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UNIFORMITY OF STABLY INTEGRAL POINTS ON PRINCIPALLY POLARIZED ABELIAN VARIETIES OF DIMENSION ≤ 2

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ABSTRACT

The purpose of this paper is to prove, assuming that the conjecture of Lang and Vojta holds true, that there is a uniform bound on the number of stably integral points in the complement of the theta divisor on a principally polarized abelian surface defined over a number field. Most of our argument works in arbitrary dimension and the restriction on the dimension ≤ 2 is used only at the last step, where we apply Pacelli's stronger uniformity results for elliptic curves.

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0. Introduction

0.1. THE CONJECTURE OF LANG AND VOJTA. Let X be a variety over a field of characteristic 0. We say that X is a variety of logarithmic general type, if there exists a desingularization $\tilde{X} \to X$, and a projective embedding $\tilde{X} \subset Y$ where $D = Y \setminus \tilde{X}$ is a divisor of normal crossings, such that the invertible sheaf $\omega_Y(D)$ is big. We note first that this property is independent of the choices of $\tilde{X}$ and Y, and that it is a *proper birational invariant*, namely, if $X' \to X$ is a proper birational morphism (or an inverse of such) then X is of logarithmic general type if and only if X' is.

Now let X be a variety of logarithmic general type defined over a number field K. Let S be a finite set of places in K and let $\mathcal{O}_{K,S} \subset K$ be the ring of S-integers. Fix a model $\mathcal{X}$ of X over $\mathcal{O}_{K,S}$. It was conjectured by S. Lang and P. Vojta (cf. [Lan86], [Voj86]) that the set of S-integral points on $\mathcal{X}$ is not Zariski dense in $\mathcal{X}$. In case X is projective, one may choose an arbitrary projective model $\mathcal{X}$ and then $\mathcal{X}(\mathcal{O}_{K,S})$ is identified with $X(K)$. In such a case, one often refers to this Lang–Vojta conjecture as *Lang's conjecture.* When $\dim X = 1$, the conjectures of Lang and Vojta reduce to Siegel's theorem and Mordell's conjecture (Faltings's theorem).

0.2. THE UNIFORMITY PRINCIPLE. In [CHM97], L. Caparaso, J. Harris and B. Mazur show that Lang's conjecture implies a uniformity result for rational points on curves of genus $g \geq 2$ over a fixed number field, which extends Faltings's theorem [Fal83]:

> *Suppose Lang's conjecture holds true. Then there exists a number $N(g,K)$ (depending only on the genus g and the number field K) such that the number of rational points $\#C(K)$ on a smooth projective curve C of genus g defined over a number field K is uniformly bounded:*
>
> $$\#C(K) < N(g,K).$$

The basic principle of [C-H-M] may be summarized by the implication

Lang's conjecture in arbitrary dimension	$\Longrightarrow$	Uniform version of Lang's conjecture in a fixed dimension (e.g. Uniform Mordell's conjecture in dimension 1).

Indeed, the results of [Has96], [ℵ97a] and [ℵV96] show that the same principle holds in higher dimensions as well.

It is only natural then to seek to show the following analogous implication, which may be considered a logarithmic generalization of the above:

The Lang–Vojta conjecture in arbitrary dimension	$\Longrightarrow$	Uniform version of the Lang–Vojta conjecture in a fixed dimension.

0.3. THE CASE OF ELLIPTIC CURVES. Let K be a number field, S a finite number of places in K, and denote by $\mathcal{O}_{K,S}$ the ring of S-integers. Let E be an elliptic curve defined over K, with origin 0, and let P be a K-rational point of $E\setminus\{0\}$, i.e., $P \in (E\setminus\{0\})(K)$. Fix a model $\mathcal{E}$ of E over $\mathcal{O}_{K,S}$, and denote by $\overline{\{0\}}$ the "zero section". We say that P is S-integral if $P \in (\mathcal{E}\setminus\overline{\{0\}})(\mathcal{O}_{K,S})$. Siegel's theorem, which may be considered a logarithmic version of Mordell's conjecture, states that the number of S-integral points on $\mathcal{E}\setminus\overline{\{0\}}$ is finite. We can view this as a case of the Lang–Vojta conjecture, regarding $E\setminus\{0\}$ as a curve of logarithmic general type. Thus according to the principle in 0.2, one might naively expect that, assuming the Lang–Vojta conjecture, a uniform version of Siegel's theorem in the following form would hold:

> Could there exist a number $N(K,S)$ (depending only on the number field and the finite number of places S in K) such that for any elliptic curve defined over K and any model $\mathcal{E}$ over $\mathcal{O}_{K,S}$, the number of S-integral points in the complement of the zero section is uniformly bounded:
> $$\#(\mathcal{E}\setminus\overline{\{0\}})(\mathcal{O}_{K,S}) < N(K,S)?$$

but the statement in this naive form fails to hold. Indeed, take an elliptic curve E with an infinite number of K-rational points. Let

$$y^2 = x^3 + Ax + B$$

be an affine equation for E with $A, B \in \mathcal{O}_{K,S}$ (here the origin of E is the point at infinity). This equation gives an integral model $\mathcal{E}$ over $\mathcal{O}_{K,S}$. Note that for an arbitrary n-tuple of K-rational points $P_1,\ldots,P_n \in (E\setminus\{0\})(K)$, one can find $c \in \mathcal{O}_{K,S}$ such that

$$c^2x(P_i),\ c^3y(P_i) \in \mathcal{O}_{K,S}$$

where $x(P_i)$ and $y(P_i)$ are x- and y-coordinates of the point P_i. By changing coordinates $x_1 = c^2x, y_1 = c^3y$, one obtains a new model $\mathcal{E}'$ of E with a different defining equation over $\mathcal{O}_{K,S}$:

$$y_1^2 = x_1^3 + c^4Ax_1 + c^6B$$

where all the points P_i are now S-integral points in $\mathcal{E}'$. This example shows that *even for a fixed elliptic curve* defined over K one may have an arbitrarily large number of S-integral points on varying models over $\mathcal{O}_{K,S}$, and hence the number is not uniformly bounded.

We observe that this unboundedness is caused, as demonstrated in the example above, by allowing some coordinate changes. Geometrically, these coordinate changes correspond to some blowing up centered at the zero points in some fibers of $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_{K,S}$, possibly followed by some blowing down. From the Lang–Vojta point of view, such a procedure may introduce a curve F in a fiber with negative intersection with the logarithmic relative dualizing sheaf:

$$\omega_{\mathcal{E}/\operatorname{Spec}\mathcal{O}_{K,S}}(\overline{\{0\}}) \cdot F \leq 0.$$

Such a component fails to be "hyperbolic" and thus may "leave space" for more integral points.

In order to avoid such a situation one may wish to impose some positivity condition on the models one takes. This lead the first author to the notion of *stably* S-integral points: a K-rational point $P \in (E \smallsetminus \{0\})(K)$ is called stably S-integral if for any finite extension L of K, with T being the set of places above S, such that E has a stable model $\mathcal{E}_{L,T}$ over $\mathcal{O}_{L,T}$, we have $P \in (\mathcal{E}_{L,T} \smallsetminus \overline{\{0\}})(\mathcal{O}_{L,T})$.

Using this definition, the following assertion was shown in [ℵ97b]:

> Assume that the Lang–Vojta conjecture holds true. Then there exists a number $N(K,S)$ such that for any elliptic curve defined over K the number of stably S-integral points $E(K,S)^{stable}$ is uniformly bounded:
> $$\#E(K,S)^{stable} < N(K,S).$$

0.4. Abelian varieties. The purpose of this paper is to extend the result of 0.3 to the higher dimensional case, according to the uniformity principle of 0.2. Let A be a principally polarized abelian variety with theta divisor Θ, defined over a number field K. Let S be a finite set of places in K, and $\mathcal{O}_{K,S}$ the ring of S-integers. It is a theorem due to Faltings [Fal91] that if $\mathcal{A} \to \mathcal{O}_{K,S}$ is a model of A over $\mathcal{O}_{K,S}$, and $\overline{\Theta}$ the closure of Θ, then

$$\#(\mathcal{A} \smallsetminus \overline{\Theta})(\mathcal{O}_{K,S}) < \infty.$$

As observed in 0.3 for the case of dimension 1, one cannot expect that the number of S-integral points be uniformly bounded without imposing some positivity condition, or equivalently, without restricting oneself to some notion of stably integral points. In Definition 3.1.1 below we define stably integral points as points which are integral on the complement of $\overline{\Theta}$ on the stable model of a principally polarized abelian variety, after taking a finite extension of K. The existence of such stable models is provided by recent results of Alexeev and Nakamura (see [Al396], [AN96], [Ale98]), in which the moduli of principally polarized abelian varieties is compactified by the moduli of stable quasi-abelian pairs.

Here is our main arithmetic result on abelian surfaces:

MAIN THEOREM: *Assume that the conjecture of Lang and Vojta holds true. Then there exists a number $N(K,S)$, such that for any principally polarized abelian* surface *with a theta divisor (A,Θ), the number of the stably S-integral points of $A \setminus \Theta$ is uniformly bounded:*

$$\#(A \setminus \Theta)(K,S)^{stable} < N(K,S).$$

We expect a similar result to hold for abelian varieties of any fixed dimension with a divisor of arbitrary fixed polarization degree.

