ISRAEL JOURNAL OF MATHEMATICS 121 (2001), 61–84

A CONJECTURE ON ARITHMETIC FUNDAMENTAL GROUPS*

BY

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ABSTRACT

The conjecture is the following: Over an algebraic variety over a finite field, the geometric monodromy group of every smooth $\overline{\mathbb{F}_\ell((t))}$-sheaf is finite. We indicate how to prove this for rank 2, using results of Drinfeld. We also show that the conjecture implies that certain deformation rings of Galois representations are complete intersection rings.

1. Introduction

Let X be a curve over a finite field k of characteristic p. In [2], Deligne formulates a conjecture on lisse $\bar{\mathbb{Q}}_\ell$-sheaves $\mathcal{E}$ over X, where $\ell \neq p$. A part of this conjecture can be stated as follows. If $\mathcal{E}$ is irreducible and has trivial determinant then the eigenvalues of the Frobenii on the stalks of $\mathcal{E}$ should be algebraic numbers (Weil numbers).

In this article we formulate a similar conjecture for lisse $\overline{\mathbb{F}_\ell((t))}$-sheaves over X. In this case the algebraicity of the eigenvalues of Frobenii implies the finiteness of the image of the monodromy representation, see Proposition 2.8. Thus the conjecture can be stated simply as follows.

* This material is based upon work supported by the National Science Foundation under Grant No. 9970049.
Received November 18, 1998

1.1 CONJECTURE (see 2.3): *For any lisse $\overline{\mathbb{F}_\ell((t))}$-sheaf $\mathcal{E}$ over X the geometric monodromy group is finite.*

In [3], Drinfeld proved part of Deligne's conjectures for lisse $\bar{\mathbb{Q}}_\ell$-sheaves of rank 2. He proved more, namely he attached an unramified cusp form (eigenform) to any such $\mathcal{E}$. In Section 4 we indicate how these arguments give a similar result for our conjecture.

1.2 THEOREM (see 4.10): *The conjecture above holds for lisse sheaves of rank 2.*

In Section 4 we spell out the link between Conjecture 1.1 and the Langlands program: A strong result towards the Langlands correspondence mod ℓ for $n \geq 3$ implies the conjecture for sheaves of rank n. The results which have been obtained in the literature on the Langlands correspondence (Drinfeld, Deligne, Laumon and others) are much stronger, but of a slightly different flavor from what we are trying to do here. For example, we find (Theorem 2.17) that one needs only to prove our conjecture over **projective** curves, hence we need to study only unramified cusp forms.

The author thinks the conjecture is interesting because it implies a result on deformation rings of Galois representations. In [9] it was shown that certain deformation rings are complete intersections. In this paper we show that deformation rings of representations of $\pi_1(X)$ are often complete intersections, see Theorem 3.5. The proof of this theorem gives a geometric explanation of this phenomenon. We briefly formulate a special case of the result. Let $\rho_0\colon \pi_1(X) \to \mathrm{SL}_n(\mathbb{F}_\ell)$ be a residual representation. We assume that $\rho_0|_{\pi_1(\bar{X})}$ is absolutely irreducible and that ℓ does not divide n. Let $\rho_{\mathrm{univ}}\colon \pi_1(X) \to \mathrm{SL}_n(R_{\mathrm{univ}})$ be the universal deformation of ρ_0.

1.3 THEOREM (see 3.5): (i) *If the conjecture above holds, then R_{univ} is a complete intersection finite and flat over $\mathbb{Z}_\ell$.*
(ii) *If $n = 2$, then R_{univ} is a complete intersection finite and flat over $\mathbb{Z}_\ell$.*

As a corollary we obtain (for $n = 2$) that any such ρ_0 can be lifted to a representation $\rho\colon \pi_1(X) \to \mathrm{SL}_2(\bar{\mathbb{Q}}_\ell)$, see Remark 3.6. Undoubtedly, the motivation behind Deligne's conjectures is that one hopes to find a family of motives corresponding to any irreducible lisse $\bar{\mathbb{Q}}_\ell$-sheaf. Therefore, the above seems to indicate that we can guess the existence of a family of motives over X, by observing a single sufficiently irreducible residual representation.

ACKNOWLEDGEMENT: During January 1998 the author was the Ordway Visitor to the Mathematics Department of the University of Minneapolis. The material of this article reproduces the content of a series of lectures given during his stay.

2. The conjecture

2.1 Notation: In this section we will use the following notations:

p is a prime number,
k is a finite field of characteristic p,
$\bar{k}$ is an algebraic closure of k,
X is a variety over k, i.e., an integral scheme, separated and of finite type over $\operatorname{Spec} k$,
$\overline{X}$ is the fibre product $X \times_{\operatorname{Spec} k} \operatorname{Spec} \bar{k}$,
ℓ is a prime number different from p, and
F is a local field of characteristic ℓ. Thus for some finite field $\mathbb{F}$ of characteristic ℓ we have $F \cong \mathbb{F}((t))$.

2.2. We want to understand representations of the arithmetic fundamental group $\pi_1(X, \bar{x})$ of X over local fields of characteristic ℓ. Recall that, if $\bar{x}$ is a point $\operatorname{Spec} \bar{k} \to \overline{X}$, there is a complex of pro-finite groups

$$1 \longrightarrow \pi_1(\overline{X}, \bar{x}) \longrightarrow \pi_1(X, \bar{x}) \longrightarrow \operatorname{Gal}(\bar{k}/k) \longrightarrow 1.$$

It is exact if X is geometrically connected over $\operatorname{Spec} k$. We often omit the base point $\bar{x}$ from the notation, e.g. $\pi_1(X) = \pi_1(X, \bar{x})$ and $\pi_1(\overline{X}) = \pi_1(\overline{X}, \bar{x})$. The conjecture will tell us something about the action of $\operatorname{Gal}(\bar{k}/k) \cong \hat{\mathbb{Z}}$ on $\pi_1(\overline{X})$.

Let V be a finite dimensional F-vector space. We are going to consider continuous representations

$$\rho \colon \pi_1(X) \longrightarrow \operatorname{GL}(V).$$

The topologies involved are the pro-finite topology on $\pi_1(X)$ and the topology induced from the topology on the local field F on $\operatorname{GL}(V) \subset \operatorname{End}(V)$.

2.3 CONJECTURE: *Let X, ρ be as above, and assume that X is a normal scheme. Then $\rho(\pi_1(\overline{X}))$ is finite.*

In words: any representation defined on the **arithmetic** fundamental group of X is finite when restricted to the **geometric** fundamental group. Let us remark on the assumptions of the conjecture.

If X is an elliptic curve over k, for example, then $\pi_1(\overline{X})$ surjects onto $\mathbb{Z}_\ell$ for any ℓ. Hence there exist representations $\bar{\rho}$: $\pi_1(\overline{X}) \to \mathrm{GL}_1(F)$ with infinite image. Just compose the surjection $\pi_1(\overline{X}) \to \mathbb{Z}_\ell$ with the map $\mathbb{Z}_\ell \to \mathbb{F}_\ell((t))^*$, mapping 1 to $1+t \in \mathbb{F}_\ell((t))^*$. Thus the conjecture does not hold for representations defined on $\pi_1(\overline{X})$.

Since $\pi_1(X)$ surjects onto $\mathbb{Z}_\ell$, there are many representations with $\rho(\pi_1(X))$ infinite. This explains why we look at the image of the geometric fundamental group.

If we do not assume that X is normal then the conjecture does not hold either. For example, suppose that X is the curve that one obtains from $\mathbb{P}^1_k$ by identifying 0 with ∞. In this case $\pi_1(\overline{X}) \cong \hat{\mathbb{Z}}$ and the exact sequence of 2.2 is split. Again it is easy to make a counterexample to the conjecture in this case.

If we allow $\ell = p$ then the conjecture is false also. For example, take $X = \mathbb{A}^1_k$. It is well known that $\mathrm{Hom}(\pi_1(X), \mathbb{F}_p)$ is (countably) infinite. We can use this to construct a continuous surjective homomorphism $\pi_1(X) \to (\mathbb{F}_p[[t]], +)$. However, there is a continuous injective homomorphism of groups

$$(\mathbb{F}_p[[t]], +) \longrightarrow \mathrm{GL}_2(\mathbb{F}_p((t))),\ x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

The composition gives the desired counter example.

If we do not assume that the local field F has finite characteristic, then the conjecture is not true either. If the characteristic is $(0, \ell)$, then one can take as a counter example the monodromy representation on the Tate module of a family of elliptic curves with nonconstant j-invariant. In fact, in this case, the image of $\pi_1(\overline{X})$ is an open subgroup of $\mathrm{SL}_2(\mathbb{Q}_\ell)$. [In the case $(0,p)$ this works also. If one considers a family of ordinary elliptic curves with nonconstant j-invariant then the image of $\pi_1(\overline{X})$ is open in $\mathrm{GL}_1(\mathbb{Q}_p)$.]

2.4 PROPOSITION: *The conjecture holds if* $\dim V = 1$.

Proof: In this case, ρ can be viewed as a continuous homomorphism $\pi_1(X) \to F^*$. We have $F^* \cong (\text{pro-}\ell) \times \mathbb{F}^* \times \mathbb{Z}$. By [2, Theorem 1.3.1] we have that $\pi_1(X)^{ab} \cong (\text{pro-}p) \times (\text{finite}) \times \hat{\mathbb{Z}}$. Furthermore, the geometric fundamental group of X maps onto the first two factors of the decomposition. Thus the proposition follows. ∎

2.5. In the rest of this section we are going to discuss a few properties of this conjecture. We will show that it suffices to consider the cases where $\rho|_{\pi_1(\overline{X})}$ is irreducible, and that it suffices to consider the cases where X is a projective curve.

