = \lambda_0(\tau(u)w)$ and denote by L_0 the image of N. It follows from Proposition 1 that $L_0 \neq \{0\}$. As φ_1 commutes with $SU_{\mathbf{A}}$ then Proposition 3 is proved.

Let χ be a character on $U_{1,k} \backslash U_{1,\mathbf{A}}$. We will denote by φ_x the map from $W_{0,\mathbf{A}}$ to the space L_U of smooth functions on $SU_k \backslash SU_{\mathbf{A}}$

$$\varphi_x(w)(u) = \int_{U_{1,k} \backslash U_{1,\mathbf{A}}} (\chi(\dot{u}_1)\lambda_0(\tau(u_1 u)g)d\dot{u}_1.^*$$

We will denote by L_x the image of φ_x.

* The integral makes sense because $U_{1,k} \backslash U_{1,\mathbf{A}}$ is compact.

It is clear that L_χ is $SU_{\mathbf{A}}$ invariant subspace in L_u. It follows from some unpublished results of Howe that L_χ are irreducible subspaces in L_u.

From Proposition 3 we obtain easily

Corollary. *Let $l_1, \cdots, l_n$ be integers such that $W_{l_i} \neq \{0\}$, then there exists a character χ on $U_{1,k}\backslash U_{1,\mathbf{A}}$ such that*

a) *for every i, $1 \leqq i \leqq r$ the restriction of χ on U_{1,∞_i} is equal to χ_{l_i},*

b) $L_\chi \neq \{0\}$.

K is purely imaginary, and therefore U_{1,∞_i} is compact for every infinite place ∞_i of k.

§4. Cohomology of discrete subgroups in $SU(n-1,1)$

Let us now consider the case when $k \neq \mathbf{Q}$ the form Q has a type $(n-1,1)$ in one infinite place, say ∞_1, and is definite in all others. For the simplicity of notations we suppose that Q is positive in all these places.

We can consider SU_Q as an algebraic group. The group of real points $SU_Q(\mathbf{R})$ is isomorphic to the product $\Pi_{1\leqq i\leqq r} SU_{\infty_i} = SU(n-1,1) \times SU(n)^{r-1}$. The group of integer points P_0 is a discrete subgroup in $SU_Q(\mathbf{R})$ which we can consider as a subgroup in $SU(n-1,1)$. The quotient space $\Gamma_0\backslash SU(n-1,1)$ is compact (because Q_{∞_2} and consequently Q is anisotropic).

Theorem 1. *There exists a congruence subgroup $P_0 \subset \Gamma$ such that $H^1(\Gamma, \mathbf{R}) \neq 0$.*

Proof. We apply Proposition 3 in the case when $l_1 = 1$, $l_i = 0$, $2 \leqq i \leqq r$. As $SU_k\backslash SU_{\mathbf{A}}$ is compact we obtain the $SU_{\mathbf{A}}$ invariant subspace $L_0 \subset L^2(SU_k\backslash SU_{\mathbf{A}})$, $L_0 \neq \{0\}$ such that the corresponding representation of $SU(n-1) \times SU(n)^{r-1}$ in L_0 is multiple of $\tau_1 \otimes (\mathrm{Id})^{r-1}$. Let C be an open subgroup in $\otimes_{\Pi-\{\infty_i\}} U_p$ such that the space of L_0^C of C-invariant vectors $\neq 0$. Denote by Γ the intersection $\Gamma = SU_k \cap SU_Q(\mathbf{R}) \times C$. It follows from the strong approximation theorem [2] that $(L^C) = \{$space of smooth functions on $\Gamma\backslash SU(\mathbf{R})\}$ and L_0^C is non-zero subspace of L^C. As every element in L_0^C is invariant under $U(n)^{r-1}$ we can consider L_0^C as a $SU(n-1,1)$ invariant subspace of $L^2(\Gamma\backslash SU(n-1,1))$.* From the properties of L_0 it follows that the representation of $SU(n-1,1)$ in L_0^C is a multiple of τ_1. Now the theorem follows from Proposition 2e) and [3].

* We consider Γ simultaneously as a subgroup in $SU_{\mathbf{A}}$, $SU(\mathbf{R})$ and $SU(n-1,1) = SU_{\infty_1}$.

§5. Some examples

Now let K be as in §4, $[K:\mathbf{Q}] \geqq 3$ and Q has type $(n-1,1)$ in two points ∞_1, ∞_2 and positive in all others. If we apply Proposition 3 in the case $l_1 > 1-n$ and $l_2 \leqq 1-n$, $l_i = 0$ for $i > 2$ then by the same arguments as before we obtain

Theorem 2. *There exists a discrete subgroup* $\Gamma \subset SU(n-1,1) \times SU(n-1)$ *and* $V \subset L^2(\Gamma \backslash (SU(n-1,1) \times SU(n-1,1))$ *such that* 1) *the quotient space is compact,* 2) *the projection of* Γ *on every component is dense,* 3) V *is* $SU(n-1,1) \times SU(n-1,1)$ *invariant subspace,* 4) V *is irreducible*; $V = V_1 \otimes V_2$ *where* V_1 *and* V_2 *are irreducible representations of* $SU(n-1,1)$, 5) V_1 *is not tempered and* V_2 *is in the discrete series.*

Now we consider the case when $\dim V = 3$ and Q is isotropic (so quasi-split). Let χ be a character such that $L_\chi \neq 0$.

