ACCRETIVE SETS AND DIFFERENTIAL EQUATIONS IN BANACH SPACES*

BY

H. BREZIS AND A. PAZY

ABSTRACT

This paper is concerned with global solutions of the initial value problem

(1) $$du/dt + Au \ni 0, \qquad u(0) = x$$

where A is a (nonlinear) accretive set in a Banach space X. We show that various approximation processes converge to the solution (whenever it exists). In particular we obtain an exponential formula for the solutions of (1).

Assuming X^* is uniformly convex, we also prove the existence of a solution under weaker assumptions on A than those made by previous authors (F. Browder, T. Kato).

1. Introduction

Let X be a real Banach space and let X^* be the dual space of X. The value of $x^* \in X^*$ at $x \in X$ will be denoted by (x, x^*). The duality map of X is the subset F of $X \times X^*$ defined by

(1.1) $$F = \{[x, x^*];\ x \in X,\ x^* \in X^* \text{ and } (x, x^*) = |x|^2 = |x^*|^2\}$$

where $|x|$ (respectively $|x^*|$) denotes the norm of x (resprctively x^*) in X (respectively X^*). If S is a nonvoid subset of X we define $\| S \| = \mathrm{Inf}_{x \in S} |x|$

A subset A of $X \times X$ is called *accretive* if for each $\lambda > 0$ and $[x_i, y_i] \in A$, $i = 1, 2$, we have

(1.2) $$|x_1 + \lambda y_1 - (x_2 + \lambda y_2)| \geqq |x_1 - x_2|$$

or equivalently (see Kato [6] Lemma 3.2) A is accretive if and only if for every $[x_i, y_i] \in A$, $i = 1, 2$, there exists $f \in F(x_1 - x_2)$ such that $(y_1 - y_2, f) \geqq 0$.

If A is a subset of $X \times X$ and $x \in X$ we define $Ax = \{z; [x, z] \in A\}$, $D(A) = \{x; Ax \neq \emptyset\}$ and $R(A) = \bigcup_{x \in D(A)} Ax$.

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This paper is concerned with global solutions of the initial value problem

$$(1.3)\qquad \begin{cases} \dfrac{du}{dt} + Au \ni 0 \text{ a.e. on } (0, +\infty) \\ \qquad u(0) = x \end{cases}$$

where A is a given accretive set in $X \times X$. A function $u(t)$ defined on $[0, +\infty]$ with values in X is a solution of (1.3) if $u(t)$ is lipschitz in t, $u(t)$ is differentiable a.e. on $(0, +\infty)$, $u(t) \in D(A)$ a.e. on $(0, +\infty)$ and u satisfies (1.3). (Note that if X is reflexive and $u(t)$ is lipschitz then u is differentiable a.e. on $(0, +\infty)$; see Komura [7] appendix). From the accretiveness of A it follows easily that the solution of (1.3) is unique.

We start Section 2 with some preliminary results concerning accretive sets in $X \times X$ and the initial value problem (1.3). Assuming that (1.3) has a solution we show that various approximation processes converge to this solution.

In Section 3 we suppose that X^* is uniformly convex and obtain the existence of a solution (1.3) under a condition on A (condition I) which is weaker than the "m-accretive" assumption made by previous authors (see F. Browder [2] and T. Kato [6]).

The authors are indebted to M. Crandall for several improvements over an earlier version of this paper.

2. **Approximation Processes for the Initial Value Problem** (1.3)

If A is accretive one can define for each $\lambda > 0$ a single valued operator $J_\lambda = (I + \lambda A)^{-1}$ with $D(J_\lambda) = R(I + \lambda A)$ and $R(J_\lambda) = D(A)$. It follows from (1.2) that J_λ is a contraction i.e.

$$|J_\lambda x - J_\lambda y| \leqq |x - y| \text{ for every } x, y \in D(J_\lambda).$$

We set $A_\lambda = \lambda^{-1}(I - J_\lambda)$ for every $\lambda > 0$. Clearly A_λ is lipschitz (with constant $2\lambda^{-1}$), $D(A_\lambda) = D(J_\lambda) = R(I + \lambda A) = D_\lambda$.

In the two following lemmas we collect some elementary properties of J_λ, A_λ and the solution of (1.3).

LEMMA 2.1. *Let A be accretive then*

(i) *A_λ is accretive,*

(ii) *For $x \in D_\lambda$, $A_\lambda x \in AJ_\lambda x$ and $\| AJ_\lambda x \| \leqq |A_\lambda x|$,*

(iii) *For $x \in D_\lambda \cap D(A)$, $|J_\lambda x - x| \leqq \lambda \| Ax \|$ and hence $|A_\lambda x| \leqq \| Ax \|$.*

For a proof of Lemma 2.1 see Kato [6].

LEMMA 2.2. *Let A be accretive and let $u(t)$ be the solution of the initial value problem* (1.3) *with $x \in D(A)$. Then*

$$\left|\frac{du}{dt}(t)\right| = \|Au(t)\| \leqq |Ax| \quad \text{a.e. on } (0, +\infty) \tag{2.1}$$

PROOF: Let Ω be the set of all values of t for which u is differentiable, $u(t) \in D(A)$ and $du/dt + Au \ni 0$.

