KOEBE DOMAINS FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

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1. Introduction. Let S denote the class of functions $f(z)=z+a_2z^2+\cdots$ which are analytic and univalent in the unit disc K, and let M denote a non-empty subclass of S. Then we define the *Koebe domain* of M to be the point-set $\mathscr{K}(M)\equiv\bigcap_{f\in M} f(K)$. In this note we determine Koebe domains for some well-known subclasses of S.

Since M above may consist of elements of S chosen in an arbitrary manner no general statements concerning $\mathscr{K}(M)$ can be made apart from some trivial ones. However, important subclasses M of S satisfy the following two conditions. (i) M is compact. (ii) For each r, $0<r<1$, if $f\in M$, then $f_r(z)\equiv(1/r)f(rz)$ belongs to M. For M satisfying those two conditions, we can say that $\mathscr{K}(M)$ is a simply connected domain that is starshaped with respect to the origin such that (a) the open disc with center at the origin and radius $\frac{1}{4}$ is contained in $\mathscr{K}(M)$, and (b) $\mathscr{K}(M)\subset K$. The assertion (a) follows from the well-known Koebe $\frac{1}{4}$-theorem, and the assertion (b) follows from the fact that $f(z)\equiv z$ is a member of M because M is compact. To prove the remaining part of the assertion, we reason as follows. If $w_1\in f(K)$ for all $f\in M$, then $w_1\in f_r(K)$ for all $f\in M$ and all r, $0<r<1$. Hence $rw_1\in f(K)$ for all $f\in M$ and all r, $0<r<1$. Hence $\mathscr{K}(M)$ is starshaped and simply connected.

If S_R denotes the subclass of S all of whose Taylor coefficients are real, then $\mathscr{K}(S_R)$ is not only simply connected and starshaped with respect to the origin, but it is also symmetric with respect to the coordinate axes and its boundary $\partial\mathscr{K}(S_R)$ meets the coordinate axes at $w=\pm\frac{1}{4}$ and $w=\pm\frac{1}{2}i$. This last result is due to Jenkins [2]. In a recent note, McGregor [5] determined the Koebe domains for the following subclasses of S_R: (i) the convex functions,

(ii) the functions starlike with respect to the origin, and (iii) the functions convex in the direction of the imaginary axis.

In this note we determine the Koebe domains for the following subclasses M of S: (i) $M \equiv Y$ is the class of circularly symmetric functions with real coefficients, introduced by Jenkins [1], (ii) $M \equiv S^\alpha$ is the class of functions $f(z)$, with real coefficients, such that $[w \mid \operatorname{Im} w > 0] \cap f(K)$ is convex in the direction $e^{i\pi\alpha}$, $0 \leqq \alpha \leqq 1$, (iii) $M \equiv L_R$ is the class of close-to-convex functions, with real coefficients, (iv) $M \equiv S_1^c$ is the class of odd convex functions, with real coefficients, and (v) $M \equiv S_1^*$ is the class of odd starlike functions, with respect to the origin, with real coefficients.

2. The class Y. For the class Y of normalized univalent functions with real coefficients and circularly symmetric with respect to the positive real axis, we have the following result.

Theorem 1. *The Koebe domain $\mathscr{K}(Y)$ is a domain that is symmetric with respect to the real axis and whose boundary $\partial\mathscr{K}(Y)$ has the equation*

$$r = \tfrac{1}{4}(1 + \cos\tfrac{1}{2}\theta)^2, \qquad -\pi \leqq \theta \leqq \pi, \tag{1}$$

where (r, θ) are polar coordinates.