Let us first discuss a special case where the conjecture is equivalent to the finiteness of $\rho(\pi_1(X))$. In the following, a continuous representation $\rho\colon \Gamma \to \mathrm{GL}(V)$ of a topological group over F is called (absolutely) irreducible if and only if the representation ρ is (absolutely) irreducible as a representation of the abstract group Γ.

2.6. Some of our results will be general results on extensions of profinite groups

$$1 \longrightarrow \Gamma \longrightarrow G \longrightarrow \hat{\mathbb{Z}} \longrightarrow 1.$$

2.7 Lemma: *In the abstract setting 2.6, suppose that $\rho\colon G \to \mathrm{GL}(V)$ is a continuous representation of G such that*

(i) $\det\rho = 1$, *i.e.,* $\rho(G) \subset \mathrm{SL}(V)$, *and*

(ii) $\rho|_\Gamma$ *is absolutely irreducible.*

Then we have

$$\#\rho(\Gamma) < \infty \Leftrightarrow \#\rho(G) < \infty.$$

Proof: The implication $\Leftarrow$ is trivial. Hence, let us assume that $H = \rho(\Gamma) \subset \mathrm{SL}(V)$ is finite. Pick $g \in \rho(G)$, a topological generator of the pro-cyclic group $\rho(G)/H$. Note that $gHg^{-1} = H$; thus conjugation by g induces an automorphism of H. Since H is finite, this automorphism has finite order. We deduce that there exists an $e \in \mathbb{N}$ such that $h = g^e h g^{-e}$ for all $h \in H$. Thus g^e commutes with H, hence by (ii) and Schur's lemma we get $g^e = \lambda 1_V$ for some $\lambda \in F$. By (i) we get $\lambda^{\dim V} = 1$. Therefore g has finite order and so does $\rho(G)$. ∎

Thus we would like to have a criterion that tells us when $\rho(\pi_1(X))$ is finite. This turns out to be equivalent to the algebraicity of the eigenvalues of Frobenii in the representation ρ. Let us write $|X|$ for the set of closed points of X. For any $x \in |X|$, we let $F_x \in \pi_1(X)$ denote a geometric Frobenius element corresponding to x. We remind the reader that F_x is well defined up to conjugacy.

2.8 Proposition: *Suppose we have X and ρ as in the conjecture. Then the following two assertions are equivalent:*

(i) $\#\rho(\pi_1(X)) < \infty$, *and*

(ii) *for every $x \in |X|$ the characteristic polynomial of $\rho(F_x)$ has coefficients in the finite subfield $\mathbb{F}$ of F.*

Proof: We remark that if $F \cong \mathbb{F}((t))$ then $\mathbb{F} \subset F$ is the set of elements of F that are algebraic over the prime field of F. Thus the implication (i) $\Rightarrow$ (ii) is trivial, as the eigenvalues of an element of finite order are algebraic.

Assume (ii). Consider the functions f_i on $\pi_1(X)$ given by the formulae $\gamma \mapsto \mathrm{Trace}(\rho(\gamma), \bigwedge^i V)$, for $i = 1, \ldots, \dim(V)$. These function are continuous and they take values in $\mathbb{F}$ on F_x for $x \in |X|$. By the Chebotarev density theorem, the F_x are dense in $\pi_1(X)$. Thus we see that f_i has values in $\mathbb{F}$ on $\pi_1(X)$ entirely. By continuity and the fact the $\mathbb{F}$ is finite, we deduce that each f_i is locally constant. Thus there is an open subgroup $H \subset \pi_1(X)$ such that these functions are constant on H, with value $\mathrm{Trace}(1_V, \bigwedge^i V)$. We conclude that $\rho(h)$ is unipotent for all $h \in H$.

We apply [1, Theorem on page 87, Chapter I, Section 4], and we deduce that there exists a basis of V such that $\rho(H)$ lies in the upper unipotent subgroup $U_n(F)$ of $\mathrm{GL}_n(F)$. (Here $n = \dim V$.) Since F has characteristic ℓ, the group $U_n(F)$ has exponent ℓ^n. Thus the map $H \to U_n(F)$ factors through the maximal pro-ℓ quotient H^ℓ of H.

Note that $H \cong \pi_1(Y)$ for some finite étale morphism $Y \to X$. Thus by Lemma 2.9 below we know that $H^\ell = \pi_1^\ell(Y)$ is topologically finitely generated. Hence, by Lemma 2.10 below we see that $\rho(H) = \rho(H^\ell)$ is finite. Therefore, $\rho(\pi_1(X))$ is finite. ∎

2.9 Lemma: *Let Y be a connected scheme of finite type over k or $\bar{k}$. Then the maximal pro-ℓ quotient $\pi_1^\ell(Y)$ of $\pi_1(Y)$ is topologically finitely generated.*

Proof: By [7, Section 4.2], we have to show that the cohomology group

$$H^1(\pi_1^\ell(Y), \mathbb{Z}/\ell\mathbb{Z}) = \mathrm{Hom}(\pi_1^\ell(Y), \mathbb{Z}/\ell\mathbb{Z})$$

is finite. This group is equal to the étale cohomology group $H^1(Y, \mathbb{Z}/\ell\mathbb{Z})$ which is finite, see e.g. [13, finitude]. ∎

2.10 Lemma: *Let $\Gamma \subset U_n(F)$ be a closed subgroup. If Γ is topologically finitely generated, then Γ is finite.*

Proof: Set $\Gamma_{i+1} = [\Gamma, \Gamma_i]$ (take topological closure) and $\Gamma_0 = \Gamma$. From the structure of $U_n(F)$ we get that $\Gamma_n = \{1\}$ and that Γ_{i+1}/Γ_i is annihilated by ℓ^n. Thus Γ is a nilpotent pro-ℓ group, and Γ_i is topologically finitely generated. Clearly this implies that Γ is finite. ∎

2.11 Proposition: *Fix $n \in \mathbb{N}$. Suppose Conjecture 2.3 holds in all cases where $\dim V \leq n$, and X is a curve. Then the conjecture holds in all cases where $\dim V \leq n$.*

Proof: We assume that the hypothesis of the proposition hold. Pick X, V with $\dim V \leq n$ and ρ as in the conjecture. We may assume by induction that Conjecture 2.3 has been proven in all cases (Y, V', ρ'), with $\dim V' \leq n$ and $\dim Y < \dim X$.

Since X is normal, the map $\mathrm{Gal}(k(\overline{X})^{\mathrm{sep}}/k(\overline{X})) \to \pi_1(\overline{X})$ is surjective. Hence for any dominant morphism $X' \to X$ of varieties over k (for example, an open immersion) the image of $\pi_1(\overline{X}') \to \pi_1(\overline{X})$ is open. Therefore, we may assume that X is smooth affine. Furthermore, after replacing X by a finite étale covering (which replaces Image(ρ) by an open subgroup), we may assume that Image(ρ) is a pro-ℓ group.

By [12, Exposé XI, Section 3], we may assume that X is an elementary fibration: There is a diagram

$$X \xrightarrow{j} X' \xrightarrow{f} Y,$$

such that j is an open immersion, f is a smooth projective morphism of relative dimension 1 and such that $X' \setminus X$ is finite étale over Y. In addition, after replacing Y by a nonempty scheme étale over it, we may even assume that the morphism $X \to Y$ has a section $s: Y \to X$. In this situation, by [11, Exposé XII, Proposition 4.3 and Exemples 4.4], there is an exact sequence of pro-ℓ fundamental groups

$$1 \longrightarrow \pi_1^\ell(X_{\bar{y}}, \bar{x}) \longrightarrow \pi_1^\ell(X, \bar{x}) \longrightarrow \pi_1^\ell(Y, \bar{y}) \longrightarrow 1.$$

Here $\bar{y}$ is the geometric point of Y induced from the geometric point $\bar{x}$ of X. We may assume that $\bar{y}$ lies over a closed point y of Y. Thus the restriction of ρ to $\pi_1(X_{\bar{y}}, \bar{x})$ is finite by our assumption applied to the restriction of ρ to $\pi_1(X_y, \bar{x})$.

By induction hypothesis, the composition $\pi_1(\bar{Y}) \xrightarrow{s_*} \pi_1(\overline{X}) \xrightarrow{\rho} \mathrm{GL}(V)$ has finite image. The reader easily deduces from these two statements that the conjecture holds for ρ. ∎

2.12 Lemma: *Let $\mathcal{G}$ be the absolute Galois group of a complete discretely valued field K of characteristic p. Assume the residue field k of K is of finite type over its prime field. Let $I \subset \mathcal{G}$ be the inertia subgroup. Let $\rho: \mathcal{G} \to \mathrm{GL}(V)$ be a continuous representation of $\mathcal{G}$ over F. Then $\rho(I)$ is finite.*

Proof: This is Grothedieck's argument, compare [8, Appendix]. Let $P \subset I$ denote the wild inertia subgroup. Since the topology on $\mathrm{GL}(V)$ is ℓ-adic and the group P is a (compact) pro-p group, the image $\rho(P)$ of P is finite. We also know that I/P is pro-cyclic; thus we can pick an element $c \in \rho(I)$ such that c

topologically generates $\rho(I)/\rho(P)$. We leave it to the reader as an exercise in group theory that a suitable power c^m centralizes the finite group $\rho(P)$ for some $m \in \mathbb{N}$.

