ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF ELLIPTIC EQUATIONS I: LIOUVILLE-TYPE THEOREMS FOR LINEAR AND NONLINEAR EQUATIONS ON $\mathbf{R}^n$

By

LEON KARP

§1. Introduction

This paper deals with the asymptotic behavior of solutions of second-order partial differential equations in $n \geqq 2$ independent variables. We are specifically concerned with the oscillation ($=\sup - \inf$) of global C^2 solutions of equations (resp. inequalities) of the form

$$\operatorname{div} \boldsymbol{A}(x, u, u_x) = B(x, u, u_x) \quad (\text{resp. } \geqq B(x, u, u_x)). \tag{1.1}$$

Here $\boldsymbol{A}$ is a given C^1 vector function of the variables (x, u, u_x), B is a given continuous scalar function of the same variables, and u_x denotes the gradient of the dependent variable $u = u(x) = u(x_1, \cdots, x_n)$.

The structure of (1.1) is determined by the functions $\boldsymbol{A}(x, s, p)$ and $B(x, s, p)$ which we take to be defined on $\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$. We assume, for simplicity, in this section that $\boldsymbol{A}$ and B satisfy

$$\begin{cases} |\boldsymbol{A}(x, s, p)| \leqq M(x)|p|^{\alpha-1}, \\ \langle \boldsymbol{A}(x, s, p), p\rangle \geqq m(x)|p|^{\alpha}, \\ B(x, s, p) \geqq f(x) - g(x)|p|^{\alpha-1} - c(x)|s|^{\beta} - h(x)|p|^{\alpha}, \end{cases} \tag{1.2}$$

for some $\alpha \geqq n$, $\beta > 0$, and for all $(x, s, p) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n$. Here $\langle \cdot, \cdot \rangle$ and $|\cdot|$ denote, respectively, the usual inner product and norm on $\mathbf{R}^n$; all functions that appear are locally integrable and non-negative and $m(x) > 0$. Moreover, we assume that $c + (g^{\alpha}/m^{\alpha-1}) \in L^1$, $h/m \in L^{\infty}$, $f \not\equiv 0$ if $g + c + h \not\equiv 0$, and — again, for simplicity in this introductory section — that $\sup M^{\alpha/(\alpha-1)}/m < \infty$.

It should be remarked here that the Euler–Lagrange equations of various (not necessarily regular) multiple integral variational problems are of the form (1.1) where $\boldsymbol{A}$ and B satisfy conditions which take the general form of (1.2) (cf. Serrin [21]). In particular, the structural conditions imposed above include, when $\alpha = n = 2$, the second-order linear elliptic equation in divergence form

(1.3) $\partial_j a^{ij}(x)\partial_i u + b^i\partial_i u + cu = f \geqq 0 \qquad (\not\equiv 0 \text{ if } |b|^2 + |c| \not\equiv 0)$

if, say, $m(x)|p|^2 \leqq a^{ij}(x)p_ip_j \leqq \mu_0 m(x)|p|^2 \ \forall (x,p) \in \mathbf{R}^2 \times \mathbf{R}^2$, $0 < m \in L^\infty$, $\mu_0 \geqq 1$, and $|b|^2 + |c| \in L^1$.

Actually, our results are valid under weaker conditions which are discussed in detail in §§2, 3 below. In particular, in §2 we define for $f \in L^1_{\text{loc}}$ an extended real number $P(f)$ that measures the "positivity" of f and is such that $P(f) = +\infty$ if $f \geqq 0$ a.e. (cf. Definition 2.3). In (1.2) it suffices to assume that $P(f) > 0$ in place of $f \geqq 0$. Moreover, the number $P(f)$ turns out to be of crucial importance for a precise understanding of the asymptotic behavior of global solutions of (1.1). In addition, we are able to handle the case $\alpha < n$ — but under different conditions on $M^{\alpha/(\alpha-1)}/m$ than that imposed above.

The investigation of the local behavior of solutions of (1.1) under conditions similar to (1.2) was initiated by Serrin [22–25] and continued by Trudinger [28]. The asymptotic behavior at infinity of solutions of $\operatorname{div} \boldsymbol{A}(x, u, u_x) = 0$ was treated by Serrin in [24]. Our results cover the case $\operatorname{div} \boldsymbol{A}(x, u, u_x) \geqq 0$ but are often of a less specific nature. As special cases of the conclusions obtained below, we record here the following theorems (in which we assume that $\boldsymbol{A}$ and B satisfy conditions (1.2) and set $\operatorname{osc} u = \sup u - \inf u$):

Theorem 1.1. *Let $g = c = 0$ in (1.2) and set $H = \sup(h/m)$. If u is a global nonconstant solution of (1.1) then $\operatorname{osc} u \geqq (1/H)P(f)$ (where $a/0 = +\infty$ if $a > 0$) even if $\{x : f(x) < 0\}$ is not assumed to be empty.*

Corollary 1.1. *Let $\boldsymbol{A}$ and B be as in Theorem 1.1. If $P(f) = +\infty$ (e.g., $f \geqq 0$) then every bounded global solution of $\operatorname{div} \boldsymbol{A} \geqq B$ is a constant.*

Theorem 1.2. *Let $g = c = h = 0$ in (1.2) and suppose that $f^- \in L^1$ and $P(f) > 0$. If u is a global solution of $\operatorname{div} \boldsymbol{A} \geqq B$ such that $\sup u < \infty$ then $u \equiv$ constant even if $\{x : f(x) < 0\}$ is not assumed to be empty.*

Theorem 1.3. *There exists a positive extended-real number γ_0 such that every nonconstant global solution of (1.1)–(1.2) satisfies $\sup |u| \geqq \gamma_0$. If $c = 0$ in (1.2) then there exists a positive extended-real number γ_1 such that every nonconstant global solution of (1.1)–(1.2) satisfies $\operatorname{osc} u \geqq \gamma_1$. The values of γ_0 and γ_1 may be given explicitly in terms of the coefficients in (1.2). Finally, $\gamma_0 = \gamma_1 = +\infty$ if $g = c = 0$.*

It should be observed that each of the theorems above gives a generalization of Liouville's theorem for subharmonic functions on $\mathbf{R}^2$. Liouville's theorem, in this form, fails to be true if $\alpha < n$ in (1.2) (cf. §6 below) and this explains the restriction $\alpha \geqq n$. However, some corresponding results can be obtained even for $\alpha < n$ (which includes the case of linear elliptic equations in $n \geqq 3$ variables) and some of these are treated in §3 below and also in a companion paper [13].

We remark that Liouville-type theorems for general linear and nonlinear elliptic equations have been treated by many people. Explicit mention should be made of the work of Bernstein [1], Bers–Nirenberg [2], Bohn–Jackson [3], Finn–Gilbarg [5], Gilbarg–Serrin [7], Ivanov [11], Moser [16], Peletier–Serrin [17], Redheffer [19], Serrin [25], Tavgelidze [26] and the survey article of Gilbarg [6]. Also, after the work described here was completed, there appeared the paper [14] of Meier. Meier obtains, among other results, the special case of Theorem 1.1 in which $\alpha = n = 2$, $f \geqq 0$, and m and M are constant — but for weak solutions. Although we assume (for simplicity) that $u \in C^2$, all our results are also valid for weak solutions. In this more general situation the proofs would follow the methods of this paper together with those of the author's earlier paper [12]. Meier's methods are similar. We also mention that Meier obtains (for m and M constant) a Liouville theorem for solutions of $\operatorname{div} \boldsymbol{A} = B$ on $\mathbf{R}^n$ for B satisfying more restrictive conditions than (1.2) (but without the restriction $\alpha \geqq n$) by means of Trudinger's Harnack inequality [27]. He remarks that this extends an earlier result of Hildebrandt and Widman [8].

The plan of the paper is as follows: The definition of $P(f)$ and some preliminary material is given in §2, the statement of the main results is given in §3, and proofs are given in §4. Some auxiliary applications (to the geometry of graphs over $\mathbf{R}^2$ and to nonlinear equations with only linear solutions of slow growth) are given in §5. The sixth section is devoted to a number of examples that shed some light on the need for the various hypotheses made in §3 and the sharpness of the estimates that we obtain.

§2. Preliminaries

This section introduces some new concepts that play a fundamental role in the asymptotic behavior of solutions of elliptic equations.

We begin by letting $D_B(\mathbf{R}^n)$ denote the C^1-diffeomorphisms of $\mathbf{R}^n$ that have uniformly bounded differentials along with their inverses:

$$D_B(\mathbf{R}^n) = \left\{ \varphi \in C^1\text{-Diff}(\mathbf{R}^n) \,\middle|\, \sup_{x \in \mathbf{R}^n} (\| d\varphi_x \| + \| d\varphi_x^{-1} \|) < \infty \right\}$$

for some norm $\|\cdot\|$ on $\mathbf{R}^n$.

Let $B(p;r)$ denote the ball of radius r centered at $p \in \mathbf{R}^n$ and let $B(r) = B(0;r)$. For each C^1-diffeomorphism φ of $\mathbf{R}^n$ the family of sets $\{\varphi(B(p;r))\}_{r>0}$ covers $\mathbf{R}^n$ and may be thought of as centered at $\varphi(p)$.

Definition 2.1. A *pseudo-spherical exhaustion of* $\mathbf{R}^n$ is a family of sets of the form $\{\varphi(B(r))\}_{r>0} = E_\varphi$ for some $\varphi \in D_B(\mathbf{R}^n)$. Moreover, we set $\text{Exh}^{(n)} =_{\text{def}} \{E_\varphi : \varphi \in D_B(\mathbf{R}^n)\}$, and when φ is fixed we set $K(r) =_{\text{def}} \varphi(B(r))$.

Remark. It is clear that $\text{Exh}^{(n)}$ contains every exhaustion of the form $\{\varphi(B(p,r))\}_{r,0}$ for $\varphi \in D_B(\mathbf{R}^n)$, $p \in \mathbf{R}^n$. In particular, $\text{Exh}^{(n)}$ contains the spherical exhaustions $\{B(p;r)\}_{r>0}$, $p \in \mathbf{R}^n$.

Definition 2.2. If $f \in L^1_{loc}(\mathbf{R}^n)$ then a *decomposition of* f is an ordered pair $\{g, h\}$ of a.e. non-negative, locally integrable functions g and h such that $f = g - h$. The *canonical decomposition of* f is the ordered pair $\{f^+, f^-\}$ where $f^+ = \max(0, f)$ and $f^- = \max(0, -f)$.

For any decomposition $\{g, h\}$ of f we have $g \geqq f^+$ and $h \geqq f^-$.

Definition 2.3. (a) For $f \in L^1_{loc}$, $\varphi \in D_B(\mathbf{R}^n)$ and a decomposition $\{g, h\}$ of f,

$$P(f;\varphi;\{g,h\}) =_{\text{def}} \begin{cases} \liminf_{r\to\infty} \log\left[\int_{K(r)} g \Big/ \int_{K(r)} h\right] & \text{if } h \not\equiv 0 \text{ a.e.} \\ +\infty & \text{if } h \equiv 0 \text{ a.e.} \end{cases}$$

where $K(r) = \varphi(B(0;r))$. Note that $\log[\int_{K(r)} g / \int_{K(r)} h]$ is defined for sufficiently large values of r if $h \not\equiv 0$.

(b) $P_0(f;\varphi) =_{\text{def}} P(f;\varphi;\{f^+, f^-\})$.

(c) $P(f;\varphi) =_{\text{def}} \sup_{\{g,h\}} P(f;\varphi;\{g,h\})$.

(d) $P(f) =_{\text{def}} \sup_{\varphi \in D_B(\mathbf{R}^n)} P(f;\varphi)$.

(e) $P_0(f) =_{\text{def}} \sup_{\varphi \in D_B(\mathbf{R}^n)} P_0(f;\varphi)$.

For later reference we record some elementary facts and examples:

Proposition 2.1. *For any pseudo-spherical exhaustion E_φ and functions f, f_1 and $f_2 \in L^1_{loc}$ we have:*

(a) $P_0(f;\varphi) > 0$ *iff* $P(f;\varphi) > 0$.

(b) *If* $P_0(f;\varphi) > 0$ *then* $P_0(f;\varphi) = P(f,\varphi)$.

(c) *If* $0 \neq \lambda \in \mathbf{R}$ *then* $P(\lambda f, \varphi) = (\operatorname{sgn}\lambda) P(f,\varphi)$.

(d) *If* $P(f_1;\varphi) > 0$ *and* $P(f_2;\varphi) > 0$ *then*

$$P(f_1 + f_2;\varphi) \geqq \min\{P(f_1;\varphi), P(f_2;\varphi)\}.$$

Proof. (a) Let $\{g, h\}$ be a decomposition of f and set $I^{\pm}(r) = \int_{K(r)} f^{\pm}$, $J^+(r) = \int_{K(r)} g$, and $J^-(r) = \int_{K(r)} h$. Then $J^{\pm}(r) = I^{\pm}(r) + A(r)$ with $A(r) \geqq 0$. It suffices to prove that $P(f,\varphi) > 0 \Rightarrow P_0(f;\varphi) > 0$ and so we assume that $\limsup_{r\to\infty}(J^-(r)/J^+(r)) < 1$ ($f \equiv 0$ trivially implying that $P_0(f;\varphi) > 0$). Since

$$(J^-(r)/J^+(r)) = (I^-(r) + A(r)/I^+(r) + A(r)) = (I^-/I^+)[1 - (A/A + I^+)] + (A/A + I^+)$$

and $(A/A + I^+)(r_k)$ has a convergent subsequence for any sequence $r_k \to \infty$, it is a simple matter to conclude that $\limsup_{r\to\infty}(I^-(r)/I^+(r)) < 1$, i.e. $P_0(f,\varphi) > 0$.