Proof. Since for each $f \in Y$ the set $f(K)$ is circularly symmetric with respect to the positive real axis, we can make the following statement. If $re^{i\theta} \notin f(K)$, then the arc Γ of the circle $|w| = r$ the endpoints $re^{i\theta}$, $re^{-i\theta}$, and bisected by the negative real axis is contained in the complement of $f(K)$ too. Because of symmetry, we may restrict θ to the range $0 < \theta \leqq \pi$. Since $f(K)$ is simply connected, it follows that the ray $[-\infty, -r]$ lies in the complement of $f(K)$. Now we can apply the principle of subordination, just as McGregor did [5], to conclude that for each r, there exists a minimum for θ such that the union of Γ and the ray lie in the complement of $f(K)$. If we call this minimum $\theta_0 = \theta_0(r)$, then θ_0 is uniquely determined by the condition that the inner radius of $f(K)$ at the origin is unity. The mapping function for this last domain can be given explicitly [6], and from it we obtain

$$\pi - \theta_0 = 2\arcsin(2\sqrt{r} - 1) \qquad \tfrac{1}{4} \leqq r < 1,$$

which implies that the open arc

$$\gamma_r: \qquad |\arg w| < \theta_0 = \pi - 2\arcsin(2\sqrt{r} - 1)$$

is contained in $f(K)$ for each $f \in Y$ and for each r, $\frac{1}{4} \leqq r < 1$. The Koebe domain is then seen to be the union of the set $\cup [\gamma_r | \frac{1}{4} \leqq r < 1]$ and the open disc of radius one quarter with center at the origin. Since $\theta_0(r)$ is monotonic in r, we then find (1) for $0 \leqq \theta \leqq \pi$, from the relation $\pi - \theta_0 = 2\arcsin(2\sqrt{r} - 1)$, $\frac{1}{4} \leqq r < 1$.

An elementary calculation shows that the argument of the tangent vector at the point $re^{i\theta}$ is $\frac{5}{4}\theta + \frac{\pi}{2}$. Hence it follows that the upper half of the boundary of $\mathscr{K}(Y)$ is a convex arc, with an angular point $\frac{1}{4}e^{i\pi}$ where one-sided tangents meet at right angles.

3. The subclasses S^α. We recall that the class S^α, for α fixed, $0 \leqq \alpha \leqq 1$, consists of the functions $f(z) = z + a_2 z^2 + \cdots$ which are analytic and univalent in the unit disc K, have real coefficients, and have the property that if $w_1 \notin f(K)$ and if $\operatorname{Im} w_1 \geqq 0$, then the ray $[w | \arg(w - w_1) = \pi\alpha]$ lies in the complement of $f(K)$. It follows from a result due to Lewandowski [4] that S^α is a subclass of L_R, certain close-to-convex functions we defined in §1. If $0 < \alpha < \frac{1}{2}$, then it follows from Kaplan's notion of close-to-convexity [3] that for each $f \in S^\alpha$, there exists a Carathéodory function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, with real p_k and positive real part in K, and a normalized convex function $\phi(z) = z + c_2 z^2 + \cdots$ such that $f'(z) \equiv p(z)\phi'(z)$ in K. Elementary geometric considerations show that $\phi(z)$ may be chosen to be independent of $f \in S^\alpha$: indeed, we find

$$\phi(z) \equiv \frac{1}{2(1 - 2\alpha)} \left[1 - \left(\frac{1 - z}{1 + z}\right)^{1 - 2\alpha}\right],$$

which maps K onto an angular domain with vertex at $1/[2(1 - 2\alpha)]$ and interior angle $\pi(1 - 2\alpha)$. Hence $f(z)$ has the form

$$f(z) \equiv \int_0^z p(\zeta) \frac{1}{(1 + \zeta)^2} \left(\frac{1 + \zeta}{1 - \zeta}\right)^{2\alpha} d\zeta,$$

where $p(z)$ is analytic and has a positive real part in K. It is easy to verify that this last formula is valid for the cases $\alpha = 0$ and $\alpha = \frac{1}{2}$ too.