0.5. In Section 1, we review the proof of the uniformity statement on rational points on varieties of general type, as we will apply several methods which have been used in that context. In Section 2 we review the construction of complete moduli of stable quasi-abelian pairs as described by Alexeev and Nakamura. In Section 3, we prove the Main Theorem. There is one difficulty in the last inductive step where we consider *families of subvarieties* of principally polarized abelian varieties, especially families of *abelian* subvarieties, which are not necessarily *principally* polarized. We can complete the argument in dimension ≤ 2 using Pacelli's stronger uniformity results for the elliptic curves, leaving the general case of dimension > 2 conjectural.

0.6. It is worth noting that a similar argument to the one we give for principally polarized abelian varieties often works for pairs of logarithmic general type (X,D) (see 0.7 below) defined over K, if a "good" moduli for the log canonical models of such pairs exists. For example, one can use such an argument to show, assuming the conjecture of Lang and Vojta, that there is a uniform bound on stably integral points for $\mathbb{P}^1 \setminus \{n \text{ points}\}$ $(n \geq 3)$, using the moduli of stable n-pointed curves of genus 0. However, at least when $n = 3$, this uniformity statement is nothing but the classical result of Siegel about the finiteness of the number of solutions of an S-unit equation (which holds regardless of the Lang–Vojta conjecture). This result of Siegel has been strengthened to a great extent in recent years; see, e.g., [Sch96].

0.7. LOGARITHMIC PAIRS. In Section 0.1 we defined what it means for a variety to be of logarithmic general type, in terms of a good compactification $\tilde{X} \subset Y$ of a desingularization $\tilde{X} \to X$. It is convenient to have a criterion which does not require choosing a desingularization. One can approach that using "singularities of pairs"; see [KMM87].

> Let $\overline{X}$ be a projective variety, $D \subset \overline{X}$ a reduced effective Weil divisor. Assume
>
> 1. the pair $(\overline{X}, D)$ has log-canonical singularities;
> 2. $\omega_{\overline{X}}(D)$ is big, and
> 3. the complement $\overline{X} \setminus D$ has canonical singularities.
>
> Then $\overline{X} \setminus D$ has logarithmic general type.

Thus it is enough to check that $\overline{X}$ has a lot of logarithmic differentials, and that its singularities are sufficiently mild.

We would like to draw the reader's attention to condition 3, which does not follow from condition 1 because of possible exceptional divisors. Many authors define the *pair* $(\overline{X}, D)$ to be of logarithmic general type if conditions 1 and 2 are satisfied — this is equivalent to the statement that the variety $\overline{X} \setminus (D \cup \mathrm{Sing}(X))$ is of logarithmic general type in our terminology.

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1. Outline of the proof of uniformity for varieties of general type

1.1. CORRELATION OF POINTS. We briefly recall the outline of the proof of uniformity of rational points on curves of genus ≥ 2 in [CHM97]. One of the main ideas of [CHM97] is to observe that, assuming Lang's conjecture holds true, the set of K-rational points on all smooth projective curves of genus $g \geq 2$ defined over a number field K is *correlated*, i.e., the collection of n-tuples of such points satisfies a nontrivial algebraic relation, for suitable n.

Let π: $X \to B$ be a projective family of smooth irreducible curves of genus $g \geq 2$ defined over a number field K. We denote by

$$\pi_n\colon X_B^n = X \times_B \cdots \times_B X \to B$$

the n-th fibered power of X over B. Denote by $\tau_n\colon X_B^n \to X_B^{n-1}$ the projection onto the first $n-1$ factors. Given a point $b \in B$ we denote by X_b the fiber $\pi^{-1}(b)$.

Similarly, given a point $Q = (P_1, \ldots, P_{n-1}) \in X_B^{n-1}$ we denote by $X_Q \subset X_B^n$ the fiber $\tau_n^{-1}(Q)$. Note that if $Q \in X_B^{n-1}(K)$ and $\pi_{n-1}(Q) = b$ then $X_Q \cong X_b$.

Assume that we are given a subset $\mathcal{P} \subset X(K)$. We denote by $\mathcal{P}_B^n \subset X_B^n$ the fibered power of $\mathcal{P}$ over B (namely the set of all n-tuples of points in $\mathcal{P}$ whose images on B are the same), and by $\mathcal{P}_b$ the points of $\mathcal{P}$ lying over b.

Definition 1.1.1: The set $\mathcal{P}$ is said to be *n-correlated* if there is a proper Zariski-closed subset $F_n \subset X_B^n$ such that $\mathcal{P}_B^n \subset F_n$.

For instance, a subset $\mathcal{P}$ is 1-correlated if and only if it is not Zariski-dense in X; in which case it is easy to see that over some nonempty open subset U in B the number of points of $\mathcal{P}$ in each fiber is uniformly bounded. This simple observation is generalized by the following lemma.

LEMMA 1.1.2 ([CHM97], Lemma 1.1; [ℵ97b], Lemma 1): *Let $X \to B$ be a projective family of smooth irreducible curves, $\mathcal{P} \subset X(K)$ an n-correlated subset. Then there exists a nonempty open subset $U \subset B$ and an integer N such that for every $b \in U$ we have $\#\mathcal{P}_b \le N$.*

We find it instructive to include a short proof of this simple lemma, in which it will be clear that the argument only works for a family of curves. A modification which does work in higher dimension will be discussed later in this article — see Sections 1.4 and 3.4.

Let $F_n = \overline{\mathcal{P}_B^n}$ be the Zariski closure, $U_n = X_B^n \smallsetminus F_n$ the complement. We now define Zariski-open and Zariski-closed subsets U_{i-1} and $F_{i-1} \subset X_B^{i-1}$ by descending induction as follows: we take $U_{i-1} = \tau_i(U_i)$, and set $F_{i-1} = X_B^{i-1} \smallsetminus U_{i-1}$ to be the complement.

Note that over U_{i-1} the map τ_i restricts to a finite map on F_i. In fact, by definition, if $x \in U_{i-1}$ then $\tau_i^{-1}(x) \not\subset F_i$ and hence $\tau_i^{-1}(x) \cap F_i$ is a finite set, since $\tau_i^{-1}(x)$ is a curve. Thus there exists $d_i \in \mathbb{N}$ such that

$$\#\tau_i^{-1}(x) \cap F_i \le d_i \quad \text{for } x \in U_{i-1}.$$

Let $U = U_0 \subset B$. We claim that over U the number of points of $\mathcal{P}$ in each fiber is bounded. Consider a point $b \in U$.

CASE 1: $\mathcal{P}_b \subset F_1$.
In this case, we have $\#\mathcal{P}_b \le d_1$.

CASE 2: There exists some $Q \in \mathcal{P}_b \smallsetminus F_1$ where $X_Q \cap \mathcal{P}_b^2 \subset F_2$.
In this case, we have $\#\mathcal{P}_b \le d_2$.

CASE i: There exists $Q = (P_1, \ldots, P_{i-1}) \in \mathcal{P}_b^{i-1} \setminus F_{i-1}$ where $X_Q \cap \mathcal{P}_b^i \subset F_i$.

In this case, we have $\#\mathcal{P}_b \leq d_i$.

As $X_Q \cap \mathcal{P}_b^n \subset F_n$ for all $Q \in \mathcal{P}_b^{n-1}$ by definition, the cases will be exhausted at some stage when $i \leq n$. Thus

$$\#\mathcal{P}_b \leq \max\{d_i\}. \quad \blacksquare$$

1.2. LANG'S CONJECTURE AND CORRELATION. The remarkable observation of the paper [CHM97] is that Lang's conjecture implies that the set of K-rational points on curves of genus $g > 1$ is n-correlated, for sufficiently large $n \in \mathbb{N}$.

PROPOSITION 1.2.1 (cf. Lemma 1.1 in [CHM97]): *Let $X \to B$ be a projective family of smooth irreducible curves of genus $g \geq 2$ over a number field K. Assume that Lang's conjecture holds true. Then $X(K)$ is n-correlated for sufficiently large $n \in \mathbb{N}$.*

In order to deduce the uniformity assertion of Section 0.2 from Proposition 1.2.1, one starts with $X \to B$, a "comprehensive" projective family of smooth irreducible curves of genus $g \geq 2$ in which "all curves appear", namely, for any projective smooth curve C of genus g defined over K there exists a morphism $\operatorname{Spec} K \to B$ satisfying

$$\begin{array}{ccc} C \simeq \operatorname{Spec} K \times X & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} K & \longrightarrow & B. \end{array}$$

Such a family always exists over a suitable Hilbert scheme, since all curves of genus $g > 1$ are canonically polarized. We set $\mathcal{P} = X(K)$. There exists a nonempty Zariski-open subset $U_0 \subset B$ and an integer N_0 such that $\#\mathcal{P}_b \leq N_0$ for $b \in U_0$ by Lemma 1.1.2. Now take $B_1 = B \setminus U_0$, and apply the lemma to the family $X_1 = X \times_B B_1 \to B_1$ to obtain a new nonempty Zariski-open subset $U_1 \subset B_1$ and an integer N_1 such that $\#\mathcal{P}_b \leq N_1$ for $b \in U_1$, and so on. By noetherian induction, we have the uniformity assertion.