(b) It suffices to prove that $P(f;\varphi) \leqq P_0(f;\varphi)$ if $P_0(f;\varphi) > 0$, and so it suffices to show that for any decomposition $\{g, h\}$ of f we have $P(f;\varphi;\{g,h\}) \leqq P_0(f;\varphi)$ if $P_0(f;\varphi) > 0$. This last inequality follows from the form of $(J^+/J^-) - (I^+/I^-)$ as in part (a) above.

(c) is obvious.

(d) Consider the particular decomposition $\{\tilde{g} = f_1^+ + f_2^+, \tilde{h} = f_1^- + f_2^-\}$ of $f_1 + f_2$. It follows from (a) and (b) that

$$P_0(f_1 + f_2; \varphi) \geqq P(f_1 + f_2; \varphi; \{\tilde{g}, \tilde{h}\}) \qquad \text{if } P(f_1 + f_2; \varphi; \{\tilde{g}, \tilde{h}\}) > 0.$$

Thus it suffices to prove that $P(f_1 + f_2; \varphi; \{\tilde{g}, \tilde{h}\}) \geqq \min(P_0(f_1, \varphi), P_0(f_2; \varphi)) > 0$ for the decomposition $\{\tilde{g}, \tilde{h}\}$. This last inequality, in turn, follows from the inequality

$$\limsup_{r\to\infty} \frac{I_1^- + I_2^-}{I_1^+ + I_2^+} \leqq \max\left(\limsup \frac{I_1^-}{I_1^+}, \limsup \frac{I_2^-}{I_2^+}\right), \tag{2.1}$$

where $I_j^{\mp}(r) = \int_{K(r)} f_j^{\mp}\, dx$. The inequality (2.1) follows easily from the identity

$$\frac{I_1^- + I_2^-}{I_1^+ + I_2^+} = \frac{I_1^+}{I_1^+ + I_2^+}\left(\frac{I_1^-}{I_1^+}\right) + \frac{I_2^+}{I_1^+ + I_2^+}\left(\frac{I_2^-}{I_2^+}\right).$$

Proposition 2.2. *If $f \in L^1_{\text{loc}}$ and $f^- \in L^1$ then $P(f) > 0$ iff f satisfies either of the following two conditions*:

(a) $f = 0$ a.e.

(b) $\int_{\mathbf{R}^n} f dx > 0$.

Proof. If $P(f) > 0$ then $P(f; \varphi) = P_0(f; \varphi) > 0$ for some $\varphi \in D_B(\mathbf{R}^n)$ (cf. Proposition 2.1 (a), (b)). If $K(r) = \varphi(B(r))$ we then have either $f = 0$ a.e. or

$$\liminf_{r\to\infty}\left(\int_{K(r)} f^+ \Big/ \int_{K(r)} f^-\right) = \lim_{r\to\infty}\left(\int_{K(r)} f^+ \Big/ \int_{K(r)} f^-\right) = \int_{\mathbf{R}^n} f^+ \Big/ \int_{\mathbf{R}^n} f^- > 1,$$

i.e. $\int_{\mathbf{R}^n} f > 0$. The converse is immediate.

Remark. For a given $f \in L^1_{\text{loc}}$ it sometimes actually occurs that $P(f; \varphi) = 0$ for some $\varphi \in D_B(\mathbf{R}^n)$ while $P(f) > 0$. For example, the function $f(x, y) = (\operatorname{sgn} x)\exp(x^2)$ in $\mathbf{R}^2$ clearly has $P_0(f, \varphi) = 0$ when φ = identity. On the other hand, it is not difficult to check that for $\varphi(x, y) = \varphi_\alpha(x, y) = (x + \alpha, y)$, $\alpha \in \mathbf{R}$, we have $P_0(f; \varphi_\alpha) = \pm\infty$ according as $\pm\alpha > 0$. It follows from Proposition 2.1 that $P(f, \text{identity}) = 0$ while $P(f; \varphi_\alpha) = \pm\infty$ according as $\pm\alpha > 0$.

We conclude this section with a technical lemma which will be required a number of times in the sequel:

Lemma 2.1. *Let $\Psi: \mathbf{R}^n \times (a, b) \to [0, \infty)$ be such that $\Psi(\cdot, \lambda)$ is measurable on $\mathbf{R}^n$ for each $\lambda \in (a, b)$, $-\infty \leqq a < b \leqq \infty$, and let $\Psi_i(\lambda) = \inf_x \Psi(x, \lambda)$ and $\Psi_s(\lambda) = \sup_x \Psi(x, \lambda) < \infty$. Assume that $\exists \lambda_0 \in (a, b)$ such that $\lim_{\lambda\to\lambda_0} \Psi(x, \lambda) = 1$ for all $x \in \mathbf{R}^n$.*

(a) *If* $P(f;\varphi)>0$ *and* $\lim_{\lambda\to\lambda_0}\Psi_s(\lambda)=1=\lim_{\lambda\to\lambda_0}\Psi_i(\lambda)$ *then* $\exists c\in(a,b)$ *and* $t_0>0$ *such that*

$$\int_{K(t)} \Psi(x,c)f(x)dx \geqq 0 \qquad \text{for all } t\geqq t_0$$

where $K(t)=\varphi(B(t))$.

(b) *If* $f^-\in L^1$, $P(f;\varphi)>0$, *and* $\lim_{\lambda\to\lambda_0}\Psi_s(\lambda)=1$ *then* $\exists c\in(a,b)$ *and* $t_0>0$ *such that*

$$\int_{K(t)} \Psi(x,c)f(x)dx \geqq 0 \qquad \text{for all } t\geqq t_0$$

where $K(t)=\varphi(B(t))$.

Proof. (a) Given $0<\varepsilon<1$ we have

$$\int_{K(t)} \Psi(x,\lambda)f(x)dx \geqq \int_{K(t)} \Psi_i(\lambda)f^+(x)dx - \int_{K(t)} \Psi_s(\lambda)f^-(x)dx$$

$$\geqq (1-\varepsilon)\int_{K(t)} f^+(x)dx - (1+\varepsilon)\int_{K(t)} f^-(x)dx$$

for all λ sufficiently close to λ_0. Since $P(f;\varphi)>0$ we have $P_0(f;\varphi)>0$ and $\liminf_{t\to\infty}(\int_{K(t)}f^+/\int_{K(t)}f^-)=L>1$ or $f^-\equiv 0$. In either case, $(1-\varepsilon)\int_{K(t)}f^+-(1+\varepsilon)\int_{K(t)}f^-\geqq 0$ for all $t\geqq$ some t_0 if ε is sufficiently small.

(b) Since $f^-\in L^1$ we have

$$\int_{K(t)} \Psi(x,\lambda)f(x)dx \geqq \int_{K(t)} \Psi(x,\lambda)f^+(x)dx - \Psi_s(\lambda)\int_{\mathbf{R}^n} f^-(x)dx.$$

If $f^+\in L^1$ then $\Psi(x,\lambda)f^+(x)$ is integrable for all λ sufficiently close to λ_0. Thus by Lebesgue's convergence theorem, $\int_{\mathbf{R}^n}\Psi(x,c')f^+(x)dx-\int_{\mathbf{R}^n}f^-(x)dx\geqq 0$ if c' is sufficiently close to λ_0 since $P(f;\varphi)>0$. Consequently, $\exists$ a neighborhood N of λ_0 and $t_0>0$ such that $\int_{K(t)}\Psi(x,c)f^+(x)dx-\Psi_s(c)\int_{\mathbf{R}^n}f^-(x)dx\geqq 0$ for all $c\in N$ and $t\geqq t_0$. On the other hand, if $f^+\notin L^1$ let $\lambda_j\to\lambda_0$. By Fatou's lemma

$$\int_{\mathbf{R}^n} f^+dx=\infty\leqq\lim\int_{\mathbf{R}^n}\Psi(x,\lambda_j)f^+(x)dx.$$

Hence $\exists c'$ such that $\int_{\mathbf{R}^n}\Psi(x,c')f^+(x)dx-\Psi_s(c')\int_{\mathbf{R}^n}f^-(x)dx\geqq 0$ and consequently $\exists c\in(a,b)$ and $t_0>0$ with the desired properties.

§3. Statement of results

In this section we consider various aspects of the asymptotic and oscillatory behavior of global solutions of equations (resp. inequalities) of the form

$$\text{(3.1)} \qquad \operatorname{div} A(x,u,u_x) = B(x,u,u_x) \quad (\text{resp. } \geqq B(x,u,u_x))$$

where $u : \mathbf{R}^n \to \mathbf{R}$, u_x is the gradient of u, and $x \in \mathbf{R}^n$. We will always require that $A(x,u,u_x)$ and $B(x,u,u_x)$ are measurable. Since we will assume (for simplicity) throughout this paper that all solutions are of class C^2, it suffices to have $A(x,s,p)$ and $B(x,s,p)$ measurable in x and continuous in s and p. (However, our results would not be affected if we only assumed that u was of class $H^{1,\alpha}_{\text{loc}}(\mathbf{R}^n)$ — with α as below — and a weak solution of (3.1), i.e.

$$\int_{\mathbf{R}^n} A(x,u,u_x)\varphi_x + B(x,u,u_x)\varphi dx = 0 \qquad \forall \varphi \in C^1_0(\mathbf{R}^n).$$

The proofs for this more general formulation would use the same estimates as below, but the differential inequalities we derive below would be replaced by difference equations obtained by choosing appropriate test functions φ — cf. [12].)

Using $\langle \cdot , \cdot \rangle$ and $|\cdot|$ to denote, respectively, the usual inner product and norm in $\mathbf{R}^n$ we now introduce the following

Definition 3.1. (1) Let $\alpha > 1$ be a real number and I a subinterval of $\mathbf{R}$. The vector field $A : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}^n$ satisfies *condition* $A(\alpha, I)$ if there exist positive measurable functions $m = m_I$ and $M = M_I$ on $\mathbf{R}^n$, and an increasing positive function $F = F_I$ on $[0,\infty)$ such that:

(a) $m_I(x)|p|^\alpha \leqq \langle A(x,s,p),p\rangle \quad \forall (x,s,p) \in \mathbf{R}^n \times I \times \mathbf{R}^n$,

(b) $|A(x,s,p)| \leqq M_I(x)|p|^{\alpha-1} \quad \forall (x,s,p) \in \mathbf{R}^n \times I \times \mathbf{R}^n$,

(c) $\displaystyle \sup_{\mathbf{R}^n}\left[\frac{1}{(|x|+1)^{(\alpha-n)/(\alpha-1)}F_I(|x|)} \cdot \frac{M_I^{\alpha/(\alpha-1)}(x)}{m_I(x)}\right] < \infty$ and $\displaystyle \int_1^\infty \frac{dr}{rF(r)} = +\infty.$

(2) A is *α-regular* on $\mathbf{R}^n$ if it satisfies condition $A(\alpha, I)$ with $I = \mathbf{R}$.

(3) A is *left α-regular* on $\mathbf{R}^n$ if it satisfies condition $A(\alpha, I)$ for every interval I of the form $I = (-\infty, N)$, $N > 0$. In this case we write $m_I = m_N$, etc.

(4) A is *finitely α-regular* on $\mathbf{R}^n$ if it satisfies condition $A(\alpha, I)$ for every interval I of the form $I = [-N, N]$, $N > 0$. In this case we write $m_I = \tilde{m}_N$, $M_I = \tilde{M}_N$.

Remarks. (1) Clearly, α-regular $\Rightarrow$ left α-regular $\Rightarrow$ finitely α-regular.

(2) Useful candidates for $F : (1,\infty) \to (0,\infty)$ are $F(r) = \log r$, $F(r) = \log r \cdot \log\log(e + r)$, etc.

(3) In condition $A(\alpha, I)$ we must have $M_I(x) \to 0$ as $|x| \to \infty$ if $\alpha < n$. In fact, $(M_I^{\alpha/(\alpha-1)}/m_I) \geqq M_I^{1/(\alpha-1)}$ so we must have $M_I^{1/(\alpha-1)}(x) \leqq \text{constant} \cdot F(|x|)/(1+|x|)^{(n-\alpha)/(\alpha-1)}$ and

$$\int_1^\infty (\inf M_I^{1/(\alpha-1)}) \frac{dr}{rF(r)} \leqq \int_1^\infty \frac{dr}{r^{1+\delta}} < \infty \qquad \text{with } \delta = (n-\alpha)/(\alpha-1) > 0.$$

On the other hand, if $\alpha \geqq n$ condition $A(\alpha, I)$ allows m_I to approach zero and M_I to tend to infinity.

