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REPRESENTATIONS OF TENSOR CATEGORIES WITH FUSION RULES OF SELF-DUALITY FOR ABELIAN GROUPS

BY

D. TAMBARA

Department of Mathematical System Science
Hirosaki University, Hirosaki 0368561, Japan
e-mail: tambara@cc.hirosaki-u.ac.jp

ABSTRACT

The tensor categories with fusion rules of self-duality for abelian groups are modeled on the representations of extraspecial 2-groups. We classify the embeddings of those categories into the category of vector spaces, by which the categories are realized as the representations of Hopf algebras.

Introduction

In order to distinguish the dihedral group D_8 and the quaternion group Q_8 by their categories of representations, Yamagami and the author studied in [3] semisimple tensor categories over a field k having the simple object set $A \cup \{m\}$, where A is a finite abelian group and m is another object, and having the fusion rule

$$a \otimes b \cong ab,$$
$$a \otimes m \cong m, \quad m \otimes a \cong m,$$
$$m \otimes m \cong \bigoplus_{a \in A} a$$

for $a, b \in A$. Instances are the categories of representations of the two kinds of extraspecial 2-groups and the 8-dimensional Hopf algebra H_8 of Kac and Paljutkin. The classification in [3] tells us that those tensor categories are parameterized by pairs (χ, τ) of nondegenerate symmetric bicharacters (bimultiplicative maps) $\chi\colon A \times A \to k^\times$ and square roots $\tau \in k$ of $|A|^{-1}$. We will denote the corresponding category by $\mathcal{C}(A, \chi, \tau)$.

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We are concerned with the question when $\mathcal{C}(A,\chi,\tau)$ can be realized as the category H-mod of representations of a Hopf algebra H, and in how many different ways. The answer will produce examples of Hopf algebra deformations of extraspecial 2-groups and also nonisomorphic Hopf algebras having the isomorphic categories of representations.

By Tannaka theory, to give a tensor category equivalence $\mathcal{C}(A,\chi,\tau) \simeq H$-mod with H a Hopf algebra is the same thing as to give a tensor functor from $\mathcal{C}(A,\chi,\tau)$ to the category of vector spaces $\mathcal{V}$ ([4]). Moreover, given two equivalences $\mathcal{C}(A,\chi,\tau) \simeq H$-mod and $\mathcal{C}(A,\chi,\tau) \simeq H'$-mod, the Hopf algebras H and H' are isomorphic if and only if the two corresponding functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ differ only by an automorphism of $\mathcal{C}(A,\chi,\tau)$. Thus our problem may be formulated as follows: Classify tensor functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ up to automorphisms of $\mathcal{C}(A,\chi,\tau)$.

Our main result gives a one-to-one correspondence between isomorphism classes of tensor functors and group theoretic invariants: pairs (σ,μ) of involutive automorphisms σ of A and quadratic forms μ on certain subquotients of A accompanied with σ (Theorem 3.5).

Using this, we settle the problem in the case where A has an odd order and the case where A is an elementary abelian 2-group. When $|A|$ is odd, there exists a tensor functor $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ if and only if $|A|$ is a square, $\tau > 0$, and χ is hyperbolic in a sense as in the theory of quadratic forms. And if it exists, it is unique up to automorphisms of $\mathcal{C}(A,\chi,\tau)$ (Proposition 4.2). The corresponding Hopf algebra is explicitly described.

When A is an elementary abelian 2-group, there exist several tensor functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ which are inequivalent under the automorphism group of $\mathcal{C}(A,\chi,\tau)$. For instance, if $|A| = 2^{2r}$, χ is alternating, and $\tau > 0$, then the number of inequivalent tensor functors is $[r/2]+1$ (Proposition 5.5). Just as the extraspecial 2-groups are central products of D_8 and Q_8, the Hopf algebras arising in the case where A is 2-elementary are composed of four Hopf algebras: the 8-dimensional algebras D_8, Q_8, H_8 and a 16-dimensional one.

In Section 1 we recall the definition of $\mathcal{C}(A,\chi,\tau)$ and determine its automorphism group. In Section 2 we collect some preparatory material. The main result is proved in Section 3. The two special cases are dealt with in Sections 4 and 5.

We work over an algebraically closed field k of characteristic zero. All vector spaces are finite dimensional. For a finite abelian group A, denote $X(A) = \mathrm{Hom}(A,k^\times)$, the dual of A. A bicharacter $\chi\colon A\times A \to k^\times$ is said to be alternating if $\chi(a,a)=1$ for all $a \in A$.

1. The tensor category $\mathcal{C}(A, \chi, \tau)$

A tensor category is a monoidal category $\mathcal{C}$ in which Hom-sets are k-vector spaces, the operations of compositions and tensor products for morphisms are k-linear, and finite direct sums exist. The monoidal structure of $\mathcal{C}$ is specified by the tensor product functor $\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, the unit object I, the associativity isomorphisms $\mathbf{a}_{X,Y,Z}: (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$, and the unit isomorphisms $\mathbf{l}_X: I \otimes X \to X$, $\mathbf{r}_X: X \otimes I \to X$ ([2]). A tensor functor $\mathcal{C} \to \mathcal{C}'$ between tensor categories $\mathcal{C}$ and $\mathcal{C}'$ consists of a k-linear functor $F: \mathcal{C} \to \mathcal{C}'$, isomorphisms $t_{X,Y}: F(X) \otimes F(Y) \to F(X \otimes Y)$ for all $X, Y \in \mathcal{C}$ and an isomorphism $u: I \to F(I)$ such that $t_{X,Y}$ are natural in X, Y and the following diagrams are commutative.

$$\text{(F1)} \qquad \begin{array}{ccc} (F(X) \otimes F(Y)) \otimes F(Z) & \xrightarrow{\mathbf{a}_{F(X),F(Y),F(Z)}} & F(X) \otimes (F(Y) \otimes F(Z)) \\ {\scriptstyle t_{X,Y} \otimes 1}\downarrow & & \downarrow{\scriptstyle 1 \otimes t_{Y,Z}} \\ F(X \otimes Y) \otimes F(Z) & & F(X) \otimes F(Y \otimes Z) \\ {\scriptstyle t_{X \otimes Y,Z}}\downarrow & & \downarrow{\scriptstyle t_{X,Y \otimes Z}} \\ F((X \otimes Y) \otimes Z) & \xrightarrow[F(\mathbf{a}_{X,Y,Z})]{} & F(X \otimes (Y \otimes Z)) \end{array}$$

$$\text{(F2)} \qquad \begin{array}{ccc} I \otimes F(X) & \xrightarrow{u \otimes 1} & F(I) \otimes F(X) \\ {\scriptstyle \mathbf{l}_{F(X)}}\downarrow & & \downarrow{\scriptstyle t_{I,X}} \\ F(X) & \xleftarrow[F(\mathbf{l}_X)]{} & F(I \otimes X) \end{array}$$

$$\text{(F3)} \qquad \begin{array}{ccc} F(X) \otimes I & \xrightarrow{1 \otimes u} & F(X) \otimes F(I) \\ {\scriptstyle \mathbf{r}_{F(X)}}\downarrow & & \downarrow{\scriptstyle t_{X,I}} \\ F(X) & \xleftarrow[F(\mathbf{r}_X)]{} & F(X \otimes I) \end{array}$$

If F is an equivalence, the tensor functor (F, t, u) is called a tensor equivalence.

For two tensor functors $(F, t, u), (F', t', u'): \mathcal{C} \to \mathcal{C}'$, a morphism $(F, t, u) \to (F', t', u')$ is a natural transformation $q: F \to F'$ making the following diagrams commute.

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{q_X \otimes q_Y} & F'(X) \otimes F'(Y) \\ {\scriptstyle t_{X,Y}}\downarrow & & \downarrow{\scriptstyle t'_{X,Y}} \\ F(X \otimes Y) & \xrightarrow[q_{X \otimes Y}]{} & F'(X \otimes Y) \end{array} \qquad \begin{array}{ccc} I & \xrightarrow{u} & F(I) \\ \| & & \downarrow{\scriptstyle q_I} \\ I & \xrightarrow[u']{} & F'(I) \end{array}$$

Hence we have the notion of isomorphisms between tensor functors.

The set of isomorphism classes of tensor equivalences $\mathcal{C} \to \mathcal{C}$ is a group under composition, which we denote by $\overline{\mathrm{Aut}}\,\mathcal{C}$.

Now we recall from [3] the definition of the tensor category $\mathcal{C}(A, \chi, \tau)$ associated with a finite abelian group A, a nondegenerate symmetric bicharacter $\chi\colon A \times A \to k^\times$, and a square root $\tau \in k$ of $|A|^{-1}$. Make the disjoint union $S = A \cup \{m\}$ of A and a one-point set $\{m\}$. Objects are finite direct sums of elements of S. Hom-spaces are given by

$$\mathrm{Hom}(X, Y) = \begin{cases} k1_X & \text{if } X = Y, \\ 0 & \text{if } X \neq Y, \end{cases}$$

for $X, Y \in S$. Tensor products of elements of S are given by

$$a \otimes b = ab, \quad a \otimes m = m, \quad m \otimes a = m, \quad m \otimes m = \bigoplus_{a \in A} a$$

for $a, b \in A$, and the unit object is 1. The associativity isomorphisms $\mathbf{a}_{X,Y,Z}\colon (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ for $X, Y, Z \in S$ are given by

$$\begin{aligned}
&\mathbf{a}_{a,b,c} = 1_{abc}\colon && abc \to abc \\
&\mathbf{a}_{a,b,m} = \mathbf{a}_{m,a,b} = 1_m\colon && m \to m \\
&\mathbf{a}_{a,m,b} = \chi(a,b)1_m\colon && m \to m \\
&\mathbf{a}_{a,m,m} = \mathbf{a}_{m,m,a} = \bigoplus_{b \in A} 1_b\colon && \bigoplus_{b \in A} b \to \bigoplus_{b \in A} b \\
&\mathbf{a}_{m,a,m} = \bigoplus_{b \in A} \chi(a,b)1_b\colon && \bigoplus_{b \in A} b \to \bigoplus_{b \in A} b \\
&\mathbf{a}_{m,m,m} = (\tau\chi(a,b)^{-1}1_m)_{a,b}\colon && \bigoplus_{b \in A} m \to \bigoplus_{a \in A} m
\end{aligned}$$

for $a, b, c \in A$. The unit isomorphisms $\mathbf{l}_X\colon 1 \otimes X \to X$, $\mathbf{r}_X\colon X \otimes 1 \to X$ for $X \in S$ are identity maps.

