ABELIAN QUOTIENTS
OF THE TEICHMÜLLER MODULAR GROUP*

By

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Let D be the Teichmüller space of genus g: it is a $3g$-3-dimensional complex analytic manifold isomorphic to a bounded and contractible domain of holomorphy in $\mathbb{C}^{3g-3}$. Let Γ be the Teichmüller modular group: it is a discrete group acting discontinuously on D. This is the analytic side of the moduli problem.

On the other hand, let $M = D/\Gamma$: this analytic space has a canonical structure of algebraic variety. In fact, it turns out to be a Zariski-open subset of a projective variety: a so-called *quasi-projective* variety. Therefore, for all subgroups $\Gamma' \subset \Gamma$ of finite index, the analytic spaces $M(\Gamma') = D/\Gamma'$, being coverings of M, are also quasi-projective algebraic varieties. This follows from the

Generalized Riemann existence theorem.** *If X is any normal algebraic variety, Y any normal analytic space, and $f: Y \to X$ is a proper holomorphic map with finite fibres, and if there is a Zariski-open set $U \subset X$ such that $f^{-1}(U)$ is dense in Y and* res $f: f^{-1}(U) \to U$ *is unramified, then Y has one and only one structure of algebraic variety making f into a morphism.*

Thus from the algebraic standpoint, one has an inverse system of quasi-projective varieties:

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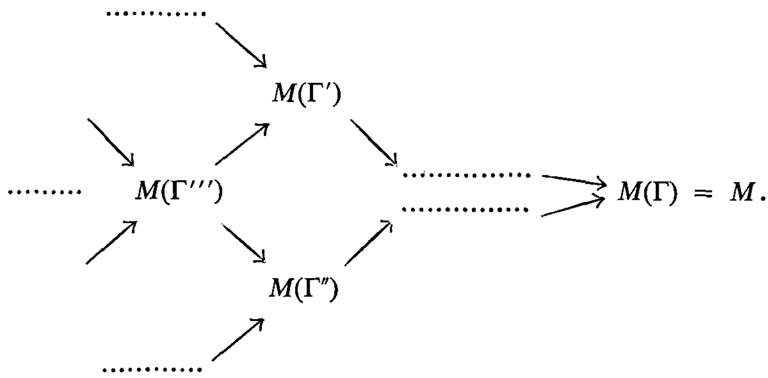
** In this form, the theorem is due to J. P. Serre and M. Artin. It can be reduced to the comparison theorem of GAGA [9], using either the paper of Grauert-Remmert [2], or resolution of singularities and elementary arguments; or, by methods of Artin and Grothendieck it can be reduced to the 1-dimensional case where it is classical.



Moreover all sufficiently small $\Gamma' \subset \Gamma$ of finite index act freely on D, hence all $M(\Gamma')$ sufficiently "far up" in this inverse system are non-singular varieties, with D as their common universal covering space. The purpose of this note is to prove two closely related results:

Theorem 1. *If* $[\Gamma,\Gamma]$ *is the commutator subgroup of* Γ, *then* $\Gamma/[\Gamma,\Gamma]$ *is a finite cyclic group, whose order divides* 10. (?!)

Theorem 2. *The Albanese variety of M is trivial, i.e. there are no non-trivial rational maps from M to an abelian variety.**

Moreover, in the terminology of [6], these results are also equivalent to:

Theorem. *The Picard group of the moduli problem is a finitely generated abelian group isomorphic to* $H^2(\Gamma,\mathbf{Z})$.

(Cf. §7 of [6]). Analogy with the many calculations that have been made for arithmetic groups acting on symmetric domains, as well as the general feeling that M should be quantitatively similar to projective space, leads me to conjecture that the rank of $H^2(\Gamma,\mathbf{Z})$ is *one.*

I want to thank Professor Magnus for a very informative letter acquainting me with the literature on the Teichmüller modular group and explaining what seems to be really proven, and what is not too clear. I also want to

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* It is equivalent to consider morphisms or rational maps from M to abelian varieties 0, since M is locally a quotient of a non-singular variety by a finite group.

thank Professors Maskit and Grothendieck for helpful comments and for checking my computations.

§1. The connection between Theorem 1, 2.

Recall that if V is any algebraic variety, and $x \in V$ is a base point, then among all morphisms:

$$\phi : V \to A$$

$$\phi(x) = 0$$

where A is an abelian variety, O its origin, there is a "universal" one, i.e. one such morphism:

$$\phi_0 : V \to A_0,$$

such that if $\psi : V \to B$ is any other, there is a unique morphism $\chi : A_0 \to B$ (which is necessarily a homomorphism) such that $\psi = \chi \circ \phi_0$. A is called the *Albanese* variety of V. We want to describe the Albanese variety of M.