Pick an element $F \in \mathcal{G}$ which maps, under the composition $\mathcal{G} \to \mathcal{G}/I \cong \mathrm{Gal}(k^{\mathrm{sep}}/k) \to \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ to a power of Frobenius. This is possible by our assumption on k. Then we have

$$\rho(F)c^m\rho(F)^{-1} = c^{mp^a},$$

for some $a \in \mathbb{N}$. This follows as $I/P \cong \prod_{\ell' \neq p} \mathbb{Z}_{\ell'}(1)$, as a $\mathcal{G}/I$-module. The displayed equation implies that all eigenvalues of c^m are roots of unity. Thus c^n is unipotent for some $n \in \mathbb{N}$. Since F has finite characteristic this implies that c^n has finite order. Thus $\rho(I)$ has finite order. ∎

2.13 Proposition: *If Conjecture 2.3 holds in all cases where X is a projective curve and $\dim V \leq n$, then the conjecture holds in all cases where $\dim V \leq n$.*

Proof: Pick X, V with $\dim V \leq n$ and ρ as in the conjecture. By Proposition 2.11 we may assume that X is a curve. Let $X \subset X'$ be a projective completion. Write $X' \setminus X = \{x_1, \ldots, x_r\}$. Let $\mathcal{O}_i$ be the complete local ring of X' at x_i, and let K_i be its fraction field. There is a natural morphism $\mathrm{Spec}\, K_i \to X$, hence by composition, ρ induces Galois representations ρ_i: $\mathrm{Gal}_{K_i} \to \mathrm{GL}(V)$. By Lemma 2.12 we know that the image of the inertia subgroup $I_i \subset \mathrm{Gal}_{K_i}$ under ρ_i is finite.

We choose an isomorphism $F = \mathbb{F}((t))$. Let $\Lambda \subset V$ be a $\mathbb{F}[[t]]$-lattice in V, stable under ρ. We can choose a large integer $N \in \mathbb{N}$ such that the finite groups $\rho(I_i)$ act faithfully on $\Lambda/t^N\Lambda$. Let $Y \to X$ be the finite étale Galois covering which corresponds to the homomorphism

$$\pi_1(X) \xrightarrow{\rho} \mathrm{GL}(\Lambda) \longrightarrow \mathrm{GL}(\Lambda/t^N\Lambda).$$

Let $Y \subset Y'$ be the projective completion of Y, which is a ramified Galois cover of X'. By our choice of Y, the inertia groups at the points of Y' not in Y, act trivially on V. Thus ρ, when restricted to $\pi_1(Y)$, extends to a representation ρ' of $\pi_1(Y')$. It is clear that if the conjecture holds for ρ', then it holds for ρ. This proves the theorem. ∎

2.14. *Lie-irreducibility.* Let Γ be a pro-finite group and let ρ: $\Gamma \to \mathrm{GL}(V)$ be a continuous representation of Γ over F. We say that ρ is **Lie-irreducible** if for any open subgroup $U \subset \Gamma$ the representation $\rho|_U$ is irreducible. Similarly, there is the concept of **absolutely Lie-irreducible**: for all $U \subset \Gamma$ open, $\rho|_U$: $U \to \mathrm{GL}(V)$ is absolutely irreducible over F.

There is a criterion for (absolute) Lie-irreducibility in terms of the Zariski closure $G_\rho \subset \mathrm{GL}(V)$ of the image of ρ. Let G_ρ^0 denote the connected component of the algebraic group G_ρ. Then ρ is (absolutely) Lie-irreducible if and only if the representation of G_ρ^0 on V is (absolutely) irreducible. We leave the proof of this fact to the reader.

2.15 LEMMA: *Assume we have $\Gamma \subset G$ as in 2.6 . Let $\rho\colon G \to \mathrm{GL}(V)$ be a representation. Then there exist a finite extension $F \subset F'$, an open subgroup $H \subset G$ and an H-stable filtration*

$$(0) \subset W_1 \subset W_2 \subset \cdots \subset W_m = V \otimes_F F'$$

by F' vector spaces such that W_{i+1}/W_i is absolutely Lie-irreducible as a representation of $H \cap \Gamma$ over F'.

Proof: Suppose the representation is an extension $0 \to V' \to V \to V'' \to 0$. If the lemma holds for V' and V'', then the lemma holds for V. (Indeed, take $H = H' \cap H''$ and as extension of F some finite extension containing both F' and F''.) Furthermore, we may replace F by any finite extension of F during the course of our proof. Finally, we may replace G by any open subgroup H of G and Γ by $H \cap \Gamma$.

Thus we may assume that V is absolutely Lie-irreducible as a representation of G. Let G_ρ be the Zariski closure of $\rho(G)$ as in 2.14 . Thus we have that G_ρ^0 acts absolutely irreducible on V. Let us choose $H \subset G$ open so that $\rho(H) \subset G_\rho^0(F)$. It is a general fact that if an algebraic group K acts absolutely irreducibly on a vector space, then so does its derived group K^{der}. Since $\rho(H)$ is Zariski dense in G_ρ^0, we see that $\rho([H,H])$ is Zariski dense in $(G_\rho^0)^{der}$. But $[H,H] \subset H \cap \Gamma$. We conclude that the Zariski closure of $\rho(H \cap \Gamma)$ contains $(G_\rho^0)^{der}$, which acts absolutely irreducibly. Therefore we are done by the discussion in 2.14 . ∎

2.16 LEMMA: *Suppose in Conjecture 2.3 the representation V is an extension $0 \to V' \to V \to V'' \to 0$ of representations. If the conjecture holds for V' and V'', then the conjecture holds for V.*

Proof: Let $H \subset \pi_1(\overline{X})$ be the open subgroup which acts trivially on V' and V''. Then ρ maps H to the unipotent subgroup $1_V + \mathrm{Hom}(V', V'')$ of $\mathrm{GL}(V)$. Since $H \cong \pi_1(Y)$ for some finite étale morphism $Y \to \overline{X}$, we see that the image of H is finite by Lemma 2.10 . ∎

2.17 THEOREM: *Suppose that Conjecture 2.3 holds in every case* (X, V, ρ), *where* X *is a smooth projective curve,* $\dim V \leq n$, *and* $\rho|_{\pi_1(\overline{X})}$ *is absolutely Lie-irreducible, and* $\det(\rho) = 1$. *Then the conjecture holds in all cases where* $\dim V \leq n$.

Proof: By Proposition 2.13 , we may assume that X is a projective curve. We may still replace F by a finite extension, and X by a finite étale covering. Thus, by Lemma 2.15, we may assume that V has a filtration by subrepresentations $0 \subset V_1 \subset \cdots \subset V_m = V$ such that the representation of $\pi_1(\overline{X})$ on V_i/V_{i-1} is absolutely Lie-irreducible. If the conjecture holds for these quotients, then the conjecture for ρ follows by Lemma 2.16. Thus we may assume that ρ is absolutely Lie-irreducible. By Proposition 2.4 we see that $\det(\rho)|_{\pi_1(\overline{X})}$ has finite image. Thus, after replacing X by a finite étale covering, $\det(\rho)$ will factor as $\pi_1(X) \to \mathrm{Gal}(k^{\mathrm{sep}}/k) \to F^*$. We leave it to the reader to see that in this situation we can find (after extending F) a character $\epsilon: \pi_1(X) \to F^*$ such that $\det(\rho \otimes \epsilon) = 1$. If we prove the conjecture for the absolutely Lie-irreducible representation $\rho \otimes \epsilon$, then the conjecture for ρ follows. This proves the theorem. ∎

3. Deformation rings and the conjecture

3.1 Notation: In this section we use the following notation in addition to the notation fixed in Subsection 2.1.

Γ is a profinite group,

$\mathbb{F}$ is a finite field of characteristic ℓ, and

$W(\mathbb{F})$ is the Witt ring of $\mathbb{F}$.

$\mathcal{O}$ is a finite extension of $W(\mathbb{F})$, which is a complete discrete valuation ring with uniformizer $\pi_{\mathcal{O}}$ such that $\mathcal{O}/\pi_{\mathcal{O}} \cong \mathbb{F}$.

$\mathcal{C}_{\mathcal{O}}$ is the category of complete Noetherian local $\mathcal{O}$-algebras R with $R/\mathfrak{m}_R \cong \mathbb{F}$.

3.2. Suppose that we are given a continuous 'residual' representation $\rho_0: \Gamma \to \mathrm{GL}_n(\mathbb{F})$ and a continuous character $\epsilon: \Gamma \to \mathcal{O}^* = \mathrm{GL}_1(\mathcal{O})$ such that $\epsilon \bmod \pi_{\mathcal{O}} = \det(\rho_0)$. In this case the deformation functor

$$\mathrm{Def}(\Gamma, \rho_0, \epsilon)$$

is the functor $\mathcal{C}_{\mathcal{O}} \to \mathit{Set}$ which maps R to the set of equivalence classes of continuous representations

$$\rho_R: \Gamma \longrightarrow \mathrm{GL}_n(R)$$

such that $\rho_R \bmod \mathfrak{m}_R = \rho_0$ and $\det(\rho_R) = (\mathcal{O}^* \to R^*) \circ \epsilon$. Here equivalence is defined as follows: $\rho_R \sim \rho'_R$ if and only if there exists an element $g \in \mathrm{GL}_n(R)$ such that $\rho'_R = \mathrm{inn}_g \circ \rho_R$, i.e., for all $\gamma \in \Gamma$ we have $\rho'_R(\gamma) = g^{-1}\rho_R(\gamma)g$.

