DECAY AT INFINITY OF SOLUTIONS TO HIGHER ORDER PARTIAL DIFFERENTIAL EQUATIONS: REMOVAL OF THE CURVATURE ASSUMPTION

BY

WALTER LITTMAN

ABSTRCT

In an earlier paper a generalization of Rellich's theorem on the Helmholz equation was obtained for a large class of higher order equations $P(1/i\partial/\partial x)u=f$, subject to the condition that the Gaussian curvature of $P(\xi) = 0$ never vanish. This restriction is removed in this note.

For a complex function $u(x)$ defined in R^N, set $D = -i\partial/\partial x$,

$$\| u \|_R^2 = \int_{R<|x|<2R} |u(x)|^2 dx.$$

Then the main goal of this note may be stated as follows:

THEOREM. *Let $P(\xi_1, \xi_2, \cdots, \xi_N)$ be a polynomial with real coefficients each of whose (complex) irreducible factors $P_j(\xi)$ has an $N-1$ dimensional real solution set S_j on which* $\operatorname{grad} P(\xi) \neq 0$. *Suppose $P(D)u = f$, f has compact support, and $\| u \|_R = o(R^{\frac{1}{2}})$ as R approaches infinity; then u has compact support.*

NOTE. The gradient may be allowed to vanish at isolated points on the S_j (cf. [3], Section 8.)

In an earlier version of the theorem, proved in [3], it was necessary to assume in addition that the Gaussian curvature of the S_j did not vanish. The main point of the present note is the removal of this restriction.

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Let us remark that an analogous theorcm for systems follows from the theorem stated above for a single equation.

Originally we had a different proof of the theorem stated above, which was more differential geometric in nature. However, we have since noticed that the methods of Peetre [4], where his theory of interpolation spaces is used to estimate the fundamental solution of elliptic equations, can be combined very efficiently with our proof in [3] to give a neater proof of the theorem. This is the proof presented here.

Apart from the definition of $\| \quad \|_R$ introduced earlier, we need some additional notation. We let $H_s = H_s(R^N)$ be the usual Sobolev spaces (s real) with norms $| \quad |_s$, defined as usual by means of Fourier transforms. We shall also need a number of other spaces. $H^{\frac{1}{2},\infty}$ is the space of functions u belonging to $L^2(R^N)$ and satisfying $|u(x+h)-u(x)|_0 < C|h|^{\frac{1}{2}}$, and the space $H^{-\frac{1}{2}}$, $(\equiv(\Delta-1)^{\frac{1}{2}}H^{\frac{1}{2},\infty})$ of sums of functions and first derivatives of functions in $H^{\frac{1}{2},\infty}$. Alternately, f belongs to $H^{-\frac{1}{2},\infty}$ if and only if $K(t,f) \leqq Ct^{\frac{1}{2}}$, $(0 < t \leqq 1)$, where $K(t,f)$ is defined as the infimum of the quantity $|g|_{-1} + t|h|_0$, the infimum being taken over all decompositions $f = g + h$.

Note. For the relevant information on the spaces introduced above the reader may refer to example to [1] (especially section 4.3), or to the original papers of Peetre and Lions referred to there and in [4].

Lemma 1. *Suppose u and v are infinitely differentiable functions satisfying $\|u\|_R = o(R^{\frac{1}{2}})$, $\|D^J v\|_R = O(R^{\frac{1}{2}})$ for $|J| \leqq m$, as R approaches infinity, m being the degree of $P(\xi)$; then*

$$(P(D)v,u) = (v,P(D)u)$$

provided both sides exist as finite L^2 inner products.