Theorem 3. *Let V be a non-singular projective variety and let*

$$f : V \to V_0$$

be a birational morphism of V onto a normal projective variety V_0. Let $Z \subset V$ be a Zariski closed subset such that

$$\operatorname{codim} f(Z) \geqq 2.$$

Then

$$H_1(V - Z, \mathbf{Q}) \xrightarrow{\sim} H_1(V, \mathbf{Q}).$$

Proof. Since $H_1(V - Z, \mathbf{Z})$ and $H_1(V, \mathbf{Z})$ are finitely generated abelian groups, it suffices to show that for almost all primes p,

$$H_1(V - Z, \mathbf{Z}/p\mathbf{Z}) \xrightarrow{\sim} H_1(V, \mathbf{Z}/p\mathbf{Z}).$$

But, for any reasonable topological space X,

$$H_1(X, \mathbf{Z}/p\mathbf{Z}) \cong \pi_1(X)/\pi_1(X)^p \cdot [\pi_1(X), \pi_1(X)].$$

Therefore the isomorphism we need amounts to showing:

(a) all p-cyclic unramified coverings of $V - Z$ extend to p-cyclic unramified coverings of V

(b) a p-cyclic unramified covering of V that splits over $V - Z$ (i.e., restricted to $V - Z$, it is isomorphic to the disjoint union of p copies of $V - Z$), also splits over V.

However, p-cyclic unramified topological coverings of V and $V - Z$ are canonically algebraic varieties in view of the generalized Riemann Existence Theorem already quoted. Therefore if V is normal, connected coverings of V are obtained by taking the normalization of V in suitable finite extension fields L of the function field $\mathbb{C}(V)$. But such coverings do not split over *any* Zariski-open subset of V. Therefore (b) holds for all p.

To prove (a), we need a preliminary step:

Lemma (Matsumura). *If $D_1, \cdots, D_n$ are the components of Z of codimension 1, then the fundamental classes of the D's are independent in $H^2(V, \mathbf{Z})$.*

Proof. Suppose there was a relation

$$\sum_{i=1}^{n} n_i \cdot \text{class}(D_i) = 0.$$

Assume $n_i \neq 0$, if $1 \leqq i \leqq k$, and $n_i = 0$, $i > k$. Let $Z^* = f(D_1 \cup \cdots \cup D_k)$. Then so long as $\dim(Z^*) > 0$, cutting V_0 by a sufficiently general hyperplane H, we can replace V_0 by $V_0 \cap H$, V by $f^{-1}(V_0 \cap H)$, and Z by $f^{-1}(f(Z) \cap H)$, and obtain the same pathology with a V of lower dimension. If $\dim Z^* = 0$, then so long as $\dim(V) \geqq 3$ we can cut V by a sufficiently general hyperplane H, replace V by $V \cap H$, V_0 by the normalization of $f(V \cap H)$ and Z by the exceptional locus of

$$\text{res} f\colon V \cap H \to f(V \cap H)$$

(which still includes $D_1 \cap H, \cdots, D_k \cap H$ since they are collapsed to point in V_0). Finally, we obtain a situation contradicting the lemma, with $\dim V = \dim V_0 = 2$. Then by [7], p. 6, the intersection matrix (D_i, D_j) is negative definite, hence the D's must be homologically independent. Q.E.D.

To complete the proof of (a), we use Kummer Theory. If

$$\pi: W' \to V - Z$$

is an unramified algebraic p-cyclic extension, it is obtained by normalizing $V - Z$ in the extension of $\mathbb{C}(V)$ given by $\sqrt[p]{f}$, some $f \in \mathbb{C}(V)$. Define W to be the normalization of V in this extension field. Let $D_1, \cdots, D_n$ be the divisor components of Z, and let

$$(f) = \sum_{i=1}^{n} n_i D_i + \sum_{i=1}^{m} m_i E_i .$$

Since $\sqrt[p]{f}$ defines an unramified covering outside of Z, it follows that $p \mid m_i$ all i. Therefore in $H^2(V, \mathbf{Z})$,

$$0 = \text{class}((f)) = \Sigma n_i \cdot \text{class}(D_i) + p \cdot \text{class}\left(\Sigma \frac{m_i}{p} E_i\right).$$

But since by the lemma, class (D_i), $1 \leqq i \leqq n$, in $H^2(V, \mathbf{Z})$ are independent, for almost all primes p this relation implies that p divides all the n_i as well. In that case

$$(f) = p \cdot D$$

for some divisor D on V, and $\sqrt[p]{f}$ defines an everywhere unramified covering of V, i.e. W/V is unramified. Q.E.D.