In order to determine the Koebe domain $\mathscr{K}(S^{\alpha})$, we first determine certain curves Γ_{α}, $0 \leqq \alpha \leqq 1$. These curves are obtained by moving two symmetric points W_0 and $\bar{W}_0$ (with $\operatorname{Im} W_0 \geqq 0$) in such a way that the two symmetric and infinite rays l_0 and l_0^* that emanate from W_0 and $\bar{W}_0$ and have directions $e^{i\pi\alpha}$ and $e^{-i\pi\alpha}$ respectively, determine a simply connected domain whose boundary is the union of l_0 and l_0^* and whose inner radius with respect to the origin is unity. If the prolongations of l_0 and l_0^* toward the origin meet at the origin, then the corresponding domain is starshaped and the mapping function has the form

$$w = \sigma(z) \equiv \frac{z}{(1+z)^2}\left(\frac{1+z}{1-z}\right)^{2\alpha}, \tag{2}$$

In this particular case $W_0 = \sigma(z_0) \equiv w_0$, where z_0 satisfies $\sigma'(z_0) = 0$, with $\operatorname{Im} z_0 \geqq 0$, so that we find

$$z_0 = 1 - 2\alpha + i2\sqrt{\alpha(1-\alpha)}, \qquad w_0 = \frac{e^{i\pi\alpha}}{4\alpha^{\alpha}(1-\alpha)^{1-\alpha}}.$$

If the prolongations of l_0 and l_0^* meet at u_0 on the real axis, then the function $f \in S^{\alpha}$ that maps K onto the slit domain determined by l_0 and l_0^* has the form

$$A[f(z) - u_0] \equiv \sigma\left(\frac{z+x}{1+xz}\right),$$

where x is real, $-1 < x < 1$, A is real and satisfies

$$A = (1-x^2)\sigma'(x) \equiv \frac{\sigma(x)}{x}[(1-x)^2 + 4\alpha x] \equiv \left(\frac{1+x}{1-x}\right)^{2\alpha}\frac{(1-x)^2 + 4\alpha x}{(1+x)^2},$$

and where $u_0 = -[\sigma(x)/A]$. Here $\sigma(z)$ is the particular star function (2). Since $f(K)$ and $\sigma(K)$ can be mapped onto one another by a linear transformation of the form $A(W - u_0) = w$, it follows that the endpoint W_0 of the ray

l_0 is connected to the point w_0 by the relation $W_0 = (w_0/A) + u_0$. Hence we have the following equation for the upper half of Γ_α:

$$W_0 = W_0(x) = \left(\frac{1-x}{1+x}\right)^{2\alpha} \frac{(1+x)^2}{(1-x)^2 + 4\alpha x} \cdot \frac{1}{4} \frac{e^{i\pi\alpha}}{\alpha^\alpha(1-\alpha)^{1-\alpha}} - \frac{x}{(1-x)^2 + 4\alpha x}, \quad -1 \leqq x \leqq 1.$$

For $x = 1$ and $x = -1$ we obtain the vertices $W_0 = 1/[4(1-\alpha)]$ and $W_0 = -[1/(4\alpha)]$ of two angular domains whose interior angles are $2\pi(1-\alpha)$ and $2\pi\alpha$, respectively. Obviously $\mathscr{K}(S^\alpha)$ lies in the intersection of these two angular domains. Since the inner radius with respect to a point is a monotonic function of domains, it follows that the upper half of Γ_α has at most one point in common with each line parallel to the direction $e^{i\pi\alpha}$.