3.3. We want to use the Schlessinger–Mazur theorem [6, Section 1.2] to insure that the functor $\mathrm{Def}(\Gamma, \rho_0, \epsilon)$ is representable on the category $\mathcal{C}_{\mathcal{O}}$. For this we need to make two kinds of assumptions: one of these assures the rigidity of the problem, and the other is a natural finiteness condition:

(a) Any $T \in M_n(\mathbb{F})$ commuting with $\mathrm{Image}(\rho_0)$ is a scalar matrix. This is true, for example, if ρ_0 is absolutely irreducible (Schur's lemma).

(b.1) Γ is topologically finitely generated.

(b.2) $H^1(\Gamma', \mathbb{F})$ is finite for every open subgroup Γ' of Γ.

If we have (a) and either of the conditions (b.1) or (b.2), then there exists a universal pair $(R_{\mathrm{univ}}, \rho_{\mathrm{univ}})$: For every (R, ρ_R) there is a unique ψ: $R_{\mathrm{univ}} \to R$ such that $\rho_R \sim \psi \circ \rho_{\mathrm{univ}}$.

3.4 Deformation rings of Galois representations. Let k be a finite field of characteristic p, with $p \neq \ell$. Let X be a curve over $\operatorname{Spec} k$, by which we mean a smooth geometrically connected scheme of dimension 1. We take $\Gamma = \pi_1(X)$ to be the arithmetic fundamental group, and we assume given

$$\rho_0\colon \pi_1(X) \to \mathrm{GL}_n(\mathbb{F}), \quad \text{and} \quad \epsilon\colon \pi_1(X) \longrightarrow \mathcal{O}^*$$

as before. We remark that we can view ρ_0 and any deformation ρ_R as a Galois representation of $k(X)$ unramified at all places dominating X. Thus the structure of the universal deformation ring of ρ_0 is related to the structure of this Galois group, and vice versa.

3.5 Theorem: *In the situation above, assume*

(i) *Conjecture 2.3 holds for X and n,*

(ii) *the restriction $\rho_0|_{\pi_1(\overline{X})}$ is absolutely irreducible, and*

(iii) *ℓ does not divide n.*

Note that (i), (ii) *and* (iii) *imply that the existence result of 3.3 can be applied to* $\mathrm{Def}(\pi_1(X), \rho_0, \epsilon)$, *hence we get a universal deformation ring R_{univ}. Then*

$$\mathcal{O} \longrightarrow R_{\mathrm{univ}}$$

is a finite flat complete intersection morphism.

3.6 Remarks: (a) In the work by Wiles and Taylor [10, 9] the complete intersection statement occurs as a theorem in some cases where $\Gamma = \mathrm{Gal}(\mathbb{Q}_\Sigma/\mathbb{Q})$, and

Σ is a finite set of primes, ϵ is the cyclotomic character, $n = 2$ and only deformations that are crystalline or ordinary at ℓ are allowed. Part of the motivation for the Conjecture 2.3 comes from the fact that it 'explains' the complete intersection behaviour in the function field case.

(b) In particular the conjecture implies: Any ρ_0 satisfying (ii) and (iii) can be lifted (in at least one way and at most finitely many ways) to a representation $\rho\colon \pi_1(X) \to \mathrm{GL}_n(\bar{\mathbb{Q}}_\ell)$. This is really quite a strong assertion!

(c) Since we prove the conjecture in case $n = 2$, we see that the deformation rings of 2 dimensional representations ρ are complete intersection rings.

3.7. The rest of this section is devoted to the proof of the theorem. We will use the exact sequence

$$1 \longrightarrow \pi_1(\overline{X}) \longrightarrow \pi_1(X) \longrightarrow \mathrm{Gal}(\bar{k}/k) \longrightarrow 1$$

to study the relation between the deformation rings of ρ_0 and of $\rho_0|_{\pi_1(\overline{X})}$.

To be more precise, let us write (R, ρ) for the universal deformation associated to the deformation functor $\mathrm{Def}(\pi_1(X), \rho_0, \epsilon)$. Further, let us write $(\bar{R}, \bar{\rho})$ for the universal pair associated to the deformation functor $\mathrm{Def}(\pi_1(\overline{X}), \rho_0|_{\pi_1(\overline{X})}, \epsilon|_{\pi_1(\overline{X})})$. We remark that the assumptions (a) and (b.2) of 3.3 are satisfied for both deformation problems.

3.8. The pair $(R, \rho|_{\pi_1(\overline{X})})$ is a deformation of $\rho_0|_{\pi_1(\overline{X})}$ with determinant $\epsilon|_{\pi_1(\overline{X})}$. Hence by the unversal property of deformation rings we get a unique map of $\mathcal{O}$-algebras

$$\psi\colon \bar{R} \longrightarrow R$$

and an element $g \in \mathrm{GL}_n(R)$ such that $\psi(\bar{\rho}(\gamma)) = g^{-1}\rho(\gamma)g$ for all $\gamma \in \pi_1(\overline{X})$. We may replace ρ by $\mathrm{inn}_g \circ \rho$, which is still universal. Then the formula above simplifies to

$$\psi(\bar{\rho}(\gamma)) = \rho(\gamma).$$

3.9. Choose an element $F \in \pi_1(X)$ which gets mapped to the Frobenius element of $\mathrm{Gal}(\bar{k}/k)$. Choose an element $h_1 \in \mathrm{GL}_n(\bar{R})$ such that $h_1 \bmod \mathfrak{m}_{\bar{R}} = \rho_0(F)$. We define the representation

$$\begin{array}{cccc} \bar{\rho}^F\colon & \pi_1(\overline{X}) & \longrightarrow & \mathrm{GL}_n(\bar{R}) \\ & \gamma & \longmapsto & h_1\bar{\rho}(F^{-1}\gamma F)h_1^{-1}. \end{array}$$

With these definitions we have $\bar{\rho}^F \bmod \mathfrak{m}_{\bar{R}} = \rho_0|_{\pi_1(\overline{X})}$, and $\det(\bar{\rho}^F) = \epsilon|_{\pi_1(\overline{X})}$. We conclude that $(\bar{R}, \bar{\rho}^F)$ is a deformation of $\rho_0|_{\pi_1(\overline{X})}$ with determinant $\epsilon|_{\pi_1(\overline{X})}$. By the universal property of deformation rings we get an $\mathcal{O}$-algebra homomorphism

$$\Phi\colon \bar{R} \longrightarrow \bar{R}$$

and an element $h_2 \in \mathrm{GL}_n(\bar{R})$ such that $\Phi(\bar{\rho}(\gamma)) = h_2^{-1}\bar{\rho}^F(\gamma)h_2$ for all $\gamma \in \pi_1(\overline{X})$.

3.10 LEMMA: (i) *Φ is an automorphism of $\bar{R}$ over $\mathcal{O}$.* (ii) *We have $\Phi \circ \psi = \psi$.*

Proof: By considering the twist $\bar{\rho}^{F^{-1}}$ similar to $\bar{\rho}^F$ above, one constructs the inverse to Φ. The second statement follows from universality, and the fact that the F-twist of ρ is equivalent to ρ (use $g = \rho(F)$). ∎

3.11 LEMMA: *$\bar{R} \cong \mathcal{O}[[x_1, \ldots, x_s]]$ for some $s \in \mathbb{N}$.*

Proof: It is well known that a deformation ring is a formal power series if the obstructions vanish, see [6, Section 1.6]. In this case the obstruction space in question is

$$H^2(\pi_1(\overline{X}), \mathfrak{sl}_n(\mathbb{F})).$$

The action of $\pi_1(\overline{X})$ on $\mathfrak{sl}_n(\mathbb{F})$ is given by ρ_0 combined with the adjoint action of GL_n on $\mathfrak{sl}_n$. Since $\overline{X}$ is a $K(\pi_1, 1)$ for étale cohomology, we see that

$$H^2(\pi_1(\overline{X}), \mathfrak{sl}_n(\mathbb{F})) = H^2_{ét}(\overline{X}, \mathcal{F})$$

where $\mathcal{F}$ is the locally constant étale sheaf associated to $\mathfrak{sl}_n(\mathbb{F})$ with $\mathrm{ad}^0(\rho_0|_{\pi_1(\overline{X})})$-action. If $\overline{X}$ is affine then the cohomological dimenion of $\overline{X}$ is 1, so this group is zero. If $\overline{X}$ is projective, then we can use Poincaré duality. The sheaf $\mathcal{F}$ is self-dual as the Killing form is a nondegenerate invariant self-duality on $\mathfrak{sl}_n(\mathbb{F})$. Here we use that ℓ does not divide n. The Poincaré dual of $H^2_{ét}(\overline{X}, \mathcal{F})$ is $H^0_{ét}(\overline{X}, \mathcal{F})$, which we want to show is zero. This is equivalent to having no nonzero $\mathrm{ad}^0(\rho_0|_{\pi_1(\overline{X})})$-invariants in $\mathfrak{sl}_n(\mathbb{F})$. This follows from condition (ii) of the proposition. ∎

3.12 Definition: We let I_Φ be the ideal of $\bar{R}$ generated by expressions of the form $r - \Phi(r)$, where $r \in \bar{R}$.