To describe the behavior of B we introduce (for a fixed real number $\alpha > 1$, a fixed set of real numbers $\beta^{(q)} =_{\text{def}} \{\beta_j\}_{j=1}^q$ with $0 \leqq \beta_1 \leqq \cdots \leqq \beta_q < \alpha$, and a fixed continuous function $\psi : \mathbf{R} \to \mathbf{R}$ with $\psi(0) = 0$) the following

Definition 3.2. (1) For an interval $I \subseteq \mathbf{R}$, the function $B : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ satisfies condition $B(\alpha, I, \beta^{(q)}, \psi, A)$ if A satisfies $A(\alpha, I)$ and $\exists$ locally integrable functions $f_I, g_I \geqq 0$, $h_{j,I} \geqq 0$ $(j = 1, \cdots, q)$, and $h_I(x) \geqq 0$ such that:

(a) $B(x, s, p) \geqq f_I(x) - g_I(x)\psi(u) - \Sigma_{j=1}^q h_{j,I} |p|^{\beta_j} - h_I |p|^\alpha$ for all $(x, s, p) \in \mathbf{R}^n \times I \times \mathbf{R}^n$.

(b) $\sup_{\mathbf{R}^n} h_I(x)/m_I(x) < \infty$ (with m_I as in condition $A'(\alpha, I)$).

(c) $\exists \varphi = \varphi_I \in D_B(\mathbf{R}^n)$ such that

(i) $P(f_I; \varphi_I) > 0$ (cf. §2),

(ii) $\limsup_{r\to\infty} \{\int_{K_I(t)} [h_{j,I}^\alpha / m_I^{\beta_j}]^{1/(\alpha-\beta_j)} / \int_{K_I(t)} f_I^+\} < \infty$ (for each $j = 1, \cdots, q$), and

(iii) $\limsup [\int_{K_I(t)} g / \int_{K_I(t)} f^+] < \infty$, where $K_I(t) = \varphi_I(B(t))$.

Here we assume $f_I \not\equiv 0$ if either g_I or some $h_{j,I}$ is $\not\equiv 0$.

(2) If A is α-regular and B satisfies condition $B(\alpha, I, \beta^{(q)}, \psi, A)$ for some choice of $\beta^{(q)}$ and ψ when $I = \mathbf{R}$ then $\operatorname{div} A \geqq B$ is *α-regular.*

(3) $\operatorname{div} A \geqq B$ is *strongly α-regular* if it is α-regular with the choice $\psi = 0$.

(4) $\operatorname{div} A \geqq B$ is *finitely α-regular* if A is finitely α-regular and, for some fixed choice of $\beta^{(q)}$ and ψ_I, B satisfies condition $B(\alpha, I, \beta^{(q)}, \psi; A)$ for each I of the form $I = [-N, N]$, $N > 0$. (In this case we write, when $I = [-N, N]$, $f_I = f_N$, etc.)

Remarks. (1) A discussion of the natural occurrence of conditions such as those imposed on A and B above can be found in Serrin's papers [21] and [23].

(2) Observe that if $h_{I,j} \in L^{(\alpha-\beta_j)/\beta_j}$ (for $j = 1, \cdots, q$), h_I is constant and $g_I \in L^1$ while $P(f_I) > 0$ and $m_I(x) > \text{const} > 0$, then the conditions in Definition 3.2 [parts 1b(ii)–(iii)] are satisfied.

(3) The local structure of solutions of equations $\operatorname{div} A = B$, with A and B satisfying conditions similar to those above, has been studied, for example, by Serrin [21–24] and Trudinger [27]. We are interested in the behavior near ∞ of global solutions of $\operatorname{div} A \geqq B$, and the conditions imposed here are slightly weaker than those imposed on A in Serrin's study [22] of solutions of $\operatorname{div} A = 0$ in a neighborhood of infinity.

Since the conditions imposed above are complicated by their generality it is well to consider some explicit examples of some interest. We, therefore, note the following

Example. Let the vector $b(x)=(b^1(x), b^2(x))$ and the function $c(x)$ satisfy $\int_{\mathbf{R}^2}|b|^2<\infty$ and $\int_{\mathbf{R}^2}|c|\,dx<\infty$. Let the matrix $(a^{ij}(x,s,p))$ satisfy $0<\lambda(x)|p|^2 \leqq a^{ij}(x,s,p)p_ip_j$ and $\|a^{ij}(x,s,p)\|=\Lambda(x)<\infty\ \forall(x,s,p\neq 0)$ where $\|\cdot\|$ denotes some matrix norm. If

$$\sup_{\mathbf{R}^2}\left[\left(\frac{1}{\log(1+|x|)}\right)\frac{\Lambda^2(x)}{\lambda(x)}\right]<\infty, \qquad P(f)>0 \tag{3.2}$$

(e.g. $f\geqq 0$) and $f\not\equiv 0$ then $\partial_i a^{ij}(x,u,u_x)\partial_j u+b^i(x)\partial_i u+c(x)u\geqq f(x)$ (where $\partial_i=\partial/\partial x_i$) is a 2-regular inequality. If $c\equiv 0$ it is even strongly 2-regular.

With these preliminaries out of the way we can now list our main results. We first consider theorems of Liouville-type for global solutions u that satisfy growth restrictions only from above:

Theorem 3.1′. *Let A be left α-regular on $\mathbf{R}^n$. If u is a global solution of* $\operatorname{div} A(x,u,u_x)\geqq 0$ *on $\mathbf{R}^n$ with* $\sup_{\mathbf{R}^n} u<\infty$ *then* $u\equiv$ constant.

This theorem, which generalizes the classical Liouville theorem for solutions of $\Delta u\geqq 0$ on $\mathbf{R}^2$, is — in turn — a special case of each of the following two theorems:

Theorem 3.2. *Suppose that for each $N>0$ $\exists$ a non-negative $P(f_N)>0$, and $B(x,s,p)\geqq f_N(x)$ $\forall(x,s,p)\in\mathbf{R}^n\times(-\infty,N]\times\mathbf{R}^n$. Let A be left α-regular on $\mathbf{R}^n$ and let u be a global solution of* $\operatorname{div} A(x,u,u_x)\geqq B(x,u,u_x)$. *If* $\sup u<\infty$ *then* $u\equiv$ constant.

Theorem 3.2. *Suppose that for each $N>0$ $\exists$ a locally locally integrable function $h_N(x)$ such that $B(x,s,p)\geqq -h_N(x)|p|^\alpha$ for $\forall(x,s,p)\in\mathbf{R}^n\times(-\infty,N]\times\mathbf{R}^n$. Let A be left α-regular on $\mathbf{R}^n$ and such that* $\sup_{\mathbf{R}^n} h_N(x)/m_N(x)=_{\text{def}}H_N<\infty$ *for all $N>0$. If u is a global solution of* $\operatorname{div} A(x,u,u_x)\geqq B(x,u,u_x)$ *such that* $\sup_{\mathbf{R}^n} u<\infty$ *then* $u\equiv$ constant.

If the conditions on A and B in Theorem 3.2 are made more restrictive then the conditions on u may be weakened as in

Theorem 3.3. *Let A be α-regular in $\mathbf{R}^n$ with $\alpha\geqq n$, and assume that* $\sup_{\mathbf{R}^n} M^{\alpha/(\alpha-1)}/m<\infty$. *Suppose that $B(x,s,p)\geqq -h(x)|p|^\alpha$ for some non-negative function $h\in L^\infty$ that satisfies $H=\sup_{\mathbf{R}^n} h(x)/m(x)<\infty$ and let u be a global solution of* $\operatorname{div} A(x,u,u_x)\geqq B(x,u,u_x)$. *If $\alpha=n$ and* $\limsup_{r\to\infty} u/\log F(r)<(\alpha-1)/H$ *(with $1/0=+\infty$) for some positive, increasing function F satisfying* $\int_1^\infty dr/rF(r)=+\infty$ *then* $u\equiv$ constant. *If $\alpha>n$ and* $\limsup_{r\to\infty} u/\log r<(\alpha-n)/H$ *(with $1/0=+\infty$) then* $u=$ constant.

Remarks. (1) After the work described here was completed, there appeared the paper [14] of M. Meier which contains (among other results) a special case of Theorem 3.2. Meier assumes that $m(x)\geqq\text{const}>0$, $h(x)\leqq\text{const}<\infty$, and $\alpha=n=$

2, and he concludes that a weak solution u is constant if $\sup u < \infty$. As remarked above, our results are also valid for weak solutions although we do not give the proof here.

(2) Note that if A is, say, α-regular and $\sup M^{\alpha/(\alpha-1)}/m < \infty$ then $m(x) \leqq M(x) \leqq$ constant $< \infty$ and so $\sup_{\mathbf{R}^n} h(x)/m(x) < \infty \Rightarrow h \in L^\infty$. This explains the change in the hypotheses concerning $h(x)$ between Theorems 3.2 and 3.3.

We now turn to restrictions on the oscillation of global solutions that are bounded from both sides. Before stating our results, we remark that the relation of the theorems below to the classical theory, say for $\Delta u \geqq 0$ on $\mathbf{R}^2$, is clarified by reformulating a version of Liouville's theorem for subharmonic functions as follows: If u is a global solution of $\Delta u = f \geqq 0$ on $\mathbf{R}^2$ then either osc u = oscillation of u ($=_{\text{def}} \sup u - \inf u$) is infinite or $u \equiv$ constant (in which case we must have $f \equiv 0$). We generalize this (and at the same time, explain how the fine structure of f plays a role) in the following analogues of Theorems 3.1 and 3.2:

Theorem 3.4. *Let A be finitely α-regular on $\mathbf{R}^n$ and suppose that for each $N > 0$ $\exists$ locally integrable functions $h_N (\geqq 0)$ and f_N such that: $P(f_N) > 0$, $\bar{H}_N = \sup_{\mathbf{R}^n} h_N(x)/\bar{m}_N(x) < \infty$ and $B(x, s, p) \geqq f_N(x) - h_N(x)|p|^\alpha$ for all $(x, s, p) \in \mathbf{R}^n \times [-N, N] \times \mathbf{R}^n$. If u is a global solution of* $\operatorname{div} A(x, u, u_x) \geqq B(x, u, u_x)$ *then*

$$\text{either} \quad u \equiv \text{constant} \quad \text{or} \quad \sup|u| \geqq \sup_N \min\left\{\frac{1}{2H_N} P(f_N), N\right\}.$$

Corollary 3.4.1. *Suppose that for each $N > 0$ $\exists f_N \in L^1_{\text{loc}}$ such that $P(f_N) > 0$ and $B(x, s, p) \geqq f_N(x)$ for all $(x, s, p) \in \mathbf{R}^n \times [-N, N] \times \mathbf{R}^n$. If A is finitely α-regular on $\mathbf{R}^n$ then any nonconstant global solution of* $\operatorname{div} A \geqq B$ *has infinite oscillation.*

As another consequence of Theorem 3.4 we have

Corollary 3.4.2. *Let A be α-regular on $\mathbf{R}^n$ and suppose that for $\forall(x, s, p)$ B satisfies $B(x, s, p) \geqq f(x) - h(x)|p|^\alpha$ with $f \in L^1_{\text{loc}}$, $P(f) > 0$ and $H = \sup_{\mathbf{R}^n} h/m < \infty$. If u is a global solution of* $\operatorname{div} A(x, u, u_x) \geqq B(x, u, u_x)$ *then either u =* constant *or* osc $u \geqq (1/H)P(f)$ *(with $a/0 = +\infty$). Thus, if $P(f) = +\infty$ (e.g. $f \geqq 0$) every bounded solution is a constant.*

Moreover, we have the following general results — the proofs of which may be reduced to that of Theorem 3.4.

Theorem 3.5. *If the differential inequality* $\operatorname{div} A(x, u, u_x) \geqq B(x, u, u_x)$ *is finitely α-regular on $\mathbf{R}^n$, then $\exists$ an explicit positive extended-real number $\gamma_0 = \gamma_0(A, B)$ with the following property: If u is a global solution of* $\operatorname{div} A \geqq B$ *and* $\sup|u| < \gamma_0$ *then u =* constant.

Theorem 3.6. *If the differential inequality* $\operatorname{div} A(x, u, u_x) \geqq B(x, u, u_x)$ *is* *strongly α-regular on* $\mathbf{R}^n$ *then* $\exists$ *an explicit positive extended-real number* $\gamma_1 = \gamma_1(\mathbf{A}, B)$ *with the following property: If u is a global solution of* $\operatorname{div} \mathbf{A} \geqq B$ *and* $\operatorname{osc} u < \gamma_1$ *then* $u \equiv$ constant.

Corollary 3.6.1. *Let* $\operatorname{div} \mathbf{A}(x, u, u_x) \geqq B(x, u, u_x)$ *be strongly α-regular on* $\mathbf{R}^n$, *and suppose that* $C : \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ *satisfies* $C(x, s, p) \geqq -k(x)|p|^{\alpha+\delta}$ *where* $\delta > 0$ *and* $\sup k(x)/m(x) < \infty$. *Given* $\omega > 0$, $\exists$ *an explicit positive extended-real number* γ_2 *with the following property: If u is nonconstant, bounded global solution of* $\operatorname{div} \mathbf{A}(x, u, u_x) \geqq B(x, u, u_x) + C(x, u, u_x)$ *with* $\sup |u_x| \leqq \omega$ *then* $\operatorname{osc} u \geqq \gamma_2$. *Moreover,* $\gamma_2(\mathbf{A}, B, C) \leqq \gamma_1(\mathbf{A}, B)$.