We shall show that (2.1) holds for all $t \in \Omega$. Let $s \geqq 0$ be fixed such that $u(s) \in D(A)$. Then we have

$$|u(t) - u(s)| \frac{d}{dt} |u(t) - u(s)| = \left(\frac{du}{dt}(t), f\right)$$

for almost all $t \geqq 0$ and all $f \in F(u(t) - u(s))$ (see Kato [5] Lemma 1.3). Let $y(t) = -du(t)/dt \in Au(t)$. For every $y \in Au(s)$ there exists $f_0 \in F(u(t) - u(s))$ such that

$$\tfrac{1}{2}(d/dt)|u(t) - u(s)|^2 = -(y(t), f_0) \leqq -(y, f_0) \leqq |y| \; |u(t) - u(s)| \tag{2.2}$$

From (2.2) we deduce that $|u(t) - u(s)| \leqq |y|(t-s)$ holds true for every $y \in Au(s)$ and $t \geqq s$. Hence

$$|u(t) - u(s)| \leqq \|Au(s)\|(t - s) \quad \text{for } t \geqq s. \tag{2.3}$$

If $s \in \Omega$, (2.3) implies $|du(s)/dt| \leqq \|Au(s)\|$ but $du(s)/dt \in -Au(s)$ and therefore $|du(s)/dt| = \|Au(s)\|$ for all $s \in \Omega$.

Let $h > 0$ and let $v(t) = u(t + h)$. Clearly $v(t)$ satisfies

$$dv/dt + Av \ni 0 \text{ a.e. on } (0, +\infty), \; v(0) = u(h).$$

We have for almost all $t > 0$

$$\tfrac{1}{2} d/dt |v(t) - u(t)|^2 = |v(t) = u(t)| \; d/dt \, |v(t) - u(t)| = (y(t) - z(t), f) \leqq 0$$

where $y(t) \in Au(t)$, $z(t) \in Av(t)$ and $f \in F(u(t) - v(t))$. Therefore $|v(t) - u(t)|$ is a momotonically nonincreasing function of t. In particular

$$|u(t + h) - u(t)| = |v(t) - u(t)| \leqq |v(0) - u(0)| = |u(h) - x| \leqq h \|Ax\|.$$

The last inequality follows from (2.3) taking $s = 0$. Thus for all $t \in \Omega$ we have $du(t)/dt| \leqq \|Ax\|$.

The accretiveness of A assures the uniqueness of the solution of (1.3). In order to define approximation processes and prove the existence of a solution we impose further conditions on A. We shall usually assume that A is accretive and satisfies

CONDITION I. For every $x \in D(A)$ there exist a neighborhood U_x of x and a sequence $\varepsilon_n \downarrow 0$ such that

$$\bigcap_n R(I + \varepsilon_n A) \supset U_x \cap D(A). \tag{2.4}$$

Condition I is weaker than the notion of locally m-accretive sets introduced by T. Kato [6] (an accretive set A is locally m-accretive on $D(A)$ if for every $x \in D(A)$, there exists U_x and $\varepsilon_n \to 0$ such that $\bigcap_n R(I + \varepsilon_n A) \supset U_x$). Note, for example, that the set $\{[x,0]; x \in D\}$ where D is any subset of X, $D \neq X$, satisfies condition I but is not locally m-accretive. In the rest of this paper we shall assume that the neighborhood U_x is an open vall $B(x, \rho(x))$. For some of our results we shall need a condition stronger than condition I, namely

CONDITION II. For every $\lambda > 0$, $R(I + \lambda A) \supset \overline{\operatorname{conv} D(A)}$.

Condition II was used previously by the authors [1] in Hilbert space.

LEMMA 2.3. *Let A be accretice and satisfying condition I. If $x \in D(A)$ then $u_{n,j} = (I + \varepsilon_n A)^{-j} x$ is defined for every n and $0 \leqq j < \rho(x)/\varepsilon_n \| Ax \|$. Moreover $u_{n,j} \in U_x \cap D(A)$ and*

$$|u_{n,j} - x| \leqq j\varepsilon_n \| Ax \|. \tag{2.5}$$

PROOF. We fix n and prove (2.5) by induction on j. For $j = 1$, (2.5) follows from Lemma 2.1(iii). Assume (2.5) is true for j and that $j + 1 < \rho(x)/\varepsilon_n \| Ax \|$, then $u_{n,j+1} - x = (I + \varepsilon_n A)^{-1} u_{n,j} - x$. By the induction hypothesis $u_{n,j+1}$ is well defined and

$$\begin{aligned} |(I + \varepsilon_n A)^{-1} u_{n,j} - x| &\leqq |(I + \varepsilon_n A)^{-1} u_{n,j} - (I + \varepsilon_n A)^{-1} x| \\ &+ |(I + \varepsilon_n A)^{-1} x - x| \leqq |u_{n,j} - x| + |(I + \varepsilon_n A)^{-1} x - x| \\ &\leqq j\varepsilon_n \| Ax \| + \varepsilon_n \| Ax \| \leqq (j+1)\varepsilon_n \| Ax \|. \end{aligned}$$

Thus (2.5) is true for $j + 1$.

Let A be accretive and satisfying Condition I. Let $x \in D(A)$. We define a sequence of step functions $u_n(t)$ on the interval $[0, T]$ where $T < \rho(x)/\| Ax \|$ by

$$u_n(t) = (I + \varepsilon_n A)^{-[t/\varepsilon_n]} x. \tag{2.6}$$

By Lemma 2.3 $u_n(t)$ is well defined for $0 \leqq t \leqq T$ and $u_n(t) \in D(A)$. We define on $[0, T]$ a second sequence of functions $v_n(t)$ as follows

If $0 \leqq t \leqq \varepsilon_n[T/\varepsilon_n]$ let

$$v_n(t) = u_n(j\varepsilon_n) + 1/\varepsilon_n(t-j\varepsilon_n)[u_n((j+1)\varepsilon_n) - u_n(j\varepsilon_n)] \quad \text{for } j\varepsilon_n \leqq t \leqq (j+1)\varepsilon_n,\ j = 0,1,\cdots,[T/\varepsilon_n]-1, \tag{2.7}$$

and if $\varepsilon_n[T/\varepsilon_n] \leqq t \leqq T$ let $v_n(t) = u_n(t)$.