1.3. THE FIBERED POWER THEOREM. It is easy to see that Proposition 1.2.1 follows from the *Fibered Power Theorem:*

THEOREM 1.3.1: *Let $\pi\colon X \to B$ be a projective family of varieties of general type, with B irreducible, defined over a field K of characteristic 0. Then there* *exists a positive integer* n, *a variety of general type* W_n *over* K *with* $\dim W_n > 0$, *and a dominant rational map*

$$r\colon X_B^n \dashrightarrow W_n.$$

To see how Proposition 1.2.1 follows from this theorem, first note that we may replace B by an irreducible component. Next, by Lang's conjecture, there exists a proper Zariski-closed subset G_n of W_n which contains all the K-rational points of W_n. Let $(X_B^n)_{dom}$ be the domain of the rational map $X_B^n \dashrightarrow W_n$. Denote $\mathcal{P} = X(K)$. Then for a point $P \in \mathcal{P}_B^n \subset X_B^n$, we have either $P \in X_B^n \smallsetminus (X_B^n)_{dom}$ or $P \in r^{-1}(G_n)$. Then we only have to set

$$F_n = [X_B^n \smallsetminus (X_B^n)_{dom}] \cup [r^{-1}(G_n)].$$

In the case of curves, the Fibered Power Theorem was proven in [CHM97] Theorem 1.3, using the following observation: let $X \to B$ be a family of curves of genus $g > 1$ as above. For each n we have a rational map $X_B^n \dashrightarrow \mathbf{M}_{g,n}$. Denote by W_n the image of this map. Then an argument is given in [CHM97] which in effect proves the following:

For large n, *the image variety* W_n *is a variety of general type.*

The argument in [CHM97] uses the compactification $\overline{\mathbf{M}}_g$, the moduli space of *stable* curves, in an essential way. A similar argument was used in [Has96], Theorem 1, for the case of surfaces. This is precisely the line of proof we will take in this paper for abelian varieties (see Theorem 3.2.2). In higher dimension, it was necessary to give a different argument in [ℵ97a], Theorem 0.1, since complete moduli spaces of stable varieties in dimension > 2 are not known to exist in general. All proofs use deep results about *weak positivity* of the push forward of pluricanonical sheaves; in the present paper we use such a result due to Kawamata [Kaw85] about abelian varieties.

1.4. Uniformity in higher dimension. In order to prove a uniformity result in dimension > 1, one needs to modify the *statement* appropriately, and then adjust the proof.

First, as a variety of general type may contain a subvariety which is not of general type, on which there may be infinitely many K-rational points, we have to modify the uniformity statement (this issue does not come up in this paper). Such a subvariety is called *exceptional*. Given a family $X \to B$ of varieties of general type, it is natural to restrict attention to points in $X(K)$ which do not lie on any exceptional subvariety, which we denote $X(K)^{nex}$ ("nex" for non-exceptional). We arrive at the following statement (see [ℵV96], Theorem 1.5):

Assume Lang's conjecture holds true. Fix a family of varieties of general type $X \to B$. Then there exists a number $N(X \to B, K)$ depending only on the given family and the number field K such that, for any $b \in B(K)$, the number of non-exceptional points on the fiber X_b is bounded:

$$\#X(K)^{nex} < N(X \to B, K).$$

Second, we still need to modify Lemma 1.1.2 for higher dimension. Such a modification was given in [ℵ95], [Pac97a], and [ℵV96], and is essential in this paper as well. We will discuss some aspects of it in the course of the proof of the Main Theorem (see Section 3.4). First, we would like to adjust the notion of correlation for the higher dimensional case:

Definition 1.4.1: Let $\tau\colon X \to B$ be a projective surjective morphism of reduced schemes of finite type over a field K, with B irreducible. Denote the dimension of the generic fiber of $\tau\colon X \to B$ by d. Fix a subset $Q \subset X(K)$, and denote by G_k the Zariski closure of Q_B^k in the fibered power X_B^k. We say that Q is *strongly k-correlated* with respect to $\tau\colon X \to B$, if every irreducible component of G_k which dominates B has relative dimension $< kd$ over B.

We will now see how to reduce a question of strong correlation to a question of correlation. Let $X = X_1 \cup \cdots \cup X_c$ be a decomposition into irreducible components. Let $X_i' \to X_i$ be the normalization, and let $X_i' \to B_i' \to B$ be the Stein factorization. There is a dense open set $U_i \subset X_i$ over which $X_i' \to X$ is an isomorphism. Therefore the set $Q_i' = U_i \cap Q$ sits naturally in $X_i'(K)$.

PROPOSITION 1.4.2: *Assume that for each X_i of relative dimension d over B, the set Q_i' is k_i'-correlated with respect to $X_i' \to B_i'$. Then for large k, the set Q is strongly k-correlated for $X \to B$.*

Proof: We make a number of reduction steps leading to the proposition.

1.4.1. Let $B' \subset B$ be a nonempty open set, $X' \subset \tau^{-1}B'$ a dense open set, and $Q' = Q \cap X'$. Assume Q' is strongly k-correlated. Then Q is strongly k-correlated as well.

This is immediate from the definition.

In particular, we may as well assume that B is normal and $\tau\colon X \to B$ is flat. We may also replace X by a birational modification, since we can restrict to the points of Q where this modification is an isomorphism. Therefore we may assume X is normal, thus it is the disjoint union of normal irreducible components.

1.4.2. Let $X_1, \ldots, X_{c'}$ be the irreducible components of X of relative dimension d over B. Let $X' = X_1 \cup \cdots \cup X_{c'}$ and let $Q' = Q \cap X'$. If Q' is strongly k-correlated then so is Q.

One can write

$$X_B^k = \bigcup_{1 \le i_j \le c} X_{i_1} \times_B \cdots \times_B X_{i_k}.$$

If some component X_i has relative dimension $< d$, then any term involving X_i in $X_B^k = \bigcup_{1\le i_j \le c} X_{i_1} \times_B \cdots \times_B X_{i_k}$ has relative dimension $< kd$.

Thus we may as well assume that all fibers of $X \to B$ have pure dimension d.

1.4.3. Let $Q_i = X_i \cap Q$. Assume Q_i is strongly l-correlated. Then for $k = l \cdot c$, Q is strongly k-correlated.

By the box principle, every term in the expression

$$X_B^k = \bigcup_{1 \le i_j \le c} X_{i_1} \times_B \cdots \times_B X_{i_k}$$

has at least one i_j appearing at least l times. Considering the projection on those factors, it follows that Q is strongly k-correlated.

1.4.4. Denote by c_i the degree of $B_i' \to B$. Assume Q_i' in the proposition is k_i'-correlated with respect to $X_i' \to B_i'$. Then for $k = c_i k_i'$ we have that Q_i is strongly k-correlated with respect to $X_i \to B$.

Let G be an irreducible component of X'^k_{iB}. A point on G corresponds to a k-tuple of points on a fiber of X_i' over B, which fall into the c_i different components of this fiber, which are identified as fibers of X_i' over B_i'. By the box principle there is a subset $J \subset \{1, \ldots, k\}$ of size at least $k/c_i = k_i'$ such that for a point $(P_1, \ldots, P_k) \in G$, all the P_j for $j \in J$ lie in the same component of the fiber. In other words, the projection $G \to (X_i)_B^J$ to the factors in J maps G onto the closed subset $(X_i)^J_{B_i'}$. Since Q_i' is k_i'-correlated for $X_i' \to B_i'$, we have that $(Q_i')_B^k \cap G$ is not dense in G, which implies that Q_i' is strongly k-correlated with respect to $X_i' \to B$. Step 1.4.1 in the proof implies that Q_i is strongly k-correlated as well. ∎

2. Moduli of stable quasi-abelian pairs

Both the statement and proof of our main theorem depend on the existence of a good compactification of the moduli space of abelian varieties. We now review some essential facts about these spaces which we will utilize.

2.1. ABELIAN SCHEMES. Recall that a principally polarized abelian scheme $(A \to S, \lambda)$ is an abelian scheme $A \to S$ (with a zero section $S \to A$) and an *isomorphism* $\lambda\colon A \to \mathbf{Pic}^0(A/S)$ which *locally over* S is a polarization induced by a relatively ample invertible sheaf.

The moduli category $\mathfrak{A}_g$ of principally polarized abelian schemes (morphisms given by fiber diagrams) is a Deligne–Mumford stack. It admits a coarse moduli scheme $\mathbf{A}_g$, which is quasi-projective over $\operatorname{Spec}\mathbb{Z}$ (see [MFK94], [FC90]). In analogy with the moduli spaces of curves, one would like to have a good compactification of $\mathfrak{A}_g$ in a canonical way, and possibly also an analogue of $\overline{\mathcal{M}}_{g,n}$.

Beginning with [AMRT75] and through the work of many authors (see [FC90]), an infinite collection of toroidal compactifications "$\overline{\mathfrak{A}}_g$" of $\mathfrak{A}_g$ was constructed, depending on choices of "cone decompositions". In general these compactifications are not moduli stacks of any explicitly described families of "stable objects". It is however shown in [FC90] that each of these compactifications carries a family of semiabelian varieties. If one takes the formal completion of such a $\overline{\mathfrak{A}}_g$ at a point, one can apply *Mumford's construction* (see [Mum72], [FC90]) and get a toroidal compactification of the family of semiabelian schemes, but this compactification again depends on "degeneration data", and one has a serious problem in gluing these together.

These issues were recently resolved in the work of Alexeev and Nakamura [AN96], [Ale98]. See also the related [Nak98].