Proof. Let $\psi(t)$ be an infinitely differentiable function of a single variable equaling one for $t < 1/3$ and zero for $t > 2/3$. Set $\phi(x) = v(x)\psi_h(x)$, where $\psi_h(x) = \psi(r/h)$, $r = |x|$. Setting $P(D) \equiv L$, we have

$$L(v\psi_h) = Lv \cdot \psi_h + \sum a_{IJ} D^J v \cdot D^I \psi_h,$$

where I, J are multi-indices, and where the summation is extended for $|J| < m$ $|I| > 0$, $|I| + |J| \leqq m$. It follows that

$$L(\phi) - L(v) = Lv \cdot (\psi_h - 1) + \sum a_{IJ} D^J v \cdot D^I \psi_h,$$

$$(L(\phi - v), u) = ((\psi_h - 1)Lv, u) + \sum a_{IJ}(D^J v \cdot D^I \psi_h, u).$$

Now since $D^I \psi_h = O(h^{-|I|})$,

$$(D^J v \cdot D^I \psi_h, u) = O(h^{\frac{1}{2}}) \cdot O(h^{-|I|}) \cdot o(h^{\frac{1}{2}}) = o(1) \qquad \text{as } h \to \infty,$$

since $|I| > 0$ in the above summation. It follows that

$$(L\phi, u) - (Lv, u) \to 0 \quad \text{as } h \to \infty.$$

But

$$(L\phi, u) = (v\psi_h, Lu) \to (v, Lu) \quad \text{as } h \to \infty,$$

which yields the conclusion of the lemma.

Let us note that the cut-off function ψ_h used here us similar to the one used by Grušin [2].

Denoting the Fourier transform of a function (or tempered distribution) v by $\hat{v}$, we state

LEMMA 2.* *$\hat{v} \in H^{-\frac{1}{2},\infty} \Rightarrow \| v \|_R = O(R^{\frac{1}{2}})$; and, if in addition $\hat{v}$ has compact support, $\| D^J v \|_R = O(R^{\frac{1}{2}})$ for any fixed multi-index J, as $R \to \infty$.*

PROOF. We shall not prove the estimate for $D^J v$, since it is an immediate consequence of the estimate for v. If

$$K(t,f) = \underset{f = g + h}{\text{infimum}}(|g|_{-1} + t|h|_0)$$

then $\hat{v}$ belongs to $H^{-\frac{1}{2},\infty}$ if and only if $K(t,f) \leqq Ct^{\frac{1}{2}}$ $(0 < t \leqq 1)$; hence for each $0 < t = R^{-1} \leqq 1$ there exists a decomposition $\hat{v} = \hat{v}_{-1} + \hat{v}_0$ such that

$$R^{\frac{1}{2}}(|\hat{v}_{-1}|_{-1} + R^{-1}|\hat{v}_0|_0) \leqq C.$$

On the other hand, for any real s,

$$\| v \|_R \leqq CR^{-s}\left(\int (1 + |x|^2)^s |v(x)|^2 dx\right)^{\frac{1}{2}} \leqq CR^{-s}|\hat{v}|_s,$$

and thus

$$\| v \|_R \leqq \| v_{-1} \|_R + \| v_0 \|_R \leqq C(R|\hat{v}_{-1}|_{-1} + |\hat{v}_0|_0) = O(R^{\frac{1}{2}}).$$

LEMMA 3. *Let S be an infinitely differentiable $N-1$ surface embedded smoothly in R^N, and let $\alpha = \alpha(\xi)$ be a C^∞ function defined on S having compact support there; then the distribution μ (having compact support) defined by*

* The proofs of lemmas 2 and 4 are essentially contained in Peetre [4]. However since these lemmas are not stated explicitly there, and as a convenience to the reader, we present them here.

$$\langle \phi, \mu \rangle = \int_{\xi \in S} \phi(\xi)\alpha(\xi)dS_\xi$$

belongs to $H^{-\frac{1}{2},\infty}$.