Our object is to classify tensor functors from $\mathcal{C}(A, \chi, \tau)$ to the category of vector spaces $\mathcal{V}$. Before going into the task, let us first determine the group $\overline{\mathrm{Aut}}\,\mathcal{C}(A, \chi, \tau)$.

Let $\mathrm{Aut}(A, \chi)$ denote the group of automorphisms of the pair (A, χ), that is, automorphisms of A preserving χ. We have an obvious homomorphism $\mathrm{Aut}(A, \chi) \to \overline{\mathrm{Aut}}\,\mathcal{C}(A, \chi, \tau)$.

PROPOSITION 1: *The map* $\mathrm{Aut}(A,\chi) \to \overline{\mathrm{Aut}}\,\mathcal{C}(A,\chi,\tau)$ *is an isomorphism.*

Proof: Let (F,t,u) be a tensor self-equivalence of $\mathcal{C}(A,\chi,\tau)$. Clearly $F(a) \in A$ for all $a \in A$ and $F(m) = m$. Let $f\colon A \to A$ be the map $a \mapsto F(a)$. f is a group automorphism. In view of (F1) for $(X,Y,Z) = (a,m,b)$, one sees that f preserves χ. Thus $f \in \mathrm{Aut}(A,\chi)$. The map $(F,t,u) \mapsto f$ yields a homomorphism $\overline{\mathrm{Aut}}\,\mathcal{C}(A,\chi,\tau) \to \mathrm{Aut}(A,\chi)$, which is a right inverse to the map in question.

It remains to show that if $f = \mathrm{id}$, then (F,t,u) is isomorphic to the identity functor. In this case we can write

$$\begin{aligned} t_{a,b} &= \theta(a,b)1_{ab}, \\ t_{a,m} &= \phi(a)1_m, \\ t_{m,a} &= \psi(a)1_m, \\ t_{m,m} &= \bigoplus_a \omega(a)1_a, \\ u &= \lambda 1_1, \end{aligned}$$

for $a,b \in A$ with $\theta(a,b), \phi(a), \psi(a), \omega(a), \lambda \in k^\times$. Commutativity of (F1) for $(X,Y,Z) = (m,a,m)$, (a,b,m), (a,m,m), and that of (F2) for $X = m$ amount to the equations

$$\phi = \psi, \quad \theta = \partial\phi, \quad \omega\phi = \text{const}, \quad \lambda\phi(1) = 1,$$

where ∂ is the coboundary operator. Put $c = \omega(1)\phi(1)$. Define an isomorphism of functors $q\colon F \to \mathrm{id}$ by $q_a = \phi(a)1_a$ for $a \in A$ and $q_m = \sqrt{c}1_m$. Then one verifies that q is an isomorphism of tensor functors $(F,t,u) \to \mathrm{id}$. ∎

2. Preliminaries

In this section we collect some auxiliary definitions and properties concerning (1) involutions of matrix algebras, (2) strongly A-graded algebras, (3) the Schur multiplier of an abelian group, (4) quadratic forms over $\mathbb{F}_2$, and (5) symmetric bilinear forms over $\mathbb{F}_2$. Most of them are standard and well-known.

(1) INVOLUTIONS OF MATRIX ALGEBRAS.

Let V be a k-vector space and $R = \mathrm{End}\,V$. Let $f\colon R \to R$ be an algebra anti-automorphism. If $f^2 = 1$, f is called an involution. We begin with an elementary fact:

PROPOSITION 2.1: *There is a nonzero bilinear form $\gamma\colon V \times V \to k$, unique up to scalar, such that*

$$\gamma(x(v), v') = \gamma(v, f(x)(v'))$$

for all $x \in R$, $v, v' \in V$. And γ is nondegenerate. If $f^2 = 1$, then γ is either symmetric or alternating.

Proof: Define $f'\colon \operatorname{End} V \to \operatorname{End} V^*$ by $f'(x) = f(x)^*$, where $*$ stands for the dual. This is an algebra automorphism, hence there exists an isomorphism $g\colon V \to V^*$ such that $f'(x) = gxg^{-1}$. Such g is unique up to scalar. Let $\gamma(v, v') = \langle g(v), v' \rangle$. Then

$$\gamma(x(v), v') = \langle g(x(v)), v' \rangle = \langle f'(x)(g(v)), v' \rangle = \langle g(v), f(x)(v') \rangle = \gamma(v, f(x)(v')).$$

Assume $f^2 = 1$. If γ^T denotes the transpose of γ, i.e., $\gamma^T(v, v') = \gamma(v', v)$, then

$$\gamma^T(x(v), v') = \gamma(v', x(v)) = \gamma(v', ff(x)(v)) = \gamma(f(x)(v'), v) = \gamma^T(v, f(x)(v')).$$

Thus γ^T has the same property as γ. Hence $\gamma^T = c\gamma$ with $0 \neq c \in k$. Clearly $c = \pm 1$. ∎

Suppose $f^2 = 1$.

Definition 2.2:

$$\operatorname{sgn}(f) = \begin{cases} +1 & \text{if } \gamma \text{ is symmetric,} \\ -1 & \text{if } \gamma \text{ is alternating.} \end{cases}$$

LEMMA 2.3:

$$\operatorname{trace}(f\colon R \to R) = \operatorname{sgn}(f) \dim V.$$

Proof: If γ is symmetric, we can take an orthonormal basis $\{e_1, \ldots, e_n\}$ of V. With identification $R = M_n(k)$, f is just the transpose $x \mapsto x^T$. Hence $\operatorname{trace}(f) = n$.

If γ is alternating, we can take a basis $\{e_1, \ldots, e_{2m}\}$ of V for which γ is given by the matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Then $f(x) = J^{-1}x^T J$. If we write

$$x = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix},$$

then

$$J^{-1}x^T J = \begin{pmatrix} x_4^T & -x_2^T \\ -x_3^T & x_1^T \end{pmatrix}.$$

It follows that $\operatorname{trace}(f) = -2m$. ∎

(2) STRONGLY A-GRADED ALGEBRAS.

Let A be a finite abelian group. A strongly graded A-algebra is a k-algebra R with decomposition $R = \bigoplus_{a\in A} R_a$ such that $\dim R_a = 1$, $R_a R_b = R_{ab}$ for all $a, b \in A$, and $1 \in R_1$. Take $0 \neq x_a \in R_a$ for each $a \in A$. Then

$$x_a x_b = \xi(a,b) x_{ab}$$

with $\xi(a,b) \in k^\times$, and ξ: $A \times A \to k^\times$ is a 2-cocycle. Another choice of a basis $\{x'_a\}_{a\in A}$ yields a 2-cocycle ξ' cohomologous to ξ. Conversely, any 2-cocycle determines a strongly A-graded algebra. Thus we have a bijection between the set of isomorphism classes of strongly A-graded algebras and the Schur multiplier $H^2(A, k^\times)$.

The following is known.

PROPOSITION 2.4: *Any strongly A-graded algebra is semisimple.*

Proof: A strongly A-graded algebra is a quotient of the group algebra of a finite central extension of A. ∎

The commutator form of a strongly A-graded algebra R is the function α: $A \times A \to k^\times$ defined by

$$xy = \alpha(a,b) yx \quad \text{for } x \in R_a,\ y \in R_b.$$

This is multiplicative in each variable and alternating. If ξ is as above, then

$$\alpha(a,b) = \frac{\xi(a,b)}{\xi(b,a)}.$$

The center $Z(R)$ of R is the sum of R_a for $a \in A$ such that $\alpha(a, A) = 1$. Thus $Z(R) = k$ if and only if the form α is nondegenerate. By virtue of the semisimplicity, we know

PROPOSITION 2.5: *A strongly A-graded algebra is simple if and only if its commutator form is nondegenerate.*

(3) THE SCHUR MULTIPLIER OF AN ABELIAN GROUP.

For a finite abelian group A let us denote

$$X^2(A) = \{\text{bicharacters } A \times A \to k^\times\},$$
$$X^2_s(A) = \{\text{symmetric bicharacters of } A\},$$
$$X^2_a(A) = \{\text{alternating bicharacters of } A\},$$

and

$$\begin{aligned} Z^2(A) &= \{\text{2-cocyles } A\times A\to k^\times\},\\ B^2(A) &= \{\text{2-coboundaries } A\times A\to k^\times\},\\ H^2(A) &= Z^2(A)/B^2(A). \end{aligned}$$

We have $X^2(A)\subset Z^2(A)$. And the map alt: $Z^2(A)\to X^2_a(A)$ is defined by

$$\mathrm{alt}(\xi)(a,b)=\frac{\xi(a,b)}{\xi(b,a)}\quad \text{for } \xi\in Z^2(A),\ a,b\in A.$$

The following is known.

PROPOSITION 2.6: *The inclusion $X^2(A)\subset Z^2(A)$ and the map* alt: $Z^2(A)\to X^2_a(A)$ *induce isomorphisms*

$$X^2(A)/X^2_s(A)\cong H^2(A)\cong X^2_a(A).$$

Proof: That $\mathrm{alt}(\xi)\in X^2_a(A)$ if $\xi\in Z^2(A)$ follows from that $\mathrm{alt}(\xi)$ is the commutator form of the strongly A-graded algebra associated with ξ. And clearly $\mathrm{alt}(\xi)=1$ if $\xi\in B^2(A)$, so alt induces the map $H^2(A)\to X^2_a(A)$. If $\xi\in Z^2(A)$ and $\mathrm{alt}(\xi)=1$, then the central extension of A defined by ξ is abelian. But $k^\times$ being a divisible group, every abelian extension of A by $k^\times$ splits. So ξ is a coboundary. Thus $H^2(A)\to X^2_a(A)$ is injective. This shows also $X^2_s(A)\subset B^2(A)$, hence the map $X^2(A)/X^2_s(A)\to H^2(A)$ is defined.

Finally, that the map alt: $X^2(A)/X^2_s(A)\to X^2_a(A)$ is an isomorphism can be seen by decomposing A into a direct sum of cyclic groups and using the fact that $X^2_a(A)=1$ and $X^2(A)=X^2_s(A)$ if A is cyclic. ∎

(4) QUADRATIC FORMS OVER $\mathbb{F}_2$.