We remark that I_Φ is generated by the elements $x_i - \Phi(x_i)$.

3.13 LEMMA: *The map ψ: $\bar{R} \to R$ identifies R with $\bar{R}/I_\Phi$:*

$$R \cong \mathcal{O}[[x_1, \ldots, x_s]]/(x_1 - \Phi(x_1), \ldots, x_s - \Phi(x_s)).$$

Proof: We first prove that the map ψ is surjective. For this it suffices to show that

$$\mathfrak{m}_{\bar{R}}/(\mathfrak{m}_{\bar{R}}^2 + \mathfrak{m}_{\mathcal{O}}\bar{R}) \longrightarrow \mathfrak{m}_R/(\mathfrak{m}_R^2 + \mathfrak{m}_{\mathcal{O}}R)$$

is surjective. These are $\mathbb{F}$-vector spaces, whose duals correspond to the tangent spaces associated to the deformation problems. Thus we have to show that the map

$$H^1(\pi_1(X), \mathfrak{sl}_n(\mathbb{F})) \longrightarrow H^1(\pi_1(\overline{X}), \mathfrak{sl}_n(\mathbb{F}))$$

is injective. By looking at the six-term exact sequence of cohomology associated to the exact sequence of 3.7, we see that it suffices to prove that

$$H^1\Big(\mathrm{Gal}(\bar{k}/k), H^0\big(\pi_1(\overline{X}), \mathfrak{sl}_n(\mathbb{F})\big)\Big) = (0).$$

This is true as $H^0(\pi_1(\overline{X}), \mathfrak{sl}_n(\mathbb{F}))$ is zero by assumption (ii).

Next, we construct a representation of $\pi_1(X)$ over the ring $\bar{R}/I_\Phi$. To do this, recall that

$$(*) \qquad \Phi(\bar{\rho}(\gamma)) = h_2^{-1}\bar{\rho}^F(\gamma)h_2 = h_2^{-1}h_1\bar{\rho}(F^{-1}\gamma F)h_1^{-1}h_2.$$

Note that modulo $\mathfrak{m}_{\bar{R}}$ we have

$$\rho_0(\gamma) = \bar{\rho}(\gamma) = \Phi(\bar{\rho}(\gamma)) = h_2^{-1}h_1\bar{\rho}(F^{-1}\gamma F)h_1^{-1}h_2 = h_2^{-1}\rho_0(\gamma)h_2,$$

by our choice of h_1. Thus $h_2 \bmod \mathfrak{m}_{\bar{R}}$ commutes with $\mathrm{Image}(\rho_0|_{\pi_1(\overline{X})})$ and hence h_2 is scalar modulo $\mathfrak{m}_{\bar{R}}$, say $h_2 = \mathrm{diag}(\lambda, \lambda, \ldots, \lambda) \bmod \mathfrak{m}_{\bar{R}}$ for some $\lambda \in \bar{R}$. Then we get that $\det(h_2^{-1}h_1) \cong \lambda^n \det(\rho_0(F)) \cong \lambda^n \epsilon(F)$. By Hensel's lemma applied to the equation $T^n = \det(h_2^{-1}h_1)\epsilon(F)^{-1}$, we find an element $\mu \in \bar{R}^*$ such that $\mu^n = \det(h_2^{-1}h_1)\epsilon(F)^{-1}$. Here we use (iii), namely $\ell \nmid n$. At this point we define

$$\rho_\Phi \colon \pi_1(X) \longrightarrow \mathrm{GL}_n(\bar{R}/I_\Phi)$$

by the formula

$$\gamma F^n \longmapsto \bar{\rho}(\gamma)(\mu h_2^{-1}h_1)^n.$$

We leave it to the reader to see that this makes sense for $n \in \hat{\mathbb{Z}}$ and that this is a homomorphism by equation $(*)$. By our choice of μ, we have $\det(\rho_\Phi) = \epsilon$.

Hence this is a deformation of ρ_0 over $\bar{R}/I_\Phi$ of determinant ϵ. Thus the universal property of (R, ρ) gives us a unique homomorphism $\psi' \colon R \to \bar{R}/I_\Phi$. It is easy to see that the composition

$$\bar{R} \xrightarrow{\psi} R \xrightarrow{\psi'} \bar{R}/I_\Phi$$

is the canonical reduction map. This, and the fact that ψ is surjective, proves the desired result. ∎

3.14. We come to the end of the proof of the proposition. We have obtained the isomorphism

$$R \cong \mathcal{O}[[x_1, \ldots, x_s]]/(x_1 - \Phi(x_1), \ldots, x_s - \Phi(x_s)).$$

A ring R of this shape will be a complete intersection finite and flat over $\mathcal{O}$ if (and only if) $R/\pi_{\mathcal{O}}R$ is zero dimensional. Indeed, by [5, (16.B)], the equality $\dim \mathcal{O}[[x_1, \ldots, x_s]]/(f_1, \ldots, f_s, \pi) = 0$ implies that $\pi, f_1, \ldots, f_s$ is a regular sequence. Thus $\bar{f}_1, \ldots, \bar{f}_s$ is a regular sequence in $\mathcal{O}/\pi\mathcal{O}[[x_1, \ldots, x_s]]$. This implies by [5, (20.F)] that $\mathcal{O}[[x_1, \ldots, x_s]]/(f_1, \ldots, f_s)$ is flat over $\mathcal{O}$. Finiteness follows from Nakayama's lemma, as R is π-adically complete.

Let us assume to the contrary that $\dim R/\pi_{\mathcal{O}}R > 0$. Then there exists a 1 dimensional quotient $R/\pi_{\mathcal{O}}R \to A$. We may assume that A is a domain. The normalization A' of A is a complete discrete valuation ring of characteristic ℓ with finite residue field. Hence $A' \cong \mathbb{F}'[[t]]$. The composition of ρ with the map $R \to A'$ produces a deformation of $\rho_0 \otimes \mathbb{F}'$ to a representation ρ' over A'. By assumption (iii) of the proposition, we may apply our conjecture to the representation ρ', and we obtain that $\rho'(\pi_1(\overline{X}))$ is finite. By the group theoretical lemma below we conclude that $\rho'|_{\pi_1(\overline{X})} \cong \rho_0|_{\pi_1(\overline{X})} \otimes \mathbb{F}'[[t]]$. Using the last lemma of this section, we deduce that the map $\bar{R} \to A'$ is the map $\bar{R} \to \mathbb{F} \to A'$. However this contradicts the construction of A as a quotient of $R = \bar{R}/I_{\Phi}$. After checking the proofs of the next two lemmas, the reader will have completed the proof of Theorem 3.5.

3.15 LEMMA: *Let G be a finite group and let k be any field. Let $\rho\colon G \to \mathrm{GL}_n(k[[t]])$ be a representation. If $\rho_0 := \rho \bmod (t)$ is absolutely irreducible, then $\rho \cong \rho_0 \otimes_k k[[t]]$.*

Proof: We make some remarks on finite dimensional k-algebras A, to be applied to $A = k[G]$. For any field extension $k \subset K$, the algebra $A_K = A \otimes_k K$ has only finite number of maximal two-sided ideals $\mathfrak{m}$. (By the Chinese remainder theorem, for example.) Each $A_K/\mathfrak{m}$ is a matrix algebra over a skew field $D_{\mathfrak{m}}$, which is a finite extension of K. The intersection $\mathfrak{m}_0 = A \cap \mathfrak{m}$ is a maximal two-sided ideal of A, and $D_{\mathfrak{m}}$ is a quotient of $D_{\mathfrak{m}_0} \otimes_k K$. If $D_{\mathfrak{m}_0} \cong k$, then $\mathfrak{m} = \mathfrak{m}_0 \otimes K$.

Let $M = k[[t]]^n$ be the representation space of ρ, let $M_0 = M/tM$. By Burnside's theorem ρ_0 induces a surjection $k[G] \to \mathrm{End}(M_0) = M_n(k)$. By Nakayama's lemma we see that ρ induces a surjection $k[[t]][G] \to \mathrm{End}(M) = M_n(k[[t]])$. Hence the kernel $\mathfrak{m}$ of $k((t))[G] \to M_n(k((t)))$ is a maximal two-sided ideal in $k((t))[G] = A_{k((t))}$, with $D_{\mathfrak{m}} = k((t))$.