Corollary 1. *Let V be a non-singular algebraic variety. Assume that V is isomorphic to $V^* - Z^*$ where V^* is a projective variety and* $\operatorname{codim} Z^* \geqq 2$. *Let*

$$\phi: V \to A$$

be the Albanese morphism of V. Then

$$\phi_*: H_1(V,Q) \xrightarrow{\sim} H_1(A,Q).$$

Proof. Let V' be a desingularization of V^* obtained by blowing up only points of Z^*, and let

$$f: V' \to V$$

be the birational morphism. Let $Z' = f^{-1}(Z^*)$. Then $V' - Z' \cong V$. The morphism ϕ extends to a morphism ϕ':

$$\begin{array}{ccc} V' & & \\ \cup & \overset{\phi'}{\searrow} & \\ V' - Z' & & \\ \wr\| & & \\ V & \xrightarrow[\phi]{} & A \end{array}$$

since V' is non-singular ([3], p. 20), and ϕ' is the Albanese morphism for V'. Then it is well-known# that

$$\phi': H_1(V',Q) \xrightarrow{\sim} H_1(A,Q).$$

This, plus the Theorem applied to $f: V' \to V$, and Z', imply the Corollary. Q.E.D.

This Corollary does not apply directly to the moduli variety M since M is not non-singular. However, if $g \geqq 3$ M does meet the other condition of the Corollary. In fact, let M_A denote the moduli variety for g-dimensional principally polarized abelian varieties, and let M_A^* denote the Satake compactification of M_A. Let

$$\theta: M \to M_A$$

This is proven, for example by constructing A as

$$H_1(V',\mathbb{R})/\text{Image } [H_1(V', \boldsymbol{Z})]$$

and constructing ϕ' by integrating the differentials $H^0(V', \Omega^1_{V'})$ cf. [10], p. 82.

be the morphism induced by associating the Jacobian variety and its Θ-polarization to each curve of genus g. In [8], A. Mayer and I determined the set $\overline{\theta(M)} - \theta(M)$ (closure in M_A^*). It turns out to be what you expect — g-dimensional products of lower dimensional jacobians in M_A itself, as Matsusaka and Hoyt had shown; h-dimensional products of lower dimensional jacobians $(h < g)$, in $M_A^* - M_A$. In particular, for each type of product decomposition we get a locally closed subset of $\overline{\theta(M)} - \theta(M)$. There is no piece of dimension $3g-4$ (codimension 1). The only piece of dimension $3g-5$ is the locus corresponding to products

$$J_1 \times J_{g-1}$$

J_1, J_{g-1} being 1 and $(g-1)$-dimensional jacobians respectively. The other pieces, such as:

a) $J_i \times J_j$, $i+j=g$, $1 < i, j < g$

b) J_{g-1} (giving a point in $(M_A^* - M_A) \cap \overline{\theta(M)})$ have codimension at least 3.

To make the Corollary apply, choose a normal subgroup $\Gamma' \subset \Gamma$ of finite index that acts freely on the Teichmüller space D. Let $M' = D/\Gamma'$ and let $\pi: M' \to M$ be the canonical morphism. Then M' is non-singular *and* compactifiable in codimension 2 (take the normalization of $\overline{\theta(M)}$ in the function field of M'). Therefore, we get

$$(\Gamma'/[\Gamma',\Gamma']) \otimes \boldsymbol{Q} \cong H_1(M',\boldsymbol{Q}) \cong H_1(A',\boldsymbol{Q})$$

if $\phi': M' \to A'$ is the Albanese morphism. Furthermore, the finite group (Γ/Γ') acts on $\Gamma'/[\Gamma',\Gamma'] \otimes \boldsymbol{Q}$, on M', and hence on A'.

(I). The canonical homomorphim

$$(\Gamma'/\Gamma',\Gamma']) \otimes \boldsymbol{Q} \to (\Gamma/[\Gamma,\Gamma]) \otimes \boldsymbol{Q},$$

is surjective and its kernel is generated by elements

$$x - x^{\gamma},\ \gamma \in \Gamma/\Gamma',\ x \in (\Gamma'/[\Gamma',\Gamma']) \otimes \boldsymbol{Q}.$$

(This is well-known; for example, use the Lyndon Spectral Sequence, [4], p. 354, formula (10.6)).

(II). If $\phi: M \to A$ is the Albanese of M, there is a canonical diagram:

$$\begin{array}{ccc} M' & \xrightarrow{\phi'} & A' \\ \pi \downarrow & & \downarrow \chi \\ M & \xrightarrow[\phi]{} & A \end{array}$$

for some homomorphism χ. χ: $A' \to A$ can be characterized as the universal homomorphism $\psi: A' \to B$, B abelian, such that $\psi(x^\gamma) = \psi(x)$, all $x \in A'$, $\gamma \in \Gamma/\Gamma'$. Therefore

$$A \cong A'/[\text{Subgroup generated by points } x^\gamma - x,\, x \in A',\, \gamma \in \Gamma/\Gamma'].$$

Using this, it is easy to see that

$$H_1(A,Q) \cong H_1(A',Q)/[\text{Subgroup generated by points } x^\gamma - x,\, x \in H_1, \gamma \in \Gamma/\Gamma'].$$

Putting all this together, we finally conclude:

Corollary 2. *If $\phi: M \to A$ is the Albanese morphism of the moduli variety M and $g \geqq 3$, then there is a canonical isomorphism*:

$$(\Gamma/[\Gamma,\Gamma]) \otimes Q \xrightarrow{\sim} H_1(A,Q).$$

§2. Dehn's presentation of Γ.