Let $F=\mathbb{F}_2$, the field with two elements.

Definition 2.7: A quadratic form on an F-vector space V is a map q: $V\to F$ such that the map b: $(x,y)\mapsto q(x)+q(y)-q(x+y)$ is a bilinear form on V. We say q is nondegenerate if b is.

As b is alternating, its nondegeneracy implies that $\dim V$ is even. The standard definition of the nondegeneracy of q is different from ours, but both coincide when V is even dimensional.

The following classification of quadratic forms is well-known ([1]).

PROPOSITION 2.8: *Let* $\dim V = 2m > 0$. *The set of nondegenerate quadratic forms on* V *is divided into two orbits under the action of* $\operatorname{Aut} V$. *With coordinates* $x_1, \ldots, x_{2m}$, *representatives of the orbits are given by*

$$q(x_1, \ldots, x_{2m}) = \sum_{i=1}^{m} x_{2i-1}x_{2i}$$

and

$$q(x_1, \ldots, x_{2m}) = \sum_{i=1}^{m} x_{2i-1}x_{2i} + x_1^2 + x_2^2.$$

Definition 2.9: We define the signature of a nondegenerate quadratic form q to be

$$\operatorname{sgn}(q) = \begin{cases} +1 & \text{if } q \text{ belongs to the first orbit,} \\ -1 & \text{if } q \text{ belongs to the second orbit.} \end{cases}$$

When $V = 0$, we set $\operatorname{sgn}(q) = +1$ for the zero form q.

The following will be used in Section 3.

LEMMA 2.10: *Let* $q\colon V \to F$ *be a nondegenerate quadratic form. Then*

$$\sum_{v \in V} (-1)^{q(v)} = \sqrt{|V|}\operatorname{sgn}(q).$$

Proof: Follows from

$$\sum_{x,y=0,1} (-1)^{xy} = 1 + 1 + 1 - 1 = +2,$$
$$\sum_{x,y=0,1} (-1)^{xy+x^2+y^2} = 1 - 1 - 1 - 1 = -2.$$

(5) SYMMETRIC BILINEAR FORMS OVER $\mathbb{F}_2$.

Here we concern ourselves with symmetric bilinear forms on vector spaces over the two-element field $F = \mathbb{F}_2$. The results will be used in Section 5.

Let $\mathcal{S}$ be the category whose objects are pairs (V, b) of F-vector spaces V and nondegenerate symmetric bilinear forms $b\colon V \times V \to F$, and morphisms are isomorphisms in an obvious sense. Direct sums of objects are defined in the usual way. Two special objects are

$$L = (F, (1)), \quad H = (F^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}),$$

where we express bilinear forms by matrices.

The following classification will be known.

PROPOSITION 2.11: *Every object M of $\mathcal{S}$ is isomorphic to $H^r \oplus L^k$ with $r \geq 0$, $k = 0, 1, 2$ uniquely determined by M.*

Proof: Let $(V, b) \in \mathcal{S}$. Suppose first that b is alternating. Then b is expressed by the matrix

$$\begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix},$$

where I_r is the identity matrix of degree r. Hence $(V, b) \cong H^r$.

Suppose next that b is not alternating. Then $q\colon v \mapsto b(v, v)$ is a nonzero linear form $V \to F$. Let $V_0 = \operatorname{Ker} q$. Then b is alternating on V_0. Let W be the radical of $b|V_0$:

$$W = \{x \in V_0 |\ b(x, V_0) = 0\} = V_0 \cap V_0^{\perp}.$$

As $\dim(V/V_0) = 1$, we have $\dim V_0^{\perp} = 1$, hence either $\dim W = 0$ or 1. If $W = 0$, then $b|V_0$ is nondegenerate alternating and $V = V_0 \oplus V_0^{\perp}$. Thus

$$(V, b) \cong H^r \oplus L$$

with $r = \frac{1}{2} \dim V_0$.

Assume $W = Fw \neq 0$. Take a complement V_1 of W in V_0. Then $b|V_1$ is nondegenerate and $V = V_1 \oplus V_1^{\perp}$. Take $v \in V_1^{\perp} \setminus W$ so that $V_1^{\perp} = \langle w, v \rangle$. Then

$$b(w, w) = 0, \quad b(w, v) \neq 0, \quad b(v, v) \neq 0.$$

Thus $b|V_1^{\perp}$ can be expressed by the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

But this is similar to I_2, because

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Thus $(V_1^{\perp}, b) \cong L^2$. Hence $(V, b) \cong H^r \oplus L^2$ with $r = \frac{1}{2}(\dim V_0 - 1)$. ∎

COROLLARY 2.12: $L^3 \cong H \oplus L$.

COROLLARY 2.13: *Let $M, M' \in \mathcal{S}$.*

(i) $M \oplus H \cong M' \oplus H \implies M \cong M'$.

(ii) $M \oplus M \cong M' \oplus M' \implies M \cong M'$.

Proof: (i) Clear from the classification. (ii) Since

$$\begin{aligned} M \cong H^r & \implies M^2 \cong H^{2r}, \\ M \cong H^r \oplus L & \implies M^2 \cong H^{2r} \oplus L^2, \\ M \cong H^r \oplus L^2 & \implies M^2 \cong H^{2r+1} \oplus L^2, \end{aligned}$$

M can be recovered from M^2. ∎

3. Classification of tensor functors

In this section we classify tensor functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ by means of group theoretic invariants. We firstly interpret tensor functors in terms of simple A-graded algebras R and involutions $f\colon R \to R$. Then we relate R and f to involutions $\sigma\colon A \to A$ and quadratic forms μ on certain subquotients of A.

Let $F\colon \mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ be a tensor functor with structure maps

$$t_{X,Y}\colon F(X) \otimes F(Y) \xrightarrow{\sim} F(X \otimes Y), \quad u\colon k \xrightarrow{\sim} F(1).$$

Put $F(a) = R_a$ for $a \in A$ and $F(m) = M$. The equalities

$$\begin{aligned} a \otimes b &= ab, \\ a \otimes m &= m, \\ m \otimes a &= m, \\ m \otimes m &= \bigoplus_{a \in A} a \end{aligned}$$

in $\mathcal{C}(A,\chi,\tau)$ combined with $t_{X,Y}$ give rise to isomorphisms

(i1)	$R_a \otimes R_b \cong R_{ab}$	$x \otimes y \mapsto xy$,
(i2)	$R_a \otimes M \cong M$	$x \otimes w \mapsto x.w$,
(i3)	$M \otimes R_a \cong M$	$w \otimes x \mapsto w.x$,
(i4)	$M \otimes M \cong \bigoplus_{a \in A} R_a$	$w \otimes w' \mapsto ([w,w']_a)_a$

in $\mathcal{V}$, where the maps are denoted as in the right sides.

Commutativity of (F1) for $X, Y, Z \in A \cup \{m\}$ amounts to the following:

$$\begin{aligned}
&(1) && x(yz) = (xy)z && \text{for } x \in R_a,\ y \in R_b,\ z \in R_c,\\
&(2) && x.(y.w) = (xy).w && \text{for } x \in R_a,\ y \in R_b,\ w \in M,\\
&(3) && x.(w.y) = \chi(a,b)(x.w).y && \text{for } x \in R_a,\ y \in R_b,\ w \in M,\\
&(4) && w.(x.y) = (w.x).y && \text{for } x \in R_a,\ y \in R_b,\ w \in M,\\
&(5) && x[w,w']_b = [x.w,w']_{ab} && \text{for } x \in R_a,\ w,w' \in M,\\
&(6) && [w,x.w']_b = \chi(a,b)[w.x,w']_b && \text{for } x \in R_a,\ w,w' \in M,\\
&(7) && [w,w'.x]_{ba} = [w,w']_b x && \text{for } x \in R_a,\ w,w' \in M,\\
&(8) && w.[w',w'']_a = \tau\sum_{b\in A}\chi(a,b)^{-1}[w,w']_b.w'' && \text{for } w,w',w'' \in M.
\end{aligned}$$

Put $1 = u(1) \in R_1$. Then commutativity of (F2) and (F3) for $X \in A \cup \{m\}$ amounts to the following:

$$\begin{aligned}
&(9) && 1x = x && \text{for } x \in R_a,\\
&(10) && 1.w = w && \text{for } w \in M,\\
&(11) && x1 = x && \text{for } x \in R_a,\\
&(12) && w.1 = w && \text{for } w \in M.
\end{aligned}$$

Conversely, vector spaces R_a for all $a \in A$ and M together with isomorphisms (i1)–(i4) and $u\colon k \to R_1$ satisfying (1)–(12) give rise to a tensor functor $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$. Isomorphism classes of tensor functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ are thus in one-to-one correspondence with isomorphism classes of such data $(R_a, M, \dots)$.

Now (1), (9), (11) mean that $R = \bigoplus_{a\in A} R_a$ is an associative algebra with multiplication $(x,y) \mapsto xy$ and identity $1 \in R_1$. (2), (10) (resp. (4), (12)) mean that M is a left (resp. right) R-module with action $(x,w) \mapsto x.w$ (resp. $(w,x) \mapsto w.x$). (i1) tells us that R is a strongly A-graded algebra (Section 2 (2)). In particular R is semisimple (Proposition 2.4).

(5) (resp. (7)) says that the isomorphism of (i4) is left (resp. right) R-linear. It follows that R is simple and M is the unique simple left (resp. right) R-module. Indeed, put $d = \dim M$. If $S_1, \dots, S_r$ are the simple left R-modules and $d_i = \dim S_i$, then $d^2 = \dim R = \sum_i d_i^2$. The isomorphism $M \otimes M \cong R$ being left linear, M must contain all S_i, hence $d \geq \sum_i d_i$. Then we have $r = 1$ and $d = d_1$.

We have also $|A| = \dim R = d^2$. As $\tau^2|A| = 1$, we may write $\tau = \epsilon d^{-1}$ with $\epsilon = \pm 1$.

By (5), $[-,-]_a$ is determined by $[-,-]_1$ as

$$[w,w']_a = x[x^{-1}.w,w']_1$$

for any $0 \neq x \in R_a$. Similarly by (7)

$$[w,w']_a = [w,w'.x^{-1}]_1 x.$$

Put $[w,w']_1 = \gamma(w,w')1$ with $\gamma(w,w') \in k$. Then

$$\gamma(x.w,w') = \gamma(w,w'.x) \tag{13}$$

for any $x \in R$, $w,w' \in M$.