Since the case of linear elliptic equations is of special interest, it is worthwhile to record some of the previous results, as well as some new ones, for linear equations in two variables. For this purpose we introduce the inequality

$$\partial_j a^{ij}(x, u, u_x)\partial_i u + b^i(x)\partial_i u + c(x)u \geqq f(x) \tag{3.3}$$

where the coefficients are assumed, for simplicity, to satisfy the following *general conditions* (GC):

(a) $\exists$ positive functions $\lambda(x) \leqq \Lambda(x)$ such that:

(i) $\lambda(x)|p|^2 \leqq a^{ij}(x, s, p)p_i p_j \leqq \Lambda(x)|p|^2$ $\forall (x, s, p) \in \mathbf{R}^2 \times \mathbf{R} \times \mathbf{R}^2$,

(ii) $\sup \Lambda^2(x)/m(x) < \infty$ (or more generally: $\sup_{|x|=r} \Lambda^2(x)/\lambda(x) =_{\text{def}} \mu(r)$ satisfies $\int_1^\infty dr/r\mu(r) = +\infty$);

(b) $P(f) > 0$, and $f \not\equiv 0$ if $|b|^2 + |c| \not\equiv 0$;

(c) $\exists \varphi \in D_B(\mathbf{R}^n)$ such that $P(f, \varphi) > 0$ and, with $\varphi(B(t)) = K(t)$, $Q_\varphi(b) =_{\text{def}} \limsup_{t\to\infty}[(\int_{K(t)}|b|^2/m)/\int_{K(t)} f^+] < \infty$, and $R_\varphi(c) =_{\text{def}} \limsup_{t\to\infty}[\int_{K(t)}|c|/\int_{K(t)} f^+] < \infty$. We set

$$E = \exp(-P(f)),$$

$$Q = \sup\{Q_\varphi(b) : \varphi \text{ satisfies } P(f; \varphi) > 0\}, \tag{3.4}$$

$$R = \sup\{R_\varphi(c) : \varphi \text{ satisfies } P(f; \varphi) > 0\}.$$

We then have

Theorem 3.7. *If the coefficients* a^{ij}, b^i, c, *and* f *in inequality* (3.3) *satisfy conditions* (GC) *above, then every bounded, nonconstant global solution of the inequality* (3.3) *satisfies*

$$\sup|u| \geqq \gamma_0 \qquad \text{where } \gamma_0 = \min\left\{\frac{1-E}{2R}, \frac{(1-E)}{2Q} \cdot \ln\left(\frac{4}{3+E}\right)\right\}$$

and E, P and Q are as in (3.4).

The proof of this result gives the following sharpened version if $b \equiv 0$:

Corollary 3.7.1. *Let (a^{ij}) and f be as in Theorem 3.7. If u is a global solution of*

$$\partial_i a^{ij}(x, u, u_x)\partial_j u + c(x)u \geqq f$$

then $\sup_{\mathbf{R}^2} |u| \geqq 1/R$ *where R is as in* (3.4).

Theorem 3.8. *Let the matrix (a^{ij}) and the vector function $b(x) = (b^1(x), b^2(x))$ satisfy the condition* (GC), *and let $f \in L^1_{\mathrm{loc}}$ satisfy $P(f) \neq 0$ and $f \not\equiv 0$. Let $\tilde{E}$ and $\tilde{Q}$ be defined as are E and Q, after first replacing f with* $[\operatorname{sgn} P(f)] \cdot f$. *If u is a global solution of*

$$\partial_i a^{ij}(x, u, u_x)\partial_j u + b^i\partial_i u = f$$

on $\mathbf{R}^2$, *then*

$$\operatorname{osc} u \geqq \gamma_1 \qquad \text{where } \gamma_1 \text{ satisfies} \left[\ln\left(\frac{4}{3+\tilde{E}}\right)\right]\frac{(1-\tilde{E})}{\tilde{Q}} \leqq \gamma_1 \leqq \frac{4}{\tilde{Q}e}.$$

Moreover, if $\tilde{E} < 1/e$ we may take $\gamma_1 \geqq (4/Q)(1/e - \tilde{E})$. Thus if $\tilde{E} = 0$ (i.e. $P(f) = \pm\infty$) we may take $\gamma_1 = 4/\tilde{Q}e$.

If it is not necessary to control terms of the form $b^i\partial_i u$, then we can obtain similar results even if the matrix is allowed to be more degenerate:

Theorem 3.9. *Suppose that the trace of the non-negative 2×2 matrix $\mathcal{A}(x, s, p) = (\mathcal{A}^{ij}(x, s, p))$ satisfies $\operatorname{tr}(\mathcal{A}^{ij}(x, s, p)) \leqq T(x)$ for all $(x, s, p) \in \mathbf{R}^2 \times \mathbf{R} \times \mathbf{R}^2$. Let u be a global solution of $\partial_i(\mathcal{A}^{ij}(x, u, u_x)\partial_j u) \geqq f(x)$ where $P(f) > 0$. Then $\mathcal{A}(x, u, u_x)u_x \equiv 0$ (in particular, $u \equiv$ constant if $\ker \mathcal{A} = 0$, i.e. if the equation is elliptic) if any of the following conditions are satisfied*:

(a) $\sup u < \infty$, $f^- \in L^1$ *and* $\sup_{|x| \geqq 1}[T(x)/F(|x|)] < \infty$ *where $F > 0$ satisfies* $\int_1^\infty dr/rF(r) = +\infty$,

(b) $\sup |u| < \infty$, *and* $\sup_{|x| \geqq 1}[T(x)/F(|x|)] < \infty$ *with F as in* (a),

(c) $\sup T(x) < \infty$, $f \geqq 0$ *and* $\limsup(u(x)/\log(1 + F(|x|))) < \infty$ *with F as in* (a).

Example. As an example of a matrix $\mathcal{A}(x, u, u_x) = (\mathcal{A}^{ij}(x, u, u_x))$ that satisfies the hypotheses of the Theorem 3.9, has a nontrivial kernel and (yet) for which $\mathcal{A}(x, u, u_x)u_x \equiv 0 \Rightarrow u \equiv$ constant, we may take

$$\mathcal{A}_0 = \frac{1}{u_{x_1}^2 + u_{x_2}^2 + 1}\begin{bmatrix} u_{x_1}^2 & u_{x_1}u_{x_2} \\ u_{x_1}u_{x_2} & u_{x_2}^2 \end{bmatrix}. \quad \mathcal{A}_0 \text{ has eigenvectors } \begin{bmatrix} u_{x_1} \\ u_{x_2} \end{bmatrix} \text{ and } \begin{bmatrix} -u_{x_2} \\ u_{x_1} \end{bmatrix}$$

with eigenvalues $\lambda_1 = |u_x^2|/(1 + |u_x^2|)$ and $\lambda_2 = 0$, respectively. (So $\mathcal{A}_0 \cdot u_x \equiv 0 \Rightarrow u_x \equiv 0$.) Thus, if u is a semibounded (i.e. bounded above or below) solution of

$$\left(\frac{u_{x_1}^3 + u_{x_1}u_{x_2}^2}{W^2}\right)_{x_1} + \left(\frac{u_{x_1}^2u_{x_2} + u_{x_2}^3}{W^2}\right)_{x_2} = 0,$$

where we have set $W^2 = 1 + |u_x|^2$, then $u = \text{constant}$. Of course, every linear function $u = ax_1 + bx_2 + c$ is also a solution.

Remark. In [7] Gilbarg and Serrin prove a result similar to Theorem 3.9: If $\mathcal{A} > 0$, $\sup T(x) < \infty$, $f = 0$, and $|u| = O((\log r)^{1-\delta})$ then $u \equiv \text{constant}$. (However, it seems that their proof, which uses an idea of Finn [4] as does ours, actually works even for $f \geqq 0$ if $|u| = O((\log r)^{1-\delta})$.) For $\sup T(x) < \infty$ our result requires only the upper bound on u given in part (c) of the theorem.

It is hardly necessary to point out that these results are weak analogues of a celebrated theorem of S. Bernstein [1] (cf. [9], [15]): If u is a bounded solution of $A(x, y)u_{xx} + 2B(x, y)u_{xy} + C(x, y)u_{yy} = 0$ and $AC - B^2 > 0$ (pointwise) then $u \equiv$ constant. It is well-known (Hopf [10]) that such a result is false (i) for $\mathbf{R}^n$, $n > 2$ and (ii) under the weaker hypothesis $\sup u < \infty$ even for $\mathbf{R}^2$. On the other hand, if $\partial_i(\mathcal{A}^{ij}(x)\partial_j u) = f \equiv 0$ is uniformly elliptic on $\mathbf{R}^n$, $n \geqq 2$, then any semi-bounded global solution is constant (Moser [16]). For other theorems of Liouville-type, we refer to the papers mentioned in the introduction.

Broadly speaking, there seem to be three techniques for proving theorems of Liouville-type: (a) via Harnack-type inequalities, (b) via derivation of pointwise-interior estimates for $|u_x|$ (e.g. by a maximum principle), and (c) via the method of differential and/or difference inequalities (the method exploited in this paper). It should be observed that, even for uniformly elliptic equations of divergence type $\operatorname{div} \mathbf{A} = B$, strong Harnack inequalities are valid only if $B(x, s, p)$ has special behavior with respect to its growth in x and p (cf. [2], [7], [20–22], and [28]). In fact, there is no strong Harnack inequality for $\Delta u \geqq 0$ in $\mathbf{R}^2$ [18] — the prototype of the equations treated in this paper. Similarly, the approach via derivation of *a priori* estimates for $\sup |u_x|$ also fails for the equation $\Delta u \geqq 0$ on $\mathbf{R}^2$ — while it works well for equations of the form $\mathcal{A}_{ij}u_{ij} = f(u, u_x)$ even in $\mathbf{R}^n$ (cf. [17], [25]).

The method exploited in this paper is also applicable in certain cases in which the function $B(x, s, 0)$ is not $\geqq 0$ as a function of x (even in a generalized sense) but rather has some special properties from the point of view of its dependence on s. We conclude this section with several results of this nature:

Theorem 3.10. *Let* $\mathbf{A}$ *be* α*-regular on* $\mathbf{R}^n$ *and suppose that*: (a) $B(x, s, p) = B_1(x, s, p) + B_2(x, s, p)$, (b) $\operatorname{sgn}(s) \cdot B_1(x, s, p) \geqq 0$ *for* $\forall(x, s, p)$, *and* $\exists h(x) \geqq 0$ *such that* $H = \sup_{\mathbf{R}^n} h(x)/m(x) < \infty$ *and* $|B_2(x, s, p)| \leqq h(x)|p|^\alpha$ *for* $\forall(x, s, p)$. *If* u *is a bounded global solution of* $\operatorname{div} \mathbf{A} = B$ *then* $u \equiv$ constant.

Corollary 3.10.1. *Let* u *be a bounded global solution of the divergence form equation*

$$\partial_i(\mathcal{A}^{ij}(x, u, u_x)\partial_j u) = f(u) + g(x, u_x) \qquad \text{in } \mathbf{R}^2.$$

If $\mathscr{A}'' = \mathscr{A}''$, $\exists C_0$ *such that* $C_0|u_x|^2 \leqq \mathscr{A}''(x,u,u_x)\partial_i u \partial_j u \leqq (1/C_0)|u_x|^2$ *for all* x, $sf(s) \geqq 0$, *and* $|g(x,p)| \leqq \text{const}|p|^2$ *then* $u \equiv \text{constant}$.

Theorem 3.11. *Let* A *and* B *be as in Theorem* 3.10 *and let* $\mu =_{\text{def}} 1/(\omega - 1)$ *where* ω *satisfies* $1/(\omega - 1) = \exp(\omega)$. *If* u *is a global solution of* $\operatorname{div} A(x,u,u_x) = B(x,u,u_x)$ *such that* $-1/\mu H < \inf u$ *and* $\exists F:(0,\infty)\to(0,\infty)$ *satisfying Definition* 3.1(c)

$$\int_1^\infty \frac{dt}{tF(t)^2} = +\infty \quad \textit{and} \quad \limsup_{|x|=r\to\infty} \frac{u(x)}{\log F(r)} < \frac{(\alpha-1)}{\omega}|\inf u|$$

then $u \equiv \text{constant}$.

Remarks. (1) Below, in §6, a number of examples are presented which show that, by and large, the theorems of this section are sharp.

(2) In a companion paper [13] we give other results on the global behavior of solutions of equations (resp. inequalities) of the form (3.1) for cases in which $\alpha < n$ and A satisfies a weaker version of α-regularity which only requires $\sup_{\mathbf{R}^n} M^{\alpha/(\alpha-1)}/m < \infty$.

§4. The proofs of the theorems

In this section we prove Theorems 3.1–3.11. The proofs all use, essentially, the same basic idea: Under the hypotheses of each of the theorems of §3 a choice is made, in each case, of a 1-parameter family of increasing C^1 functions $\Phi_\lambda : \mathbf{R}^1 \to \mathbf{R}^1$ and a continuous function $\psi(x,t) \geqq 0$ such that $\psi(x,0) = 0$ and $\psi(x,t) > 0$ if $t \neq 0$. It is then shown that either $u_x \equiv 0$ or for some diffeomorphism $\varphi \in D_B$, some $t_0 > 0$, and some $\lambda = \lambda_0$ a differential inequality can be derived for $J(t) = \int_{\varphi(B(t))} \Phi_\lambda(u)\psi(x,|u_x|^2)dx$ (on $t_0 \leqq t < \infty$) that leads — after an integration — to a contradiction.