2.2. STABLE QUASI-ABELIAN PAIRS. A first important step is to change the original moduli problem in a way which surprisingly simplifies the situation. Instead of working with principally polarized abelian schemes, one forgets the zero section of the abelian scheme and instead one insists that the polarization come from a global relatively ample invertible sheaf; in fact, since we work with principal polarizations, this sheaf has a unique divisor Θ. To this end, a smooth principally polarized *quasi-abelian scheme* $(P \to S, \Theta)$ is a *torsor* $P \to S$ on an abelian scheme $A \to S$, and a relatively ample divisor $\Theta \subset P$ which behaves like a principal polarization, in the sense that its Hilbert polynomial is n^g. (We note that in [Ale98], Alexeev changed the terminology to "*abelic* pairs".) The moduli stack of smooth principally polarized quasi-abelian schemes is canonically isomorphic to $\mathfrak{A}_g$ ([Ale98], Corollary 3.0.8). It admits a universal family which we denote $(\mathfrak{A}_{g,1} \to \mathfrak{A}_g, \tilde{\Theta})$; the added subscript 1 indicates that $\mathfrak{A}_{g,1}$ is the moduli stack for smooth principally polarized quasi-abelian schemes *with one marked point.* (It should not be confused with Mumford's notation for level structure.)

Next, Alexeev and Nakamura make a canonical choice of degeneration data in Mumford's construction. Over a discrete valuation ring, this gives a canonical way to compactify a torsor on a semiabelian scheme with smooth principally polarized quasi-abelian generic fiber. The fibers (P, Θ_P) of this construction are called *stable principally polarized quasi-abelian varieties*, and they can be characterized explicitly; see [Ale98], Definition 1.1.5. (The reader is advised not to be confused by this nomenclature: a smooth quasi-abelian variety is by definition also a stable quasi-abelian variety.)

Finally, it follows from Alexeev's work (see [Ale98], 1.2 (H)) that the category of stable principally polarized quasi-abelian schemes is a Deligne–Mumford stack $\overline{\mathfrak{A}}_g$ admitting a projective coarse moduli scheme $\overline{\mathbf{A}}_g \to \operatorname{Spec}\mathbb{Z}$. On the level of geometric points, Alexeev shows that $\overline{\mathfrak{A}}_g$ agrees with the so-called "Second Voronoi Compactification" $\overline{\mathfrak{A}}'_g$, which is a very special toroidal compactification of $\mathfrak{A}_g$ (see [Ale98], Theorem 1.2.17). Indeed, there is a morphism $\overline{\mathfrak{A}}'_g \to \overline{\mathfrak{A}}_g$ which is one-to-one on geometric points.

Again, we denote the universal family by $(\overline{\mathfrak{A}}_{g,1}, \bar{\tilde{\Theta}}) \to \overline{\mathfrak{A}}_g$. We denote by $\overline{\mathfrak{A}}_{g,n}$ the fibered power $(\overline{\mathfrak{A}}_{g,1})^n_{\overline{\mathfrak{A}}_g}$. (This is, to some extent, in analogy with the space of stable pointed curves $\overline{\mathcal{M}}_{g,n}$, although we do not use Knudsen's stabilization.) Denote by $p_i\colon \overline{\mathfrak{A}}_{g,n} \to \overline{\mathfrak{A}}_{g,1}$ the projection to the i-th factor. We have a natural relatively ample divisor $\bar{\tilde{\Theta}}_n \subset \overline{\mathfrak{A}}_{g,n}$ defined by $\bar{\tilde{\Theta}}_n = \sum_i p_i^* \bar{\tilde{\Theta}}$.

We denote by $\overline{\mathbf{A}}_{g,n}$ the coarse moduli spaces of $\overline{\mathfrak{A}}_{g,n}$, and by $\overline{\Theta}_n \subset \overline{\mathbf{A}}_{g,n}$ the image of $\bar{\tilde{\Theta}}_n$. A-priori these are Artin algebraic spaces (see [KM97]), but since some multiple $m\bar{\tilde{\Theta}}_n$ descends to a Cartier divisor on $\overline{\mathbf{A}}_{g,n}$ and is relatively ample, these are projective schemes over $\operatorname{Spec}\mathbb{Z}$.

2.3. Properties of stable pairs. We now collect a few properties of stable principally polarized quasi-abelian schemes, which we will use in the next section.

To save words, we will refer to a stable principally polarized quasi-abelian scheme (P, Θ) (always assumed flat over a base scheme S) as a *stable pair.*

The first two items are included in [Ale98], Definition 1.1.5.

2.3.1. For a stable pair (P, Θ) over a field, the underlying stable quasi-abelian variety P is proper and reduced, and Θ is an ample Cartier divisor.

2.3.2. Let $(P \to S, \Theta)$ be a stable pair over S. Let $P_0 \subset P$ be an open subset, consisting of exactly one irreducible component of the smooth locus in every fiber.

Then $A = \mathrm{Aut}^0(P_0/S)$ is semiabelian, P_0 is an A-torsor, and A independent of the choice of P_0. Over a field, P is stratified by finitely many orbits of A.

2.3.3. Let $U_S \subset S$ be a toroidal embedding over a field, and let $(\pi: P \to S, \Theta)$ be a stable pair over S, such that $P \to S$ is smooth over the open set U_S. Let $U_P = \pi^{-1}U_S$. Then $U_P \subset P$ is a toroidal embedding, and $P \to S$ is a toroidal morphism. (Toroidal morphisms are defined in [ℵK97], Definition 1.2.)

See [Ale98], 5.6. Indeed, Mumford's construction is by definition toroidal!

(If one is working in mixed characteristics, one only needs to replace "toroidal" by "log-smooth" in the sense of K. Kato.)

2.3.4. Let (P, Θ) be a stable pair over a field. Let $p_i: P^n \to P$ be the projection onto the i-th factor, and consider the divisor $\Theta_n = \sum_i p_i^*\Theta$. Then (P^n, Θ_n) is a stable pair.

This follows immediately from [Ale98], Definition 1.1.5.

2.3.5. In the situation of 2.3.3, suppose S has Gorenstein singularities. Then the scheme P has Gorenstein singularities.

This is a general fact about toroidal morphisms with *reduced fibers* and *no horizontal divisors.* The proof is easy using the associated polyhedral complexes; see [ℵK97], Lemma 6.1 (which applies when S is nonsingular) and [ℵK97], Lemma 6.3 (which reduces to that case).

We note that, since a toroidal embedding has rational singularities, it follows that P has rational Gorenstein singularities, hence *canonical* singularities. This is a refinement of Alexeev's [Ale96], Lemma 3.8.

2.3.6. Suppose the base field has characteristic 0. In the situation of 2.3.5, the pair (P, Θ) has log-canonical singularities.

Indeed, the proof of [Ale96], Theorem 3.10 applies word-for-word, if we do not add the central fiber P_0 (this only makes the proof simpler).

Finally, we have the following crucial extension property, proved in [Ale98], 5.7:

2.3.7. Let S be the spectrum of a discrete valuation ring R, with generic point η. Let (P_η, Θ_η) be a stable pair. Then there exists a finite separable extension of discrete valuation rings $R \subset R_1$, with spectrum S_1 and generic point η_1, and a stable pair $(P_1 \to S_1, \Theta_1)$ extending $(P_{\eta_1}, \Theta_{\eta_1})$.

This result immediately extends to dedekind domains.

We call $(P_1 \to S_1, \Theta_1)$ a *stable quasi-abelian model* of $(P_{\eta_1}, \Theta_{\eta_1})$.

2.4. Relation with the Néron model. Let R be a discrete valuation ring, $S = \operatorname{Spec} R$, with generic point η and special point $\mathfrak{p}$. Let $(P \to S, \Theta)$ be a stable pair, and assume $A = P_\eta$ is smooth. For simplicity we also assume there is a section $s: S \to P$ landing in the smooth locus of $P \to S$. This makes A into an abelian variety. Denote by $\mathcal{N}_A \to S$ the Néron model of A.

Proposition 2.4.1: *There is a unique morphism*

$$f: \mathcal{N}_A \to P$$

extending the isomorphism

$$(\mathcal{N}_A)_\eta \xrightarrow{\sim} A.$$

Proof: The formation of $\mathcal{N}_A \to S$ commutes with étale base change. Once we prove the proposition after such a base change, the uniqueness implies that we can descend back to S. Thus we may replace R by its strict henselization.

Now by construction, the stable quasi-abelian model $(P \to S, \tilde{\Theta})$ can be viewed as a "compactification" of a semi-abelian scheme $\mathcal{A}_0 = \operatorname{Aut}^0(P/S) \to S$ with the origin identified with the section s. Note that the construction of the stable quasi-abelian model gives an action

$$\mathcal{A}_0 \times P \to P$$

extending the addition law on $\mathcal{A}_0$.

Note that there is also a natural inclusion $\mathcal{A}_0 \hookrightarrow \mathcal{N}_A$ as the zero component.

Denote by

$$M_i, \quad i = 1, \ldots, t$$

the components of the fiber $(\mathcal{N}_A)_\mathfrak{p}$ of the Néron model over $\mathfrak{p}$. For each i, we have an open neighborhood

$$\mathcal{N}_i = \mathcal{N}_A \setminus \bigcup_{j \neq i} M_j.$$

We may choose the numbering so that $\mathcal{N}_1 = \mathcal{A}_0$. We have

$$\mathcal{N}_A = \bigcup_i \mathcal{N}_i$$

and

$$\mathcal{N}_i \cap \mathcal{N}_j = (\mathcal{N}_A)_\eta \quad \forall i \neq j.$$

Since R is strictly henselian, we can choose, for each i, a section

$$s_i: S \to \mathcal{N}_i$$

such that $s_1 = s$. The schemes $\mathcal{N}_i$ can be viewed as $\mathcal{A}_0$-torsors, and the choice of the s_i gives a trivialization of these torsors.