PROOF. It suffices to prove this lemma for μ with sufficiently small support (since we can always decompose μ into a finite sum of such, using a partition of unity). Hence we may assume that the support of μ is contained in a local coordinate system in which the first coordinate vanishes on S. Since $H^{-\frac{1}{2},\infty}$ is invariant under smooth changes of coordinates, it suffices to show that $\delta(\xi_1)\phi(\xi')$ belongs to $H^{-\frac{1}{2},\infty}$, where $\phi \in \mathcal{D}$, and where $\xi' = (\xi_2, \cdots, \xi_N)$. Denoting by β the Heaviside step function, this is implied by the statement $\beta(\xi_1)\phi(\xi') \in H^{\frac{1}{2},\infty}_{\text{loc}}(R^N)$. The latter inclusion follows from the fact that $w \in H^{\frac{1}{2},\infty}$ is equivalent to $w \in L^2$ and $|w(\xi + h) - w(\xi)|_{L^2} \leqq C|h|^{\frac{1}{2}}$, which is easily checked for the particular function in question.

COROLLARY 1. *The function* $v(x)$ *defined by the formula*

$$v(x) \equiv \int_{\xi \in S} e^{ix\xi}\alpha(\xi)\,dS_\xi$$

satisfies the estimate $\| D^J v \|_R = O(R^{\frac{1}{2}})$, *for any fixed multi-index* J.

If the function f is smooth in the complement of the smooth surface S, and has a Cauchy type singularity on S, we may define a distribution P.V. f by

$$\langle \text{P.V.} f, \phi \rangle = \text{P.V.} \int f(\xi)\phi(\xi)\ d\xi,$$

where by "P.V." is meant the Cauchy principal value of the integral. Such integrals are discussed in [3].

LEMMA 4. *Let* $\gamma(\xi) \in \mathcal{D}$. *Then the distribution (with compact support)*

$$\text{P.V.} \frac{\gamma(\xi)}{P(\xi)},$$

where $P(\xi)$ *is the polynomial of the theorem, belongs to* $H^{-\frac{1}{2},\infty}$.

PROOF. The proof of this lemma can be reduced to that of the previous lemma, as is done in a somewhat analogous situation in [3], or can be done directly as in Lemma 3 (cf. Peetre [4]), relying ultimately on the fact that in one dimension the distribution P.V. $1/\xi$ is the derivative of the function $\log|\xi|$; and on the convergence of the improper integral $\int |\log|1 + \xi^{-1}||^2 d\xi$, integrated over R^1.

COROLLARY 2. *Suppose $\hat{w}(\xi) \in \mathscr{D}$. Then the function*

$$v(x) = \text{P.V.} \int e^{ix\xi} \frac{\hat{w}(\xi)}{P(\xi)} d\xi$$

satisfies $\| v(x) \|_R = O(R^{\frac{1}{2}})$.

PROOF OF THEOREM: The proof of the theorem now follows the proof of theorem 1 in [3], replacing lemmas 1, 2 and 3 of that paper with Lemma 1, and Corollaries 2 and 1 of this paper respectively. We give the simple proof for the case $f \equiv 0$. For $\hat{w} \in \mathscr{D}$, let v be given as in Corollary 2. It is easily established that $v(x)$ satisfies the equation $P(D)v = w$. Then by Lemma 1 and Corollary 2, we have $(w, u) = (v, Lu) = 0$ for all w with $\hat{w}$ in $\mathscr{D}$. Since these are dense, u must vanish identically.

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REFERENCES

1. P. L. Butzer and H. Berens, *Semi-groups of operators and approximation*, Springer Verlag, 1967.

2. V. V. Grušin, *On Sommerfeld type conditions*, Mat. Sb. (N.S.) **61** (103) (1963), 147–174. (Also Amer. Math. Soc. Translations, series 2, vol 51).

3. W. Littman, *Decay at infinity of solutions to partial differential equations with constant coefficients*, Trans. Amer. Math. Soc., **123** (1966), 449–459.

4. J. Peetre, *Estimates for eigenfunctions*, Studia Math. **28** (1967), 169–182.

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