We now turn to the

Proof of Theorems 3.1–3.2. Suppose that u satisfies $\operatorname{div} A \geqq B$. Choose a C^1 function $\Phi : \mathbf{R} \to (0,\infty)$ such that $\Phi'(t) > 0$, and diffeomorphism $\varphi \in D_B(\mathbf{R}^n)$. Let $\lambda \in (0,\infty)$. An integration of $\operatorname{div}[\Phi(\lambda u)A]$ over the set $K(t) = \varphi(B(t))$ immediately yields the inequality

$$\lambda \int_{K(t)} \Phi'(\lambda u)\langle u_x, A\rangle dx + \int_{K(t)} \Phi(\lambda u) B dx \leqq \int_{\partial K(t)} |\Phi(\lambda u)||A(x,u,u_x)|\,d\sigma. \tag{4.1}$$

If A is left α-regular and $N = \max\{1, \sup u\}$ we have

$$\lambda \int_K \Phi'(\lambda u)\langle u_x, A\rangle dx \geqq \lambda \int_K m_N(x)\Phi'(\lambda u)|u_x|^\alpha dx \tag{4.2}$$

and

$$\int_{\partial K} \Phi(\lambda u)|A|\,d\sigma \leqq \int_{\partial K} \Phi(\lambda u)M_N(x)|u_x|^{\alpha-1}d\sigma. \tag{4.3}$$

It follows from Hölder's inequality and (4.1)–(4.3) that

$$\begin{aligned} &\left[\lambda\int_K m_N\cdot\Phi'(\lambda u)|u_x|^\alpha dx + \int_K \Phi(\lambda u)B\,dx\right]^{\alpha/(\alpha-1)} \\ &\leqq \left[\int_{\partial K} m_N\cdot\Phi'(\lambda u)|u_x|^\alpha d\sigma\right]\left[\int_{\partial K}\left(\frac{M_N^\alpha}{m_N^{\alpha-1}}\right)\cdot\frac{[\Phi(\lambda u)]^\alpha}{[\Phi'(\lambda u)]^{\alpha-1}}\,d\sigma\right]^{1/(\alpha-1)}. \end{aligned} \tag{4.4}$$

Choosing Φ such that Φ/Φ' is bounded on $(-\infty, \lambda N)$ and setting $J(t) = \int_{K(t)} m_N(x)\Phi'(\lambda u)|u_x|^\alpha dx$, we conclude that

$$\left[\lambda J(t) + \int_{K(t)} \Phi(\lambda u)B\,dx\right]^{\alpha/(\alpha-1)} \leqq C_{N,\lambda}J'(t)\cdot G(t) \tag{4.5}$$

with

$$C_{N,\lambda} = \sup_{x\in\mathbf{R}^n}\left\{\left|\frac{\Phi(\lambda u)}{\Phi'(\lambda u)}\right|\cdot\Phi(\lambda u)^{1/(\alpha-1)}\right\}$$

and, for some constant γ_φ (depending on φ),

$$G(t) = \gamma_\varphi \sup_{x\in\partial K(t)}\left\{\frac{M_N^{\alpha/(\alpha-1)}}{m_N}(x)\right\}\left[\int_{\partial K(t)} d\sigma\right]^{1/(\alpha-1)}$$

Note that under the hypotheses of the theorem $G(t) \leqq C_\varphi(1+t)F_N(C_\varphi(1+t))$ for some constant $C_\varphi > 0$ which depends on φ. To proceed we need the following lemma:

Lemma 4.1. *Under the hypotheses of Theorems* 3.1 *and* 3.2, *it is possible to choose a C^1 function Φ, a diffeomorphism $\varphi \in D_B(\mathbf{R}^n)$ and positive real numbers λ, δ, and t_0 such that*

$$\lambda J(t) + \int_{K(t)} \Phi(\lambda u)B(x,u,u_x)\,dx \geqq \delta J(t) \qquad \textit{for all } t \geqq t_0.$$

The proof of this lemma is postponed until the combined proof of the theorems is complete.

Using the lemma and observing that $J(t)$ is nondecreasing we may conclude that either $J(t) \equiv 0$ or $\exists$ some $t_1 \geqq t_0$ and constants C_1 and C_2 such that

$$(4.6)\qquad -\frac{d}{dt}\frac{1}{J(t)^{1/(\alpha-1)}} \geqq \frac{C_1}{(1+t)F_N(C_2(1+t))} \equiv_{\text{def}} \frac{1}{T(t)}.$$

Integrating this differential inequality from t_2 to t_3 in (t_1, ∞) yields

$$\frac{1}{J(t_2)^{1/(\alpha-1)}} \geqq \frac{1}{J(t_2)^{1/(\alpha-1)}} - \frac{1}{J(t_3)^{1/(\alpha-1)}} \geqq \int_{t_2}^{t_3} \frac{dt}{T(t)}.$$

Since $\int_1^\infty dt/T(t) = +\infty$ we may let $t_3 \to \infty$ and conclude that $J(t_2) = 0$. Since $t_2 > t_1$ is arbitrary we conclude that $J(t) \equiv 0$. Thus we must have $u_x \equiv 0$ and $u \equiv$ constant. To complete the proof of Theorems 3.1 and 3.2 we give the

Proof of Lemma 4.1. If $B(x, s, p)$ satisfies the hypotheses of Theorem 3.1 then with $N = \max\{1, \sup u\}$ (as in the proof of the theorem) $\exists f = f_N \in L^1_{\text{loc}}$ with $f_N^- \in L^1$ and $P(f, \varphi_0) > 0$ for some $\varphi_0 \in D_B$ (cf. §2). Choosing $\varphi = \varphi_0$ we have, since $\Phi > 0$,

$$\int_{K(t)} \Phi(\lambda u) B(x, u, u_x) dx \geqq \int_{\varphi_0(B(t))} \Phi(\lambda u) f dx \geqq 0$$

for all $t \geqq$ some t_0 and $0 < \lambda \leqq$ some λ_0 by part (b) of Lemma 2.1 of §2.

If B satisfies the hypotheses of Theorem 3.2 then again with $N = \max\{1, \sup u\}$ and choosing Φ as above (i.e., $\Phi > 0$ and $\sup \Phi/\Phi' < \infty$, e.g., $\Phi = \exp$) we have (with the diffeomorphism φ = identity, say)

$$\lambda J(t) + \int_{K(t)} \Phi(\lambda u) B(x, u, u_x) \geqq \int \left[\lambda - \frac{h_N(x)}{m_N(x)} \frac{\Phi(\lambda u)}{\Phi'(\lambda u)}\right] \Phi'(\lambda u) m_N |u_x|^\alpha.$$

Then it suffices to take $\lambda = H_N \cdot \sup_{t>0} \Phi(t)/\Phi'(t) + \delta$ (with $\delta > 0$ but otherwise arbitrary). This completes the proof of the lemma and hence the proof of Theorems 3.1 and 3.2.

Proof of Theorem 3.3. Choosing $\Phi = \exp u$ and φ = identity we may follow the proof of Theorems 3.1 and 3.2 above (replacing m_N and M_N with $m(x)$ and $M(x)$ — and without introducing $N > 0$ at all — since A is α-regular) to derive, in place of (4.5), the inequality

$$\left[\lambda J(t) + \int_{B(t)} e^{\lambda u} B dx\right]^{\alpha/(\alpha-1)} \leqq \omega_n J'(t) \cdot \sup_{\mathbf{R}^n} \frac{M^{\alpha/(\alpha-1)}}{m} \cdot \left[\sup_{|x|=t} e^{\lambda u} t^{n-1}\right]^{1/(\alpha-1)}$$

where $J(t) = \int_{B(t)} \exp(\lambda u) m \cdot |u_x|^\alpha dx$ and ω_n is a constant that depends only on n. Now if $\limsup u(x)/\log F(r) < (\alpha - 1)/H$ then $\exists t_0 > 0$ and $\delta > 0$ such that

$$\exp\left(\frac{H+\delta}{\alpha-1} u(x)\right) \leqq F(|x|) \qquad \text{in } |x| \geqq t_0.$$

Choosing $\lambda = H + \delta$ (with this δ) it follows as in the proof of Lemma 4.1 that $\lambda J(t) + \int_{B(t)} \exp(\lambda u) B \geqq \delta J(t)$. Consequently, either $J(t) \equiv 0$ or $\exists$ some $t_1 \geqq t_0$ and a constant $C > 0$ such that

$$-\frac{d}{dt}\left[\frac{1}{J(t)}\right]^{1/(\alpha-1)} \geqq CtF(t)$$

since $\alpha = n$. Since $\int_1^\infty dt/tF(t) = +\infty$ the proof can be completed as above for Theorem 3.1. Since a similar analysis works if $\alpha > n$ and $\limsup u/\log r < (\alpha - n)/H$ we are done.

Proof of Theorem 3.4. Let u be a global solution of $\operatorname{div} A \geqq B$ that has finite oscillation and such that

$$\sup |u| < \sup_{N'} \min\left\{N', \frac{1}{2H_{N'}} P(f_{N'})\right\}.$$

Then choosing N such that $\sup|u| \leqq N$ and $\operatorname{osc} u < (1/H_N)P(f_N)$, we may follow the proof of Theorem 3.1 (with $\Phi(t) = \exp(t)$, and $\bar{m}_N$ and $\bar{M}_N$ replacing m_N and M_N, respectively) to obtain, for some diffeomorphism $\varphi \in D_B(\mathbf{R}^n)$ and $C_{N,\varphi,\lambda} =$ constant > 0:

$$\left[\lambda J_N(t) + \int_K \exp(\lambda u) B\right]^{\alpha/(\alpha-1)} \leqq C_{N,\varphi,\lambda} \cdot J'_N(t) \cdot T_\varphi(t).$$

Here $J_N(t) = \int_{K(t)} \exp(\lambda u) \bar{m}_N(x) |u_x|^\alpha dx$, T_φ is a positive, increasing function on $(0,\infty)$ such that $\int_1^\infty [T_\varphi(t)]^{-1} dt = +\infty$, and $K(t) = \varphi(B(t))$ (cf. (4.5) and (4.6)). We now need the following

Lemma 4.2. *With N as above, $\exists$ positive numbers $\lambda = \lambda_N$, $\delta = \delta_N$, and $t_0 = t_{0,N}$, and a diffeomorphism $\varphi = \varphi_N \in D_B(\mathbf{R}^n)$ such that*

$$\lambda J_N(t) + \int_{\varphi(B(t))} \exp(\lambda u) B dx \geqq \delta J_N(t)$$

for all $t \geqq t_0$.

Assuming the validity of the lemma, it follows (as in the proof of Theorem 3.1) that $J_N(t) \equiv 0$ and $u \equiv$ constant. To complete the proof of the theorem we give the

Proof of Lemma 4.2. Since we have $\lambda J_N(t) + \int_K \exp(\lambda u) B \geqq \int_K \exp(\lambda u) f_N + \int_{K(t)} [\lambda - \bar{H}_N] \bar{m}_N(x) \exp(\lambda u) |u_x|^\alpha$, it suffices to choose $\lambda = \lambda_N = \bar{H}_N + \delta$ (with $\delta > 0$ arbitrary) and prove that

$$\exp(\lambda_N \inf u) \int_{\varphi(B(t))} f_N^+ - \exp(\lambda_N \sup u) \int_{\varphi(B(t))} f_N^- \geqq 0$$

for some $\varphi = \varphi_N$ and all $t \geqq$ some t_0. Since we may choose φ_N such that $P(f_N; \varphi_N)$ is arbitrarily close to $P(f_N)$, it follows that the desired inequality is a consequence (for δ sufficiently small) of $\exp(\lambda_N \operatorname{osc} u) < \sup_\varphi \{\liminf_{t\to\infty}[\int_{\varphi(B(t))} f_N^+ / \int_{\varphi(B(t))} f_N^-]\}$ — i.e. of $\operatorname{osc} u < (1/H_N) P(f_N)$.

Proof of Corollary 3.4.1. This is just an application of the theorem with $H_N = 0$ for all N and $1/0 = +\infty$. (This is valid as is easily verified by inspecting the proof of the theorem.)

Proof of Corollary 3.4.2. If $\operatorname{osc} u < (1/H)P(f)$ and $\operatorname{div} A \geqq B$ is α-regular we have $\operatorname{osc} u < \inf_{N'}\{(1/H_{N'})P(f_{N'})\}$, where $f_{N'}$ and $H_{N'}$ are as in the proof of Theorem 3.4. Choosing $N' = N > \sup|u|$ we may follow the proof of Theorem 3.4 to conclude that $u \equiv$ constant.