Conversely, if we are given a pairing γ: $M \times M \to k$ with this property, we can define $[-,-]_a$ by

$$[w,w']_a = \gamma(x^{-1}.w,w')x = \gamma(w,w'.x^{-1})x, \quad 0 \neq x \in R_a,$$

which satisfy both (5) and (7).

As R is simple, the two-sided R-linear map $M \otimes M \to R$: $w \otimes w' \mapsto ([w,w']_a)_a$ is bijective if it is nonzero. Thus, under the assumption of (5), (7), and the simplicity of R, the bijectivity of the map is equivalent to the nondegeneracy of the pairing γ, or even to the nontriviality of γ.

Next we look at (6). Let $w = y.v$ with $0 \neq y \in R_b$, $v \in M$. Then

$$[w,x.w']_b = y[v,x.w']_1,$$
$$\chi(a,b)[w.x,w']_b = [\chi(a,b)(y.v).x,w']_b = [y.(v.x),w']_b = y[v.x,w']_1.$$

Hence (6) reduces to the property

$$\gamma(w,x.w') = \gamma(w.x,w') \tag{14}$$

for all $w,w' \in M$ and $x \in R$.

Next consider (8). Write $w'' = u.x$ with $0 \neq x \in R_a$. Then

$$w.[w',w'']_a = w.[w',u]_1 x = (w.[w',u]_1).x,$$

while

$$\chi(a,b)^{-1}[w,w']_b.w'' = \chi(a,b)^{-1}[w,w']_b.(u.x) = ([w,w']_b.u).x.$$

So it suffices to consider (8) only for $a = 1$:

$$w.[w', w'']_1 = \tau \sum_b [w, w']_b.w''. \tag{15}$$

Put $\phi(w, w', w'') = \sum_b [w, w']_b.w''$. By (2) and (5), we have

$$\phi(x.w, w', w'') = x.\phi(w, w', w'').$$

As M is a simple R-module, the maps $\phi(-, w', w''): M \to M$ must be scalar for all $w', w'' \in M$. By taking the traces of both sides of (15) relative to the variable w, (15) is equivalent to the identity

$$d\gamma(w', w'') = \tau \operatorname{trace}(\phi(-, w', w'')).$$

For each $b \in A$, take $0 \neq x \in R_b$. Then

$$[w, w']_b.w'' = \gamma(w, w'.x^{-1})x.w'',$$

so

$$\operatorname{trace}([-, w']_b.w'': W \to W) = \gamma(x.w'', w'.x^{-1}) = \gamma(w'', w')$$

by (13). Hence

$$\operatorname{trace}(\phi(-, w', w'')) = |A|\gamma(w'', w').$$

Thus (8) is equivalent to

$$d\gamma(w', w'') = \tau|A|\gamma(w'', w'),$$

or

$$\gamma(w', w'') = \epsilon\gamma(w'', w'). \tag{16}$$

It remains to consider (3). As R is a simple ring and M is a simple left (resp. right) R-module, the module action gives an algebra isomorphism (resp. anti-isomorphism) $R \to \operatorname{End} M$. So we have an algebra anti-automorphism $f: R \to R$ such that

$$w.x = f(x).w$$

for $w \in M$, $x \in R$. Then (3) is rewritten as

$$xf(y).w = \chi(a, b)f(y)x.w$$

for $x \in R_a$, $y \in R_b$, $w \in M$. Hence

$$xf(y) = \chi(a,b)f(y)x. \tag{17}$$

(13) is then rewritten as

$$\gamma(x.w, w') = \gamma(w, f(x).w'). \tag{18}$$

Thus γ is a pairing which the anti-automorphism f of the simple ring R induces on the simple left R-module M, as in Proposition 2.1.

(14) is rewritten as

$$\gamma(w, x.w') = \gamma(f(x).w, w').$$

Combined with (18) and the nondegeneracy of γ, this amounts to the identity

$$f^2 = 1.$$

Then by Definition 2.2 we have the invariant $\operatorname{sgn}(f) = \pm 1$ so that

$$\gamma(w', w) = \operatorname{sgn}(f)\gamma(w, w').$$

Comparing with (16), one has $\operatorname{sgn}(f) = \epsilon$.

Summarizing above, with a tensor functor F: $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$ we associate a pair (R, f) of a simple strongly A-graded algebra R and an involutive anti-automorphism f of R such that

$$xf(y) = \chi(a,b)f(y)x \quad \text{for } x \in R_a,\ y \in R_b$$

and

$$\operatorname{sgn}(f) = \epsilon.$$

Conversely, suppose given such a pair (R, f). Choose a simple left R-module M, with action denoted as $(x, w) \mapsto x.w$. Proposition 2.1 tells us that there exists a nondegenerate pairing γ: $M \times M \to k$, unique up to scalar, such that

$$\gamma(x.w, w') = \gamma(w, f(x).w').$$

Then setting

$$\begin{aligned} w.x &= f(x).w, \\ [w, w']_a &= \gamma(w, w.x^{-1})x \quad \text{with } 0 \neq x \in R_a, \end{aligned}$$

we obtain $(R_a, M, \dots)$ satisfying (1)–(12) and (i1)–(i4), hence a tensor functor $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$. This establishes

PROPOSITION 3.1: *There is a bijection between isomorphism classes of tensor functors F and isomorphism classes of pairs (R, f).*

Let us analyze (R, f). Take $0 \neq x_a \in R_a$ for each $a \in A$ and write

$$x_a x_b = \xi(a,b) x_{ab}$$

with $\xi(a,b) \in k^\times$. Then ξ is a 2-cocycle. Put

$$\alpha(a,b) = \frac{\xi(a,b)}{\xi(b,a)}$$

so that

$$yx = \alpha(a,b) xy$$

for all $x \in R_a$, $y \in R_b$. The map α: $A \times A \to k^\times$ is an alternating form on A, and as R is simple, α is nondegenerate (Proposition 2.5). The form χ is nondegenerate as well, so there exists an automorphism σ: $A \to A$ such that

$$\chi(a,b) = \alpha(a, \sigma(b))$$

for all $a, b \in A$.

By the nondegeneracy of α, we have

$$R_c = \{z \in R |\ xz = \alpha(a,c) zx \text{ for all } x \in R_a,\ a \in A\}$$

for any $c \in A$. Hence (17) means that

$$f(y) \in R_{\sigma(b)} \quad \text{if } y \in R_b.$$

So we may write $f(x_a) = \nu(a) x_{\sigma(a)}$ with $\nu(a) \in k^\times$. Then

$$f(x_a x_b) = f(\xi(a,b) x_{ab}) = \xi(a,b) \nu(ab) x_{\sigma(ab)},$$
$$f(x_b) f(x_a) = \nu(a) x_{\sigma(b)} \nu(a) x_{\sigma(a)} = \nu(a)\nu(b)\xi(\sigma(b), \sigma(a)) x_{\sigma(b)\sigma(a)}.$$

Hence, that f is an anti-automorphism means that

$$\frac{\nu(a)\nu(b)}{\nu(ab)} = \frac{\xi(a,b)}{\xi(\sigma(b), \sigma(a))}.$$

Also

$$f^2(x_a) = f(\nu(a) x_{\sigma(a)}) = \nu(a) \nu(\sigma(a)) x_{\sigma^2(a)}.$$

Hence $f^2 = 1$ means that

$$\nu(a)\nu(\sigma(a)) = 1, \quad \sigma^2 = 1.$$

Finally by Lemma 2.3

$$\operatorname{sgn}(f)d = \operatorname{trace}(f\colon R \to R) = \sum_{\substack{a\in A \text{ s.t.} \\ \sigma(a)=a}} \nu(a).$$

Hence the condition $\operatorname{sgn}(f) = \epsilon$ is expressed as

$$\sum_{\substack{a\in A \text{ s.t.} \\ \sigma(a)=a}} \nu(a) = \epsilon d.$$

If we choose another basis $\{x'_a = \phi(a)x_a\}_{a\in A}$ of R with $\phi(a) \in k^\times$, then the corresponding ξ', ν' are given by

$$\xi'(a,b) = \frac{\phi(a)\phi(b)}{\phi(ab)}\xi(a,b), \quad \nu'(a) = \frac{\phi(a)}{\phi(\sigma(a))}\nu(a).$$

Any 2-cocycle ξ whose anti-symmetrization α is nondegenerate conversely defines a simple strongly A-graded algebra. Thus we obtain

Proposition 3.2: *Let $|A| = d^2$, $\tau = \epsilon d^{-1}$ with $d \in \mathbb{N}$, $\epsilon = \pm 1$. Let $\chi\colon A \times A \to k^\times$ be a nondegenerate symmetric bicharacter. Then isomorphism classes of tensor functors $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$ are in one-to-one correspondence with equivalence classes of triples (σ, ξ, ν) consisting of an involutive automorphism $\sigma\colon A \to A$, a 2-cocycle $\xi\colon A \times A \to k^\times$, and a map $\nu\colon A \to k^\times$ satisfying*

$$\chi(a,b) = \alpha(a, \sigma(b)),$$
$$\frac{\nu(a)\nu(b)}{\nu(ab)} = \frac{\xi(a,b)}{\xi(\sigma(b), \sigma(a))},$$
$$\nu(a)\nu(\sigma(a)) = 1,$$
$$\sum_{\substack{a\in A \text{ s.t.} \\ \sigma(a)=a}} \nu(a) = \epsilon d,$$

where

$$\alpha(a,b) = \frac{\xi(a,b)}{\xi(b,a)}.$$

Here two triples (σ,ξ,ν), (σ',ξ',ν') are said to be equivalent if $\sigma=\sigma'$ and there exists a map ϕ: $A\to k^\times$ such that

$$\xi'(a,b)=\frac{\phi(a)\phi(b)}{\phi(ab)}\xi(a,b),\quad \nu'(a)=\frac{\phi(a)}{\phi(\sigma(a))}\nu(a).$$

Let $J(A,\chi,\tau)$ be the set of such triples (σ,ξ,ν), and $\bar{J}(A,\chi,\tau)$ the quotient set of $J(A,\chi,\tau)$ under the above equivalence relation.