If we make the change of variable $\tau = \left(\frac{1+x}{1-x}\right)^2$ in the equation for the upper half of Γ_α, then we obtain

$$W = e^{i\pi\alpha}A(\tau) + B(\tau), \qquad 0 \leqq \tau \leqq \infty, \tag{3}$$

where

$$A(\tau) \equiv \frac{1}{4} \frac{\tau^{1-\alpha}}{1-\alpha+\alpha\tau} \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}},$$

$$B(\tau) \equiv \frac{1}{4} \frac{1-\tau}{1-\alpha+\alpha\tau}.$$

To show that the upper half of Γ_α is convex, it is sufficient to show that $\arg W'(\tau) \equiv \operatorname{Im}\log W'(\tau)$ is monotonic, that is, to show

$$\operatorname{Im}\frac{W''(\tau)}{W'(\tau)} \equiv \operatorname{Im}\frac{(e^{i\pi\alpha}A'' + B'')(e^{-i\pi\alpha}A' + B')}{|e^{i\pi\alpha}A' + B'|^2} \equiv \frac{\sin\pi\alpha(A''B' - A'B'')}{|W'|^2} > 0$$

holds. But this last is easy to show, because $A'(\tau)/B'(\tau)$ is monotonic increasing. Hence the upper half of Γ_α is convex.

For the particular case $\alpha = 0$ we see that the inner radius of the strip domain

$|\operatorname{Im} w| < \pi/4$ with respect to the origin is unity, so that Γ_0 must be contained in this strip. It is easy to see that the function

$$f_0(z) \equiv \frac{1}{4}\left[\log\frac{1+z}{1-z} + \frac{2z}{(1+z)^2}\right]$$

satisfies the identity $[f_0'(z)/\phi'(z)] \equiv (1+z^2)/(1-z^2)$, with $\phi(z) \equiv z/(1+z)$ a convex univalent function in K. Hence $f_0(z)$ is close-to-convex and univalent in K. A calculation now shows that $f_0(K)$ is the plane slit along two rays that are parallel to the positive real axis and have endpoints $\frac{1}{4} \pm i(\pi/8)$. If we use the methods of the preceding discussion, then we can show that the upper half of Γ_0 has the equation

$$(4)\quad W = \frac{\frac{1}{4} + i\frac{\pi}{8} - f_0(x)}{(1-x^2)f_0'(x)} = \frac{1}{4}\,\frac{1+\tau-\tau\log\tau+i\pi\tau}{1+\tau},\ 0 \leqq \tau = \left(\frac{1+x}{1-x}\right)^2 \leqq \infty.$$

For the case $\alpha = 1$, we can get Γ_1 by reflecting Γ_0 about the imaginary axis. The equation for Γ_1 is

$$(5)\qquad W = \frac{1}{4}\,\frac{\tau\log\tau - 1 - \tau + i\pi\tau}{1+\tau}, \qquad 0 \leqq \tau \leqq \infty.$$

The following result asserts that we have obtained the boundary of $\mathscr{K}(S^\alpha)$, $0 \leqq \alpha \leqq 1$.

Theorem 2. *The Koebe domain* $\mathscr{K}(S^\alpha)$, $0 \leqq \alpha \leqq 1$, *is a domain that is symmetric with respect to the real axis; its boundary consists of the convex arc* Γ_α *and the reflection of* Γ_α *in the real axis. Equations for* Γ_α *are given by* (3), (4) *and* (5).

Proof. Let $W_0 \in \Gamma_\alpha$ and consider the plane minus the rays from W_0 and $\bar{W}_0$ toward infinity and in the directions $e^{i\pi\alpha}$ and $e^{-i\pi\alpha}$, respectively; let $g \in S^\alpha$ be the function that maps K onto this slit domain. It follows from the definitions of Γ_α that the inner radius of this slit domain with respect to the origin is unity. Now let $f \in S^\alpha$. If the prolongations of those slits, from W_0

and $\bar{W}_0$ "toward the origin," did not lie wholly in $f(K)$, then the inner radius of $f(K)$ with respect to the origin would be less than unity; this last assertion follows from the monotonic character of the inner radius. But the inner radius of $f(K)$ with respect to the origin is unity. Hence we may conclude that the prolongations of the slits from W_0 and $\bar{W}_0$ "toward the origin" lie in $f(K)$, for each $f \in S^\alpha$. This completes the proof of the theorem.