Everything that follows depends on the fundamental paper [1] of Dehn. Let F be a fixed differentiable surface of genus g. We shall picture F as follows:

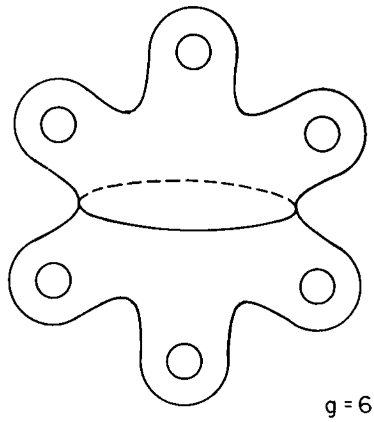


Figure 1

The first basic fact is that:

$$\Gamma \cong \frac{\{\text{Group of orientation preserving homeomorphisms of } F\}}{\{\text{Those homotopic to identity}\}}$$

$$\cong \frac{\{\text{Group of automorphisms of } \pi_1(F,p),\ \det = +1\}}{\{\text{Inner Automorphisms}\}}.$$

Here the determinant of an automorphism $\alpha:\pi_1(F)\to\pi_1(F)$ refers to the determinant of the induced map on $\pi_1/[\pi_1,\pi_1]$, which is a lattice of rank $2g$. To obtain generators of Γ, Dehn introduced particular homeomorphism classes of F into itself that he called "*Schraubungen*" — we shall call them "*screw maps*". Let γ be a simple closed curve on F: Draw a small collar around γ, i.e. choose a continuous injective orientation-preserving map $f:[0,1]\times S^1\to F$ such that $\gamma = f[(\frac{1}{2})\times S^1]$. The screw map S_γ associated to γ is defined by

$$\begin{cases} S_\gamma(x) = x & \text{if } x\notin \operatorname{Im}(f). \\ S_\gamma(f(\alpha,\theta)) = f(\alpha,\theta+2\pi\alpha). \end{cases}$$

In other words, if γ_α is the curve $f[(\alpha)\times S^1]$, we rotate γ_α through an angle $2\pi\alpha$, ranging from 0 on γ_0, to π on γ itself, to 0 again on γ_1*. The main theorem

* Note that this screw map depends, modulo homotopy, only on γ and the orientation of F, and *not* on an orientation for γ.

of his paper is that Γ is generated by a finite number of these screw maps (p. 203). To get more precise information about the group Γ — we should have called it Γ_g, where $g =$ genus — it is not enough to refer to the groups Γ_h, where $h < g$. More generally, we have to look at an oriented surface $F_{g,n}$ of genus g *with* n *holes*. For all g, n the appropriate Teichmüller modular group $\Gamma_{g,n}$ is:

$$\Gamma_{g,n} = \left\{ \frac{\text{Group of homeomorphisms of } F_{g,n} \text{ leaving the } n \text{ boundary curves pointwise fixed}}{\text{Those homotopic to identity, leaving the boundary pointwise fixed along the way}} \right\}.$$

Note that whenever we choose a continuous, injective orientation preserving map $f: F_{h,n} \to F$, we obtain a map $f_*: \Gamma_{h,n} \to \Gamma$, since homeomorphisms of $F_{h,n}$ extend uniquely to homeomorphisms of F which are the identity on $F - f(F_{h,n})$.

Dehn's main result is that $\Gamma_{g,n}$ is generated by the screw maps along the set of curves β_k and γ_{ij}:

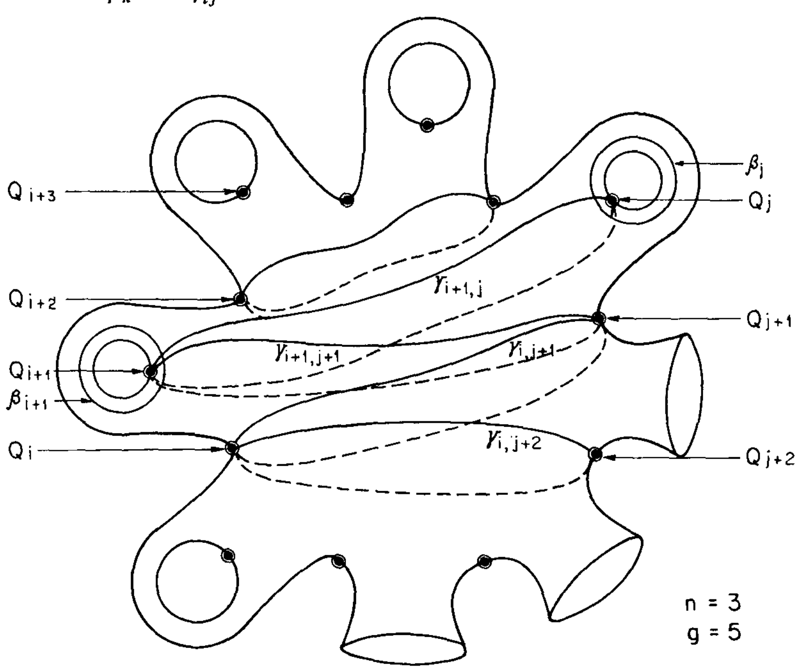


Figure 2

This result can easily be improved. On our surface F, consider the following piece homeomorphic to $F_{2,1}$:

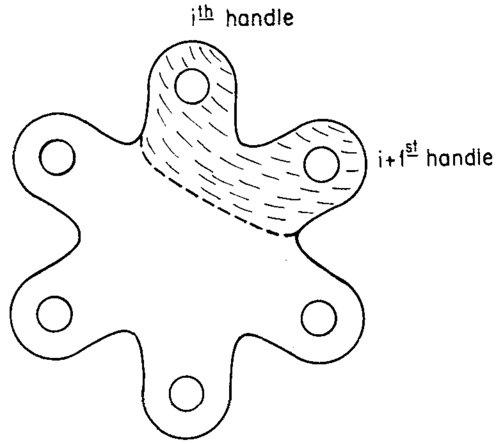


Figure 3

For all i, $1 \leqq i \leqq g$, this defines $f_{i,*}: \Gamma_{2,1} \to \Gamma$.

Theorem 4. *Γ is generated by the images of $\Gamma_{2,1}$ under $f_{1,*}, \cdots, f_{g,*}$.*

Before beginning to prove this, let me list what is known about the groups $\Gamma_{g,n}$ for low g and n:

(a) $\Gamma_{0,0} = \Gamma_{0,1} = (e)$

(b) $\Gamma_{0,2} = \boldsymbol{Z}$: $F_{0,2}$ is just a collar, and $\Gamma_{0,2}$ is a free group on the screw map along its midline.

(c) $\Gamma_{0,3} = \boldsymbol{Z} \oplus \boldsymbol{Z} \oplus \boldsymbol{Z}$, with generators given by screw maps along curves which are the 3 boundary curves, pulled slightly in:

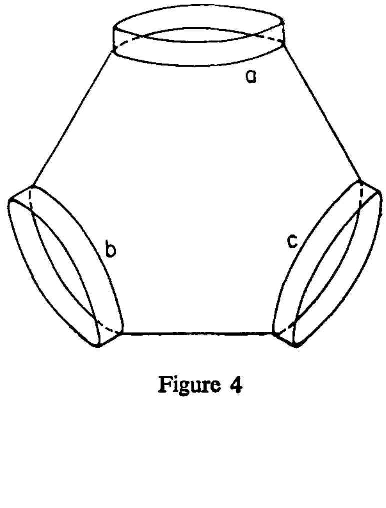


Figure 4

(d) $\Gamma_{1,1}$ = free group on screw maps Δ_a, Δ_b along a, b, modulo the relation

$$\Delta_b\Delta_a\Delta_b\Delta_a^{-1}\Delta_b^{-1}\Delta_a^{-1} = e$$

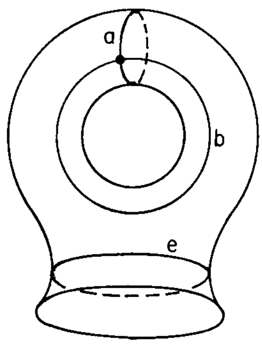


Figure 5

Moreover, $\Delta_e = (\Delta_a\Delta_b\Delta_a)^4$. (Dehn, pp. 156 and 172).

(e) $\Gamma_{0,4}$ is generated by screw maps Δ_a, Δ_b, Δ_{r_1}, Δ_{r_2}, Δ_{r_3}, Δ_{r_4} where the Δ_{r_i}'s are in the center. It appears that there are no other relations, i.e. $\Gamma_{0,4} \cong \mathbf{Z}^4 \oplus$ (free group on 2 elements).