Let us consider in general a situation where χ is a nondegenerate symmetric bicharacter of A and σ is an involutive automorphism of A such that $\alpha=\chi\circ(1\times\sigma)$ is alternating. ($\circ$ stands for the composition of maps.) Then $\chi=\alpha\circ(1\times\sigma)$. Also by transposing, $\alpha^{-1}=\chi\circ(\sigma\times 1)$. Hence

$$\alpha\circ(\sigma\times\sigma)=\alpha^{-1}. \tag{19}$$

Put

$$A_\sigma=\{a\sigma(a)|\ a\in A\},$$
$$A^\sigma=\{a\in A|\ \sigma(a)=a\}.$$

Then $A_\sigma\le A^\sigma\le A$ and A^σ/A_σ has exponent 2.

Also $A_\sigma^\perp=A^\sigma$, where $\perp$ is taken with respect to χ. Indeed, for any $a,b\in A$,

$$\begin{aligned}\chi(a\sigma(a),b)&=\chi(a,b)\chi(\sigma(a),b)\\&=\chi(a,b)\chi(b,\sigma(a))\\&=\chi(a,b)\chi(a,\sigma(b))^{-1}\\&=\chi(a,b\sigma(b)^{-1}).\end{aligned}$$

Hence, by the nondegeneracy of χ, we have

$$b\in A_\sigma^\perp \iff b\sigma(b)^{-1}=1 \iff b\in A^\sigma.$$

It follows that χ induces a nondegenerate pairing

$$\bar{\chi}\colon A^\sigma/A_\sigma\times A^\sigma/A_\sigma\to k^\times.$$

As $\chi=\alpha$ on A^σ, $\bar{\chi}$ is alternating. In particular, the rank of the elementary abelian 2-group A^σ/A_σ must be even. We may write $|A^\sigma/A_\sigma|=\bar{d}^2$, $\bar{d}\in\mathbb{N}$ so that $d=|A_\sigma|\bar{d}$.

Suppose given a triple $(\sigma, \xi, \nu) \in J(A, \chi, \tau)$. Then $\alpha = \chi \circ (1 \times \sigma) = \mathrm{alt}(\xi)$ is alternating, so the above constructions can apply.

The condition

$$\nu(a)\nu(\sigma(a)) = 1 \quad \text{for all } a \in A$$

may be replaced by

$$\nu | A_\sigma = 1,$$

because

$$\nu(a\sigma(a)) = \nu(a)\nu(\sigma(a)) \frac{\xi(\sigma^2(a), \sigma(a))}{\xi(a, \sigma(a))} = \nu(a)\nu(\sigma(a)).$$

On A^σ we have

$$\frac{\nu(a)\nu(b)}{\nu(ab)} = \frac{\xi(a,b)}{\xi(b,a)} = \alpha(a,b) = \chi(a,b).$$

In particular $\nu|A^\sigma$ is constant on each coset of A_σ, hence $\nu|A^\sigma$ factors through a map $\bar{\nu}$: $A^\sigma/A_\sigma \to k^\times$. And the condition

$$\sum_{a \in A^\sigma} \nu(a) = \epsilon d$$

may be rephrased as

$$\sum_{a \in A^\sigma/A_\sigma} \bar{\nu}(a) = \epsilon \bar{d}.$$

We may regard A^σ/A_σ as a vector space over $\mathbb{F}_2$. That $\bar{\chi}$ is an alternating form on A^σ/A_σ and the equation

$$\frac{\bar{\nu}(a)\bar{\nu}(b)}{\bar{\nu}(ab)} = \bar{\chi}(a,b)$$

imply that $\bar{\nu}(1) = 1$ and $\bar{\nu}(a) = \pm 1$. Regarding $\{\pm 1\}$ as $\mathbb{F}_2$, we may say $\bar{\nu}$ is a quadratic form on A^σ/A_σ (Definition 2.7). So the invariant $\mathrm{sgn}(\bar{\nu})$ is defined (Definition 2.9). By Lemma 2.10, the equation

$$\sum_{a \in A^\sigma/A_\sigma} \bar{\nu}(a) = \epsilon \bar{d}$$

is rephrased as

$$\mathrm{sgn}(\bar{\nu}) = \epsilon.$$

In summary, $J(A,\chi,\tau)$ consists of (σ,ξ,ν) such that

$$\begin{aligned}
&\sigma \in \mathrm{Aut}(A), \sigma^2 = 1,\\
&\xi\colon A \times A \to k^\times \text{ is a 2-cocycle},\\
&\chi \circ (1 \times \sigma) = \mathrm{alt}(\xi),\\
&\nu\colon A \to k^\times,\\
&\partial\nu = \frac{\xi}{\xi^T \circ (\sigma \times \sigma)},\\
&\nu | A_\sigma = 1,\\
&\mathrm{sgn}(\bar{\nu}) = \epsilon,
\end{aligned}$$

where ∂ is the coboundary operator, ξ^T denotes the transpose of ξ, i.e., $\xi^T(a,b) = \xi(b,a)$, and $\mathrm{alt}(\xi) = \xi/\xi^T$. Triples (σ,ξ,ν) and (σ,ξ',ν') are equivalent if there exists a map $\phi\colon A \to k^\times$ such that

$$\xi' = \xi\,\partial\phi, \quad \nu' = \nu\frac{\phi}{\phi\circ\sigma}.$$

Note also that in this case $\nu = \nu'$ on A^σ, hence $\bar{\nu} = \bar{\nu}'$.

Now let $J'(A,\chi,\tau)$ be the set of pairs (σ,μ) such that

$$\begin{aligned}
&\sigma \in \mathrm{Aut}(A), \sigma^2 = 1,\\
&\chi \circ (1 \times \sigma) \text{ is alternating (so we can speak of } A_\sigma, A^\sigma, \bar{\chi}),\\
&\mu\colon A^\sigma / A_\sigma \to k^\times,\\
&\partial\mu = \bar{\chi} \text{ (so } \mu \text{ is a quadratic form and we can speak of } \mathrm{sgn}(\mu)),\\
&\mathrm{sgn}(\mu) = \epsilon.
\end{aligned}$$

The preceding discussion shows that $(\sigma,\xi,\nu) \mapsto (\sigma,\bar{\nu})$ yields a map

$$\bar{J}(A,\chi,\tau) \to J'(A,\chi,\tau).$$

PROPOSITION 3.3: *This map is bijective.*

Proof: Injectivity: Firstly we observe that if $(\sigma,\xi,\nu) \in J(A,\chi,\tau)$ and $\phi\colon A \to k^\times$ is a map, then (σ,ξ',ν') given by

$$\xi' = \xi\,\partial\phi, \quad \nu' = \nu\frac{\phi}{\phi\circ\sigma}$$

belongs to $J(A,\chi,\tau)$ (and is equivalent to (σ,ξ,ν)). Indeed, $\mathrm{alt}(\xi)=\mathrm{alt}(\xi')$, and

$$\begin{aligned}\partial\nu' &= \partial\nu\frac{\partial\phi}{\partial\phi\circ(\sigma\times\sigma)}\\ &= \frac{\xi}{\xi^T\circ(\sigma\times\sigma)}\frac{\partial\phi}{\partial\phi\circ(\sigma\times\sigma)}\\ &= \frac{\xi\cdot\partial\phi}{(\xi\cdot\partial\phi)^T\circ(\sigma\times\sigma)}\\ &= \frac{\xi'}{\xi'^T\circ(\sigma\times\sigma)},\end{aligned}$$

and $\nu|A^\sigma=\nu'|A^\sigma$, so $\bar\nu=\bar\nu'$. Thus $(\sigma,\xi',\nu')\in J(A,\chi,\tau)$.

Now suppose $(\sigma,\xi,\nu),(\sigma,\xi',\nu')\in J(A,\chi,\tau)$ and $\bar\nu=\bar\nu'$. We have

$$\mathrm{alt}(\xi)=\chi\circ(1\times\sigma)=\mathrm{alt}(\xi'),$$

but $\mathrm{alt}\colon H^2(A)\to X^2_a(A)$ is an isomorphism (Proposition 2.6), so ξ,ξ' are cohomologous. Replacing (σ,ξ,ν) by an equivalent triple $(\sigma,\xi\partial\phi,\nu\frac{\phi}{\phi\circ\sigma})$, we may assume $\xi=\xi'$. Then

$$\partial\nu=\frac{\xi}{\xi^T\circ(\sigma\times\sigma)}=\partial\nu',$$

hence $\nu'/\nu\in X(A)=\mathrm{Hom}(A,k^\times)$. As $\bar\nu=\bar\nu'$, ν'/ν is trivial on A^σ. Now by Lemma 3.4 below, we can find $\phi\in X(A)$ such that

$$\frac{\phi}{\phi\circ\sigma}=\frac{\nu'}{\nu}.$$

Hence

$$\xi\partial\phi=\xi=\xi',\quad \nu\frac{\phi}{\phi\circ\sigma}=\nu'.$$

Thus (σ,ξ,ν), (σ,ξ',ν') are equivalent.

Surjectivity: Let $(\sigma,\mu)\in J'(A,\chi,\tau)$. Put $\alpha=\chi\circ(1\times\sigma)$. By Proposition 2.6, we can take $\xi\in X^2(A)$ such that $\mathrm{alt}(\xi)=\alpha$. Then

$$\frac{\xi}{\xi^T\circ(\sigma\times\sigma)}\in X^2_s(A),$$

because

$$\mathrm{alt}\left(\frac{\xi}{\xi^T\circ(\sigma\times\sigma)}\right)=\mathrm{alt}(\xi)\cdot\big(\mathrm{alt}(\xi)\circ(\sigma\times\sigma)\big)=\alpha\cdot(\alpha\circ(\sigma\times\sigma))=1$$

by (19). Therefore, by Proposition 2.6 again, there exists a map $\nu'\colon A \to k^\times$ such that

$$\partial\nu' = \frac{\xi}{\xi^T \circ (\sigma \times \sigma)}.$$

Then over $A^\sigma \times A^\sigma$ we have

$$\partial\nu' = \frac{\xi}{\xi^T} = \alpha = \chi = \bar{\chi} \circ (\pi \times \pi) = \partial(\mu \circ \pi),$$

where $\pi\colon A^\sigma \to A^\sigma/A_\sigma$ is the projection. Hence

$$\frac{\mu \circ \pi}{\nu'|A^\sigma} \in X(A^\sigma).$$

Extend this to $\psi \in X(A)$ and set $\nu = \nu'\psi$. Then

$$\partial\nu = \partial\nu' = \frac{\xi}{\xi^T \circ (\sigma \times \sigma)}$$

and $\nu|A^\sigma = \mu \circ \pi$. So $(\sigma, \xi, \nu) \in J(A, \chi, \tau)$ and $\bar{\nu} = \mu$. ∎

LEMMA 3.4:
$$\left\{\frac{\phi}{\phi \circ \sigma} | \phi \in X(A)\right\} = (A^\sigma)^\perp,$$

where $\perp$ is taken relative to the canonical pairing $A \times X(A) \to k^\times$.