Theorem 3. a) $L_\chi \subset L^2(SU_k \backslash SU_{\mathbf{A}})$.

b) *If* $\chi_{\mathbf{R}} = \chi_l$ *for* $l < 0$ *then* L_χ *is a cuspidal subspace.*

c) *For every* $l \geqq 0$ *there exists a character* χ *on* $U_k \backslash U_{1,\mathbf{A}}$ *such that* $\chi_{\mathbf{R}} = \chi_l$ *and* $L_\chi \neq \{0\}$ *and is not in the space of cuspidal forms.*

In our case, this is equivalent to the form

$$Q(X, Y, Z; X_1, Y_1, Z_1) = X\bar{Z}_1 + Z\bar{X}_1 + Y\bar{Y}_1$$

where X, Y, Z are coordinates corresponding to a basis $l_1, l_2, l_3 \in V$ and $\bar{\ }$ is the automorphism of K over k. The split torus T of U is isomorphic to the multiplicative group. It acts on V in the following way: $l_1 \to tl_1, l_2 \to l_2, l_3 \to t^{-1}l_3$ where t is a natural parameter on T. The unipotent subgroup $N \subset SU$ is isomorphic to the group of matrices

$$\begin{pmatrix} 1 & \beta & \gamma \\ 0 & 1 & -\bar{\beta} \\ 0 & 0 & 1 \end{pmatrix}$$

where $\beta \in K$ and $\operatorname{Re} \gamma = -N(\beta)/2$ in the basis (l_1, l_2, l_3). Let $\alpha \in K$, $\alpha \neq 0$, be an element such that $\bar{\alpha} = -\alpha$. Then $(l_1, l_2, l_3, \alpha l_1, \alpha l_2, \alpha l_3)$ is a basis of V over k. Let $M, M' \subset V$ be the subspaces generated by $(l_1, \alpha l_1, l_2)$ and $(l_3, \alpha l_3, \alpha l_2)$. It is clear that they are isotropic subspaces for $\operatorname{Im} Q$. So we can realize $W_{\mathbf{A}}$ as the space of Schwartz–Bruhat functions $S(M_{\mathbf{A}})$ on $M_{\mathbf{A}}$, and we can easily describe the action of

$\tau(T)$ and $\tau(N)$ on $W_{\mathbf{A}}$. In particular, if (a_1, a_2, a_3) are coordinates in $M_{\mathbf{A}}$ corresponding to the basis $(l_1, \alpha l_1, l_2)$, then

$$(\tau(t)\varphi)(a_1, a_2, a_3) = |t|^{-1}\varphi(t^{-1}, a_1, t^{-1}a_2, a_3), \qquad t \in T,$$

(1)

$$\left(\tau\begin{pmatrix}1 & 0 & \gamma\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\varphi\right)(a_1, a_2, a_3) = \psi\left(\frac{(a_1^2 + k_0a_2^2)\gamma}{\alpha}\right)\varphi(a_1, a_2, a_3),$$

for $\gamma \in K$, $\bar{\gamma} = -\gamma$ where $k_0 = \alpha^2 \in k$,

(2)
$$\left(\tau\begin{pmatrix}1 & \beta & -\beta\bar{\beta}/2\\ 0 & 1 & -\beta\\ 0 & 0 & 1\end{pmatrix}\varphi\right)(a_1, a_2, a_3)$$

$$= \psi(B_\beta(a_1, a_2, a_3))\varphi(a_1 - \operatorname{Re}\beta a_3, a_2 - k_0\operatorname{Re}\beta a_3, a_3)$$

where B_β is a quadratic form which is easy to write explicitly. We will use the following fact:

(3)
$$B_\beta(0,0,1) \equiv 0.$$

Now let $w \in W_{\mathbf{A}}$ be a function on $M_{\mathbf{A}}$. From (1) it follows that $|\varphi_w(t)| = |\lambda_0(\tau(t)w)| < C_{wt}$. On the other hand we know ([1]) that we can write $SU_{\mathbf{A}}$ as the product of SU_k and a Siegel domain $S = T \cdot N^0T_rC$, where T_{r_0} is the subset in $T_{\mathbf{A}}$ consisting of t with $|t| > r_0$, $N^0 \subset N_{\mathbf{A}}$ is the set of elements with $|\beta|, |\gamma| \leqq 1$, and C is the compactum. The right invariant measure on S is $dndt \cdot dc/|t|^5$. So

$$\int_{SU_k\backslash \widetilde{SU_{\mathbf{A}}}} |\varphi_w(u)|^2 du \leqq \int_{u\in S} |\varphi_w(u)|^2 du \leqq C'_w \int_{r_0}^{\infty} \frac{dr}{r^3} < \infty.$$

Now we have to determine when $L_\chi \subset L^2$ is cuspidal.