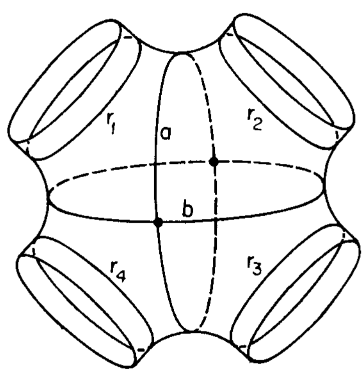


Figure 6

(f) $\Gamma_{1,2}$ is generated by screw maps Δ_a, Δ_b, Δ_c, Δ_e and Δ_f:

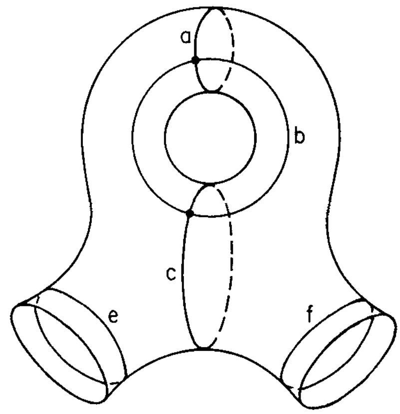


Figure 7

(Dehn, p. 188). The relations among these were worked out by Magnus [5], in a somewhat different form. To begin with, one has the obvious relations:

$$\Delta_e, \Delta_f \in \text{Center}$$
$$\Delta_a\Delta_c = \Delta_c\Delta_a.$$

Much subtler is the following*

$$\Delta_e\Delta_f = (\Delta_a\Delta_b)^2\Delta_c\Delta_b(\Delta_a\Delta_b)^2\Delta_b\Delta_c\Delta_b\Delta_c^{-1}.$$

There are still further relations that we will not list. $\Gamma_{1,2}$ is the first really complicated group and it is related to Artin's Braid group.

Proof of Theorem 4. Because of (f), the mapping class group $\Gamma_{1,2}$ is generated by Δ_a, Δ_b, Δ_c and *either* Δ_e or Δ_f, whichever is convenient. We use this to prove:

Lemma. *$\Gamma_{2,2}$ is generated by*

(a) $\Gamma_{2,1}$

(b) Δ_{c_1} *or* Δ_{c_2}, *whichever you want*

(c) Δ_{e_1} *or* Δ_{e_2} *whichever you want* (cf. Figure 8).

* This can be readily derived from the formula

$$[\Theta^{-1}u\Theta,\ \Theta^{-1}a\Theta,\ \Theta^{-1}b\Theta] \doteq s^{-4}\rho\tau^{-1}\rho^{-1}\tau$$

on p. 638 of Magnus' article.

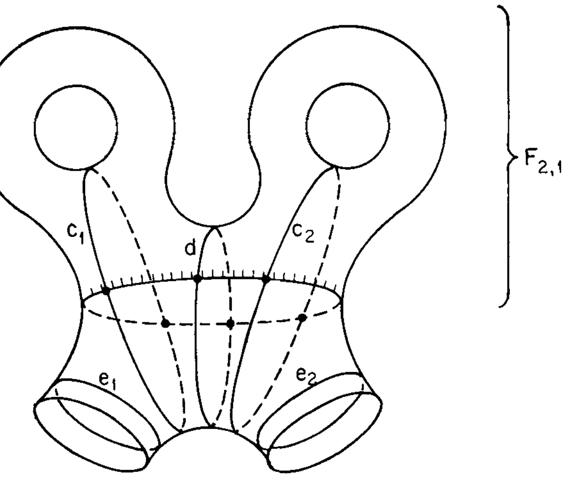
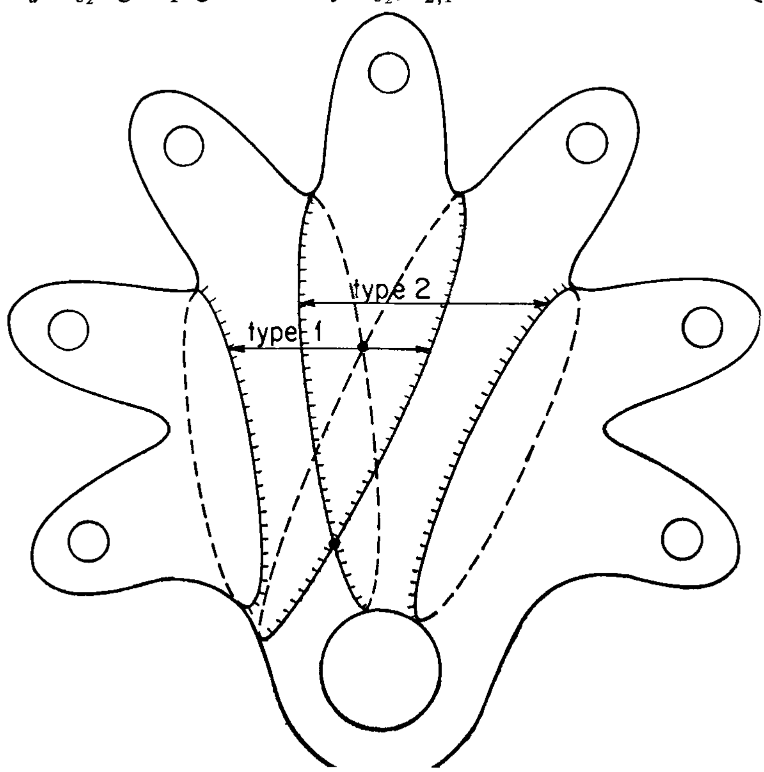