Proof: Let $\sigma^*\colon X(A) \to X(A)$ be the dual of σ. Let σ/id denote the endomorphism $a \mapsto \sigma(a)/a$ and σ^*/id the similar one for $X(A)$. Then

$$A^\sigma = \mathrm{Ker}\,\frac{\sigma}{\mathrm{id}} = \left(\mathrm{Im}\,\frac{\sigma^*}{\mathrm{id}}\right)^\perp.$$

Taking $\perp$, we have the required identity. ∎

Combining Propositions 3.2 and 3.3, we obtain

THEOREM 3.5: *Let $|A| = d^2$, $\tau = \epsilon d^{-1}$ with $d \in \mathbb{N}$, $\epsilon = \pm 1$. Let $\chi\colon A \times A \to k^\times$ be a nondegenerate symmetric bicharacter. Then isomorphism classes of tensor functors $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$ are in one-to-one correspondence with pairs (σ, μ) such that*

$\sigma \in \mathrm{Aut}(A), \quad \sigma^2 = 1,$

$\chi \circ (1 \times \sigma)$ *is alternating,*

$\mu\colon A^\sigma/A_\sigma \to k^\times$ *is a map,*

$\partial\mu = \bar{\chi},$

$\mathrm{sgn}(\mu) = \epsilon,$

where

$$A_\sigma = \{a\sigma(a) |\ a \in A\},$$
$$A^\sigma = \{a \in A |\ \sigma(a) = a\},$$

$\bar{\chi}$: $A^\sigma/A_\sigma \times A^\sigma/A_\sigma \to k^\times$ *is the pairing induced from* χ, *and* $\mathrm{sgn}(\mu)$ *is the signature of the quadratic form* μ.

4. Case where $|A|$ is odd

When $|A|$ is odd, we can complete the classification of tensor functors $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$. As before, let $|A| = d^2$, $d \in \mathbb{N}$, $\tau = \epsilon d^{-1}$. Let (σ, μ) be a pair of Theorem 3.5. We have $A^\sigma/A_\sigma = 1$. The only quadratic form on A^σ/A_σ is the zero map, which has signature $+1$ by definition. So if $\epsilon = -1$, there does not exist a pair (σ, μ).

If $\epsilon = +1$, it is enough to take care of only σ. Since $|A|$ is odd and $\sigma^2 = \mathrm{id}$, we have the decomposition

$$A = A_+ \times A_-,$$
$$A_+ = \{a \in A |\ \sigma(a) = a\} = A^\sigma,$$
$$A_- = \{a \in A |\ \sigma(a) = a^{-1}\}.$$

That χ is symmetric and $\chi \circ (1 \times \sigma)$ is alternating implies

$$\chi(A_+, A_+) = \chi(A_-, A_-) = 1.$$

Conversely, if we are given a decomposition $A = B_0 \times B_1$ such that χ is trivial on both B_0 and B_1, then we have the involution σ: $(b_0, b_1) \mapsto (b_0, b_1^{-1})$ of A and $\chi \circ (1 \times \sigma)$ is alternating. Thus involutions in question are in one-to-one correspondence with such decompositions $A = B_0 \times B_1$. By Theorem 3.5 we obtain

PROPOSITION 4.1: (i) *If* $\epsilon = -1$, *there exists no tensor functor* $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$.

(ii) *If* $\epsilon = 1$, *isomorphism classes of tensor functors* $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$ *correspond bijectively to pairs* (B_0, B_1) *of subgroups of* A *such that*

$$A = B_0 \times B_1, \quad \chi(B_0, B_0) = \chi(B_1, B_1) = 1.$$

Let (B_0, B_1) be as above. Then χ induces an isomorphism of B_1 onto the dual $X(B_0)$. Thus the pair (A, χ) is isomorphic to $(B_0 \times X(B_0), \omega)$, where

$$\omega((b, \beta), (b', \beta')) = \beta(b')\beta'(b).$$

We will say such χ is hyperbolic.

If (B_0', B_1') is another pair with the same property, then $A \cong B_0 \times X(B_0) \cong B_0 \times B_0$ and $A \cong B_0' \times B_0'$. It follows that $B_0 \cong B_0'$. And $B_1 \cong X(B_0) \cong X(B_0') \cong B_1'$. The isomorphisms $B_i \to B_i'$ thus obtained induce the automorphism

$$A = B_0 \times B_1 \to B_0' \times B_1' = A,$$

which obviously preserves χ. We conclude that if χ is hyperbolic, the group $\mathrm{Aut}(A, \chi)$ acts transitively on the set of pairs (B_0, B_1) of the preceding proposition. Thus we have the following.

PROPOSITION 4.2: *Suppose $\epsilon = 1$. There exists a tensor functor $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$ if and only if χ is hyperbolic. In this case the group $\overline{\mathrm{Aut}}\, \mathcal{C}(A, \chi, \tau)$ acts transitively on the set of isomorphism classes of tensor functors $\mathcal{C}(A, \chi, \tau) \to \mathcal{V}$.*

By the theory of Tannakian categories ([4]), a tensor functor $F\colon \mathcal{C}(A, \chi, \tau) \to \mathcal{V}$ determines a Hopf algebra H so that F factors as the composite of a tensor equivalence $\mathcal{C}(A, \chi, \tau) \simeq H$-mod and the forgetful functor H-mod $\to \mathcal{V}$, where H-mod is the category of finite dimensional H-modules. ($\mathcal{C}(A, \chi, \tau)$ is rigid as shown in [3].) Let us describe explicitly the Hopf algebra H in the case of the preceding proposition.

We assume $\epsilon = 1$ and (A, χ) is hyperbolic. Specifically we take

$$\begin{aligned} &A = B \times X(B),\\ &\chi\colon ((b, \beta), (b', \beta')) \mapsto \langle b, \beta' \rangle \langle b', \beta \rangle,\\ &\sigma\colon (b, \beta) \mapsto (b, \beta^{-1}),\\ &\mu = 1, \end{aligned}$$

where $\langle -, - \rangle$ is the canonical pairing between B and $X(B)$.

We will keep the notation in Section 3. We have $\chi \circ (1 \times \sigma)\colon ((b, \beta), (b', \beta')) \mapsto \langle b, \beta' \rangle^{-1} \langle b', \beta \rangle$, so we can take the 2-cocycle ξ to be

$$\xi\colon ((b, \beta), (b', \beta')) \mapsto \langle b', \beta \rangle.$$

Then

$$\frac{\xi}{\xi^T \circ (\sigma \times \sigma)} = \chi,$$

so we can take ν to be

$$\nu\colon (b, \beta) \mapsto \langle b, \beta \rangle^{-1}.$$

Hence the A-graded algebra R is given by

$$R = \bigoplus_{b\in B, \beta\in X(B)} R_{(b,\beta)}, \quad R_{(b,\beta)} = kx_b x_\beta$$

with relations

$$x_b x_{b'} = x_{b'} x_b, \quad x_\beta x_{\beta'} = x_{\beta'} x_\beta, \quad x_\beta x_b = \langle \beta, b\rangle x_b x_\beta,$$

and the involution f is given by

$$f(x_b) = x_b, \quad f(x_\beta) = x_{\beta^{-1}}.$$

As the simple left R-module M, we take $M = \bigoplus_{b\in B} kv_b$ with action

$$x_b.v_{b'} = v_{bb'}, \quad x_\beta.v_{b'} = \langle \beta, b'\rangle v_{b'}.$$

The pairing γ on M is given by

$$\gamma(v_b, v_c) = \delta_{bc,1}$$

so that the isomorphism $M \otimes M \to R$ is given by

$$v_b \otimes v_c \mapsto \sum_{\beta\in X(B)} \langle \beta, c\rangle x_{bc} x_\beta$$

with inverse

$$x_b x_\beta \mapsto \frac{1}{|B|} \sum_{c\in B} \langle \beta, c\rangle^{-1} v_{bc^{-1}} \otimes v_c.$$

We have thus specified the isomorphisms (i1)–(i4) of Section 3, which determine a tensor functor $F\colon \mathcal{C}(A, \chi, \tau) \to \mathcal{V}$.

Now we describe the Hopf algebra H for F. We let $H = \prod_{a\in A} \operatorname{End} R_a \times \operatorname{End} M$ as an algebra. The spaces R_a for $a \in A$ and M are simple left H-modules. Let $f_a \in H$ for $a \in A$ be the central idempotent in the factor $\operatorname{End} R_a \cong k$, and let $e_{b,b'} \in H$ for $b, b' \in B$ be the matrix units in the factor $\operatorname{End} M$ relative to the basis $\{v_b\}$ of M, i.e., $e_{b,b'}(v_{b''}) = \delta_{b',b''} v_b$.