Clearly $v_n(t)$ is differentiable on $[0,T]$ except for a finite number of points and we have

$$\begin{aligned} dv_n/dt(t) &= 1/\varepsilon_n[u_n(j+1)\varepsilon_n) - u_n(j\varepsilon_n)] = -A_{\varepsilon_n}u_n(j\varepsilon_n) \\ &= -A_{\varepsilon_n}u_n(t) = -A_{\varepsilon_n}v_n(j\varepsilon_n) \quad \text{for } j\varepsilon_n < t < (j+1)\varepsilon_n. \end{aligned} \tag{2.8}$$

From the definitions of A_{ε_n} and u_n we have $A_{\varepsilon_n}u_n(j\varepsilon_n) \in Au_n((j+1)\varepsilon_n)$ and by induction, using Lemma 2.1, it is easy to show that

$$|A_{\varepsilon_n}u_n(j\varepsilon_n)| \leqq \|Ax\|. \tag{2.9}$$

Therefore

$$\left|\frac{dv_n}{dt}\right| \leqq \|Ax\| \text{ a.e. on } (0,T). \tag{2.10}$$

Finally using Lemma 2.1 again we also obtain

$$|v_n(t) - u_n(t)| \leqq \varepsilon_n\|Ax\| \quad \text{for } 0 \leqq t \leqq T. \tag{2.11}$$

THEOREM 2.1. *Let A be accretive satisfying Condition I and let $x \in D(A)$. If the initial value problem* (1.3) *has a solution $u(t)$ then the sequences $u_n(t)$ and $v_n(t)$ converge uniformly to u on $[0,T]$.*

PROOF. From the definition of $u_n(t)$ we have for $\varepsilon_n \leqq t \leqq T$

$$\varepsilon_n^{-1}(u_n(t) - u_n(t-\varepsilon_n)) + y_n(t) = 0 \quad \text{where } y_n(t) \in Au_n(t). \tag{2.12}$$

In order for (2.12) to hold for all $0 \leqq t \leqq T$ it is convenient to define $u_n(t)$ for $t < 0$ as $u_n(t) = x + \varepsilon_n y$ with any $y \in Ax$. Let $y(t) = -du(t)/dt$, $y(t) \in Au(t)$ a.e. on $(0,T)$. We extend $u(t)$ as x for $t < 0$. The accretiveness of A implies that a.e. on $(0,T)$

$$\begin{aligned} \left|\frac{du}{dt}(t) - \frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right| &= \left|\frac{u_n(t)-u_n(t-\varepsilon_n)}{\varepsilon_n} - \frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n} + y_n(t) - y(t)\right| \\ &\geqq \left|\frac{u_n(t)-u(t)}{\varepsilon_n} + y_n(t) - y(t)\right| - \left|\frac{u_n(t-\varepsilon_n)-u(t-\varepsilon_n)}{\varepsilon_n}\right| \\ &\geqq \left|\frac{u_n(t)-u(t)}{\varepsilon_n}\right| - \left|\frac{u_n(t-\varepsilon_n)-u(t-\varepsilon_n)}{\varepsilon_n}\right|. \end{aligned}$$

Integrating this inequality on $(0,\theta)$ with $\varepsilon_n \leqq \theta \leqq T$ we obtain

$$-\int_{-\varepsilon_n}^{0} \varepsilon_n^{-1}\left|u_n(t)-u(t)\right|dt+\int_{\theta-\varepsilon_n}^{\theta}\varepsilon_n^{-1}\left|u_n(t)-u(t)\right|dt$$

$$\leqq \int_0^{\theta}\left|\frac{du}{dt}-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right|dt$$

or

$$(2.13)\quad \begin{cases} \varepsilon_n^{-1}\displaystyle\int_{\theta-\varepsilon_n}^{\theta}\left|u_n(t)-u(t)\right|dt \leqq \int_0^{\theta}\left|\frac{du}{dt}-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right|dt \\ \qquad\qquad\qquad\qquad + \varepsilon_n^{-1}\displaystyle\int_{-\varepsilon_n}^{0}\left|u_n(t)-x\right|dt \\ \qquad\qquad \leqq \displaystyle\int_0^{T}\left|\frac{du}{dt}-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right|dt+\varepsilon_n\left|y\right|. \end{cases}$$

Adding these inequalities for $\theta=\varepsilon_n, 2\varepsilon_n,\cdots,N\varepsilon_n$, $N=[T/\varepsilon_n]$ yields

$$\varepsilon_n^{-1}\int_0^{N\varepsilon_n}\left|u_n(t)-u(t)\right|dt\leqq N\int_0^{T}\left|\frac{du}{dt}-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right|dt+N\varepsilon_n\left|y\right|$$

and therefore

$$\int_0^{N\varepsilon_n}\left|u_n(t)-u(t)\right|dt\leqq T\int_0^{T}\left|\frac{du}{dt}-\frac{u(t)-u(t-\varepsilon_n)}{\varepsilon_n}\right|dt+\varepsilon_n T\left|y\right|$$

Since $[u(t)-u(t-\varepsilon_n)]/\varepsilon_n \to du/dt$ a.e. on $(0,T)$ as $\varepsilon_n\to 0$ and $\left|du/dt-[u(t)-u(t-\varepsilon_n)]/\varepsilon_n\right|\leqq 2\|Ax\|$ by Lemma 2.2 we conclude that $u_n\to u$ in $L^1(0,T;X)$. Therefore also $v_n\to v$ in $L^1(0,T;X)$ by (2.11). But v_n is differentiable a.e. and

$$\frac{d}{dt}\left|v_n(t)-u(t)\right|\leqq\left|\frac{d}{dt}(v_n(t)-u(t))\right|\leqq 2\|Ax\| \text{ a.e.}$$

here we used (2.10) and Lemma 2.2. Therefore

$$\frac{1}{2}\frac{d}{dt}\left|v_n(t)-u(t)\right|^2\leqq 2\|Ax\|\;\left|v_n(t)-u(t)\right| \text{ a.e.}$$

or

$$\left|v_n(t)-u(t)\right|^2\leqq 4\|Ax\|\int_0^t\left|v_n(s)-u(s)\right|ds\leqq 4\|Ax\|\int_0^T\left|v_n(s)-u(s)\right|ds$$

$$\text{for } 0\leqq t\leqq T$$

which implies $v_n(t)\to u(t)$ uniformly in $[0,T]$. By (2.11) the same holds for u_n.