4. The class L_R. We now determine the Koebe domain for the class of normalized close-to-convex functions $f(z) = z + a_2 z^2 + \cdots$ with real coefficients.

Since L_R is compact, it follows that to each point w_0 of $\partial\mathscr{K}(L_R)$, the boundary of $\mathscr{K}(L_R)$, there corresponds at least one $g \in L_R$ such that $w_0 \notin g(K)$. Moreover, it follows from the starshapedness of $\mathscr{K}(L_R)$ that all points on the ray from w_0 to the origin, except for w_0, belong to $f(K)$ for each $f \in (L_R)$. The intersection of $\mathscr{K}(L_R)$ with the real axis is the open segment $(-\frac{1}{4}, \frac{1}{4})$; hence we may assume $\operatorname{Im} w_0 > 0$. Now it follows from Biernacki's characterization of close-to-convex functions [4] that the complementary set of $g(K)$ contains two infinite non-intersecting rays l_0 and l_0^* with endpoints w_0 and $\bar{w}_0$ which are parallel to the directions $e^{i\pi\alpha}$ and $e^{-i\pi\alpha}$, respectively; here $0 \leqq \alpha \leqq 1$. If we let $h(z)$ denote the function that maps K conformally onto the domain bounded by $l_0 \cup l_0^*$ subject to the conditions $h(0) = 0$ and $h'(0) > 0$, then we see that $g(z)$ is subordinate to $h(z)$. If $h'(0) \equiv \lambda > 1$, then the functions $h(z)$ is not only a member of L, but $h(z)/\lambda$ does not take on the value w_0/λ. This contradicts the basic property of w_0. Hence $h(z) \equiv g(z)$. It follows that for each β, $0 \leqq \beta \leqq 1$, with $\arg w_0 = \beta\pi$, there exists a function $g_\beta \in S^\alpha \equiv S^{\alpha(\beta)}$ such that $\mathscr{K}(L_R) = \bigcap_\beta g_\beta(K)$; this last implies $\bigcap_\alpha \mathscr{K}(S^\alpha) \subset \mathscr{K}(L_R)$. On the other hand, for each α, $0 \leqq \alpha \leqq 1$, we have $S^\alpha \subset L_R$, so that

$$\bigcap_{f \in L_R} f(K) = \mathscr{K}(L_R) \subset \bigcap_{f \in S^\alpha} f(K) = \mathscr{K}(S^\alpha)$$

and $\mathscr{K}(L_R) \subset \bigcap_\alpha \mathscr{K}(S^\alpha)$ hold. Hence $\mathscr{K}(L_R) = \bigcap_\alpha \mathscr{K}(S^\alpha)$. Now it follows from this last statement and Theorem 2 that the upper half and the lower half of $\mathscr{K}(L_R)$ are convex domains.

An equation for the boundary of $\mathscr{K}(L_R)$ can be obtained in the following way. For $W = U + iV$ on the boundary of $\mathscr{K}(S^\alpha)$, it follows from (3) that

$$
\begin{aligned}
4U \equiv U_1 &= \frac{\cos \pi\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \frac{\tau^{1-\alpha}}{1-\alpha+\alpha\tau} + \frac{1-\tau}{1-\alpha+\alpha\tau}, \\
4V \equiv V_1 &= \frac{\sin \pi\alpha}{\alpha^\alpha(1-\alpha)^{1-\alpha}} \frac{\tau^{1-\alpha}}{1-\alpha+\alpha\tau}
\end{aligned}
\tag{6}
$$

hold. We can assume $0 < \alpha < \frac{1}{2}$ and that U and V are positive because $\mathscr{K}(L_R)$ is symmetric with respect to the coordinate axes. We find

$$U_1 = \sqrt{1 - V_1^2} + \frac{1-\tau}{1-\alpha+\alpha\tau}.$$

We now wish to find the value of

$$\inf \left[\frac{1-\tau}{1-\alpha+\alpha\tau} \;\middle|\; V_1 \equiv \text{constant},\ 0 < V_1 < 2 \right].$$