We denote by

$$(s_i)_\eta, \quad i = 1, \ldots, t$$

the corresponding rational points on $(\mathcal{N}_A)_\eta = P_\eta$. Since $P \to S$ is proper, we can extend $(s_i)_\eta$ to sections

$$t_i \in P(S)$$

such that

$$(t_i)_\eta = (s_i)_\eta.$$

We define

$$f_i \colon \mathcal{N}_i \to P, \quad i = 1, \ldots, t$$

as follows. Given a scheme T over S and a point $z \in \mathcal{N}_i(T)$ there exists a unique point $a \in \mathcal{A}_0(T)$ such that $z = a \cdot s_i$. Here the notation $a \cdot s_i$ stands for the action $\mathcal{A}_0 \times \mathcal{N}_i \to \mathcal{N}_i$. Define $f_i(z) = a \cdot t_i$. Here the notation $a \cdot t_i$ stands for the action $\mathcal{A}_0 \times P \to P$. This rule is clearly functorial and therefore defines a morphism.

We claim that the morphisms f_i are independent of the choice of s_i and coincide on $(\mathcal{N}_A)_\eta$. In fact, given $s'_i \in \mathcal{N}_i(R)$ consider the corresponding t'_i and f'_i. There exists $b_i \in \mathcal{A}_0$ such that $s_i = b_i \cdot s'_i$. Therefore, $(t_i)_\eta = (b_i)_\eta \cdot (t'_i)_\eta$, which implies that $t_i = b_i \cdot t'_i$, since P is separated. Therefore, we conclude

$$\begin{aligned} f_i(z) =& f_i(a \cdot s_i) = a \cdot t_i = a \cdot (b_i \cdot t'_i) = (ab_i) \cdot t'_i \\ =& f'_i((ab_i) \cdot s'_i) = f'_i(a \cdot (b_i \cdot s'_i)) = f'_i(a \cdot s_i) = f'_i(z). \end{aligned}$$

The same argument shows that given a scheme T over η and a point $z \in \mathcal{N}_A(T)$ we have $f_i(z) = f_j(z)$, i.e., the morphisms f_i coincide over the intersection of their domains $(\mathcal{N}_A)_\eta$.

Since $\mathcal{N}_A$ is covered by the $\mathcal{N}_i$ and $\mathcal{N}_i \cap \mathcal{N}_j = (\mathcal{N}_A)_\eta$ whenever $i \neq j$, it follows that the f_i glue together to give a morphism $\mathcal{N}_A \to P$ as required in the Proposition. ∎

2.5. Tautological families over moduli spaces. Since $\overline{\mathbf{A}}_g$ is not a fine moduli scheme, it does not have a universal family. Our arguments below depend on the geometry of families, therefore it is useful to have some approximation of a universal family, which, following [CHM97], one calls *a tautological family.* We need such a family with a strong equivariance property. This is summarized in the following statement.

PROPOSITION 2.5.1: *Let $W_0 \subset \overline{\mathbf{A}}_g$ be a closed integral subscheme, and let $W_1 \subset \overline{\mathbf{A}}_{g,1}$ be the reduced scheme underlying its inverse image. Then there exists a projective, normal integral scheme B, and a family of g-dimensional stable pairs $(A \to B, \Theta)$, satisfying the following properties:*

1. *The natural moduli morphisms $B \to \overline{\mathbf{A}}_g$ and $A \to \overline{\mathbf{A}}_{g,1}$ are finite and generically étale, with images W_0 and W_1, respectively.*
2. *Denote $G = \mathrm{Aut}_{\overline{\mathbf{A}}_g}(A \to B, \Theta)$, namely the group of automorphisms of the gadget $(A \to B, \Theta)$ commuting with the morphism $B \to \overline{\mathbf{A}}_g$. Then the two morphisms $B/G \to \overline{\mathbf{A}}_g$ and $A/G \to \overline{\mathbf{A}}_{g,1}$ are birational onto their images.*

Moreover, in this situation consider the moduli morphism $A^n_B \to \overline{\mathbf{A}}_{g,n}$. There is a diagonal action of G on A^n_B, and $A^n_B/G \to \overline{\mathbf{A}}_{g,n}$ is again birational onto the image.

Proof: The existence of a family $(A \to B, \Theta)$ satisfying condition 1 is an immediate consequence of [Kol90], Proposition 2.7. (Kollár attributes the proof to M. Artin. See also discussion in [CHM97]. Proofs of this fact have been given by a number of authors through the years.)

We now wish to replace this family by one which satisfies the equivariance condition 2. First, we may assume that the function field extension $K(W_0) \subset K(B)$ is Galois, by going to the Galois closure. Denote the Galois group G_0. Write η for the generic point of B. Second, we may assume that the geometric points of the finite group $H = \mathrm{Aut}_\eta(A_\eta, \Theta_\eta)$ are all rational over $K(B)$ — simply pass to a suitable finite extension and take Galois closure again. The morphism $\mathrm{Aut}_B(A, \Theta) \to B$ is proper, since the moduli stack is separated. Therefore the automorphisms of (A_η, Θ_η) extend to automorphisms of the family $(A \to B, \Theta)$ over B. Third, in a similar manner we can ensure that for any $g \in G_0$ we have a B-isomorphism $(A \to B, \Theta) \xrightarrow{\sim} g^*(A \to B, \Theta)$. Now consider the set

$$G = \{(g, h) | g \in G_0, h\colon (A \to B, \Theta) \xrightarrow{\sim} g^*(A \to B, \Theta)\}.$$

Note that for any $(g, h) \in G$ and $g' \in G_0$ we can define

$$h^{\{g'\}}\colon g'^*(A \to B, \Theta) \xrightarrow{\sim} g'^* g^*(A \to B, \Theta).$$

We can now define

$$(g, h)(g', h') = (gg', h^{\{g'\}} h').$$

We leave it to the reader to check that this is a group. It is now immediate to verify condition (2) and the fact that $A^n_B/G \to \overline{\mathbf{A}}_{g,n}$ is birational onto the image.

3. Proof of the Main Theorem

3.1. STABLY S-INTEGRAL POINTS. Let A be a principally polarized abelian variety, with theta divisor Θ, defined over a number field K. Let S be a finite number of places in K, and denote by $\mathcal{O}_{K,S}$ the ring of S-integers. For an extension L of K, we denote by S_L the set of places in L over S.

Definition 3.1.1: A K-rational point $P \in (A \setminus \Theta)(K)$ is called a *stably S-integral point* if *there exists* a finite extension $L \supset K$, and a stable quasi-abelian model $(\mathcal{P} \to \operatorname{Spec} \mathcal{O}_{L,S_L}, \tilde{\Theta})$, such that $P \in (\mathcal{P} \setminus \tilde{\Theta})(\mathcal{O}_{L,S_L})$, namely P is integral on the complement of $\tilde{\Theta}$ in $\mathcal{P}$.

The following result shows that the existential quantifier is not important in the definition:

PROPOSITION 3.1.2: *Suppose $P \in (A \setminus \Theta)(K)$ is stably S-integral. Let $L' \supset K$ be any finite extension for which there exists a stable quasi-abelian model $(\mathcal{P}' \to \operatorname{Spec} \mathcal{O}_{L',S_{L'}}, \tilde{\Theta}')$.*

Then $P \in (\mathcal{P} \setminus \tilde{\Theta}')(\mathcal{O}_{L',S_{L'}})$.

Proof: Let $L \supset K$ be a finite extension satisfying the conditions in the definition. Take a Galois extension $M \supset L'$ which contains L, with Galois group $G = \operatorname{Gal}(M/L')$. Then by the functoriality of the stable model

$$\begin{aligned} P &\in \{(\mathcal{P} \setminus \tilde{\Theta})(\mathcal{O}_{L,S_L}) \underset{L}{\otimes} M\} \cap \{(A \setminus \Theta)(K) \underset{K}{\otimes} M\} \\ &\subset (\mathcal{P}_M \setminus \tilde{\Theta}_M)(\mathcal{O}_{M,S_M})^G \\ &= (\mathcal{P} \setminus \tilde{\Theta})(\mathcal{O}_{L',S_{L'}}). \end{aligned}$$

This completes the proof. ∎

Stably integral points have a nice characterization in terms of moduli:

PROPOSITION 3.1.3: *Let $P \in (A \setminus \Theta)(K)$. Consider the associated moduli morphism P_m: $\operatorname{Spec} K \to A \to \overline{\mathbf{A}}_{g,1}$. Then P is stably S-integral if and only if P_m is an S-integral point on $\overline{\mathbf{A}}_{g,1} \setminus \overline{\Theta}$.*

Proof: Clearly P_m is a rational point on $\overline{\mathbf{A}}_{g,1} \setminus \overline{\Theta}$, so to check that it is S integral we may pass to a finite extension field. Let $L \supset K$ be an extension such that there exists a stable quasi-abelian model $(\mathcal{P} \to \operatorname{Spec} \mathcal{O}_{L,S_L}, \tilde{\Theta})$. Since $\mathcal{P}$ and $\overline{\mathbf{A}}_{g,1}$ are proper, we have morphisms $\tilde{P}$: $\operatorname{Spec} \mathcal{O}_{L,S_L} \to \mathcal{P}$ and $\tilde{P}_m$: $\operatorname{Spec} \mathcal{O}_{L,S_L} \to \overline{\mathbf{A}}_{g,1}$. Note that by the coarse moduli property, $\tilde{\Theta}$ is the set theoretic inverse image of

$\overline{\Theta}$. Then P is stably S-integral if and only if $\tilde{P}$ is disjoint from $\tilde{\Theta}$, if and only if $\tilde{P_m}$ is disjoint from $\overline{\Theta}$. ∎

3.2. Reduction to moduli. Fix a family $(\mathfrak{A} \to \mathfrak{B}, \Xi)$ of smooth principally polarized quasi-abelian varieties with theta divisors. We say that a point $P \in (\mathfrak{A} \setminus \Xi)(K)$ is *stably S-integral*, if it is stably S-integral on the fiber of $\mathfrak{A} \to \mathfrak{B}$ on which it lies.