Figure 8

Proof. According to Dehn's result, $\Gamma_{2,2}$ is generated by $\Gamma_{2,1}$, Δ_{c_1}, Δ_{c2}, Δ_d, Δ_{e_1} and Δ_{e_2}. But first of all Δ_{c_1} and Δ_{c_2} are conjugate with respect to a suitable element of $\Gamma_{2,1}$ (Dehn, §10, (1), p. 200; it is not hard to work this case out explicitly). Secondly, suppose you cut the surface open along d: then $F_{2,2} = F'_{1,2} \cup F''_{12}$. Applying the results of (f) to each piece, it follows that

1) $\Delta_{e_1} \cdot \Delta_d \in$ group generated by $\Delta_{c_1}, \Gamma_{2,1}$
2) $\Delta_d \cdot \Delta_{e_2} \in$ group generated by $\Delta_{c_2}, \Gamma_{2,1}$. Q.E.D.



Apply this lemma to the $\Gamma_{2,2}$'s of types 1, 2 in Figure 9. It follows that any of the screw maps $\Delta_{\gamma_{ij}}$ of Fig. 2 can be written successively as products involving various elements of the $f_{k,*}(\Gamma_{2,1})$'s and screw maps $\Delta_{\gamma_{i',j'}}$ where $|i'-j'|<|i-j|$. Loosely speaking, modulo terms in the $f_{k,*}(\Gamma_{2,1})$'s, one replace $\Delta_{\gamma_{ij}}$ by $\Delta_{\gamma_{i'j'}}$'s with $\gamma_{i',j'}$ wrapping round a smaller part of F. This evetually eliminates the Δ_γ's altogether. Q.E.D.

Corollary 1. *Γ is generated by the four homeomorphisms Δ_a, Δ_b, Δ_d and R* (cf. Fig. 10).

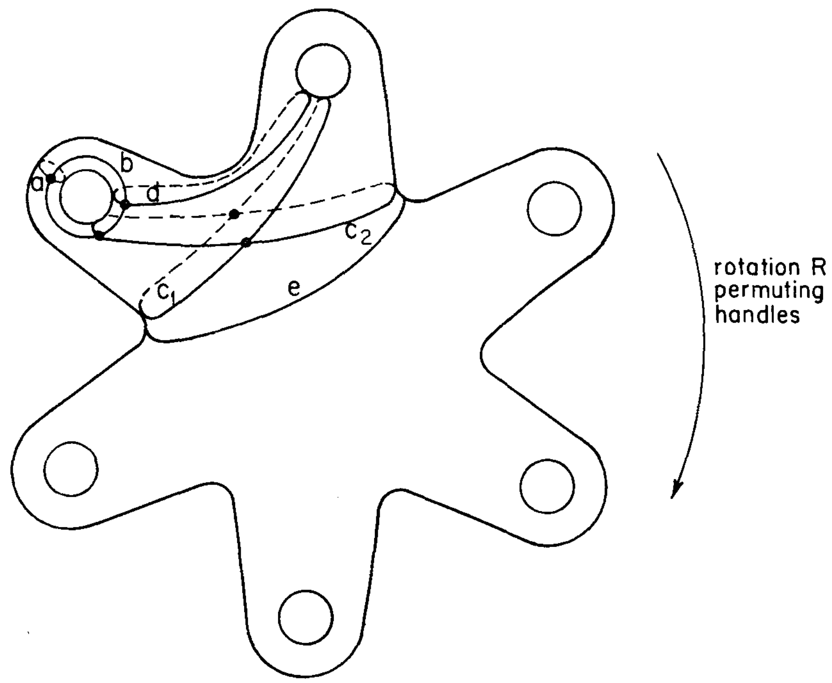
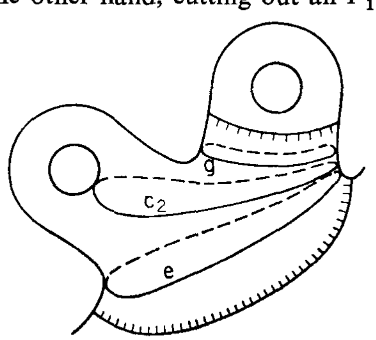


Figure 10

Proof. According to the Theorem, R plus Δ_a, Δ_b, Δ_{c_1}, Δ_{c_2}, Δ_d and Δ_e certainly suffice. On the other hand, cutting out an $F_{1,2}$ as follows:



it follows from (f) that Δ_e can be expressed as a product of the elements Δ_a, Δ_b, Δ_{c_2} and Δ_g; and Δ_g is a product of the elements $R\Delta_a R^{-1}$, $R\Delta_b R^{-1}$. This gets rid of Δ_e. Secondly, cutting out an $F_{1,2}$ like this:

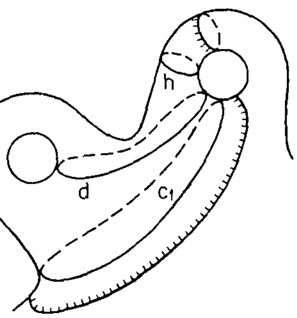


Figure 12

it follows that Δ_{c_1} can be expressed as a product of the elements Δ_a, Δ_b, Δ_d and Δ_h. But $\Delta_h = R\Delta_a R^{-1}$. Q.E.D.