The comultiplication $\Delta\colon H \to H \otimes H$ is determined by the requirement that the isomorphisms (i1)–(i4) are H-linear. For example, $f_{(b,\beta)}$ acts on R as the projection onto $R_{(b,\beta)}$. Translating this via the isomorphism $M \otimes M \cong R$, one finds that $f_{(b,\beta)}$ acts on $M \otimes M$ as

$$v_{c'} \otimes v_{d'} \mapsto \frac{1}{|B|} \sum_d \delta_{b,c'd'} \langle \beta, d'd^{-1}\rangle v_{bd^{-1}} \otimes v_d.$$

This tells us that the $\operatorname{End} M \otimes \operatorname{End} M$-component of $\Delta(f_{(b,\beta)})$ is given by

$$\frac{1}{|B|}\sum_{d,d'\in B}\langle\beta, d^{-1}d'\rangle e_{bd^{-1},bd'^{-1}}\otimes e_{d,d'}.$$

Working with the other isomorphisms likewise, one finds that the comultiplication Δ of H is given by

$$\Delta(f_{(b,\beta)}) = \sum_{(b',\beta')\in A} f_{(bb'^{-1},\beta\beta'^{-1})}\otimes f_{(b',\beta')} + \frac{1}{|B|}\sum_{d,d'\in B}\langle\beta, d^{-1}d'\rangle e_{bd^{-1},bd'^{-1}}\otimes e_{d,d'},$$

$$\Delta(e_{d,d'}) = \sum_{(b,\beta)\in A}\left(\langle\beta, d^{-1}d'\rangle f_{(b,\beta)}\otimes e_{b^{-1}d,b^{-1}d'} + \langle\beta, dd'^{-1}\rangle e_{b^{-1}d,b^{-1}d'}\otimes f_{(b,\beta)}\right).$$

5. Case where A is an elementary abelian 2-group

For the case where $|A|$ is even, we give a complete classification of tensor functors $\mathcal{C}(A,\chi,\tau)\to\mathcal{V}$ only when A is an elementary abelian 2-group.

Here elementary abelian 2-groups are regarded as vector spaces over the field $F=\mathbb{F}_2$. Let $\mathcal{S}$ be the category whose objects are pairs (A,χ) of F-vector spaces A and nondegenerate symmetric bilinear forms χ on A, and morphisms are isomorphisms in an obvious sense. Let $\mathcal{T}$ be the category whose objects are triples (A,χ,σ) of F-vector spaces A, nondegenerate symmetric bilinear forms χ on A, and involutions $\sigma\in\operatorname{Aut}(A)$ such that $\chi\circ(1\times\sigma)$ are alternating, and whose morphisms are isomorphisms in an obvious sense. Direct sums of objects are defined componentwise.

For $(A,\chi)\in\mathcal{S}$, we have the object $(A\oplus A,\chi\oplus\chi,T)\in\mathcal{T}$, where T: $(a,b)\mapsto(b,a)$. The automorphism $(a,b)\mapsto(a,a+b)$ of $A\oplus A$ yields the isomorphism

$$(A\oplus A,\chi\oplus\chi,T)=(A\oplus A,\begin{pmatrix}\chi&0\\0&\chi\end{pmatrix},\begin{pmatrix}0&1\\1&0\end{pmatrix})\cong(A\oplus A,\begin{pmatrix}0&\chi\\\chi&\chi\end{pmatrix},\begin{pmatrix}1&1\\0&1\end{pmatrix})$$

in $\mathcal{T}$, where the third component of the right side stands for the automorphism $(a,b)\mapsto(a+b,b)$. If χ is alternating, we have also the object $(A,\chi,\mathrm{id})\in\mathcal{T}$.

Conversely, let $(A,\chi,\sigma)\in\mathcal{T}$. Let $A_\sigma\le A^\sigma\le A$ be as in Section 3. Let $\rho(a)=\sigma(a)-a$. Then $\rho^2=0$ and $A_\sigma=\operatorname{Im}\rho$, $A^\sigma=\operatorname{Ker}\rho$. Hence ρ induces an isomorphism $\tilde{\rho}$: $A/A^\sigma\to A_\sigma$. As $A^\sigma=A_\sigma^\perp$, the form χ induces nondegenerate pairings

$$\tilde{\chi}\colon A_\sigma\times A/A^\sigma\to F,$$
$$\bar{\chi}\colon A^\sigma/A_\sigma\times A^\sigma/A_\sigma\to F,$$

and $\bar{\chi}$ is alternating. Since both $\chi \circ (1 \times \sigma)$ and χ are symmetric, we have $\chi(a, \rho(b)) = \chi(b, \rho(a))$. Let

$$\chi_\sigma = \tilde{\chi} \circ (1 \times \tilde{\rho}^{-1}) \colon A_\sigma \times A_\sigma \to F$$

so that $\chi_\sigma(a, \rho(b)) = \chi(a, b)$ for $a \in A_\sigma$, $b \in A$. Then

$$\chi_\sigma(\rho(a), \rho(b)) = \chi(\rho(a), b) = \chi(\rho(b), a) = \chi_\sigma(\rho(b), \rho(a)),$$

so χ_σ is symmetric.

Thus we have functors $P_1, P_2 \colon \mathcal{T} \to \mathcal{S}$ taking (A, χ, σ) to (A_σ, χ_σ), $(A^\sigma / A_\sigma, \bar{\chi})$, respectively.

PROPOSITION 5.1: *Every object $(A, \chi, \sigma) \in \mathcal{T}$ is isomorphic to*

$$(A_1 \oplus A_1, \chi_1 \oplus \chi_1, T) \oplus (A_2, \chi_2, \mathrm{id})$$

with $(A_1, \chi_1), (A_2, \chi_2) \in \mathcal{S}$ and χ_2 alternating. The isomorphism classes of (A_1, χ_1) and (A_2, χ_2) are uniquely determined.

Proof: The uniqueness follows from that the functors P_1, P_2 take the object

$$(A_1 \oplus A_1, \chi_1 \oplus \chi_1, T) \oplus (A_2, \chi_2, \mathrm{id})$$

to $(A_1, \chi_1), (A_2, \chi_2)$, respectively.

Let $(A, \chi, \sigma) \in \mathcal{T}$. Then $\alpha = \chi \circ (1 \times \sigma)$ is a nondegenerate alternating form on A and $\alpha(A_\sigma, A_\sigma) = 0$ and $A^\sigma = \{a \in A | \ \alpha(a, A_\sigma) = 0\}$. By Lemma 5.2 below, there exists $A_3 \leq A$ such that $A = A^\sigma \oplus A_3$ and $\alpha(A_3, A_3) = 0$. Put $A_2 = (A_\sigma + A_3)^\perp = A^\sigma \cap A_3^\perp$ ($\perp$ is taken relative to χ). Then $A^\sigma = A_\sigma \oplus A_2$. Indeed,

$$A_\sigma \cap A^\sigma \cap A_3^\perp = A_\sigma \cap A_3^\perp = (A^\sigma)^\perp \cap A_3^\perp = (A^\sigma + A_3)^\perp = A^\perp = 0$$

and

$$\dim A^\sigma / A_2 = \dim (A_\sigma + A_3)/A_\sigma = \dim A_3 = \dim A/A^\sigma = \dim A_\sigma.$$

Thus $A = A_\sigma \oplus A_2 \oplus A_3$.

The map ρ induces an isomorphism $A_3 \cong A_\sigma$. We have $\chi(a, b) = \chi_\sigma(a, \rho(b))$ for $a \in A_\sigma$, $b \in A_3$. As $\alpha(A_3, A_3) = 0$, we have $\chi(a, \sigma(b)) = 1$ for $a, b \in A_3$, hence

$$\chi(a, b) = \chi(a, \rho(b)) = \chi_\sigma(\rho(a), \rho(b)).$$

Let χ_2 be the restriction of χ to A_2. The isomorphism

$$\begin{aligned} A = A_\sigma \oplus A_2 \oplus A_3 &\cong A_\sigma \oplus A_\sigma \oplus A_2 \\ (a_1, a_2, a_3) &\mapsto (a_1, \rho(a_3), a_2) \end{aligned}$$

yields a desired isomorphism

$$(A, \chi, \sigma) \cong (A_\sigma \oplus A_\sigma, \begin{pmatrix} 0 & \chi_\sigma \\ \chi_\sigma & \chi_\sigma \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) \oplus (A_2, \chi_2, \mathrm{id}). \qquad ∎$$

LEMMA 5.2: *Let V be a vector space over F, α: $V \times V \to F$ a nondegenerate alternating form, and U a subspace of V such that $\alpha(U,U) = 0$. Then there exists a subspace W of V such that $\alpha(W,W) = 0$ and $V = U^\perp \oplus W$.*

Proof: We will inductively construct vectors $u_1, \ldots, u_k$ in U and $w_1, \ldots, w_k$ in V such that $\alpha(u_i, w_j) = \delta_{ij}$ and $\alpha(w_j, w_{j'}) = 0$ for all $1 \le i, j, j' \le k$. If $k = \dim U$, then $W = \langle w_1, \ldots, w_k \rangle$ is a required subspace. Suppose we have found such vectors for $k < \dim U$. Then $\langle u_1, \ldots, u_k \rangle \subsetneqq U$. Take $w \in \langle u_1, \ldots, u_k \rangle^\perp \setminus U^\perp$. Take $u \in U$ such that $\alpha(u, w) = 1$. Put

$$\begin{aligned} w_{k+1} &= w - \sum_{j=1}^{k} \alpha(w, w_j) u_j, \\ u_{k+1} &= u - \sum_{j=1}^{k} \alpha(u, w_j) u_j. \end{aligned}$$

Then

$$\begin{aligned} &\alpha(w_{k+1}, w_j) = 0, \alpha(u_{k+1}, w_j) = 0 \quad \text{for } 1 \le j \le k, \quad \text{and} \\ &\alpha(u_{k+1}, w_{k+1}) = 1. \end{aligned}$$

Thus $u_1, \ldots, u_{k+1}; w_1, \ldots, w_{k+1}$ have the required property. ∎

PROPOSITION 5.3: *The isomorphism class of an object (A, χ, σ) in $\mathcal{T}$ is detected by* $\mathrm{rank}(\sigma - 1)$ *and the isomorphism class of (A, χ) in $\mathcal{S}$.*

Proof: Take an isomorphism

$$(A, \chi, \sigma) \cong (A_1 \oplus A_1, \chi_1 \oplus \chi_1, T) \oplus (A_2, \chi_2, \mathrm{id}).$$

It is enough to show that the isomorphism classes of $(A_1, \chi_1), (A_2, \chi_2)$ are determined from (A, χ) and $\mathrm{rank}(\sigma - 1)$. Clearly $\dim A$ and $\mathrm{rank}(\sigma - 1)$ determine

$\dim A_1$, $\dim A_2$. Since χ_2 is a nondegenerate alternating form, (A_2, χ_2) is isomorphic to the direct sum H^s with $s = \frac{1}{2}\dim A_2$ in the notation of Section 2 (5). Hence

$$(A, \chi) \cong (A_1, \chi_1)^2 \oplus H^s.$$

By Corollary 2.13, the isomorphism class of (A_1, χ_1) is uniquely determined from that of (A, χ). ∎

Let $(A, \chi) \in \mathcal{S}$. The group $\mathrm{Aut}(A, \chi)$ acts on the set $J'(A, \chi, \tau)$. The orbit set $J'(A, \chi, \tau)/\mathrm{Aut}(A, \chi)$ is described as follows.