Let $\mathcal{P} \subset (\mathfrak{A} \setminus \Xi)(K)$ be the set of stably S-integral points.

Proposition 3.2.1: *Assume that the Lang–Vojta conjecture holds true. Then for a sufficiently large integer n, the set $\mathcal{P}$ is n-correlated.*

Proof: We may assume $\mathfrak{B}$ is irreducible (by taking irreducible components one by one) and hence is a variety.

Following [CHM97], we would like to reduce the situation to a situation on moduli spaces. There is the natural moduli morphism $\nu\colon \mathfrak{B} \to \overline{\mathbf{A}}_g$ from $\mathfrak{B}$ to the coarse moduli space of stable pairs. We denote by W_0 the image $\nu(\mathfrak{B})$ under this map. There is also a compatible dominant morphism $\mathfrak{A} \to W_1 \subset \overline{\mathbf{A}}_{g,1}$, creating a commutative diagram:

$$\begin{array}{ccccc} \mathfrak{A} & \to & W_1 & \subset & \overline{\mathbf{A}}_{g,1} \\ \downarrow & & \downarrow & & \downarrow \\ \mathfrak{B} & \to & W_0 & \subset & \overline{\mathbf{A}}_g. \end{array}$$

Recall that we have characterized stably integral points in terms of their image in moduli. Thus the Proposition follows immediately from the following purely geometric result:

Theorem 3.2.2: *Let $W_0 \subset (\overline{\mathbf{A}}_g)_{\mathbb{C}}$ be a closed subvariety, and suppose we have $W_0 \cap (\mathbf{A}_g)_{\mathbb{C}} \neq \emptyset$ (thus the generic point of W_0 parametrizes a smooth quasi-abelian variety). Let $W_n \subset (\overline{\mathbf{A}}_{g,n})_{\mathbb{C}}$ be the reduced scheme underlying the inverse image, and $\Theta_{W_n} = W_n \cap \overline{\Theta}_n$. Then for large integer n, the variety $W_n \setminus \Theta_{W_n}$ is of logarithmic general type.*

In view of Proposition 2.5.1, it suffices to prove the following:

Proposition 3.2.3: *Let $(P \to B, \Theta)$ be a generically smooth complex projective family of stable pairs over a projective base variety B. Assume $(P \to B, \Theta)$ is* of maximal variation, *namely the morphism $B \to \overline{\mathbf{A}}_g$ is generically finite. Let $G \subset \mathrm{Aut}(P \to B, \Theta)$ be a finite subgroup. Then for a sufficiently large integer n, the quotient variety $(P^n_B \setminus \Theta_n)/G$ is of logarithmic general type.*

Proof: Our first step is to replace $P \to B$ by a toroidal situation. We have a moduli morphism $B \to \overline{\mathfrak{A}}_g$, hence at least a rational map to the Voronoi compactification $B \dashrightarrow \overline{\mathfrak{A}}'_g$, which is toroidal. Unfortunately we might need finite covers to lift this to a morphism, since we are working with stacks!

In any case, there is a variety B_1 and a finite surjective morphism $B_1 \to B$ such that $B_1 \to B \dashrightarrow \overline{\mathfrak{A}}'_g$ extends to a morphism. Taking the normalization in the Galois closure of $K(B_1)$ over $K(B/G)$, we may assume that $B_1 \to B/G$ is Galois, with Galois group G'. The group

$$G_1 = G \underset{\mathrm{Aut} B}{\times} G'$$

acts on $P_1 = P \times_B B_1$ in such a way that $(P_1)^n_{B_1}/G_1 = P^n_B/G$. Thus we may replace $P \to B$ by $P_1 \to B_1$, in particular we may assume that there is a morphism $B \to \overline{\mathfrak{A}}'_g$ lifting $B \to \overline{\mathfrak{A}}_g$. We may also assume that $B \to \overline{\mathfrak{A}}'_g$ factors through a toroidal desingularization $\overline{\mathfrak{A}}''_g \to \overline{\mathfrak{A}}'_g$. We denote $\overline{\mathfrak{A}}''_{g,1} = \overline{\mathfrak{A}}'_{g,1} \times_{\overline{\mathfrak{A}}'_g} \overline{\mathfrak{A}}''_g$.

We can choose a G-equivariant resolution of singularities $B' \to B$ such that $P' = P \times_B B'$ degenerates over a normal crossings divisor. Again, we replace $P \to B$ by $P' \to B'$, so we may assume B is nonsingular and $P \to B$ degenerates over a normal crossings divisor. Denote by $U_B \subset B$ the complement of this divisor, namely the smooth locus of $P \to B$, and $U_P \subset P$ the inverse image.

Since $\overline{\mathfrak{A}}'_{g,1} \to \overline{\mathfrak{A}}'_g$ is toroidal, we have that $\overline{\mathfrak{A}}''_{g,1} \to \overline{\mathfrak{A}}''_g$ is toroidal. Since $\overline{\mathfrak{A}}''_g$ is nonsingular, we can apply [A-Karu], Lemma 6.2. This implies that $(U_P \subset P) \to (U_B \subset B)$ is also a toroidal morphism.

Under this assumption we claim

CLAIM:

1. *The pair (P^n_B, Θ_n) has log-canonical singularities, and*
2. *The complement $P^n_B \smallsetminus \Theta_n$ has only canonical singularities.*

Proof: Observe that (P^n, Θ_n) is also a family of stable pairs over a nonsingular base B_1, satisfying the toroidal assumptions. Now the assertion follows directly from 2.3.6 and 2.3.5. ∎(Claim)

We go back to the proof of the proposition.

Since B is generically finite over the moduli space and since the generic fiber of π is a smooth Abelian variety, Theorem 1.1 of [Kaw85] implies that

$$\det(\pi_* \omega^l_{P/B}) = \pi_* \omega^l_{P/B}$$

is a big invertible sheaf on B for some positive integer l. Here we used the fact that

$$h^0(\pi^{-1}(b), \omega^l_{\pi^{-1}(b)}) = 1 \quad \forall b \in B.$$

Since $\omega_{P/B}(\Theta)$ is relatively ample for π, we conclude

$$\omega_{P/B}(\Theta) \otimes \pi^*(\pi_*\omega^l_{P/B})^n$$

is big for some sufficiently large integer n. As we have the natural inclusion map

$$\omega_{P/B}(\Theta) \otimes \pi^*(\pi_*\omega^l_{P/B})^n \hookrightarrow (\omega_{P/B}(\Theta))^{1+ln},$$

we conclude that the sheaf $\omega_{P/B}(\Theta)$ is big.

Let $\Sigma \subset P$ be the locus of fixed points of nontrivial elements of the group G. Denote the sheaf of ideals $\mathcal{I}_\Sigma$. For sufficiently large n the sheaf

$$(\omega_{P/B}(\Theta))^{\otimes n} \otimes \pi^*\omega_B \otimes \mathcal{I}_\Sigma^{|G|}$$

is big as well. Taking the n-th fibered powers, we have that

$$(\omega_{P^n_B/B}(\Theta_n))^{\otimes n} \otimes \pi_n^*\omega_B^{\otimes n} \otimes \prod_{i=1}^n p_i^{-1}\mathcal{I}_\Sigma^{|G|}$$

is a big sheaf on P^n_B. Note that

$$\prod_{i=1}^n p_i^{-1}\mathcal{I}_\Sigma^{|G|} \subset (\Sigma_{i=1}^n p_i^{-1}\mathcal{I}_\Sigma^{|G|})^n,$$

where the latter ideal vanishes to order $\geq n|G|$ on the fixed points of nontrivial elements of G in P^n_B. Also note that

$$(\omega_{P^n_B/B}(\Theta_n))^{\otimes n} \otimes \pi_n^*\omega_B^{\otimes n} = (\omega_{P^n_B}(\Theta_n))^{\otimes n}.$$

Therefore, for $l >> 0$ we have many invariant sections of $\omega_{P^n_B}(\Theta_n)^{\otimes ln}$ vanishing on this fixed point locus to order $\geq ln \cdot |G|$. Let $f\colon (V, D) \to (P^n_B, \Theta_n)$ be an equivariant good resolution of singularities such that $f^{-1}(\Theta_n) = D$. Now the following lemma, together with the Claim above, imply that the invariant sections of $\omega_{P^n_B}(\Theta_n)^{\otimes ln}$ vanishing on the fixed point locus to order $\geq ln \cdot |G|$ descend to sections of the pluri-log canonical divisors of a good resolution of the quotient pair $(P^n_B/G, \Theta_n/G)$, and hence $(P^n_B \setminus \Theta_n)/G$ is of logarithmic general type.