Proofs of Theorems 3.5 and 3.6. Set $\varepsilon > 0$,

$$F_N(x) =_{\text{def}} \sum_{j=1}^{q} [h_{j,N}^{\alpha}(x)/\bar{m}_N(x)^{\beta_j}]^{1/(\alpha-\beta_j)}, \qquad C_j(\varepsilon) =_{\text{def}} \frac{\beta_j}{\alpha}\left(\frac{\alpha-\beta_j}{\varepsilon\alpha}\right)^{(\alpha-\beta_j)/\beta_j}$$

and $C(\varepsilon) = \sum_{j=1}^q C_j(\varepsilon)$ if $\operatorname{div} A \geqq B$ is finitely α-regular. Using Young's inequality in the form $ab \leqq \varepsilon a^{\alpha/(\alpha-\beta_j)} + C_j(\varepsilon) b^{\alpha/\beta_j}$, it follows that for all $(x, s, p) \in \mathbf{R}^n \times [-N, N] \times \mathbf{R}^n$ we have

$$B(x,s,p) \geqq f_N(x) - \varepsilon F_N(x) - [C(\varepsilon)\bar{m}_N(x) + h_N(x)]|p|^{\alpha} - g_N(x) \sup_{|s|\leqq N} \psi(s).$$

Since $\psi(0) = 0$ and $\limsup_{t\to\infty}[\int_{\varphi_N(B(t))} F_N(x) + g_N(x) / \int_{\varphi_N(B(t))} f_N^+(x)] < \infty$ for some $\varphi_N \in D_B(\mathbf{R}^n)$ such that $P(f_N, \varphi_N) > 0$, we may choose N_0 and $\varepsilon_0 > 0$ (depending on φ_{N_0}) such that $\tilde{f}_{N_0,\varepsilon}(x) =_{\text{def}} f_{N_0}(x) - \varepsilon F_{N_0}(x) - g_{N_0}(x)\sup_{|s|\leqq N_0}\psi(s)$ satisfies $P(\tilde{f}_{N_0,\varepsilon}; \varphi_{N_0}) > 0$ for all $0 < \varepsilon \leqq \varepsilon_0$. Setting $\tilde{h}_{N_0,\varepsilon}(x) = C(\varepsilon)\bar{m}_{N_0}(x) + h_{N_0}(x)$, so that $\sup_{\mathbf{R}^n}[\tilde{h}_{N_0,\varepsilon}(x)/\bar{m}_{N_0}(x)] = \bar{H}_{N_0,\varepsilon} < \infty$, we have

$$B(x,s,p) \geqq \tilde{f}_{N_0,\varepsilon}(x) - \tilde{h}_{N_0,\varepsilon}(x)|p|^{\alpha} \qquad \text{for all } (x,s,p) \in \mathbf{R}^n \times [-N_0, N_0] \times \mathbf{R}^n.$$

It now follows from the proof of Theorem 3.4 that if

$$\sup|u| < \sup_{\substack{\varepsilon \leqq \varepsilon_0(\varphi_{N_0}) \\ \varphi \ni P(f_{N_0}, \varphi_{N_0}) > 0}} \min_{N_0} \left\{\frac{1}{2\bar{H}_{N_0,\varepsilon}} P(\tilde{f}_{N_0,\varepsilon}; \varphi_{N_0}), N_0\right\} = \gamma_0$$

then $u \equiv$ constant.

If $\operatorname{div} A \geqq B$ is strongly α-regular then Young's inequality leads to the estimate

$$B(x,s,p) \geqq f_\varepsilon(x) - h_\varepsilon(x)|p|^{\alpha} \qquad \forall (x,s,p) \in \mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n.$$

Here $f_\varepsilon(x) =_{\text{def}} f(x) - \varepsilon F(x)$, $h_\varepsilon(x) = C(\varepsilon) \cdot m(x) + h(x)$, $F(x) =_{\text{def}} \sum_{j=1}^q [h_j^{\alpha}(x)/m^{\beta}(x)]^{1/(\alpha-\beta_j)}$ and $C(\varepsilon)$ is as above. If $\varphi \in D_B$ satisfies $P(f, \varphi) > 0$ then we can choose $\varepsilon_0 = \varepsilon_0(\varphi)$ such that $P(f_\varepsilon, \varphi) > 0$ for all $0 < \varepsilon \leqq \varepsilon_0$. For such ε we may apply Corollary 3.4.2 and conclude that if u is a global solution of div $A \geqq B$ with

$$\operatorname{osc} u < \gamma_1 =_{\text{def}} \sup_{\varphi \ni . P(f,\varphi)>0} \sup_{\varepsilon \leqq \varepsilon_0(\varphi)} \left\{ \frac{1}{H_\varepsilon} P(f_\varepsilon) \right\}$$

then $u \equiv$ constant.

Proof of Corollary 3.6.1. Let u be a global solution of div $A \geqq B + C$ with $\sup|u_x| \leqq \omega < \infty$. Since $|u_x|^{\alpha+\delta} = \omega^{\alpha+\delta}|u_x/\omega|^{\alpha+\delta} \leqq \omega^\delta |u_x|^\alpha$, u also satisfies the α-regular inequality div $A \geqq \tilde{B}$ where $\tilde{B} = f(x) - \Sigma_{j=1}^q h_j(x)|p|^{\beta_j} - [h(x) + k(x)\omega^\delta]|p|^\alpha$ and f, h_j and h satisfy $B \geqq f(x) - \Sigma_{j=1}^q h_j(x)|p|^{\beta_j} - h(x)|p|^\alpha$. Following the proof of Theorem 3.6 we find that $u \equiv$ constant if

$$\operatorname{osc} u < \gamma_2 \equiv \sup_{\varphi \ni . P(f,\varphi)>0} \sup_{\varepsilon \leqq \varepsilon_0(\varphi)} \left\{ \frac{1}{\tilde{H}_\varepsilon} P(f_\varepsilon) \right\} \quad \text{where } \tilde{H}_\varepsilon = \sup_{\mathbf{R}^n} \left\{ \frac{h(x)}{m(x)} + \omega^\delta \frac{k(x)}{m(x)} \right\}.$$

Thus $\gamma_2 = \gamma_2(A, B, C) \leqq \gamma_1(A, B)$.

Proof of Theorem 3.7. It suffices to give an explicit value for γ_0 in Theorem 3.5, since the conditions (GC) imply that (3.3) is α-regular. This immediately reduces to finding γ_0 with the following property: If $|u| \leqq \gamma_0$ then $\exists \lambda, \delta$ and $t_0 > 0$ and a diffeomorphism φ such that

$$\begin{aligned} &\lambda \int_{\varphi(B(t))} \exp(\lambda u) \cdot m \cdot |u_x|^2 + \int_{\varphi(B(t))} (f - b^i \partial_i u - cu) \exp(\lambda u) \\ &\qquad \geqq \delta \int_{\varphi(B(t))} \exp(\lambda u) \cdot m \cdot |u_x|^2, \qquad \text{for all } t \geqq t_0. \end{aligned} \tag{4.7}$$

For $\varepsilon > 0$ we have

$$f - b^i \partial_i u \geqq f - \frac{\varepsilon}{m} |b(x)|^2 - \frac{1}{4\varepsilon} m |p|^2$$

and setting $\lambda = \lambda(\varepsilon) = 1/4\varepsilon + \delta$, where $\delta > 0$, it suffices to find $\varepsilon > 0$ and $\varphi \in D_B(\mathbf{R}^n)$ such that

$$\exp[\lambda(\varepsilon) \inf u] \int_{\varphi(B(t))} f^+ dx - \exp[\lambda(\varepsilon) \sup u] \int_{\varphi(B(t))} \left[f^- + \frac{\varepsilon b^2}{m} + \gamma_0 |c| \right] dx \geqq 0$$

for all $t \geqq$ some t_0. This immediately reduces to $\sup_{\varepsilon>0}\{\exp[-\lambda(\varepsilon) \cdot \operatorname{osc} u] - \exp[-P_0(f, \varphi)] - \varepsilon Q_\varphi - \gamma_0 R_\varphi\} > 0$ with

$$Q_\varphi = \limsup_{t \to \infty} \left[\int_{\varphi(B(t))} (|b|^2/m) dx \Big/ \int_{\varphi(B(t))} f^+ \right],$$

$$R_\varphi = \limsup_{t \to \infty} \left[\int_{\varphi(B(t))} |c| dx \Big/ \int_{\varphi(B(t))} f^+ \right].$$

Replacing $1/4\varepsilon$ with η, and noticing that $\operatorname{osc} u \leqq 2\gamma_0$, it clearly suffices to choose γ_0, η and φ such that

$$\exp[-2\eta\gamma_0] > \exp[-P_0(f,\varphi)] + \frac{1}{4\eta} Q + \gamma_0 R.$$

Since $P_0(f,\varphi)$ can be taken arbitrarily close to $P(f)$, it suffices to find γ_0 and η such that

$$\exp[-2\eta\gamma_0] > E + \frac{1}{4\eta} Q + \gamma_0 R \qquad \text{where } E =_{\text{def}} \exp(-P(f)) < 1. \tag{4.8}$$

Putting $\mu =_{\text{def}} \ln(4/(3+E)) > 0$ and $2\eta\gamma_0 = \mu$ it is easily seen that (4.8) follows from the choice

$$\gamma_0 = \min\left\{\frac{1-E}{2R}, \frac{\mu(1-E)}{2Q}\right\}.$$

Proof of Corollary 3.7.1. If $b \equiv 0$ then we may let $\eta \to 0$ in (4.8) and conclude that if $\gamma_0 R < 1$ then $u \equiv$ constant.

Proof of Theorem 3.8. Since $\operatorname{osc}(u) = \operatorname{osc}(-u)$ we may assume that $P(f) > 0$. Following the proof of Theorem 3.7, it is clear that we may choose any $\gamma_1 = \gamma$ such that

$$1 > \left(E + \frac{1}{4\eta} Q\right) \exp(\eta\gamma) =_{\text{def}} F_\gamma(\eta) \qquad \text{for some } \eta > 0.$$

Define $\Gamma = \sup\{\gamma > 0 : \inf_{\eta>0} F_\gamma(\eta) < 1\}$. Let $H_\gamma(\eta) = \exp(\eta\gamma)Q/4\eta$. Then $\min_\eta H_\gamma(\eta) = \gamma Qe/4$. Since $H_\gamma \leqq F_\gamma$, $\inf F_\gamma < 1 \Rightarrow \gamma \leqq 4/Qe \Rightarrow \Gamma \leqq 4/Qe$. On the other hand, $F_\gamma(1/\gamma) = (E + \gamma Q/4)e$ so $\Gamma \geqq (4/Q)(1/e - E)$ for E sufficiently small. Finally, from the proof of Theorem 3.7 it follows that $\Gamma \geqq \mu(1-E)/Q$ in any case.

Proof of Theorem 3.9. Let $\varphi \in D_B(\mathbf{R}^n)$ and $\lambda > 0$. Integration of $\operatorname{div}[\exp(\lambda u)\mathcal{A}u_x]$ over $K(t) = \varphi(B(t))$ yields

$$\lambda \int_{K(t)} e^{\lambda u}\langle u_x, \mathcal{A}u_x\rangle dx + \int_{K(t)} e^{\lambda u} f dx = \int_{\partial K(t)} e^{\lambda u}\langle \mathcal{A}u_x, \hat{n}\rangle d\sigma$$

where $\hat{n}$ denotes the unit exterior normal to $\partial K(t)$. Using the inequality $\langle \mathcal{A}u_x, \hat{n}\rangle \leqq \langle \mathcal{A}u_x, u_x\rangle^{1/2}\langle \mathcal{A}\hat{n}, \hat{n}\rangle^{1/2}$ and setting $\int_{K(t)} e^{\lambda u}\langle u_x, \mathcal{A}u_x\rangle dx \equiv J_{\mathcal{A}}(t)$ we derive, for some constant $C = C_\varphi$,

$$\left[\lambda J_{\mathcal{A}}(t) + \int_{K(t)} e^{\lambda u} f\right]^2 \leqq C_\varphi \cdot J'_{\mathcal{A}}(t) \cdot \sup_{x \in \varphi(\partial B(t))}[T(x) \cdot e^{\lambda u}] \cdot t. \tag{4.9}$$

If (a) $\sup u < \infty$ and $f^- \in L^1$ or if (b) $\sup|u| < \infty$ then, according to Lemma 2.1, we can choose $\lambda > 0$, $\varphi \in D_B(\mathbf{R}^n)$ and $t_0 > 0$ such that $\int_{\varphi(B(t))} e^{\lambda u} f dx \geqq 0$ for all $t \geqq t_0$. With this choice of λ, t_0 and φ, and noting that since $\varphi \in D_B(\mathbf{R}^n)$ $\exists C_1 > 0$ such that $T(\varphi(y)) \leqq C_1 F(C_1(1+|y|))$ for $\forall |y| \geqq t_1$, we conclude that

$$J_{\mathscr{A}}(t)^2 \leqq C J'_{\mathscr{A}}(t) t F(C(1+t)) \qquad \forall t \geqq \text{some } t_2 \tag{4.10}$$

for some constant $C > 0$. As in the proof of Theorem 3.1 this differential inequality leads to the conclusion that $J_{\mathscr{A}}(t) = \int_{K(t)} e^{\lambda u} \langle u_x, \mathscr{A} u_x \rangle \equiv 0$. Since $\mathscr{A}$ is non-negative $\Rightarrow$ $\mathscr{A} u_x \equiv 0$.