If we use the method of Lagrange multipliers and use the function

$$\frac{1-\tau}{1-\alpha+\alpha\tau} + \lambda \log V_1,$$

and if we then differentiate with respect to α and τ and then eliminate λ, we obtain

$$\log \frac{\alpha\tau}{1-\alpha} + \frac{(\tau-1)(1-\alpha+\alpha\tau)}{\tau} = \pi \cot \pi\alpha. \tag{7}$$

The derivative of the left-hand member of (7) with respect to τ is $\frac{1}{\tau^2}(\tau+1)(1-\alpha+\alpha\tau)$, which is positive for $\tau > 0$ and $0 < \alpha < \frac{1}{2}$. Hence the left-hand member of (7) increases monotonically as τ increases. Hence for each α, $0 < \alpha < \frac{1}{2}$ there exists a unique $\tau = \tau(\alpha)$ for which (7) holds. Since the left-hand member of (7) is negative for $\tau = 1$ and $0 < \alpha < \frac{1}{2}$, it follows that $\tau = \tau(\alpha) > 1$.

An examination of the preceding discussion shows that we have proved the following result.

Theorem 3. *The Koebe domain $\mathscr{K}(L_R)$ is a domain symmetric with respect to the coordinate axes and whose upper and lower halves are convex domains. An equation for that part of the boundary of $\mathscr{K}(L_R)$ that is situated in the first quadrant is given by* (6), *where $0<\alpha<\frac{1}{2}$ and where $\tau=\tau(\alpha)$ is the unique solution of equation* (7) *for each such α.*

5. Odd convex and odd starlike functions.

Theorem 4. *The Koebe domain $\mathscr{K}(S_1^c)$ for the class of normalized odd convex functions with real coefficients is a convex domain that is symmetric with respect to the coordinate axes such that the part of that lies in the first quadrant has as supporting lines the lines*

$$\frac{x}{\cos\pi\alpha}+\frac{y}{\sin\pi\alpha}=h(\alpha)\equiv\frac{1}{4\sqrt{\pi}}\Gamma(\alpha)\Gamma(\tfrac{1}{2}-\alpha). \qquad 0<\alpha<1, \tag{8}$$

and the (*limiting*) *lines $x=\pi/4$ and $y=\pi/4$.*

Proof. It follows from the symmetry and the convexity of each set $f(K)$, for $f\in S_1^c$, that $\mathscr{K}(S_1^c)=\bigcap_{f\in S_1^c} f(K)$ is both convex and symmetric with respect to the coordinate axes. Now let β be fixed, $0<\beta<\frac{1}{2}$, and let $w_1\in\partial\mathscr{K}(S_1^c)$ such that the supporting line l_β of $\mathscr{K}(S_1^c)$ at w_1 has direction $e^{i\pi\beta}$ as its normal direction and $p(\beta)$ as its normal intercept. On the other hand, if $f\in S_1^c$, then the supporting line to $f(K)$ that has $e^{i\beta\pi}$ as its normal direction has a normal intercept of, say, $p(f;\beta)$; it is clear that $p(\beta)=\min[p(f;\beta)\,|\,f\in S_1^c]$, which follows from the compactness of S_1^c. Now let R_β denote the rhombus with angles $2\pi(1-\beta)$ and $2\pi\beta$ determined by l_β, the reflection l_β^* of l_β in the real axis, and the reflection of l_β and l_β^* in the imaginary axis, and let $f_\beta(z)$ be the element of S_1^c that minimizes the normal intercept $p(f;\beta)$, with β fixed. We see that $f_\beta(K)\subset R_\beta$, $f'_\beta(0)=1$. Now suppose that $R_\beta-f_\beta(K)$ contains interior points. Then the inner radius of R_β with respect to the origin would be, say, $\lambda > 1$. Hence if $\phi_\beta(z)$ is the convex univalent function that maps K onto R_β, with $\phi_\beta(0) = 0$ and $\phi'_\beta(0) > 0$, then $\phi_\beta(z)/\lambda$ is a member of S_1^c, and the distance from the origin to the supporting line would be $p(\beta)/\lambda$ which is less than $p(\beta)$. But $p(\beta)$ is the minimum such distance. Hence we conclude that all the supporting lines of $\mathscr{K}(S_1^c)$ that are not parallel to the coordinate axes must be sides of rhombi with angles $2\pi(1-\beta)$ and $2\pi\beta$ and whose inner radius with respect to the origin is unity. The members of S_1^c that map K onto these rhombi are given by