We apply the discussion above to our $\mathfrak{m}$. Then we see that $D_{\mathfrak{m}_0} \otimes_k k((t))$ maps onto $k((t))$, in other words $k((t))$ splits $D_{\mathfrak{m}_0}$. However, $Br(k) \to Br(k((t)))$ is injective. Hence $D_{\mathfrak{m}_0} \cong k$ and $k[G]/\mathfrak{m}_0 \cong M_r(k)$ for some r. Then $\mathfrak{m} = \mathfrak{m}_0 \otimes k((t))$ and $r = n$. It follows that the kernel of $k[G] \to End(M_0)$ is $\mathfrak{m}_0$. We conclude that $\mathrm{End}(M)$ and $\mathrm{End}(M_0) \otimes k[[t]]$ are isomorphic as $k[[t]][G]$-algebras. From this one easily derives the lemma. ∎

3.16 LEMMA: *Let $\mathcal{O}, \Gamma, \rho_0, \epsilon$ be as in Subsection 3.3, and let $(R_{\text{univ}}, \rho_{\text{univ}})$ be the universal deformation. Let A be a local complete Noetherian $\mathbb{F}$-algebra such that $\mathbb{F}' = A/\mathfrak{m}_A$ is a finite extension of $\mathbb{F}$. If ψ: $R_{\text{univ}} \to A$ is an $\mathcal{O}$-algebra homomorphism such that $\psi \circ \rho_{\text{univ}} \cong \rho_0 \otimes A$, then ψ is the map $R_{\text{univ}} \to \mathbb{F} \to A$.*

Proof: By the universal property of the pair $(R_{\text{univ}}, \rho_{\text{univ}})$ this is true if $A/\mathfrak{m}_A \cong \mathbb{F}$. Let $\mathcal{O} \subset \mathcal{O}' = \mathcal{O} \otimes_{W(\mathbb{F})} W(\mathbb{F}')$ be the finite étale extension with residue field $\mathbb{F}'$. To prove the assertion, we simply show that $R_{\text{univ}} \otimes_{\mathcal{O}} \mathcal{O}'$ is the universal ring for the deformation functor $\mathrm{Def}(\Gamma, \rho_0 \otimes \mathbb{F}', \epsilon \otimes \mathcal{O}')$. This is well known. See [6, Section 1.3 (d)] for the existence of the map comparing the two rings. It is surjective. But the equations defining these rings are the same as they lie in a cohomology group which is compatible with base change: $H^2(\Gamma, \mathfrak{sl}) \otimes \mathbb{F}' = H^2(\Gamma, \mathfrak{sl} \otimes \mathbb{F}')$. ∎

4. Unramified cusp forms over function fields

In this section we recall the definition of an unramified cusp form. After this we recall the result of Drinfeld, see [3]. We apply these results to prove the Conjecture 2.3 in the case of representations of dimension 2.

4.1. In this subsection we introduce some notations.

k finite field of characteristic p,
X projective, smooth, geometrically connected curve over k,
K the function field of X: $K = k(X)$,
v denotes a place of K,
K_v the completion of K at v,
$\mathcal{O}_v$ the ring of integers of K_v,
$\mathcal{O} = \prod_v \mathcal{O}_v \subset \mathbb{A} = \prod'_v K_v$,
Λ is any ring in which p is invertible.

4.2 Definition: An unramified cusp form on $\mathrm{GL}_2(\mathbb{A})$ with coefficients in Λ is a function

$$f\colon \mathrm{GL}_2(\mathbb{A}) \longrightarrow \Lambda$$

such that

(i) $f(x\gamma) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbb{A})$ and all $\gamma \in \mathrm{GL}_2(K)$,

(ii) $f(ux) = f(x)$ for all $x \in \mathrm{GL}_2(\mathbb{A})$ and all $u \in \mathrm{GL}_2(\mathcal{O})$,

(iii) the following integral equations hold for all $x \in \mathrm{GL}_2(\mathbb{A})$:

$$\int_{z \in \mathbb{A} \bmod K} f\left(x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \mathrm{d}z = 0.$$

We briefly explain (iii). For every $x \in \mathrm{GL}_2(\mathbb{A})$ the left coset $\mathrm{GL}_2(\mathcal{O})x$ is open in $\mathrm{GL}_2(\mathbb{A})$. Hence by (ii) we see that f is a locally constant function on the topological group $\mathrm{GL}_2(\mathbb{A})$. Thus the function on $\mathbb{A}$, appearing in the integrand of (iii), is locally constant. Furthermore, (i) implies that this function is invariant under $z \mapsto z + t$ for all $t \in K$. Thus in (iii) we are integrating a locally constant function on the compact topological group $\mathbb{A} \bmod K$. In fact, $\mathbb{A} \bmod K$ is a pro-p topological group. Let $\mathrm{d}z$ be the normalized Haar measure on it. Then $\int_U \mathrm{d}z \in p^{\mathbb{Z}}$ for every open subset $U \subset \mathbb{A} \bmod K$. Since Λ is a $\mathbb{Z}[1/p]$-algebra, the integrals in (iii) make sense.

4.3. *Hecke operators.* For v a place of X and f a cusp form, we set

$$(T_v f)(x) := \int_{g \in M_v} f(g^{-1}x)\mathrm{d}g.$$

Furthermore,

$$(U_v f)(x) := f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & \pi_v^{-1} \end{pmatrix} x\right).$$

Note that $T_v f$ and $U_v f$ are also cusp forms. We used the following notations:

π_v is a uniformizer of $\mathcal{O}_v$,

$M_v = \{h \in \mathrm{Mat}(2 \times 2, \mathcal{O}_v) |\ \det(h) \in \pi_v \mathcal{O}_v^*\}$, and

$\mathrm{d}g$ is the Haar measure on $\mathrm{GL}_2(K_v)$ normalized by the condition: $\int_{\mathrm{GL}_2(\mathcal{O}_v)} \mathrm{d}g = 1$.

Again it might be useful to explain the meaning of the integral in the definition of T_v. We will not do this, instead we give the result which will convince any reader that T_v is well defined. Let $\lambda_1, \ldots, \lambda_{q_v} \in \mathcal{O}_v$ be a system of representatives for $\mathcal{O}_v/\mathfrak{m}_v = \kappa(v)$. Then

$$(T_v f)(x) = f\left(\begin{pmatrix} \pi_v^{-1} & 0 \\ 0 & 1 \end{pmatrix} x\right) + \sum_{i=1}^{q_v} f\left(\begin{pmatrix} 1 & 0 \\ -\lambda_i \pi_v^{-1} & \pi_v^{-1} \end{pmatrix} x\right).$$

4.4 Definition: An eigenform f is a cusp form f such that some value of f is a unit, and for all v there are $t_v, u_v \in \Lambda$ with $T_v f = \lambda_v f$ and $U_v f = u_v f$.

4.5. In this case the elements $u_v \in \Lambda$ are units, and the elements $t_v \in \Lambda$ are well defined. The **central character** associated to an eigenform f is the homomorphism

$$\chi_f : \mathcal{O}^* \backslash \mathbb{A}^* / K^* \longrightarrow \Lambda^*$$

given by $\chi_f((1,\ldots,1,\pi_v,1,\ldots,1)) = u_v^{-1}$. This is well defined by the invariance properties of f.

4.6. In [3], Drinfeld proves one direction of the correspondence between representations ρ of $\pi_1(X)$, and eigenforms f. Using normalizations as in [3], ρ corresponds to f if and only if $t_v = \operatorname{trace}\rho(F_v)$, and $u_v = q_v^{-1}\det\rho(F_v)$. Since we want to study here only the case of representations ρ with trivial determinant, we are led to consider only those f such that $u_v = q_v^{-1}$.

We make an ad hoc definition:

$$C(\Lambda) = \{\text{cusp forms } f \text{ with coefficients in } \Lambda \text{ such that } U_v f = q_v^{-1} f \text{ for all } v\}.$$

This is a Λ-module. For each v we have a Λ-linear operator $T_v: C(\Lambda) \longrightarrow C(\Lambda)$.

4.7 PROPOSITION: *If Λ is Noetherian, then $C(\Lambda)$ is a finitely generated Λ-module. If Λ is a field, and $\mathbb{F} \subset \Lambda$ is its prime subfield, then the natural map*

$$C(\mathbb{F}) \longrightarrow C(\Lambda)$$

induces an isomorphism $C(\mathbb{F}) \otimes_{\mathbb{F}} \Lambda \cong C(\Lambda)$ commuting with the Hecke operators T_v.

4.8 COROLLARY: *If f is an eigenform over a field Λ with central character $q^{\deg(-)}$, then all the eigenvalues $t_v \in \Lambda$ are algebraic: there exists a subfield $E \subset \Lambda$, with $[E : \mathbb{F}] < \infty$ such that $t_v \in E$ for all v.*

Proof: Both the corollary and the proposition are well-known. We sketch the proof of the proposition; we use some of the terminology and notations of [3]. We may think of any element $f \in C(\Lambda)$ as a function on Bun_2, the set of isomorphism classes of rank 2 locally free sheaves of $\mathcal{O}_X$-modules.

Note that f is completely determined by its values on $\mathcal{L} \in \mathrm{Bun}_2$, with $\deg(\mathcal{L}) \in \{0,1\}$. Indeed, by our definition of $C(\Lambda)$ we have $f(\mathcal{L} \otimes \mathcal{N}) = q^{\deg \mathcal{N}} f(\mathcal{L})$ for every invertible $\mathcal{O}_X$-module $\mathcal{N}$. Furthermore, $\deg(\mathcal{L} \otimes \mathcal{N}) = \deg(\mathcal{L}) + 2\deg(\mathcal{N})$. Since X has a linebundle of degree 1, the result follows.