If $f \geqq 0$ we may take, for simplicity, φ = identity. If $u(x) \leqq C \log(1 + F(|x|))$ for some $C > 0$ and all x with $|x| \geqq \text{some } t_0$, then we may choose λ such that $\exp(\lambda u) \leqq 1 + F(|x|)$ in $|x| \geqq t_0$. Thus, if $\sup T(x) < \infty$, we again arrive at (4.10), and conclude that $\mathscr{A} u_x \equiv 0$. This completes the proof of the theorem.

Proof of Theorem 3.10. Suppose that $\sup|u| < C$ and let N be a positive integer. Integration of $\operatorname{div}(u^{2N+1} A)$ over the ball $B(t)$ yields, since $u^{2N+1} B_1(x, u, u_x) \geqq 0$,

$$(2N+1) \int_{B(t)} u^{2N} \langle u_x, A \rangle dx + \int_{B(t)} u^{2N+1} B_2(x, u, u_x) dx \leqq \int_{\partial B} |u|^{2N+1} |A| \, d\sigma.$$

Thus

$$\int_{B(t)} |u|^{2N} \left[(2N+1) - |u| \frac{h(x)}{m(x)} \right] m(x) |u_x|^\alpha \leqq \int_{\partial B} |u|^{2N+1} |A| \, d\sigma. \tag{4.11}$$

Using the fact that $\int_{\partial B} |u|^{2N+1} |A| \, d\sigma \leqq C^{(\alpha+2N)/\alpha} \int_{\partial B} |u|^{2N(\alpha-1)/\alpha} M |u_x|^{\alpha-1}$, we conclude that there exists a positive constant $C_{\alpha,N}$ such that

$$\begin{aligned} &\left(\int_{\partial B(t)} |u|^{2N+1} |A| \, d\sigma \right)^{\alpha/(\alpha-1)} \\ &\leqq C_{\alpha,N} \left[\int_{\partial B(t)} |u|^{2N} m |u_x|^\alpha d\sigma \right] \left[\sup_{\partial B(t)} \frac{M(x)^{\alpha/(\alpha-1)}}{m(x)} t^{(n-1)/(\alpha-1)} \right]. \end{aligned} \tag{4.12}$$

With $2N > CH$ and $J_N(t) =_{\text{def}} \int_{B(t)} |u|^{2N} m |u_x|^\alpha dx$, we deduce from (4.11), (4.12), and Definition 3.1 that

$$J_N(t)^{\alpha/(\alpha-1)} \leqq C_{\alpha,N} J'_N(t) \cdot t F(t)$$

for some $F > 0$ with $\int_1^\infty dt / t F(t) = -\infty$. This differential inequality leads to $J_N(t) \equiv 0$, i.e. $u \equiv$ constant, as in the proof of Theorem 3.1.

Corollary 3.10.1 follows immediately from Theorem 3.10 with $\alpha = 2$.

Proof of Theorem 3.11. Integration of $\operatorname{div}[(e^{\lambda u}-1)A]$ over $B(t)$ leads to the inequality

$$\lambda\int_{B(t)} e^{\lambda u}m\,|u_x|^\alpha - \int_{B(t)} (e^{\lambda u}+1)|B_2| \leqq \int_B \operatorname{div}(e^{\lambda u}-1)A \leqq \int_{\partial B(t)} (e^{\lambda u}+1)|A|\,d\sigma \tag{4.13}$$

since $(e^{\lambda u}-1)B_1(x,u,u_x)\geqq 0$.

The left-hand member of (4.13) may be estimated from below by $\int_{B(t)}[\lambda-(e^{-\lambda u}+1)H]me^{\lambda u}|u_x|^\alpha$. Let $a>0$ and set $F_a(\lambda)=(e^{\lambda a}+1)/\lambda$. It is easily seen that $\min_{\lambda>0}F_a(\lambda)=F_a(\omega/a)=\mu a$ where $\mu=1/(\omega-1)$ and $1/(\omega-1)=\exp(\omega)$. Thus if $\inf u\geqq -a>-\infty$, $a<1/H\mu$, $\lambda=\omega/a$ and $\delta=1-\mu aH>0$, we have

$$\int_{B(t)} \operatorname{div}(e^{\lambda u}-1)A \geqq \delta\int_{B(t)} e^{\lambda u}m\,|u_x|^\alpha \qquad \text{for } \forall t>0. \tag{4.14}$$

On the other hand,

$$\begin{aligned}&\left(\int_{\partial B(t)} (e^{\lambda u}+1)|A|\,d\sigma\right)^{\alpha/(\alpha-1)}\\ &\leqq \int_{\partial B(t)} e^{\lambda u}m\,|u_x|^\alpha\cdot\left(\int_{\partial B(t)} \frac{M^\alpha}{m^{\alpha-1}}(e^{-\lambda u}+1)^\alpha e^{\lambda u}\right)^{1/(\alpha-1)}.\end{aligned} \tag{4.15}$$

If

$$-a\leqq u(x)\leqq\frac{(\alpha-1)a}{\omega}\log F(|x|) \qquad \text{when } |x|\geqq \text{some } r_0,$$

then — for λ as above — we have $e^{\lambda u}\leqq F(|x|)^{1/(\alpha-1)}$. This estimate, together with (4.14) and (4.15), leads to the differential inequality

$$J(t)^{\alpha/(\alpha-1)}\leqq CJ'(t)F(t)\cdot t \qquad \text{on } t\geqq t_0 \tag{4.16}$$

for some $C>0$ and $t_0>0$. As in the proof of Theorem 3.1, the inequality (4.16) leads to $J(t)\equiv 0$, i.e., $u\equiv$constant. This completes the proof of the theorem.

§5. Applications

In this section we note some elementary consequences of the theorems of §3.

Theorem 5.1. *Let $u\in C^3$ be a global solution of the variational problem $\delta\int_{\mathbf{R}^2}F(p,q)dx=0$ where $p=u_x$ and $q=u_y$, $\sup_{p,q}|F_{pp}+F_{qq}|<\infty$, and $F_{pp}F_{qq}-F_{pq}^2>0$. If u_x and u_y are semi-bounded (not necessarily both on the same side) then u is linear.*

Proof. u satisfies $(F_p)_x+(F_q)_y=0$. Differentiation of this equation with respect to x shows that $p=u_x$ satisfies the equation

$$\operatorname{div}\mathscr{A}\cdot\operatorname{grad}p=0 \qquad \text{with } \mathscr{A}=\begin{bmatrix}F_{pp} & F_{qq}\\ F_{pq} & F_{qq}\end{bmatrix}.$$

Differentiation with respect to y shows that $q=u_y$ satisfies the same equation. The conclusion of the theorem now follows from Theorem 3.9, since we may assume $F_{pp}>0$.

Remark. As the proof shows, the conclusion of Theorem 5.1 remains valid under the weaker assumption: $\limsup_{r\to\infty}(\alpha u_x/\log\log r)<\infty$ for some choice of $\alpha=\pm1$, and $\limsup_{r\to\infty}\beta u_y/\log\log r<\infty$ for some choice of $\beta=\pm1$. For the same conclusion under the assumptions $u=0\,(|\log r|^{1-\delta})$ see Finn–Gilbarg [5].

We now consider equations on $\mathbf{R}^2$ of the general form

$$a^{ij}(x,u,\nabla u)u_{ij}=0 \tag{5.1}$$

where $\nabla u=\operatorname{grad}u$ and $u_{ij}=\partial^2u/\partial x_i\partial x_j$.

Suppose that the symmetric matrix $(a^{ij}(x,s,p))$ satisfies the following conditions:

(a) $\exists$ measurable positive functions $m(x)\leqq M(x)$ such that the eigenvalues of $a^{ij}(x,s,p)$ lie in $[m(x),M(x)]$,

(b) $\sup_{\substack{|x|=r\\(s,p)}}\{(1/a^{11}+1/a^{22})\cdot M^2(x)/m(x)\}=\mu(r)$ satisfies

$$\int_1^\infty\frac{dr}{r\mu(r)}=+\infty. \tag{5.2}$$

Theorem 5.2. *Suppose that $A=(a^{ij})$ satisfies the conditions immediately above and $u\in C^3$ satisfies (5.1). If the partial derivatives u_x and u_y are semi-bounded, not necessarily both on the same side, then u is linear.*

Proof. If u satisfies (5.1) then it is well-known and easily verified that $v=u_x$ satisfies $\operatorname{div}A_2\operatorname{grad}v=0$, and $w=u_y$ satisfies $\operatorname{div}A_1\operatorname{grad}w=0$ with

$$A_2=\begin{bmatrix}\dfrac{a^{11}}{a^{22}} & \dfrac{2a^{12}}{a^{22}}\\ 0 & 1\end{bmatrix}\quad\text{and}\quad A_1=\begin{bmatrix}1 & 0\\ \dfrac{2a^{12}}{a^{11}} & \dfrac{a^{22}}{a^{11}}\end{bmatrix}.$$

Note that for $(x,u,p)\in\mathbf{R}^2\times\mathbf{R}\times\mathbf{R}^2$,

$$\langle A_j\cdot p,p\rangle=\frac{1}{a_{jj}}(a^{ik}p_ip_k)\geqq\frac{1}{a_{jj}(x)}m(x)|p|^2$$

and, for some $C>0$,

$$|A_j\cdot p|\leqq C\frac{1}{a_{jj}}M(x)|p| \qquad (j=1,2).$$

Since $\sup_{|x|=r}(1/a^{11}+1/a^{22})\cdot M^2/m=\mu(r)$ satisfies (5.3), the vector fields $\tilde{A}_j(x,z,\xi)=A_j(x,u(x),\operatorname{grad} u(x))\cdot\xi$ — which are independent of z — are left 2-regular (cf. Definition 3.1). Since $z=u_j=\partial u/\partial x_i$ satisfies $\operatorname{div}\tilde{A}_j(x,z,\operatorname{grad} w)=0$ for $j=1,2$, the result follows from Theorem 3.1.

Along the same lines we have

Theorem 5.3. *Let a, b, and c be C^1 functions on $\mathbf{R}$ that satisfy $a(t)>0$ and $a(t)\cdot c(t)-b(t)^2>0$ for $\forall t$. If u is a solution of*

$$a(u)u_{xx}+2b(u)u_{xy}+c(u)u_{yy}\geqq 0 \qquad \text{on } \mathbf{R}^2 \tag{5.3}$$

and $\sup u<\infty$ *then* $u\equiv$ constant.

Proof. Equation (5.3) may be rewritten as

$$(a(u)u_x+b(u)u_y)_x+(b(u)u_x+c(u)u_y)_y=f(x,u,u_x,u_y)$$

where

$$\begin{aligned} f(x,u,u_x,u_y) &= a'(u)u_x^2+c'(u)u_y^2+2b'(u)u_xu_y \\ &\geqq -[|a'(u)|+|b'(u)|+|c'(u)|]|\operatorname{grad} u|^2 \\ &\geqq -\text{constant}\cdot|\operatorname{grad} u|^2 \end{aligned}$$

if u is bounded. The desired conclusion now follows from Theorem 3.3, Theorem 3.4, or Corollary 3.4.2.

In the same manner we have the following consequence of Theorem 3.9:

Theorem 5.4. *Let $\varphi:\mathbf{R}\to(0,\infty)$ be a C^1 function and let $F:(0,\infty)\to(0,\infty)$ satisfy $\int^\infty dt/tF(t)=+\infty$. If $(a^{ij}(x))$ is a positive definite (2×2)-matrix satisfying $m|\xi|^2\leqq a^{ij}(x)\xi_i\xi_j\leqq M(x)|\xi|^2$ with $M(x)\leqq\sqrt{F(|x|)}$ then every bounded solution of*

$$\partial_i(\varphi(u)a^{ij}\partial_j u)\geqq 0 \tag{5.4}$$

is a constant.

Remark. Of course, the requirement that φ be C^1 is not essential. If φ is continuous the theorem is valid for weak subsolutions of (5.4).

As a final and more geometric application we have

Theorem 5.5. *Let $\Gamma=\Gamma(u)$ denote the graph of $u:\mathbf{R}^2\to\mathbf{R}$ and let $H(x,y)$ denote the mean curvature of $\Gamma=\Gamma(u)$ at $(x,y,u(x,y))\in\mathbf{R}^3$. If u is bounded on $\mathbf{R}^2$ then $P(H)=0$ and $\int_{\mathbf{R}^2}H\,dx\,dy=0$ if u has a Lebesgue integral (which a priori may be infinite).*

Corollary 5.5.1. *If H is the mean curvature of a bounded graph over $\mathbf{R}^2$ then*

$\liminf_{r\to\infty} H \leqq 0 \leqq \limsup_{r\to\infty} H$. *Moreover,* $\int H dx dy = 0$ *if* H *does not take both positive and negative values in every neighborhood of infinity.*

Proof. It is well-known that if H is the mean curvature of the graph Γ of $u:\mathbf{R}^2\to\mathbf{R}$ then $\operatorname{div} A = \pm H$ with $A = \frac{1}{2}\operatorname{grad}(\pm u)/\sqrt{1+|\operatorname{grad} u|^2}$. It follows from Corollary 3.4.2 that $P(\pm H) = \pm P(H) \leqq 0$ (i.e. $P(H)=0$) if u is bounded. It follows that if H has Lebesgue integral (that is, if H^+ or H^- is integrable) then $\int H = 0$. The corollary follows immediately from the theorem.