$$\phi_\beta(z) \equiv \int_0^z \left(\frac{1-\zeta^2}{1+\zeta^2}\right) \frac{d\zeta}{1-\zeta^2},$$

We find

$$|\phi_\beta(i) - \phi_\beta(1)| = \frac{1}{4\sqrt{\pi}}\,\Gamma(\alpha)\Gamma(\tfrac{1}{2}-\alpha)$$

as the length of the sides of the rhombi. It now follows that the equation of the side l_β in the first quadrant is given by (8). The limiting cases $\alpha = 0$ and $\alpha = 1$ yield $x = \pi/4$ and $y = \pi/4$, as supporting lines. This completes the proof.

The analogous result for starlike maps is the following one.

Theorem 5. *The Koebe domain $\mathscr{K}(S_1^*)$ for the class of normalized odd starlike functions with real coefficients is again a domain that is symmetric with respect to the coordinate axes and is starlike with respect to the origin. That part of the boundary of $\mathscr{K}(S_1^*)$ that lies in the first quadrant has the polar equation $W = Re^{i\Phi}$, where*

$$(9)\qquad \begin{aligned} R &= \frac{1}{2}(\cos\theta)^{-\cos^2\theta}(\sin\theta)^{-\sin^2\theta}\,,\\ \Phi &= \frac{\pi}{2}\sin^2\theta \qquad 0 \leqq \theta \leqq \pi/2. \end{aligned}$$

Proof. If we use the method of subordination as used by McGregor [5], then we find that the boundary $\partial\mathscr{K}(S_1^*)$ is the set of endpoints of groups of four slits l_k, $k=1,2,3,4$; these slits are infinite, lie on the lines $\arg w = \pi \pm \alpha(\pi/2)$ and $\arg w = \pm\alpha(\pi/2)$, $0 \leqq \alpha \leqq \frac{1}{2}$, have endpoints that are equidistant from the origin, and are so chosen that the inner radius of the slit domain so determined, with respect to the origin, is unity. The corresponding mapping function, for each α, $0 \leqq \alpha \leqq \frac{1}{2}$, is

$$\phi(z) \equiv \sqrt{\sigma(z^2)} = \frac{z}{1+z^2}\left(\frac{1+z^2}{1-z^2}\right)^{\alpha}$$

where $\sigma(z)$ is the starlike function (2). To find the endpoints of the slits, we use the equation $\phi'(z)=0$, and we find that they satisfy the equation $z^2 = 1-2\alpha \pm i2\sqrt{\alpha(1-\alpha)}$. Hence $\alpha = \sin^2\theta$, where $z=e^{i\theta}$ is the endpoint of the slit in the first quadrant. This yields (9), for $0<\theta<\dfrac{\pi}{2}$. For the limiting cases $\theta=0$ and $\theta=\dfrac{\pi}{2}$, we find $w_1=\frac{1}{2}$ and $w_1=\frac{1}{2}i$, respectively. This completes the proof.

6. Concluding remarks. In a subsequent publication we shall give characterizations of the Koebe domains for certain other classes of univalent functions; these will include certain classes of bounded functions, and certain classes of meromorphic univalent functions.

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