REMARK. If the initial value problem (1.3) has only a local solution i.e. a solution for $0 \leqq t \leqq T_x$ then the conclusion of Lemma 2.2 (respectively Theorem 2.1) holds on the interval $[0, T_x]$ (respectively $[0, T_1]$ with $T_1 = \min(T, T_x)$).

COROLLARY 2.1. *Let A be accretive and satisfying Condition II. If for $x \in D(A)$ the initial value problem* (1.3) *has a solution $u(t)$, then for every sequence $\varepsilon_n \downarrow 0$ we have*

$$u(t) = \lim_{n \to +\infty} (I + \varepsilon_n A)^{-[t/\varepsilon_n]} x$$

and the limit is uniform on bounded intervals. In particular for $\varepsilon_n = t/n$ we have the exponential formula

$$u(t) = \lim_{n \to +\infty} \left(I + \frac{t}{n} A\right)^{-n} x.$$

Let A be accretive and satisfy Condition II. Let $x \in D(A)$. A standard method to solve the initial value problem (1.3) is to approximate A by lipschitz operators A_λ, then solve the equation

$$\begin{cases} \dfrac{du_\lambda}{dt} + A_\lambda u_\lambda = 0 \\ u_\lambda(0) = x \end{cases} \tag{2.14}$$

and let λ tend to zero. This method was used by K. Yosida [11] for the linear case and by Y. Komura [7], T. Kato [6] and others in the nonlinear case with A being m-accretive (i.e. $R(I + \lambda A) = X$ for every $\lambda > 0$). Our next theorem shows that if A is accretive and satisfies Condition II, then the approximated equation (2.14) has a solution u_λ which converges to u, the solution of (1.3), as $\lambda \to 0$ (assuming u exists).

THEOREM 2.2. *Let A be accretive, satisfying Condition II and let $x \in D(A)$. For every $\lambda > 0$ the initial value problem* (2.14) *has a solution $u_\lambda(t)$. If the initial value problem* (1.3) *has a solution $u(t)$ then*

$$u(t) = \lim_{\lambda \to 0} u_\lambda(t) \textit{ for every } t \geqq 0$$

and the limit is uniform on bounded intervals.

In the proof of Theorem 2.2 we shall use the following lemma.

LEMMA 2.4. *Let C be a closed convex subset of X and let T be a contraction defined on C into C. Then for every $x \in C$ the equation*

$$\left\{\begin{aligned}\frac{du}{dt}+(I-T)u &= 0\\ u(0) &= x\end{aligned}\right. \tag{2.15}$$

has a solution $u(t)\in C$ *for all* $t \geqq 0$ *and*

$$\left|u(n)-T^n x\right| \leqq \sqrt{n}\left|x-Tx\right|. \tag{2.16}$$

We prove first the following observation.

LEMMA 2.5. *Let* ϕ_n *be a sequence of functions in* $L^1_{\mathrm{loc}}(0,+\infty)$ *satisfying* $\phi_0(t) \leqq t$ *and*

$$\phi_n(t) \leqq ne^{-t} + \int_0^t e^{s-t}\phi_{n-1}(s)\,ds \tag{2.17}$$

then

$$\phi_n(t) \leqq [(n-t)^2+t]^{1/2}. \tag{2.18}$$

PROOF OF LEMMA 2.5: Inequality (2·18) clearly holds for $n=0$. If it is true for $n-1$ then

$$\phi_n(t) \leqq ne^{-t} + \int_0^t e^{s-t}[(n-1-s)^2+s]^{1/2}ds = \psi_n(t)e^{-t}.$$

In order to complete the proof by induction we show that $\psi_n(t) \leqq e^t[(n-t)^2+t]^{1/2}$. Since $\psi_n(0)=n$ it is sufficient to prove that

$$\begin{aligned}\psi_n'(t) = \ & e^t[(n-1-t)^2+t]^{1/2} \leqq e^t[(n-t)^2+t]^{1/2}\\ &+ \tfrac{1}{2}e^t[(n-t)^2+t]^{-1/2}(1-2n+2t).\end{aligned}$$

The last inequality can be easily checked noting that the right hand side is positive and comparing the squares of both sides.

PROOF OF LEMMA 2.4. Clearly equation (2.15) is equivalent to

$$u(t) = e^{-t}x + \int_0^t e^{s-t}T(u(s))\,ds. \tag{2.10}$$

Equation (2.19) can be easily solved by the Picard fixed point theorem noting that the mapping

$$\Phi_x u(t) = e^{-t}x + \int_0^t e^{s-t}T(u(s))\,ds$$

maps the closed convex set

$$\{u \in C(0, T_0; X),\ u(t) \in C \ \text{ for } 0 \leqq t \leqq T_0\}$$

nto itself and that Φ_x is lipschitz with constant $(1 - e^{-T_0})$. Using the accretiveness of $(I - T)$ it follows from Lemma 2.2 that

$$|u(t) - x| \leqq t|(I - T)x|. \tag{2.20}$$

In addition

$$u(t) - T^n x = e^{-t}(x - T^n x) + \int_0^t e^{s-t}[T(u(s)) - T^n x]ds$$

which implies

$$|u(t) - T^n x| \leqq e^{-t}|x - T^n x| + \int_0^t e^{s-t}|u(s) - T^{n-1}x|ds.$$

Also

$$|x - T^n x| = \sum_{k=1}^{n} |T^{k-1}x - T^k x| \leqq n|(I-T)x|$$

Using Lemma 2.5 with $\phi_n(t) = |u(t) - T\ x|\ |(I-T)x|^{-1}$ and substituting $t = n$ in (2.17) yields (2.16)

REMARK. Inequality (2.16) for nonlinear contractions T on X is due to Miyadera and Ôharu [9] (extending a previous result for linear contractions by P. Chernoff [3]). The existence of a solution u under the condition of Lemma 2.4 was noticed independently by M. Crandall and the authors. The simple proof that we have brought here is due to M. Crandall (see [4]).