Proposition 5.4: *Let $\dim A = 2r$. The map $(\sigma, \mu) \mapsto r - \mathrm{rank}(\sigma - 1)$ gives an injection*

$$J'(A, \chi, \tau)/\mathrm{Aut}(A, \chi) \to \mathbb{Z}$$

with image

$$\begin{aligned} \{k \in [0, r] \mid k \equiv r \bmod 2\} \quad & \text{if } \chi \text{ is alternating and } \epsilon = 1, \\ \{k \in [1, r] \mid k \equiv r \bmod 2\} \quad & \text{if } \chi \text{ is alternating and } \epsilon = -1, \\ [0, r-1] \quad & \text{if } \chi \text{ is not alternating and } \epsilon = 1, \\ [1, r-1] \quad & \text{if } \chi \text{ is not alternating and } \epsilon = -1. \end{aligned}$$

Proof: Injectivity: Let $(\sigma, \mu), (\sigma', \mu') \in J'(A, \chi, \tau)$ and suppose $\mathrm{rank}(\sigma - 1) = \mathrm{rank}(\sigma' - 1)$. By the preceding proposition, $(A, \chi, \sigma) \cong (A, \chi, \sigma')$. So we may assume $\sigma = \sigma'$. Then μ, μ' are quadratic forms on A^σ/A_σ having the same associated bilinear form $\bar{\chi}$ and the same signature ϵ. Hence they are conjugate under the group $\mathrm{Aut}(A^\sigma/A_\sigma, \bar{\chi})$. Now the natural map $\mathrm{Aut}(A, \chi, \sigma) \to \mathrm{Aut}(A^\sigma/A_\sigma, \bar{\chi})$ is surjective, as seen from the isomorphism of the preceding proposition. Hence (σ, μ), (σ, μ') are conjugate under $\mathrm{Aut}(A, \chi, \sigma)$.

Image: Suppose χ is alternating. Then $(A, \chi) \cong H^r$. For any integers $k_1, k_2 \geq 0$ with $r = 2k_1 + k_2$ let $(A_1, \chi_1) = H^{k_1}$, $(A_2, \chi_2) = H^{k_2}$ and set

$$(A', \chi', \sigma') = (A_1 \oplus A_1, \chi_1 \oplus \chi_1, T) \oplus (A_2, \chi_2, \mathrm{id}).$$

Then $(A', \chi') \cong (A, \chi)$ and $r - \mathrm{rank}(\sigma' - 1) = k_2$. Unless $\epsilon = -1$ and $k_2 = 0$, we can take a quadratic form μ' on A_2 having the associated bilinear form χ_2 and the signature ϵ. Thus we have $(\sigma', \mu') \in J'(A', \chi', \tau)$. Hence such k_2 is in the image of the map. The converse inclusion is similarly shown.

Next, suppose χ is not alternating. Then $(A, \chi) \cong H^{r-1} \oplus L^2$ by Proposition 2.11. For integers $k_1, k_2 \geq 0$ such that $2k_1 + k_2 = r - 1$ (resp. $2k_1 + 1 + k_2 = r - 1$)

consider the above (A',χ',σ') with $(A_1,\chi_1) = H^{k_1} \oplus L$ (resp. $H^{k_1} \oplus L^2$) and $(A_2,\chi_2) = H^{k_2}$. Then $(A',\chi') \cong (A,\chi)$ and $r - \operatorname{rank}(\sigma' - 1) = k_2$. Similarly to the above, k_2 is in the image unless $\epsilon = -1$ and $k_2 = 0$. ∎

By Theorem 3.5 and Proposition 1 we obtain the following.

PROPOSITION 5.5: *The number of orbits of isomorphism classes of tensor functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ under the action of $\overline{\operatorname{Aut}}\,\mathcal{C}(A,\chi,\tau)$ is given by*

$$\begin{array}{rl} \left[\dfrac{r}{2}\right]+1 & \text{if } \chi \text{ is alternating and } \epsilon = 1,\\ \left[\dfrac{r-1}{2}\right]+1 & \text{if } \chi \text{ is alternating and } \epsilon = -1,\\ r & \text{if } \chi \text{ is not alternating and } \epsilon = 1,\\ r-1 & \text{if } \chi \text{ is not alternating and } \epsilon = -1. \end{array}$$

In the rest of this section we describe Hopf algebras arising from tensor functors $\mathcal{C}(A,\chi,\tau) \to \mathcal{V}$. Firstly we make a reduction to the indecomposable case. We have a natural notion of direct sums for 5-tuples (A,χ,τ,σ,μ) with $(\sigma,\mu) \in J'(A,\chi,\tau)$:

$$(A,\chi,\tau,\sigma,\mu) \oplus (A',\chi',\tau',\sigma',\mu') = (A'',\chi'',\tau'',\sigma'',\mu''),$$

where

$$\begin{aligned} &A'' = A \times A',\\ &\chi''\colon ((a,a'),(b,b')) \mapsto \chi(a,b)\chi'(a',b'),\\ &\tau'' = \tau\tau',\\ &\sigma'' = \sigma \times \sigma',\\ &\mu''\colon (a,a') \mapsto \mu(a)\mu'(a'). \end{aligned}$$

If $F\colon \mathcal{C}(A,\chi,\tau) \to \mathcal{V}$ and $F'\colon \mathcal{C}(A',\chi',\tau') \to \mathcal{V}$ are tensor functors respectively corresponding to (σ,μ) and (σ',μ'), then the tensor functor $F''\colon \mathcal{C}(A'',\chi'',\tau'') \to \mathcal{V}$ defined by

$$\begin{aligned} F''((a,a')) &= F(a) \otimes F'(a'),\\ F''(m) &= F(m) \otimes F'(m) \end{aligned}$$

corresponds to (σ'',μ''). Let F and F' yield equivalences $\mathcal{C}(A,\chi,\tau) \simeq H$-mod and $\mathcal{C}(A',\chi',\tau') \simeq H'$-mod with H and H' Hopf algebras, respectively. Let

H'' be the factor algebra of $H \otimes H'$ determined by the simple $H \otimes H'$-modules $F(a) \otimes F(a')$ for all $a \in A$, $a' \in A'$ and $F(m) \otimes F'(m)$. As direct sums of those simple modules are closed under tensor products, H'' is a Hopf algebra. Then F'' yields an equivalence $\mathcal{C}(A'', \chi'', \tau'') \simeq H''$-mod.

By Propositions 5.1 and 5.3, any object $(A, \chi, \tau, \sigma, \mu)$ with A an elementary abelian 2-group is a direct sum of the following four objects:

$$(F^4, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, +\frac{1}{4}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, 0),$$
$$(F^2, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, +\frac{1}{2}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0),$$
$$(F^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, +\frac{1}{2}, \mathrm{id}, q_+),$$
$$(F^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, -\frac{1}{2}, \mathrm{id}, q_-),$$

where the quadratic forms q_+, q_- on F^2 are given in coordinates by

$$q_+ = x_1 x_2, \quad q_- = x_1^2 + x_2^2 + x_1 x_2.$$

Thus we may confine ourselves to describing Hopf algebras for the tensor functors corresponding to the above four objects. The last three were already treated in [3]. They correspond to the Hopf algebra H_8 of Kac and Paljutkin, the dihedral group D_8, and the quaternion group Q_8, respectively. So we consider here only the first. Switching back to the multiplicative notation, we let

$$\begin{aligned} &A = B \times B, \\ &\chi\colon ((a,b),(a',b')) \mapsto \omega(a,a')\omega(b,b'), \\ &\tau = \frac{1}{4}, \\ &\sigma\colon (a,b) \mapsto (b,a), \\ &\mu = 1, \end{aligned}$$

where B is the Klein 4-group and $\omega\colon B \times B \to \{\pm 1\} \subset k^\times$ is the unique alternating form on B.

The A-graded algebra R is given by

$$R = \bigoplus_{a,b \in B} R_{(a,b)}, \quad R_{(a,b)} = k x_a y_b$$

with relations

$$x_a x_{a'} = x_{aa'}, \quad y_b y_{b'} = y_{bb'}, \quad x_a y_b = \omega(a,b) y_b x_a.$$

The involution f interchanges x_a and y_a. The simple left R-module M is given by

$$M = \bigoplus_{b \in B} k v_b$$

with action

$$x_a . y_b = v_{ab}, \quad y_a . v_b = \omega(a,b) v_b.$$

The isomorphism $M \otimes M \to R$ is given by

$$v_c \otimes v_d \mapsto \sum_{a,b \in B} \omega(ac, bd) x_a y_b$$

with inverse

$$x_a y_b \mapsto \frac{1}{|A|} \sum_{c,d} \omega(ac, bd) v_c \otimes v_d.$$

Thus we obtain the isomorphisms in (i1)–(i4), which determine a tensor functor $F\colon \mathcal{C}(A, \chi, \tau) \to \mathcal{V}$.

Let H be the Hopf algebra for F. Then $H = \prod_{a \in A} \operatorname{End} R_a \times \operatorname{End} M$ as an algebra. Let $f_a \in H$ be the idempotent in the factor $\operatorname{End} R_a \cong k$ and let $e_{b,b'} \in H$ be the matrix units in the factor $\operatorname{End} M$ relative to the basis $\{v_b\}$ of M. Similarly to the end of Section 4, one sees that the comultiplication Δ of H is given by

$$\begin{aligned}
\Delta(f_{(a,b)}) &= \sum_{(a',b') \in A} f_{(a',b')} \otimes f_{(a'^{-1}a, b'^{-1}b)} \\
&\quad + \frac{1}{|A|} \sum_{c,c',d,d' \in B} \omega(ac, bd) \omega(ac', bd') e_{c,c'} \otimes e_{d,d'}, \\
\Delta(e_{d,d'}) &= \sum_{(a,b) \in A} \left(\omega(b, dd') f_{(a,b)} \otimes e_{ad,ad'} + \omega(a, dd') e_{bd,bd'} \otimes f_{(a,b)} \right).
\end{aligned}$$

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