Let us call an element $\mathcal{L} \in \mathrm{Bun}_2$ **very unstable** if there exists an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L} \longrightarrow \mathcal{B} \longrightarrow 0,$$

with $\mathcal{A}, \mathcal{B} \in \mathrm{Pic}(X)$ and $\deg(\mathcal{A}) - \deg(\mathcal{B}) > 2g_X - 2$. In this case we have

$$\mathrm{Ext}^1_X(\mathcal{B}, \mathcal{A}) = \mathrm{H}^1(X, \mathcal{A} \otimes \mathcal{B}^{-1}) = \mathrm{H}^0(X, \mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega^1_X)^* = 0,$$

by our degree assumption. Therefore $\mathcal{L} = \mathcal{A} \oplus \mathcal{B}$, and $\mathcal{L}$ corresponds to the element

$$x = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in \mathrm{GL}_2(\mathbb{A})$$

with $\mathcal{O}_X(\mathrm{div}(a)) \cong \mathcal{A}$ and $\mathcal{O}_X(\mathrm{div}(b)) \cong \mathcal{B}$. In this situation all the elements

$$x' = x \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

correspond to extensions of $\mathcal{B}$ by $\mathcal{A}$ as well. These therefore all correspond to the isomorphism class of $\mathcal{L}$ in Bun_2. Thus condition (iii) of the definition of cusp forms therefore implies that $f(\mathcal{L}) = 0$ for every very unstable element $\mathcal{L} \in \mathrm{Bun}_2$.

There are only finitely many elements $\mathcal{L} \in \mathrm{Bun}_2$ which are not very unstable and with $\deg(\mathcal{L}) \in \{0, 1\}$. We indicate briefly how to prove this. First, one uses Riemann-Roch on X to show that any such $\mathcal{L}$ has a maximal invertible subsheaf $\mathcal{A} \subset \mathcal{L}$ of degree not less than $-g_X$. But as $\mathcal{L}$ is also not very unstable, we have $\deg(\mathcal{A}) < g_X$. This also gives that $-g_X < \deg(\mathcal{B}) \leq g_X$, with $\mathcal{B} = \mathcal{L}/\mathcal{A}$. Thus the number of isomorphism classes of $\mathcal{A}$ and $\mathcal{B}$ is finite. In addition the Ext-groups are finite as we are working over the finite field k.

We conclude from the above that any element f of $C(\Lambda)$ is determined by its values in finitely many points. Thus we get an injection of Λ-modules $C(\Lambda) \to \Lambda^N$ for some $N \in \mathbb{N}$. This proves the first assertion. The second easily follows from this, and the fact that $C(\Lambda)$ is given by certain universal linear equations in Λ^N with coefficients in the prime field of Λ. ∎

4.9 Theorem (Compare Drinfeld [3, Main Theorem]): *Let F be a local field of characteristic $\ell \neq p$. Let $\rho\colon \pi_1(X) \to \mathrm{GL}_2(F)$ be a continuous two dimensional representation with trivial determinant. Assume that $\rho|_{\pi_1(\overline{X})}$ is absolutely irreducible. Then there exists an eigenform $f \in C(\Lambda)$ such that $t_v = \mathrm{trace}\,\rho(f_v)$ for all places v.*

4.10 Corollary: *The Conjecture 2.3 holds for two dimensional representations.*

4.11. In the rest of this section we will indicate how to modify (slightly) the arguments of Drinfeld, in order to prove the theorem. However, we first indicate how the corollary can be derived from it.

For this we use Theorem 2.17. We deduce from it that it suffices to prove the conjecture for every triple (X, F, ρ), as in Theorem 4.9. By Proposition 2.8 it suffices to show that the characteristic polynomials of the Frobenii F_v are algebraic for every v. This follows from Theorem 4.9 and Corollary 4.8.

4.12. *Additions to* [3, Section 1]. Some of this has been discussed above. As coefficient ring we are going to use $\Lambda = \mathbb{F}[[t]]$. As above we set $u_v = q_v^{-1}$ and $t_v = \operatorname{trace} \rho(F_v)$. We skip the arguments involving Fourier transform, and directly define the function $f\colon \mathrm{Flag}_2 \to \Lambda$ by the formula (6) of [3]. Note that the elements $c_v(n)$ and $r(D)$ of Λ are well defined.

We leave it to the reader to establish that f so defined is not zero, has the correct eigenvalues for T_v and U_v, and that the cusp property holds. [To see that $f \neq 0$, remark that if $\mathcal{L}$ is the nontrivial extension of $\mathcal{O}_X$ by Ω_X^1, then $f(\Omega_X^1 \subset \mathcal{L}) = -q^{2g-2}$. The principle of algebraic identities can be used to verify the other properties: first define a 'universal' f_{univ} over the ring $\mathbb{Z}[1/p, t_v]$, where Drinfeld's arguments apply, and then specialize to our particular f.]

We also leave it to the reader to check that [3, Proposition 1.1] holds with coefficient ring $\Lambda = \mathbb{F}[[t]]$.

4.13. *Additions to* [3, Section 2]. We may assume that our representation ρ actually has image in $\mathrm{SL}_2(\mathbb{F}[[t]])$. The representations $\rho \bmod t^r$ then correspond to finite locally constant sheaves of $\mathbb{F}[t]/(t^r)$-modules $\mathcal{E}_r$ over $X_{\acute{e}t}$. As in Drinfeld, we let $\mathcal{E} = \{\mathcal{E}_r\}_{r\in\mathbb{N}}$. Thus we can define $\mathcal{E}^{(n)}$ as in Drinfeld by the system $\mathcal{E}_r^{(n)} = (\varphi_* \boxtimes^n \mathcal{E}_r)^{S_n}$, etc. Note that the stalks of the sheaves $\mathcal{E}_r^{(n)}$ are flat $\mathbb{F}[t]/(t^r)$-modules. In our situation, one takes traces exactly as in the 'usual' case where $\Lambda = \bar{\mathbb{Z}}_\ell$: first one takes the trace on each finite level, giving elements in $\mathbb{F}[t]/(t^r)$, and then one takes the inverse limit. For example the identity $r(D) = \operatorname{trace}(F_D|\mathcal{E}^{(m)})$ of [3, proof of 2.1] remains true in our situation.

In the proof of [3, Proposition 2.1] we encounter our first difficulty. Here Drinfeld uses fourier transform on the finite group $\mathrm{Pic}^0 X$ to prove the identity (7). However, this identity also follows from Deligne's Theorem, which is formulated in 4.17 below. Indeed, we have to show that $\sum_{D\in T} r(D) = 0$, when D runs through a complete linear series T of degree $m > 2(2g-2)$. Now, T is the set of rational points of the fibre over some $u \in \mathrm{Pic}^m X$ of the morphism $\mathrm{jac}_m\colon \mathrm{Sym}^m X \to \underline{\mathrm{Pic}}^m X$. Since $R^j(\mathrm{jac}_m)_*\mathcal{E}_r^{(m)}$ is killed by a fixed power of t for all j and r (see 4.17), we see that $\operatorname{trace}\left(F_u \middle| R(\mathrm{jac}_m)_*\mathcal{E}_r^{(m)}\right)$ is killed by a fixed power of t (independent of r). We deduce (7) by the Frobenius trace formula and proper base change.

The rest of Section 2 of [3] goes through completely unchanged. However, even though this is perhaps not strictly necessary, we would like to explain why the sequence (8) of [3] remains exact in our setting.

4.14. The sequence for each r reads

$$0 \longrightarrow \psi^* \mathcal{E}_r^{(m)} \xrightarrow{\beta} \mathcal{E}_r^{(m-1)} \boxtimes \mathcal{E}_r \xrightarrow{\gamma} \alpha_*(\mathcal{E}_r^{(m-2)} \boxtimes \det \mathcal{E}_r) \longrightarrow 0.$$

We construct β as follows. Consider the map a: $X^m \to \mathrm{Sym}^{m-1} X \times X$ (see diagram below). We have $\psi \circ a = \varphi$. Note that $\mathcal{E}_r^{(m-1)} \boxtimes \mathcal{E}_r = (a_* \boxtimes^m \mathcal{E}_r)^{S_{m-1}}$, and $\mathcal{E}_r^{(m)} = (\varphi_* \boxtimes^m \mathcal{E}_r)^{S_m}$. By adjunction we have

$$\begin{aligned} \mathrm{Hom}(\psi^* \mathcal{E}_r^{(m)}, \mathcal{E}_r^{(m-1)} \boxtimes \mathcal{E}_r) &= \mathrm{Hom}(\mathcal{E}_r^{(m)}, \psi_*(\mathcal{E}_r^{(m-1)} \boxtimes \mathcal{E}_r)) \\ &= \mathrm{Hom}((\varphi_* \boxtimes^m \mathcal{E}_r)^{S_m}, (\varphi_* \boxtimes^m \mathcal{E})^{S_{m-1}}). \end{aligned}$$

In the last group there is a natural map, which gives β.

We construct γ as follows. In addition to the maps above, let b: $X^{m-1} \to X^m$ be the map $(x_1, \ldots, x_{m-1}) \mapsto (x_1, \ldots, x_{m-1}, x_{m-1})$. Then the diagram

$$\begin{array}{ccccc} X^{m-1} & \xrightarrow{b} & X^m & = & X^m \\ \downarrow{\scriptstyle a'} & & \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle \varphi} \\ \mathrm{Sym}^{m-2} X \times X & \xrightarrow{\alpha} & \mathrm{Sym}^{m-1} X \times X & \xrightarrow{\psi} & \mathrm{Sym}^m X \end{array}$$

is commutative, where a' is the obvious map. Consider the sequence of maps

$$\begin{aligned} \mathcal{E}_r^{(m-1)} \boxtimes \mathcal{E}_r = (a_* \boxtimes^m \mathcal{E}_r)^{S_{m-1}} \hookrightarrow a_*(\boxtimes^m \mathcal{E}_r) &\xrightarrow{a_*\gamma'} a_*(b_*(\boxtimes^{m-2} \mathcal{E}_r \boxtimes {\textstyle\bigwedge}^2 \mathcal{E}_r)) \\ = \alpha_*(a'_*(\boxtimes^{m-2} \mathcal{E}_r \boxtimes {\textstyle\bigwedge}^2 \mathcal{E}_r)). \end{aligned}$$

Here γ' is deduced from the natural map $\mathcal{E}_r \boxtimes \mathcal{E}_r \to \Delta_* \bigwedge^2 \mathcal{E}_r$, with Δ: $X \to X \times X$ as usual. It is easy to see that the composition above maps $\mathcal{E}_r^{(m-1)} \boxtimes \mathcal{E}_r$ into $\alpha_*(\mathcal{E}_r^{(m-2)} \boxtimes \bigwedge^2 \mathcal{E}_r)$.