PROOF OF THEOREM 2.2. Restricting J_λ to $C = \overline{\operatorname{conv} D(A)}$ we obtain a contraction defined on C into C and by Lemma 2.4 the equation

$$\begin{cases} \dfrac{dv_\lambda}{dt} + (I - J_\lambda)v_\lambda = 0 \\ v_\lambda(0) = x \end{cases}$$

has a solution $v_\lambda(t)$ which satisfies

$$|v_\lambda(n) - J_\lambda^n x| \leqq \sqrt{n}|(I - J_\lambda)x| \leqq \sqrt{n}\lambda|A_\lambda x|.$$

Obviously $u_\lambda(t) = v_\lambda(t/\lambda)$ is a solution of (2.14) and we have

$$|u_\lambda(n\lambda) - J_\lambda^n x| \leqq \sqrt{n}\lambda|A_\lambda x|.$$

Let $\lambda_k \downarrow 0$ and let $n_k = [t/\lambda_k]$ then $n_k\lambda_k \uparrow t$, and from Corollary 2.1 we have $J_{\lambda_k}^{n_k} x \to u(t)$ as $\lambda_k \to 0$. Also

$$|u_{\lambda_k}(n_k\lambda_k) - J_{\lambda_k}^{n_k}x| \leqq \sqrt{n_k}\lambda_k|A_{\lambda_k}x| \leqq \frac{t}{\sqrt{n_k}}\|Ax\|.$$

Finally

$$\left| u_{\lambda_k}(n_k\lambda_k) - u_{\lambda_k}(t) \right| \leqq \| Ax \| (t - n_k\lambda_k).$$

Thus $u_{\lambda_k}(t) \to u(t)$. Since $\{\lambda_k\}$ was arbitrary we have $u_\lambda(t) \to u(t)$ as $\lambda \to 0$.

3. An Existence Theorem

In the main results of this section we assume that X^* is a uniformly convex Banach space. The uniform convexity of X^* implies in particular that the duality map F of X is uniformly continuous on bounded sets of X. We start with a lemma of T. Kato.

LEMMA 3.1. *Let X be a reflexive Banach space and let $u_n(t)$ be a sequence of $L^p(0,T;X)$, $p > 1$, such that $u_n(t)$ is bounded for almost all $t \in (0,T)$. Let $V(t)$ be the set of all weak cluster points of $u_n(t)$. If u_n converges weakly to u in $L^p(0,T;X)$ then $u(t) \in \overline{\mathrm{conv}\, V(t)}$ a.e. on $(0,T)$.*

The proof of Lemma 3.1 is given in Kato [6].

DEFINITION. *A set A in $X \times X$ is called demiclosed if $[x_i, y_i] \in A$, $x_i \to x$ and $y_i \rightharpoonup y$ imply $[x,y] \in A$ ($\rightharpoonup$ denotes weak convergence).*

LEMMA 3.2. *Let X^* be uniformly convex; let A be an accretive set satisfying Condition I and let $x \in D(A)$. The sequence of functions v_n defined by (2.7) converges uniformly to u. If furthermore A is demiclosed then $u(t) \in D(A)$ for every $t \in [0,T]$.*

PROOF. We define for $0 \leqq t \leqq \mathrm{Min}\{T - \varepsilon_n, T - \varepsilon_m\}$,

$$x_{nm}(t) = v_n(t) - v_m(t)$$

$$y_{nm}(t) = u_n(t + \varepsilon_n) - u_m(t + \varepsilon_m).$$

Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left| x_{nm}(t) \right|^2 &= \left(\frac{d}{dt}(v_n(t) - v_m(t)),\ F(x_{nm}(t)) \right) \\ &= -(A_{\varepsilon_n}u_n(t) - A_{\varepsilon_m}u_m(t),\ F(x_{nm}(t))) \\ &\leqq -(A_{\varepsilon_n}u_n(t) - A_{\varepsilon_m}u_m(t),\ F(x_{nm}(t)) - F(y_{nm}(t))) \\ &\leqq 2\| Ax \| \left| F(x_{nm}(t)) - F(y_{nm}(t)) \right|; \end{aligned} \tag{3.1}$$

here we used the accretiveness of A and Lemma 2.1. Next by (2.10)

$$\begin{aligned} \left| x_{nm}(t) - y_{nm}(t) \right| &\leqq \left| v_n(t) - u_n(t + \varepsilon_n) \right| + \left| v_m(t) - u_m(t + \varepsilon_m) \right| \\ &\leqq (\varepsilon_n + \varepsilon_m) \| Ax \|. \end{aligned}$$

Since F is uniformly continuous on bounded sets

$$\left| Fx_{nm}(t) - Fy_{nm}(t) \right| — 0 \quad \text{as} \quad n, m \to +\infty .$$

Integrating (3.1) over $(0,t)$ we obtain $\left| x_{nm}(t) \right| \to 0$ as $n, m \to +\infty$. Thus v_n is a Cauchy sequence in $C(0,T;X)$. Let $v_n \to u$ in $C(0,T;X)$. Assuming that A is demiclosed we now prove that $u(t) \in D(A)$ for $t \in [0,T)$. Let $j_n\varepsilon_n \to t$ then $v_n(j_n\varepsilon_n) \to u(t)$, $\left| A_{\varepsilon_n} v_n(j_n\varepsilon_n) \right| \leqq \| Ax \|$ and $J_{\varepsilon_n} v_n(j_n\varepsilon_n) \to u(t)$. But $J_{\varepsilon_n} v_n(j_n\varepsilon_n) \in D(A)$ and $A_{\varepsilon_n} v_n(j_n\varepsilon_n) \in AJ_{\varepsilon_n} v(j_n\varepsilon_n)$ which by the demiclosedness of A implies $u(t) \in D(A)$ and $\| Au(t) \| \leqq \| Ax \|$. Using again the demiclosedness of A and the demicontinuity of u we have $u(T) \in D(A)$.