Finally, if a_1, a_2 are any 2 simple closed curves on F neither of which divides F into 2 components, it is clear that $a_1 = T(a_2)$ for some orientation preserving homeomorphism T of F (i.e. since there must be an orientation preserving homeomorphism of $F - a_1$ and $F - a_2$). Therefore Δ_{a_1} and Δ_{a_2} are always conjugate in Γ in this case.

Corollary 2. *Γ is generated by Δ_a and its conjugates* (cf. Figure 10).

Proof. In fact, Δ_a, Δ_b, Δ_d all lie in the same conjugacy class by the remark just made, since none of the curves a, b, d decomposes F. And looking at the proof of Corollary 1, we see that we generated Γ by conjugates of Δ_a, Δ_b, Δ_d with respect to powers of R. Q.E.D.

Therefore, $\Gamma/[\Gamma,\Gamma]$ has one generator, the image of the conjugacy class δ of screw maps with respect to non-decomposing simple closed curves. Now refer back to the relation in (f) on $\Gamma_{1,2}$. We can embed $F_{1,2}$ in any surface F of genus $g (g \geqq 2)$ so that a, b, c, e, f go into *non-decomposing* curves. Then Δ_a, Δ_b, Δ_c, Δ_e and Δ_f all have image δ in $\Gamma/[\Gamma,\Gamma]$, and the relation says that $10\delta = 0$. This completes the proof of Theorem 1.

Theorem 2 follows from Theorem 1 by the results of §1 if $g \geqq 3$. If $g = 2$, Theorem 1 is known from the results of Igusa [11].

Actually, when $g = 2$, it is not hard to check that $\Gamma/[\Gamma,\Gamma]$ is exactly $\boldsymbol{Z}/10\boldsymbol{Z}$. Closely related to this is the fact that the Picard group $\mathrm{Pic}(\mathcal{M})$ of the moduli problem of genus 2 is also $\boldsymbol{Z}/10\boldsymbol{Z}$.

Proof. The methods in [6], §7 carry over without modification to prove that

$$\{\text{Torsion subgroup of } \mathrm{Pic}(\mathcal{M})\} \cong \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], \boldsymbol{Q}/\boldsymbol{Z}).$$

(cf. 1st Corollary, p. 77; Corollary, p. 81, [6]). On the other hand, if M is the moduli space of genus 2, then, *modulo torsion*, $\mathrm{Pic}(M)$ and $\mathrm{Pic}(\mathcal{M})$ are isomorphic. Igusa showed ([11], p. 638) that there is a finite morphism $\pi: \boldsymbol{A}^3 \to M$. So if $d = \mathrm{degree}(\pi)$, and D is any Cartier divisor on M, then

$$\begin{aligned} d.D &= \pi(\pi^{-1}(D)) \\ &= \pi((f)), \text{ some } f \in \Gamma(A^3, o_A) \\ &= (Nmf). \end{aligned}$$

Therefore $\mathrm{Pic}(M)$ is torsion and $\mathrm{Pic}(\mathcal{M})$ and $\Gamma/[\Gamma,\Gamma]$ are dual finite abelian groups. But we can easily check that 2 and 5 divide the order of $\mathrm{Pic}(\mathcal{M})$. To do this, use the element $\delta \in \mathrm{Pic}(\mathcal{M})$ gotten by attaching to each curve C of genus 2 the one-dimensional vector space:

$$\delta(C) = \Lambda^2[H^0(C, \Omega_C^1)]$$

(Ω^1 = sheaf of 1-forms on C; cf. [6], §5). Taking C_1 to be the curve $y^2 = x^5 - 1$, the automorphisms

$$\begin{cases} x \to \eta x \\ y \to y \\ \eta^5 = 1, \ \eta \neq 1 \end{cases}$$

of C_1 all act non-trivially on $\delta(C_1)$. Therefore $5 \mid \mathrm{order}(\delta)$. Taking C_2 to be the curve $y^2 = x \cdot (x^4 - 1)$, the automorphisms

$$x \to \eta^2 \cdot x$$

$$y \to \eta \cdot x$$

$$\eta^8 = 1, \eta^4 \neq 1$$

all act non-trivially on $\delta(C_2)$, although their squares act trivially. Therefore $2 \mid \mathrm{order}(\delta)$. Q.E.D.

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