Exactness of the sequence. This can be checked on the stalks. Given the description of the stalks of $\mathcal{E}_r^{(m)} = \Gamma_{ext}^m(\mathcal{E}_r)$ in [12, t. III, exposé XVII, (5.5.8.1)], we need only to prove: for $M = \Lambda^2$, the sequence of Λ-modules

$$0 \longrightarrow \Gamma^n(M) \longrightarrow \Gamma^{n-1}(M) \otimes M \longrightarrow \Gamma^{n-2}(M) \otimes {\textstyle\bigwedge}^2 M \longrightarrow 0$$

is exact (with maps as above). This is easy to verify by hand, but it also follows from the second exact sequence of [4, t I, 4.3.1.7], taking $E = 0$, $F = M$ and $G = M$.

4.15. *Additions to* [3, Section 3]. No changes need to be made. Indeed, the vanishing cycle theorem I is a geometric statement and is proved in Section 3 of [3] for any finite coefficient ring of order prime to p.

4.16. *Additions to* [3, Section 4]. No changes need to be made. The switch to any finite coefficient ring of order prime to p is made on page 105 of the article. The arguments before this switch do not depend on the particular coefficient rings.

4.17. *Deligne's Theorem, see* [3, page 113]. Let us formulate this only for our specific local system $\mathcal{E}$. The formulation we will use in our case is the following:

Assume that $m > 2(2g-2)$; then there exists a constant $C > 0$ such that all the sheaves $R^j(\mathrm{jac_m})_*\mathcal{E}_r^{(m)}$ are annihilated by t^C. The constant C may depend on ρ and m but does not depend on j or r.

Let us make a few additional remarks that explain how we see this in our case. The arguments of [3] do provide a proof if one is sufficiently confident with inverse limits. We advise the reader to read that proof before continuing.

First, we remark that the lemma on page 113 of [3] is still correct in our setting, i.e., the sheaves $R^j(\mathrm{jac_m})_*\mathcal{E}_r^{(m)}$ are locally constant on $\underline{\mathrm{Pic}}^m X$.

Second, we are going to use symmetric Künneth formula [12, t. III, Exposé XVII, Section 5.5]:

$$R\Gamma(\mathrm{Sym}^m\overline{X}, \mathcal{E}_r^{(m)}) = L\Gamma^m_{ext}(R\Gamma(\overline{X}, \mathcal{E}_r)).$$

4.18. Third, we will use the following algebra lemma. For every $r_0, r_1, r_2 \geq 0$ and $m > r_1 - r_0 - r_2$, there exists a universal constant $c = c(r_0, r_1, r_2, m)$ with the following property: For every ring R and $x \in R$ and for every complex of finite free R-modules

$$0 \longrightarrow K^0 \xrightarrow{\alpha} K^1 \xrightarrow{\beta} K^2 \longrightarrow 0,$$

with $\mathrm{rank}(K^i) = r_i$ such that there exist $s\colon K^2 \to K^1$ and $\pi\colon K^1 \to K^0$ with $\pi \circ d^0 = x_{K^0}$ and $d^1 \circ s = x_{K^2}$ we have: x^c annihilates all homology groups of $L\Gamma^m_{ext}(K^\bullet)$. This lemma can easily be proved by reducing to the 'universal' case, where

$$R = \mathbb{Z}[x, \alpha_{ij}, \beta_{ij}, s_{ij}, \pi_{ij}]/(\text{relations}).$$

This is a Noetherian ring, hence we need only show that $L\Gamma^m_{ext}(K^\bullet) \otimes R[1/x] = 0$. But $L\Gamma^m_{ext}(K^\bullet) \otimes R[1/x = L\Gamma^m_{ext}(K^\bullet \otimes R[1/x]) = L\Lambda^m(M)$, where M is the cohomology of $K^\bullet \otimes R[1/x]$ in degree 1. See [4, t I, Proposition 4.3.2.1]. However, $L\Lambda^m(M) = \Lambda^m(M)$ as M is finite projective (by the existence of s and π). Finally, $\Lambda^m(M) = 0$, as M has rank $r_1 - r_0 - r_2 < m$.

4.19. Let $\Gamma \subset \pi_1(\overline{X})$ be the closure of the commutator subgroup of $\pi_1(\overline{X})$. Since $\rho|_{\pi_1(\overline{X})}$ over F is absolutely irreducible, we see that the representation $\rho|_\Gamma$

has no invariants, nor coinvariants over F. Thus some power t^N of t annihilates the module of coinvariants $(\mathbb{F}[[t]]^2)_\Gamma$. We fix a choice of N.

Suppose we are given a finite ring Λ, a ring homomorphism $\psi\colon \mathbb{F}[[t]] \to \Lambda$ and a continuous character $\omega\colon \pi_1(\overline{X}) \to \Lambda^*$. Then we can consider the finite locally free sheaf of Λ-modules $\mathcal{E}(\omega)$ over $\overline{X}$, obtained by twisting $\mathcal{E} = \{\mathcal{E}_r\}$ by ω. The corresponding representation of $\pi_1(\overline{X})$ is $(\psi \circ \rho) \otimes \omega$.

Note that $H^2_{\acute{e}t}(\overline{X}, \mathcal{E}(\omega))$ equals $(\Lambda^2)_{\pi_1(\overline{X})}$ (action via $(\psi \circ \rho) \otimes \omega$) which is a quotient of $(\Lambda^2)_\Gamma = (\mathbb{F}[[t]]^2)_\Gamma \otimes \Lambda$. Thus by the above, it is clear that $H^2_{\acute{e}t}(\overline{X}, \mathcal{E}(\omega))$ is annihilated by the image of t^N in Λ. Similarly for $H^2_{\acute{e}t}(\overline{X}, \mathcal{E}(\omega^{-1}))$. Recall that $R\Gamma(\overline{X}, \mathcal{E}(\omega))$ is a perfect complex of Λ-modules of amplitude $[0,2]$, whose Λ-dual is $R\Gamma(\overline{X}, \mathcal{E}(\omega^{-1}))$. Also its Euler characateristic in $K_0(\Lambda)$ is $[\Lambda^{2(2g-2)}]$. From these results it follows that $R\Gamma(\overline{X}, \mathcal{E}(\omega))$ can be representated by a complex of free modules of ranks $r_0 = 2$, $r_1 = 4g$ and $r_2 = 2$, satisfying the conditions of 4.18 with $x = t^N$. We deduce that the cohomology groups of $R\Gamma(\mathrm{Sym}^m\overline{X}, \mathcal{E}(\omega)^{(m)})$ are annihilated by t^{cN}, with c as in 4.18 .

4.20. Using the above, we can duplicate the arguments of [3, pp. 113–114]. Pick $C >> cN$, where c and N are as above, and assume by induction that C works for $j' < j$. Let M be the $\mathbb{F}[t]/(t^r)[\pi_1(\mathrm{Pic}^m\overline{X})]$-module that corresponds to $R^j(\mathrm{jac}_m)_*\mathcal{E}_r^{(m)}$. The action of $\pi_1(\mathrm{Pic}^m\overline{X}) = \pi_1(\overline{X})^{ab}$ factors through a finite quotient A. For any element $m \in M$, the element

$$s_m = \sum_{a \in A} a(m) \otimes a \in M \otimes_{\mathbb{F}[t]/(t^r)} \mathbb{F}[t]/(t^r)[A]$$

is A-invariant (diagonal action). Let us set $\Lambda = \mathbb{F}[t]/(t^r)[A]$ and let $\omega\colon \pi_1(\overline{X}) \to \Lambda^*$ be the obvious character. If t^M does not annihilated m, then t^M does not annihilate the global section s_m of

$$R^j(\mathrm{jac}_m)_*\mathcal{E}(\omega)^{(m)} \cong \left(R^j(\mathrm{jac}_m)_*\mathcal{E}_r^{(m)}\right) \otimes_{\mathbb{F}[t]/(t^r)} \Lambda(\omega).$$

Here $\Lambda(\omega)$ is the rank 1 locally free sheaf of Λ-modules corresponding to the character ω. By induction on j, the sheaves $R^{j'}(\mathrm{jac}_m)_*\mathcal{E}_r^{(m)}$ are annihilated by t^C for $j' < j$. Thus by the Leray spectral sequence there exists an element $s' \in H^j(\mathrm{Sym}^m\overline{X}, \mathcal{E}(\omega)^{(m)})$ which maps to $t^{jC}s$ in $\Gamma(\underline{\mathrm{Pic}}^m\overline{X}, R^j(\mathrm{jac}_m)_*\mathcal{E}(\omega)^{(m)})$. By the above, we see that s' is annihilated by t^{cN}, and hence s_m and m are annihilated by t^{jC+cN}. This ends the proof of Deligne's Theorem.

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