LEMMA 3.3. *Let X be a Banach space and let A be accretive and closed. Let $\tilde{A}$ be an accretive extension of A such that for every $x \in D(\tilde{A})$ there exist a neighborhood U_x of x and a sequence $\varepsilon_n \downarrow 0$ with $\bigcap_n R(I + \varepsilon_n A) \supset U_x \cap D(\tilde{A})$. Let $u(t)$ be a function such that $u(t) \in D(\tilde{A})$ for all $t \in [0,T]$, u is differentiable a.e. on $(0,T)$ and*

$$\frac{du}{dt} + \tilde{A}u \ni 0 \ \textit{a.e. on} \ (0,T). \tag{3.2}$$

Then $u(t) \in D(A)$ a.e. on $(0,T)$ and

$$\frac{du}{dt} + Au \ni 0 \ \textit{a.e. on} \ (0,T). \tag{3.3}$$

PROOF. Let $0 < t_0 < T$ be such that u is differentiable at t_0 and $du(t_0)dt + \tilde{A}u(t_0) \ni 0$. We set $u(t_0) = x$ and $\phi(t) = [u(t) - u(t_0)]/(t - t_0) - du(t_0)/dt$ For $\left| t - t_0 \right|$ small enough, $u(t) \in U_x$. Hence there exist $x_n \in D(A)$ and $y_n \in Ax_n$, such that $u(t_0 - \varepsilon_n) = x_n + \varepsilon_n y_n$. By the accretiveness of $\tilde{A}$ at x and x_n we have

$$\left(-\frac{du}{dt}(t_0) - y_n, \ F(x - x_n) \right) \geqq 0 .$$

But

$$\phi(t_0 - \varepsilon_n) = \frac{x - x_n}{\varepsilon_n} - y_n - \frac{du}{dt}(t_0).$$

Thus

$$(\phi(t_0 - \varepsilon_n) - \frac{x - x_n}{\varepsilon_n}, \ F(x - x_n)) \geqq 0$$

and

$$\left| x - x_n \right| \leqq \varepsilon_n \left| \phi(t_0 - \varepsilon_n) \right| .$$

Consequently

$$\left| y_n + \frac{du}{dt}(t_0) \right| \leqq 2\,|\phi(t_0 - \varepsilon_n)|.$$

We conclude by the closedness of A that $u(t_0) \in D(A)$ and $du(t_0)/dt + Au(t_0) \ni 0$.

THEOREM 3.1. *Let X^* be uniformly convex and let A be demiclosed, accretive and satisfying Condition I. Then for every $x \in D(A)$ there exists a unique function u such that $u(t) \in D(A)$ for every $t \geqq 0$, u is lipschitz continuous and*

$$\text{(3.4)} \qquad \begin{cases} \dfrac{du}{dt} + Au \ni 0 \ \textit{a.e. on}\ (0, +\infty) \\ u(0) = x \end{cases}$$

PROOF OF THEOREM 3.1. Define a set B in $X \times X$ as follows; $D(B) = D(A)$ and $Bx = \overline{\mathrm{conv}(Ax)}$. B is accretive since

$$(y_1 - y_2, F(x_1 - x_2)) \geqq 0 \text{ for every } [x_1, y_i] \in A,\ i = 1,2;$$

implies

$$(y_1 - \eta_2, F(x_1 - x_2)) \geqq 0 \text{ for every } x_i \in D(A),\ y_1 \in Ax_1,\ \eta_2 \in \overline{\mathrm{conv}(Ax_2)};$$

hence

$$(\eta_1 - \eta_2, F(x_1 - x_2)) \geqq 0 \text{ for every } [x_i, \eta_i] \in B,\ i = 1,2.$$

By Lemma 3.2, $v_n \to u$ in $C(0,T;X)$ and $u(t) \in D(A) = D(B)$ for all $t \in [0,T]$. Since $|v_n'(t)| \leqq |Ax|$ a.e. on $(0,T)$ and $v_n \rightharpoonup u'$ in $\mathscr{D}'(0,T;X)$ we conclude that $v_n' \rightharpoonup u'$ in $L^p(0,T;X)$ for every $1 < p < +\infty$. In addition the set of all weak cluster points of $v_n'(t)$ is contained in $-Au(t)$ (since A is demiclosed) and by Lemma 3.1,

$$-u'(t) \in \overline{\mathrm{conv}\, Au(t)} = Bu(t) \text{ a.e. on } (0,T).$$

From Lemma 3.3 we have

$$\frac{du}{dt} + Au \ni 0 \text{ a.e. on } (0,T).$$

To complete the proof we have to show that u can be extended to a solution of (3.4) for all $t \neq 0$. Let u be a solution of (3.4) on $[0, T_1)$ where T_1 is maximal. If $T_1 \neq +\infty$ let $t_n \to T_1, t_n < T_1$ then $u(t_n) \to u_0$ since $|u(t_n) - u(t_m)| \leqq |t_n - t_m|\, \|Ax\|$. Also $\|Au(t_n)\| \leqq \|Ax\|$ implies that $u_0 \in D(A)$ and by the first part of the proof u can be extended beyond T_1; this contradicts the maximality of T_1.