§6. Examples

In this section we record a number of examples that show that the theorems of §3 cannot be appreciably improved.

Example 1. If $\alpha < n$ the differential inequality

$$\operatorname{div}(|u_x|^{\alpha-2}u_x) \geqq 0 \tag{6.1}$$

which is not left α-regular has the bounded solution

$$u(x) = \begin{cases} A + Br^2 + Cr^4, & \text{in } |x| = r \leqq 1, \\ -r^p, & \text{in } |x| = r > 1, \end{cases}$$

where $p = (\alpha - n)/(\alpha - 1)$, $A = -1 + 3p/4 - p^2/8$, $B = -p + p^2/4$, $C = p/4 - p^2/8$. Thus the requirement of left α-regularity is necessary in Theorem 3.1.

Example 2. Given $\varepsilon > 0$ and $N > 0$, $\exists$ smooth functions u_ε and f_ε on $\mathbf{R}^2$ that satisfy

$$\Delta u_\varepsilon = f_\varepsilon$$

while $u_\varepsilon(x) = O(|x|^{-N})$ as $|x|\to\infty$ and f_ε satisfies either of the following conditions:

(i) $f_\varepsilon(x) \geqq 0$ in $|x| \geqq \varepsilon$,

(ii) $\int_{|x|<r} f_\varepsilon(x)dx > 0$ $\forall r > 0$ (but $P(f_\varepsilon) = 0$).

This should be compared with the statements of Theorem 3.1 and Corollary 3.4.1. To construct u_ε and f_ε it suffices to set $u_\varepsilon = (-1)^j(1+r^2)^{-k}$ and make an appropriate choice of k and j.

Example 3. For $p > 1$, the function

$$u(x) =_{\text{def}} \int_{\sqrt{2}}^{\sqrt{r^2+2}} \frac{ds}{s(\log s)^p}, \qquad r^2 = |x|^2,$$

satisfies $0 \leqq u(x) \leqq$ constant on $\mathbf{R}^2$ and also the differential inequality $\partial_i a^{ij}(x)\partial_j u \geqq 0$ with $a^{ij}(x) = (\log\sqrt{r^2+2})^p\delta^{ij}$. On the other hand, according to Theorem 3.9, if $\operatorname{tr}(a^{ij}(x)) = O(\log r)$ then any bounded solution of $\partial_i a^{ij}\partial_j u \geqq 0$ is a constant.

Example 4. According to a result of Gilbarg and Serrin [7] the equation

$$\Delta u + b^i \partial_i u = f \tag{6.2}$$

has no nonconstant bounded solution in $\mathbf{R}^2$ (and even in $\mathbf{R}^n$, $n \geqq 2$) if $b^i(x) = O(|x|^{-1})$ and $f \equiv 0$. The condition on b^i cannot be significantly relaxed since the function $u = \tan^{-1}(v)$, where α and β are constants, and $v = \alpha x + \beta y$, satisfies (6.2) with $b(x) = (b^1(x), b^2(x)) = (2v/(1+v^2))(\alpha, \beta)$ and $f \equiv 0$.

On the other hand, the conditions $b^i(x) = O(|x|^{-1})$ are *not sufficient* for the same theorem to hold with $f \geqq 0$. For example, the function $u = \tan^{-1}(r^2)$, where $x \in \mathbf{R}^2$ and $r^2 = x_1^2 + x_2^2$, satisfies

$$\Delta u + \frac{4r^2}{1+r^4}[x \cdot \operatorname{grad} u] \geqq 0.$$

Still, it might be conjectured that if the vector $b(x) = (b^1(x), b^2(x))$ decays fast enough as $|x| \to \infty$ then equation (6.2) should behave in the same manner as $\Delta u = f$. This turns out to be false: It is possible to construct vectors $b(x)$ with compact support such that the inequality $\Delta u + b^i \partial_i u \geqq 0$ has bounded solutions on $\mathbf{R}^2$. Without going into details, we remark that the function $u(x) = r^2/(1+\alpha r^4)$, $r^2 = |x|^2$, satisfies an inequality of this type if α is sufficiently small.

It follows from the discussion above that Theorem 3.8 gives the best possible result for (6.2) if $f \geqq 0$: If the coefficients $b^i = O(|x|^{-1-\delta})$ for some $\delta > 0$ (or even if $b(x) \in L^2(\mathbf{R}^2)$) then there exists a lower bound for the oscillation of global solutions of $\Delta u + b^i \partial_i u = f \geqq 0$.

Example 5. If $P(f) > 0$ and $|g(x,p)| \leqq C|p|^2$ for all $(x,p) \in \mathbf{R}^2 \times \mathbf{R}^2$ then, according to Corollary 3.4.2, the oscillation of every global solution of

$$\Delta u = g(x, \operatorname{grad} u) + f(x) \tag{6.3}$$

satisfies $\operatorname{osc} u \geqq \gamma = \gamma(g,f)$ where the constant $\gamma \geqq (1/C)P(f)$. It can be shown that in general γ does not satisfy the stronger inequality $\gamma > (2/C)P(f)$. In fact, for $t > 0$ the functions $u_t(x) =_{\text{def}} t\cos^2 x_1$ have $\operatorname{osc} u_t = t$ and satisfy $\Delta u_t + |\operatorname{grad} u_t|^2 = f_t(x) = \varphi_t(x) - \psi_t(x)$ where $\varphi_t(x) = 2t\sin^2 x_1 + t^2 \sin^2 2x_1$, $\psi_t(x) = 2t\cos^2 x_1$ and $x = (x_1, x_2)$. A short computation shows that

$$\exp P(f_t, \mathrm{id}, \{\varphi_t, \psi_t\}) = \liminf_{r\to\infty} \left(\int_{|x|<r} \varphi_t \Big/ \int_{|x|<r} \psi_t \right) = 1 + t/2 > 1$$

(cf. §2). Thus $\exp P(f_t) > 1 + t/2$. If, in general, $\gamma(g,f) \geqq (A/C)P(f)$ with $A \geqq 1$ then the computations above give $\exp(t/A) \geqq (1 + t/2)$ for all $t > 0$. Thus $A \leqq 2$. Moreover, we have constructed solutions u_t of (6.3) with $f = f_t$ and $\operatorname{osc} u_t - (1/C)P(f_t) \to 0$ as $t \to 0$.

Example 6. According to Corollary 3.7.1, the inequality

$$\Delta u + cu \geqq f \geqq 0 \qquad \text{where } f \not\equiv 0$$

has no nonconstant bounded solutions with $|u| < \liminf_{r\to\infty} [\int_{|x|<r} f / \int_{|x|<r} |c|] =_{\text{def}} 1/\mathbf{R}$. On the other hand, the equation $\Delta u + 2u = 2\sin^2 x_1$ $(x = (x_1, x_2))$ has the solution $u = \cos^2 x_1$ with oscillation = 1 while

$$\frac{1}{\mathbf{R}} = \liminf_{r\to\infty} \left(\int_{|x|<r} 2\sin^2 x_1/2\pi r^2 \right) = 1/2.$$

Thus the hypothesis $|u| \leqq 2/\mathbf{R}$ cannot replace the condition $|u| < 1/\mathbf{R}$.

Example 7. The function $u(x) = \log[1 + (1/2)\mathscr{A} \log(1 + r^2)]$ satisfies $u(x) \geqq 0$, $\limsup_{r\to\infty} (\log\log r)^{-1} u(x) = 1$ and $\Delta u + |\nabla u|^2 \geqq 0$ on $\mathbf{R}^2$. Thus the condition "$\limsup(\log\log r)^{-1} u < (\alpha - 1)/H$" of Theorem 3.3 is sharp and cannot be relaxed to $\limsup(\log\log r)^{-1} u \leqq (\alpha - 1)/H$.

Example 8. The equation $\Delta u = u$ has the solution $u(x) = -\exp x_1$. Thus the boundedness condition $\sup|u| < \infty$ of Corollary 3.10.1 cannot be relaxed to $\sup u < \infty$.

Added in proof. After this paper was type-set some additional references (J. Frehse, *Essential self adjointness of singular elliptic operators*, Bol. Soc. Brasil. Mat. **8** (1977), 87–107; S. Granlund, *Strong maximum principle for a quasilinear equation with applications*, Ann. Acad. Sci. Fenn. Ser. A **21** (1978), 1–25; J. Serrin, *Liouville theorems for quasilinear elliptic equations*, Atti Accad. Naz. Lincei **217** (1975), 207–215), which treat related problems, came to our attention. Frehse's paper is, in fact, cited in [8] and [14] as motivating some of their work. Of course, our results are not contained in these papers.

References

1. S. Bernstein, *Über ein geometrisches Theorem und seine Anwendung auf die partiellen Differential gleichungen vom Elliptisches Typus*, Math. Z. **26** (1927), 551–558.
2. L. Bers and L. Nirenberg, *On linear and nonlinear boundary value problems in the plane*, in *Atti Convegno Internazionale Sulle Equazioni Derivate Partiali, Trieste*, Edizioni Cremonese, Rome, 1955, pp. 141–167.
3. E. Bohn and L. K. Jackson, *The Liouville theorem for a quasilinear elliptic partial differential equation*, Trans. Amer. Math. Soc. **104** (1962), 392–397.
4. R. Finn, *Sur quelques généralisations du théorème de Picard*, C. R. Acad. Sci. Paris **235** (1952), 596–598.
5. R. Finn and D. Gilbarg, *Asymptotic behavior and uniqueness of plane subsonic flows*, Comm. Pure Appl. Math. **10** (1957), 23–63.
6. D. Gilbarg, *Some local properties of elliptic equations*, Proc. Symp. Pure Math., Vol. IV, Providence, RI, Amer. Math. Soc., 1961, 127–141.
7. D. Gilbarg and J. Serrin, *On isolated singularities of solutions of second order elliptic differential equations*, J. Analyse. Math. **4** (1955–1956), 309–340.
8. S. Hildebrandt and K. O. Widman, *Sätze vom Liouvillschen Typ für quasilineare elliptische Gleichungen und systeme*, Nachr. Akad. Wiss. Göttingen, No. 4 (1979), 1–19.

9. E. Hopf, *On S. Bernstein's theorem on surfaces $z(x, y)$ of nonpositive curvature*, Proc. Amer. Math. Soc. **1** (1950), 80–85.

10. E. Hopf, *Bemerkungen zu einem Satze vom S. Bernstein aus der theorie der elliptischen Differentialgleichungen*, Math. Z. **29** (1928), 744–745.

11. A. V. Ivanov, *Local estimates for the first derivatives of solutions of quasilinear second order elliptic equations and their application to Liouville type theorems*, Sem. Math. V. A. Steklov Math. Inst. Leningrad **30** (1972) 40–50 (translated as J. Soviet Math. **4** (1975), 335–344).

12. L. Karp, *Subharmonic functions on real and complex manifolds*, Math. Z., to appear.

13. L. Karp, *Asymptotic behavior of solutions of elliptic equations II*, J. Analyse Math. **39** (1981), 103–115.

14. M. Meier, *Liouville theorems for nonlinear elliptic equations and systems*, Manuscripta Math. **29** (1979), 207–228.

15. E. J. Mickle, *A remark on a theorem by S. Bernstein*, Proc. Amer. Math. Soc. **1** (1950), 86–89.

16. J. Moser, *On Harnack's theorem for elliptic differential equations*, Comm. Pure Appl. Math. **14** (1961), 577–591.

17. L. A. Peletier and J. Serrin, *Gradient bounds and Liouville theorems for quasilinear elliptic equations*, Ann. Scuola Norm. Sup. Pisa, IV, **5** (1978), 65–104.

18. M. H. Protter and H. Weinberger, *Maximum Principles in Differential Equations*, Prentice Hall, Englewood Cliffs, NJ, 1967.

19. R. M. Redheffer, *On the inequality $\Delta u \geqq f(u, |\operatorname{grad} u|)$*, J. Math. Anal. Appl. **1** (1960), 277–299.

20. J. Serrin, *On the Harnack inequality for linear elliptic equations*, J. Analyse Math. **4** (1955/56), 297–308.

21. J. Serrin, *A Harnack inequality for nonlinear equations*, Bull. Amer. Math. Soc. **69** (1963), 481–486.

22. J. Serrin, *Local behavior of solutions of quasilinear elliptic equations*, Acta Math. **111** (1964), 247–302.

23. J. Serrin, *Singularities of solutions of quasilinear elliptic equations*, Proc. Symp. Appl. Math., Vol. 17, Amer. Math. Soc., 1965, pp. 68–88.

24. J. Serrin, *Isolated singularities of solutions of quasilinear equations*, Acta Math. **113** (1965), 219–240.

25. J. Serrin, *Entire solutions of nonlinear Poisson equations*, Proc. London Math. Soc. **24** (1972), 348–366.

26. I. N. Tavgelidze, *Liouville theorems for second order elliptic and parabolic equations*, Moscow Univ. Math. Bull. **31** (1976), 70–76.

27. N. S. Trudinger, *On Harnack type inequalities and their applications to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

School of Mathematics
Institute for Advanced Study
Princeton, NJ 08540 USA

and

Fine Hall
Princeton University
Princeton, NJ 08540 USA

Current address
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109 USA

(Received February 10, 1980)