LEMMA 3.2.4 ([ℵ97b], Lemma 4; [CHM97], Lemma 4.1): *Let V be a nonsingular variety with a normal crossing divisor D, and let $G \subset \mathrm{Aut}(V, D)$ be a finite subgroup of the automorphism group of the pair (V, D). Let Σ be the locus of the fixed points of the nontrivial elements of G.*

Denote by $q\colon (V, D) \to (W = V/G, D_W = D/G)$ the natural morphism to the quotient and choose a good resolution $r\colon (\tilde{W}, D_{\tilde{W}}) \to (W, D_W)$.

Then an invariant section

$$s \in H^0(V, \omega_V(D)^{\otimes k})^G$$

such that

$$s_x \in \omega_V(D)^{\otimes k} \otimes \mathcal{I}_\Sigma^{\otimes k|G|} \quad \forall x \in \Sigma \setminus D$$

comes from $\tilde{W}$, i.e., there exists

$$t \in H^0(\tilde{W}, \omega_{\tilde{W}}(D_{\tilde{W}})^{\otimes k})$$

such that $s = q^* r_* t$.

This completes the proof of Proposition 3.2.3, which implies Theorem 3.2.2, and Proposition 3.2.1 follows. ∎

3.3. A "COMPREHENSIVE" FAMILY. In order to prove a result about *all* principally polarized abelian varieties of a given dimension, we use the following fact: there exists a projective family of principally polarized abelian varieties with theta divisors $\pi\colon (\mathfrak{A} \to \mathfrak{B}, \Xi)$ over a quasi-projective base $\mathfrak{B}$ such that for any principally polarized abelian variety with theta divisor (A, Θ) defined over K there is a morphism $\operatorname{Spec} K \to \mathfrak{B}$ satisfying

$$\begin{array}{ccc}
(A, \Theta) = \operatorname{Spec} K \times (\mathfrak{A}, \Xi) & \longrightarrow & (\mathfrak{A}, \Xi) \\
\downarrow & & \downarrow \\
\operatorname{Spec} K & \longrightarrow & \mathfrak{B}.
\end{array}$$

Such a family can be found easily by noting that any principally polarized abelian variety (A, Θ) defined over K can be embedded in a projective space using the very ample linear system $|l\Theta|$ for some fixed $l \geq 3$, and therefore one can choose an appropriate quasi projective subscheme of the Hilbert scheme as the base $\mathfrak{B}$.

3.4. FROM CORRELATION TO UNIFORMITY. Proposition 3.2.1 shows that the set $\mathcal{P}$ of stably S-integral points on $\mathfrak{A} \setminus \Xi$ is correlated. We now suggest an argument for uniformity, replacing that in Lemma 1.1.2. We follow [Pac97b] and [ℵV96]. The argument is complete in case $g \leq 2$.

Let $E_n \subset \mathcal{P}^n_{\mathfrak{B}}$ be the set of n-tuples of *distinct* stably S-integral points. Consider the tower of maps

$$\cdots \to E_{n+1} \to E_n \to E_{n-1} \to \cdots \to E_1.$$

Let $E_n^{(m)}$ be the image of E_m in E_n and set $F_n^{(m)} = \overline{E_n^{(m)}}$. We have a descending chain of closed subsets

$$F_n^{(m)} \supset F_n^{(m+1)} \supset \cdots$$

which must stabilize to a closed set F_n, i.e., there exists m_n such that

$$F_n^{(m_n)} = F_n^{(m_n+1)} = \cdots = F_n.$$

Accordingly, we have a tower of maps

$$\cdots F_{n+1} \to F_n \to F_{n-1} \to \cdots \to F_1.$$

Ultimately we would like to conclude that all the F_n are empty, which implies the uniformity.

First observe that $F_{n+1} \to F_n$ is surjective for all n. In fact, if we take $m \geq m_{n+1}, m_n$ then $E_{n+1}^{(m)} \to E_n^{(m)}$ is surjective by definition and hence $F_{n+1} = \overline{E_{n+1}^{(m)}} \to F_n = \overline{E_n^{(m)}}$ is surjective. Second, we wish to prove inductively that a fiber of $F_{n+1} \to F_n$ cannot have dimension $0, 1, \ldots, g$ and hence all the F_n are empty; in this paper we do this only for $g \leq 2$.

Denote by $\tau_{n+1}\colon F_{n+1} \to F_n$ the surjective morphisms as above. It is enough to prove the following "Inductive Statement d" for all $0 \leq d \leq g$:

> **Inductive Statement d**: *"Suppose no fiber of the map $\tau_{n+1}\colon F_{n+1} \to F_n$ has dimension $< d$ for all $n \geq 1$, where $0 \leq d \leq g$. Assume that the Lang–Vojta conjecture holds true. Then no fiber of $\tau_{n+1}\colon F_{n+1} \to F_n$ has dimension $\leq d$ for all $n \geq 1$."*

Inductive Statement 0 can be proven without assuming any conjectures:

CLAIM: *No fiber of $\tau_{n+1}\colon F_{n+1} \to F_n$ has dimension 0 for all $n \geq 1$.*

Proof of Claim: Assume that a fiber of $\tau_{n+1}\colon F_{n+1} \to F_n$ has dimension 0 for some n. Then by the upper semicontinuity of the fiber dimension, the dimension of a fiber must be 0 over some open subset U of F_n. Over U we may assume the number of points in a fiber is also a constant e. The open set U contains a point in $E_n^{(m)}$ where $m \geq m_n, m_{n+1}$ and $m - n > e$. But as E_m consists of $n + (m - n)$-tuples of distinct points, this is impossible. ∎(Claim)

We can prove the "upper end of the induction" $d = g$ in general:

CLAIM: *Assume the Lang–Vojta conjecture holds true. Suppose every fiber of $\tau_{n+1}\colon F_{n+1} \to F_n$ has dimension g for all $n \geq 1$. Then $F_n = \emptyset$ for all n.*

Proof of Claim: Take an irreducible component M_n of F_n and denote by M_{n+k} its inverse images in F_{n+k} (which is also an irreducible component in F_{n+k}) for $k \geq 1$. Then $M_{n+k} = {M_{n+1}}^k_{M_n}$, where $M_{n+1} \to M_n$ is a family of principally polarized quasi-abelian varieties of dimension g, with theta divisors $\Theta_{M_{n+1}}$. By Proposition 3.2.1, for sufficiently large integer k, the set of k-tuples of stably S-integral points $\mathcal{P}^k_{M_n}$ is not dense in the fibered power $(M_{n+1})^k_{M_n}$. On the other hand, M_{n+k} has *by definition* a dense set of stably S-integral points, a contradiction. ∎(Claim)

From now on fix $d \geq 1$, We now make a few general reduction steps for Inductive Statement d. We note that in general we have $F_{n+k} \subset {F_{n+1}}^k_{F_n}$ for $k \geq 1$. Consider an irreducible component $B \subset F_n$ over which $X = \tau_{n+1}^{-1} B$ has relative dimension d. Consider $\tau = \tau_{n+1}|_X\colon X \to B$, and $\mathcal{Q} = E_{n+1} \cap X$. In terms of Definition 1.4.1, we can reformulate our problem as follows:

LEMMA 3.4.1: *In order to prove Inductive Statement d, it suffices to show that $\mathcal{Q}$ is strongly k-correlated for some integer k.*

Indeed, we have that the dimension of every fiber of $F_{n+k} \to F_n$ is $\geq kd$. ∎

Denote by $X_i, i = 1, \ldots, c'$ the irreducible components of X of relative dimension d over B. Let $X_i' \to X_i$ be the normalization, and $X_i' \to B_i' \to B$ the Stein factorization. In order to keep things inside a family of smooth quasi-abelian varieties, consider $A = B' \times_{\mathfrak{B}} \mathfrak{A}$ and denote by X_i'' the image of X_i' in A. Note that we have a factorization $X_i' \to X_i'' \to X_i$. Consider the subset $\mathcal{Q}_i'$ as in Proposition 1.4.2. Writing $\Theta = B' \times_{\mathfrak{B}} \Xi$, we can view $\mathcal{Q}_i'$ as a subset of the set of stably integral points on $A \setminus \Theta$.

By Proposition 1.4.2, we have the following:

LEMMA 3.4.2: *In order to prove Inductive Statement d, it suffices to show that $\mathcal{Q}_i' \subset X_i''(K)$ is correlated for $X_i''(K) \to B'$ for all i.*

What can we say about correlation of $\mathcal{Q}'_i$? For simplicity of notation denote $X''_i = H \subset A$, and replace B by B'. We may even replace B by a nonempty open subset, therefore we may assume $H \to B$ is flat. We can now classify the family $H \to B$. We apply the following proposition, which is essentially due to Ueno [Uen73] and Kawamata [Kaw80], to our situation.

PROPOSITION 3.4.3: *Let $H \to B$ be a flat family of geometrically irreducible closed subvarieties of a family of smooth quasi-abelian varieties $A \to B$, defined over a number field K, where A is a torsor on an abelian scheme $\tilde{A} \to B$. Then there exists an abelian subscheme $D \to B$ such that H is invariant under translation by D and such that the generic fiber of $H/D \to B$ is a variety of general type.*

Ueno and Kawamata state the above proposition only in the case $B = \operatorname{Spec} \mathbb{C}$. The general case is immediate from [McQ94], Théorème 1.2, and Lemme 1.3 applied to the generic fiber.

Consider the family $H/D \to B$. It is a family of varieties of general type. We have two cases to consider

CASE 1: $H/D \to B$ has relative dimension > 0.