For this we have to understand when

$$\int_{U_{1,k}\backslash \tilde{U}_{1,\mathbf{A}}} \chi(u)\tau(u) \int_{N_k\backslash N_{\mathbf{A}}} \tau(n)\lambda_0 dndu_1 \neq 0.$$

From the formulas (2) and (3) it follows that $\lambda_N \stackrel{\text{def}}{=} \int_{N_k\backslash N_{\mathbf{A}}} \tau(n)\lambda_0 dn$ is the following functional on $S(M_{\mathbf{A}})$:

$$\lambda_N(\varphi) = \sum_{a\in k} \varphi(0,0,a).$$

$M_A = K_A + \mathbf{A}$, and therefore $S(M_A) = S(K_A) \otimes S(\mathbf{A})$. This decomposition agrees with the actions of U_1 on $S(K_A)$ and $S(\mathbf{A})$ (the sections come from the decomposition $V = V_1 \oplus V_2$, where $V_1 = (l_1, l_3)$, $V_2 = (l_2)$, which is invariant under U_1). Moreover, $\lambda_N = \delta \otimes \lambda_0^1$, where $\delta(\varphi_1) = \varphi_1(0)$ for $\varphi_1 \in S(K_A)$ and $\lambda_0^1(\varphi_2) = \Sigma_{a\in k}\varphi_2(a)$ for $\varphi_2 \in S(\mathbf{A})$. It is clear that δ is invariant under the action of U_1. So Theorem 3 now follows from Lemma 6 and Proposition 1.

Remark. To prove that $L \neq 0$ and is not cuspidal for a character χ on $U_k\backslash U_{1A}$, it clearly suffices to check that λ_χ^1 is not zero in the case when dim $V = 1$.

Corollary. *There exists a subgroup* $\Gamma \subset SU(2,1)$ *and a subspace* $L_\chi^0 \subset L^2(\Gamma\backslash SU(2,1))$ *such that*

1) L_χ^0 *is SU-invariant, and the corresponding representation of SU in* L_χ *is irreducible and does not lie in the principal series.*

2) L_χ^0 *is not cuspidal.*

Moreover, we may choose Γ to be in a congruence subgroup of Γ_0. Here Γ_0 is the group of all matrices in $\mathrm{SL}(3, Q(i))$ which preserve the Hermitian form $Z_1\bar{Z}_3 + Z_3\bar{Z}_1 + Z_2\bar{Z}_2$ on $\mathbf{C}^3$.

Proof. We take $k = \mathbf{Q}$, $K = \mathbf{Q}(i)$, $\chi_{\mathbf{R}} = \chi_l$, $l \geqq 0$. Let χ be a character on $U_{1,A}$ which satisfies the conditions of Theorem 3.

Let L_χ be the corresponding subspace in $L^2(SU_{\mathbf{Q}}\backslash SU_A)$. Let $\mathbf{C}$ be an open compact subgroup in $\Pi_{p\in\pi-\infty} V_p$ such that $L_\chi^l \neq 0$. Moreover, we suppose that there exists $\varphi(l) \neq 0$.

The space L_χ^l is invariant under the action of $SU_{\mathbf{R}} \simeq SU(2,1)$, and is isomorphic as $SU(2,1)$-module to $n\pi_l$, where $n \in \mathbf{Z}$, and π_l is the representation of $SU(2,1)$ corresponding to χ_l. As $l \leqq -3$, π_l is in the discrete series. Let $L_\chi^1 \subset L_\chi^l$ be an irreducible $SU(2,1)$-submodule which contains a function φ such that $\varphi(l) \neq 0$. Then $\Gamma = p^{-1}(C) \cap U_Q \subset U_{\mathbf{R}} = SU(2,1)$ and $L_\chi^0 = \{$restriction of L_χ^1 to $U_{\mathbf{R}} \subset \tilde{U}_A\}$ satisfies the conditions of the Corollary.

I want to thank Professor Sternberg, who showed me the preprint [4] which helped me very essentially, Professor Sah and Professor Howe, for very helpful discussions, and Professor Koblitz, who helped me to write this article.

References

1. A. Borel, *Reduction theory*, Proc. Sym. Pure Math. IX, Amer. Math. Soc., Providence, Rhode Island, 1966.

2. M. Kneser, *Strong approximation*, Proc. Sym. Pure Math. IX, Amer. Math. Soc., Providence, Rhode Island, 1966.

3. H. Matsushima, *A formula for Betti numbers of compact locally symmetric Riemannian manifolds*, J. Differential Geometry **1** (1967), 99–109.

4. Shlomo Sternberg and Joseph Wolf, preprint.

5. S. Tanaka, *On irreducible unitary representations of some special linear groups of the second order*, Osaka J. Math., 1966.

6. A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. **111** (1964), 143–211.

DEPARTMENT OF MATHEMATICS
HARVARD UNIVERSITY
CAMBRIDGE, MASS. 02138 USA

(Received March 3, 1977)