COROLLARY 3.1. *Let X and X^* be uniformly convex and let A be demiclosed, accretive and satisfying Condition I. Then for every $x \in D(A)$*

(i) The set Ax has a unique element of minimum norm which is denoted by A^0x.

(ii) There exists a unique function $u(t) \in D(A)$ for every $t \geqq 0$ which is lipschitz continuous and everywhere differentiable from the right satisfying

$$\left\{\begin{aligned} &\frac{d^+u}{dt} + A^0u = 0 \quad \text{for every } t \geqq 0 \\ &u(0) = x \end{aligned}\right. \tag{3.5}$$

PROOF. Let $x \in D(A)$ then

$$x = x_{\varepsilon_n} + \varepsilon_n A_{\varepsilon_n} x = x_{\varepsilon_n} + \varepsilon_n B_{\varepsilon_n} x,$$

$|A_{\varepsilon_n}x| \leqq \|Ax\|$ by Lemma 2.1 and therefore $x_{\varepsilon_n} \to x$, as $n \to +\infty$. Let ε_n' be a subsequence of ε_n for which $A_{\varepsilon'_n}x \rightharpoonup \xi$ then $[x, \xi] \in A$ by the demiclosedness of A. But $|A_{\varepsilon'_n}x| = |B_{\varepsilon'_n}x| \leqq |B^0x|$ where B^0x is the (unique) element of minimum norm in Bx and therefore $|\xi| \leqq |B_0x|$ which implies $\xi = B^0x \in Ax$. Consequently Ax has a unique element of minimum norm $A^0x = B^0x$. Therefore

$$\frac{du}{dt} + A^0u = 0 \text{ a.e. on } (0, \infty). \tag{3.6}$$

Next we prove that u is differentiable from the right for all $t \geqq 0$ and that $d^+u/dt + A^0u = 0$ for all $t \geqq 0$. First note that

$$\lim_{t \to 0} A^0u(t) = A^0x. \tag{3.7}$$

Indeed using Lemma 2.2 we have

$$|A^0u(t)| = \|Au(t)\| \leqq \|Ax\| = |A^0x| \text{ a.e. on } (0, +\infty)$$

which implies by the demiclosedness of A that

$$|A^0u(t)| \leqq |A^0x| \text{ for all } t \geqq 0.$$

Every sequence $t_k \to 0$ has a subsequence $t_{k'}$ for which $A^0u(t_{k'}) \rightharpoonup \eta$, $u(t_{k'}) \to x$ and $|\eta| \leqq |A^0x|$; thus $\eta \in Ax$ and $\eta = A^0x$. By the uniform convexity of X, $A^0u(t_{k'}) \to A^0x$. From the uniqueness of the limit (3.7) follows. Integrating (3.6) over $(0, t)$ we obtain

$$\left| \frac{u(t) - x}{t} + A_0x \right| \leqq \frac{1}{t} \int_0^t |A^0u(t) - A^0x| \, dt. \tag{3.8}$$

Letting $t \to 0$ and using (3.5) we conclude that $d^+u/dt(0)$ exists and that $d^+u(0)/dt + A^0x = 0$. Since we could start with any $u(t) \in D(A)$ as x, the proof is concluded.

COROLLARY 3.2. *Let X and X^* be uniformly convex. Let A be closed accretive and satisfying*

$$R(I + \lambda A) \supset D(A) \quad \textit{for all} \quad \lambda > 0. \tag{3.9}$$

Then for every $x \in D(A)$ we have the same conclusion as in Corollary 3.1.

PROOF. Since A is closed and accretive it is easy to see that (3.9) implies

$$R(I + \lambda A) \supset \overline{D(A)} \text{ for all } \lambda > 0.$$

Let $\tilde{A}$ be the strong-weak closure of A, i.e. the smallest demiclosed extension of A. Clearly $D(A) \subset D(\tilde{A}) \subset \overline{D(A)}$ and

$$R(I + \lambda\tilde{A}) \supset D(\tilde{A}) \text{ for all } \lambda > 0.$$

This implies that $\tilde{A}$ satisfies Condition I and since $\tilde{A}$ is obviously accretive Theorem 3.1 shows that the initial value problem

$$\begin{cases} \dfrac{d^+u}{dt} + (\tilde{A})^0u = 0, & t \geqq 0 \\ u(0) = x \end{cases}$$

has a solution.

Next we prove that $D(\tilde{A}) = D(A)$ and $(\tilde{A})^0 = A^0$. Let $x \in D(\tilde{A})$; by the assumption (3.9) there exists $[x_\lambda, y_\lambda] \in A \subset \tilde{A}$ such that

$$x = x_\lambda + \lambda y_\lambda, \qquad \lambda > 0.$$

Since $x \in D(\tilde{A})$, $x_\lambda \to x$ and $y_\lambda \to x$ and $y_\lambda = \tilde{A}_\lambda x \to A^0x$ as $\lambda \to 0$. From the closedness of A we deduce that $x \in D(A)$ and $\tilde{A}^0x \in Ax$. Therefore $D(\tilde{A}) = D(A)$ and for every $x \in D(A)$ Ax has an element norm $A^0x = (\tilde{A})^0x$. This concludes the proof of Corollary 3.2.

REMARK. If we do not assume X is uniformly convex in Corollary 3.2 it is not clear whether $D(\tilde{A}) = D(A)$. However one can still prove using Lemma 3.3 that for every $x \in D(A)$ there exists a unique function u on $[0, +\infty)$ which is lipschitz continuous such that $u(t) \in D(A)$ *a.e.* on $(0, +\infty)$ and satisfying

$$\begin{cases} \dfrac{du}{dt} + Au \ni 0 \text{ a.e. on } (0, +\infty) \\ u(0) = x \end{cases}$$

COROLLARY 3.3. *Let X^* be uniformly convex and let A be demiclosed (respectively closed) and accretive. Let C be a closed convex set, $\overline{D(A)} \subset C$, satisfying*

(3.10) *for every $x \in D(A)$ (respectively $x \in \overline{D(A)}$) there exist a neighborhood $U_x(= B(x, \rho(x)))$ of x and a sequence $\varepsilon_n \downarrow 0$ such that*

$$R(I + \varepsilon_n A) \supset C \cap U_x.$$

Then $R(I + \lambda A) \supset C$ for all $\lambda > 0$.