By the Fibered Power Theorem (Theorem 1.3.1), $(H/D)^k_B$ dominates a positive dimensional variety of general type, for some integer k. This immediately implies that H^k_B dominates a positive dimensional variety of general type. Lang's conjecture implies that $H^k_B(K)$ is not dense, which implies that $\mathcal{Q}'_i$ is k-correlated, which is what we wanted.

CASE 2: $H/D \to B$ is an isomorphism.

In this case, the generic fiber of $H \subset A$ is a translate of the generic fiber of D, which is an abelian variety. *The issue here is whether or not the set of stably S-integral points contained in this sub-family is correlated.*

3.5. THE CASE OF AN ELLIPTIC SUBSCHEME. In this section we provide an argument for case $d = 1$, completing the proof of Inductive Statement 1. The argument uses Pacelli's strong uniformity results for elliptic curves (see [Pac97b]). This completes the proof of the Main Theorem. At the end of the paper we discuss a possible line of argument which could lead to a proof in arbitrary dimension.

Thus we assume $\tau: H \to B$ is a family of elliptic curves inside of principally polarized abelian varieties. For simplicity of notation we denote $\mathcal{Q}'_i = \mathcal{R}$.

Take a point $P \in \mathcal{R}$, and let $b = \tau(P) \in B$. Choose a point $O \in E \cap \Theta$ (which may not be K-rational). Note that there is a constant a which only depends on the family $H \to B$ and such that $a \geq \operatorname{length}_K \mathcal{O}_{E\cap\Theta}$ and hence that the extension degree of the residue field at O over K is bounded by this number, $[k(O):K] \leq a$. Let $(E \times \operatorname{Spec} k(O), O)$ be the elliptic curve extended by extension to the residue field $k(O)$. Then the following lemma implies that $P \in (E \times \operatorname{Spec} k(O) \setminus \{O\})(k(O))$ is a stably $S_{k(O)}$-integral point, where $S_{k(O)}$ is the set of places in $k(O)$ over S.

LEMMA 3.5.1: *Let (A, Θ) be a principally polarized abelian variety and $E \subset A$ be an elliptic curve with the origin $O \in E \cap \Theta \subset A$ defined over a number field L. Let $Q \in E \setminus \Theta \subset A \setminus \Theta$ be an L-rational point, i.e., $Q \in (E \setminus O)(L) \subset (A \setminus \Theta)(L)$, which is a stably S_L-integral point of the pair (A, Θ).*

Then Q is also a stably S_L-integral point of the pair (E, O).

Proof: First remark that by Proposition 3.1.2 we are allowed to take any finite extension of L in order to prove the assertion.

By taking such a finite extension of L, we may assume that

$$(A \to \operatorname{Spec} L, \Theta)$$

has the stable quasi-abelian model

$$\pi\colon (\mathcal{P} \to \operatorname{Spec} \mathcal{O}_{L,S_L}, \tilde{\Theta}),$$

and that $(E, O) \to \operatorname{Spec} L$ has a stable model $\mathcal{E} \to \operatorname{Spec} \mathcal{O}_{L,S_L}$.

For any prime $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_{L,S_L}$, we need to show that the point $\overline{Q}_{\mathcal{E}}$ is disjoint from the origin O in the fiber $\mathcal{E}_{\mathfrak{p}}$ over $\mathfrak{p}$, where $\overline{Q}_{\mathcal{E}}$ is the Zariski closure of the L-rational point Q in $\mathcal{E}$. Therefore, instead of working over $\mathcal{O}_{L,S_L}$, we may work over the completion R of $\mathcal{O}_{L,S_L}$ at the prime $\mathfrak{p}$.

We denote by η the generic point of $\operatorname{Spec} R$. In a slight abuse of notation, we use the same letters for the objects over $\operatorname{Spec} R$ as for those over $\operatorname{Spec} \mathcal{O}_{L,S_L}$.

Replacing R by a finite extension we may assume there exists a section $s\colon \operatorname{Spec} R \to \mathcal{P}$ so that its image O_s sits in the locus where the morphism π is smooth. This can be done by picking a smooth point on the central fiber $\mathcal{P}_{\mathfrak{p}}$ and lifting using Hensel's lemma.

Let $\mathcal{N}_A \to \operatorname{Spec} R$ be the Néron model of A (with the origin O_s), and $\mathcal{N}_E \to \operatorname{Spec} R$ the Néron model of E (with the origin O).

By the universal property of the Néron model, we have a morphism

$$\phi\colon \mathcal{N}_E \to \mathcal{N}_A$$

extending the inclusion $E \hookrightarrow A$. By Proposition 2.4.1 we also have a morphism

$$\psi\colon \mathcal{N}_A \to \mathcal{P},$$

extending the isomorphism

$$(\mathcal{N}_A)_\eta \tilde{\to} \mathcal{P}_\eta = A.$$

Therefore, we have a morphism

$$\psi \circ \phi \colon \mathcal{N}_E \to \mathcal{P}$$

extending the inclusion

$$E \hookrightarrow A.$$

Under this morphism, the origin $O_{\mathcal{N}_E}$ maps into $\tilde{\Theta}$.

Denote by $\overline{Q}_{\mathcal{N}_E}$ and $\overline{Q}_{\mathcal{P}}$ the closures of the L-rational point Q in $\mathcal{N}_E$ and $\mathcal{P}$, respectively. Then clearly $\psi \circ \phi(\overline{Q}_{\mathcal{N}_E}) = \overline{Q}_{\mathcal{P}}$.

Since Q is a stably S_L-integral point of the pair (A, Θ), we have that $\tilde{\Theta}$ and $\overline{Q}_{\mathcal{P}}$ are disjoint in $\mathcal{P}$. A-fortiori, $O_{\mathcal{N}_E}$ and $\overline{Q}_{\mathcal{N}_E}$ are disjoint in $\mathcal{N}_E$.

Note that $\mathcal{N}_E$ and $\mathcal{E}$ are isomorphic in a neighborhood of $O_{\mathcal{N}_E}$. We conclude that $O_{\mathcal{E}}$ and $\overline{Q}_{\mathcal{E}}$ are disjoint, and hence that Q is a stably S_L-integral point of the pair (E, O). ∎(Lemma)

Going back to Inductive Statement 1, recall that the extension degree $[k(O) : K] \le a$ over the fixed number field K is uniformly bounded. In particular, $[k(O) : \mathbb{Q}] \le d = a \cdot [K : \mathbb{Q}]$. Pacelli's result [Pac97b] asserts that, assuming the Lang–Vojta conjecture, there is a uniform bound $N(d, S)$ such that for any elliptic curve $(E, \{O\})$ defined over a number field L of degree $\le d$, the number of the stably S_L-integral points is uniformly bounded,

$$\# E(L, S_L)^{stable} < N(d, S).$$

In our situation, this implies that the number of stably S-integral points on $A \setminus \Theta$ lying in E is uniformly bounded by $N(d, S)$, which in particular says that the points are $N(d, S)$-correlated.

This completes the proof of Inductive Statement 1, which completes the case $\dim A = g = 2$ and hence the Main Theorem. ∎

3.6. Towards higher dimensions. Finally we discuss a possible line of argument for higher dimensions (which would also lead to a result about arbitrary polarizations).

If one considers Lemma 3.4.2 and Case 2 in the discussion following that Lemma, one reduces to the following conjecture:

CONJECTURE 3.6.1: *Let $(A \to B, \Theta)$ be a family of smooth principally polarized quasi-abelian varieties over a number field K. Let $H \subset A$ be a family of quasi-abelian subvarieties. Let $\mathcal{P} \subset A(K)$ be the set of stably integral points. Then $\mathcal{P} \cap H$ is correlated with respect to $H \to B$.*

We would like, at least, to show that this conjecture follows from the Lang–Vojta conjecture.

Denote $\Theta_H = H \cap \Theta$. Replacing B by a nonempty open subset, we may assume Θ does not contain a fiber of $H \to B$. Then $(H \to B, \Theta_H)$ can be viewed as a family of polarized quasi-abelian varieties. The issue is, that these are not necessarily principally polarized.

In the recent preprint [Ale98], Alexeev defines a complete moduli space for such pairs as well. We call these "Alexeev stable pairs" below. This suggests the following approach to the problem:

1. Define "Alexeev stably integral points" of (H, Θ_H) to be rational points which are integral on the complement of Θ_H in an Alexeev stable model of a pair (H, Θ_H).
2. Give a criterion for Alexeev stably integral points in terms of Néron models.
3. Deduce that a stably integral point of a pair (A, Θ) is also stably integral on (H, Θ_H).
4. Assuming Lang–Vojta, reduce the problem to a problem on moduli of n-pointed Alexeev stable pairs similar to Theorem 3.2.2 and Proposition 3.2.3.
5. Prove a result analogous to Proposition 3.2.3.

All but the last step seem straightforward. The main issues in the last step are:

1. Suppose $(P \to B, \Theta)$ is an Alexeev stable pair of maximal variation, defined over a field K, over a projective irreducible nonsingular base B, with smooth generic fiber. There exists $\varepsilon > 0$ such that, for all n, the pair $(P^n_B, \varepsilon\Theta_n)$ has log-canonical singularities.
2. For such $(P \to B, \Theta)$, the sheaf $\omega^m_{P/B}(\Theta)$ is big for some $m > 0$.

We expect that these statements can be proven using Alexeev's work.

References

[ℵ95] D. Abramovich, *Uniformité des points rationnels des courbes algébriques sur les extensions quadratiques et cubiques*, Comptes Rendus de l'Académie des Sciences, Paris, Série I, Mathématique **321** (1995), 755–758.