PROOF. We assume first that A is demiclosed and satisfies (3.10) for every $x \in D(A)$. Let $z \in C$ and $\lambda > 0$. We define the set B by $D(B) = D(A)$ and $Bu = u + \lambda Au - z$. B is accretive, demiclosed and satisfies condition I. Indeed if $x \in D(B)$ and $y \in D(B) \cap B(x, \rho(x)/2)$ equation

$$u + \alpha_n Bu \ni y, \quad \alpha_n = \frac{\varepsilon_n}{\lambda - \varepsilon_n} \tag{3.11}$$

has a solution for n large enough since it can be written as

$$u + \varepsilon_n Au \ni \left(1 - \frac{\varepsilon_n}{\lambda}\right) y + \frac{\varepsilon_n}{\lambda} z.$$

Thus the initial value problem

$$\begin{cases} \dfrac{du}{dt} + u + \lambda Au - z \ni 0 \ \text{ a.e. on } (0, +\infty) \\ u(0) = u_0 \in D(A) \end{cases}$$

has a solution by Theorem 3.1. In addition a standard argument shows that

$$\left|\frac{du}{dt}\right| \leqq e^{-t} \| z - \lambda Au_0 - u_0 \| \ \text{ a.e. on } (0, +\infty).$$

Hence $\lim_{t \to +\infty} u(t) = l$ exists and satisfies

$$l + \lambda Al - z \ni 0.$$

For the case where A is closed, but (3.10) holds for every $x \in \overline{D(A)}$, we consider the strong-weak closure $\tilde{A}$ of A and we define $\tilde{B}$ by $D(\tilde{B}) = D(\tilde{A})$, $\tilde{B}u = u + \lambda \tilde{A}u - z$. Clearly $\tilde{B}$ is accretive demiclosed and satisfies condition I. Thus the initial value problem

$$\begin{cases} \dfrac{du}{dt} + \tilde{B}u \ni 0 \ \text{ a.e. on } (0, +\infty) \\ u(0) = u_0 \in D(\tilde{B}) \end{cases}$$

has a solution by Theorem 3.1. By Lemma 3.3 $u(t) \in D(B)$ a.e. on $(0, +\infty)$ and

$$\frac{du}{dt} + Bu \ni 0 \text{ a.e. on } (0, +\infty)$$

i.e.

$$u(t) \in D(A) \text{ a.e. on } (0, +\infty)$$

and

$$\frac{du}{dt} + u + \lambda Au - z \ni 0 \text{ a.e. on } (0, +\infty).$$

The proof is concluded as in the previous case.

REMARKS. 1. Theorem 3.1 extends some results of T. Kato [6] and F. Browder [2] who obtained essentially the same conclusion assuming A is m-accretive (or locally m-accretive). Their technique which consists of solving the equation

$$\text{(3.12)} \qquad \begin{cases} \dfrac{du_\lambda}{dt} = A_\lambda u_\lambda = 0, & t \geqq 0 \\ u_\lambda(0) = 0 \end{cases}$$

and passing to the limit as $\lambda \to 0$ cannot be applied under condition I since it is not clear whether or not equation (3.12) has a solution at all.

J. Mermin [8] has used, for single valued m-accretive operators, a techniqne similar to the method we used in Section 3.

2. For a general Banach space X we do not know any existence result analogous to Theorem 3.1 unless we make further assumption on A. For example if A is accretive closed, locally uniformly continuous on $D(A)$ and satisfies Condition II then the initial value problem (1.3) has a C^1-solution for every $x \in D(A)$. Also if A is m-accretive everywhere defined and continuous the initial value problem (1.3) has a C^1-solution for every $x \in X$ (see Webb [10]).

3. Assumption (3.9) is clearly stronger than Condition I but is weaker than Condition II. If X is uniformly convex, Condition II implies that $\overline{D(A)}$ is convex which is not the case for condition (3.7). Indeed let

$$D = \{x \in \overline{\operatorname{conv} D(A)}: \quad J_\lambda x \to x \text{ as } \lambda \to 0\}.$$

Since D is closed and $\overline{D(A)} \subset D$ it is sufficient to show that D is convex. Let $x, y \in D$; we have

$$\left| J_\lambda\left(\frac{x+y}{2}\right) - J_\lambda x \right| \leqq \left|\frac{x-y}{2}\right|$$

$$\left| J_\lambda\left(\frac{x+y}{2}\right) - J_\lambda y \right| \leqq \left|\frac{x-y}{2}\right|.$$

Choosing a sequence $\lambda_n \to 0$ such that $J_{\lambda_n}((x+y)/2) \rightharpoonup \eta$ we obtain $|\eta - x| \leqq |(x-y)/2|$ and $|\eta - y| \leqq |(x-y)/2|$. Thus $\eta = (x+y)/2$ and $J_\lambda((x+y)/2) \rightharpoonup (x+y)/2$ as $\lambda \to 0$ by the uniqueness of the limit. Moreover,

$$\limsup_{\lambda \to 0} \left| J_\lambda\left(\frac{x+y}{2}\right) - J_\lambda x \right| \leqq \left|\frac{x-y}{2}\right|$$

and consequently

$$J_\lambda\left(\frac{x+y}{2}\right) - J_\lambda x \to \frac{y-x}{2} \quad \text{as } \lambda \to 0.$$

So $(x+y)/2 \in D$. (This argument is due to M. Crandall.)

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The University of Paris
and
The Hebrew University of Jerusalem