ON QUASICONFORMAL GROUPS

By

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A. Introduction

A *quasiconformal group* of an open set $D \subset \bar{R}^n = R^n \cup \{\infty\}$, $n \geqq 2$, is a group of quasiconformal homeomorphisms of D such that every $g \in G$ is K-quasiconformal for some fixed K. If we wish to specify K and D we can say that G is a K-quasiconformal group of D.

We are here mainly concerned with the question, first posed by Gehring–Palka [8, p. 197], when a quasiconformal group G of $\bar{R}^n$ is of the form

$$G = fHf^{-1} \tag{A1}$$

for some group H of Möbius transformations of $\bar{R}^n$ and for some quasiconformal homeomorphism f of $\bar{R}^n$; clearly, all groups of this form are quasiconformal groups.

Our method to study this question is based on measurable conformal structures of $\bar{R}^n$ (Section D). Such a structure μ assigns to a.e. $x \in R^n$ a positive definite $n \times n$ matrix $\mu(x)$ with determinant one. This is similar to the idea of a measurable Riemannian structure but since quasiconformality is unaffected by a change of scale we can normalize and require that the determinant is 1. A quasiconformal map f of $\bar{R}^n$ is conformal in μ (or preserves μ) if for a.e. $x \in R^n$

$$\mu(x) = f'(x)[\mu(f(x))] = |\det f'(x)|^{-2/n} f'(x)^{\mathrm{T}} \mu(f(x)) f'(x),$$

T being the transpose and $\det f'(x)$ the determinant of the derivative.

We first establish that, given a quasiconformal group G of $\bar{R}^n$ (or more generally, of an open subset of $\bar{R}^n$), then there is a G-invariant conformal structure μ of $\bar{R}^n$, that is, μ is preserved by every $g \in G$ (Theorem F). This has been first proved by Sullivan [17] in the discrete case. Our proof is different and applies to all groups.

If $n = 2$, then one can realize this G-invariant conformal structure as a pull-back of the ordinary structure. Hence, for $n = 2$, every quasiconformal group of $\bar{R}^2$ is a quasiconformal conjugation of a Möbius group [17, 18].

The situation is different for $n > 2$. One knows that there are quasiconformal groups which cannot be obtained in this manner [15, 19]. However, there are some conditions guaranteeing this conclusion. For instance, it follows from [22, Section 6]

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that if a quasiconformal group contains enough Möbius transformations, for instance all orientation preserving Möbius transformations, then it is actually a Möbius group. Here we give some additional conditions for a quasiconformal group to be of the form (A1).

These conditions involve the notions of a radial point (see (G0)) and a limit point (see (H1) and (H2)) of a quasiconformal group. These are generalizations of the corresponding notions for Möbius groups (in which case radial points are sometimes called radial limit points or conical limit points). We show that if there is a conformal structure μ invariant under a quasiconformal group G, and if μ is approximately continuous (see (A2)) at a radial point of G, or if μ is continuous at a limit point of G, then G is a quasiconformal conjugation of a Möbius group (Theorems G and H2). We can do this since we can transform the action of G by elements of G into neighbourhoods of a limit point or of a radial point x. If μ is sufficiently constant near x, then G acts like a Möbius group near x. By blowing up neighbourhoods of x into $\bar{R}^n$ we now get our theorems.

We get as a corollary that if the action of G can be extended to the hyperbolic space $H^{n+1} = R^n \times (0, \infty)$ in such a way that the extension is a quasiconformal group of H^{n+1} for which H^{n+1}/G is compact, then G is again of the form (A1). More generally, G is of this form (Corollary C) if T^n/G is compact when T^n is the so-called triple space (Section C) whose elements are triples of the form $(u, v, w) \in (\bar{R}^n)^3$ and to which the action of G naturally extends. We can establish our theorem since in both of these cases every $x \in \bar{R}^n$ is a radial point of G. A similar theorem has been indicated by Gromow [9, p. 209].

Finally, we prove that radial points of quasiconformal groups have the same kind of properties as radial points of Möbius groups. Either they have zero measure or full measure in $\bar{R}^n$ and the latter case occurs if and only if the action of the group in $\bar{R}^n \times \bar{R}^n$ is ergodic (Theorem I).

We remark that the central idea of this paper is that of Theorem G which is closely related to that of [21]. We proved in [21] that if G is a Möbius group of $\bar{R}^n$ such that no $x \in \bar{R}^n$ is fixed by every $g \in G$, and if f is a G-compatible map of $\bar{R}^n$ which is differentiable at a radial point x of G, then f is a Möbius transformation. In this theorem, like in Theorem G, one first went to neighbourhoods of x and then blew up. This was the starting point of the present paper.

It is worth noting that Lelong-Ferrand's study [13] was based on a somewhat similar idea.

The situation in dimension $n = 1$ is quite different from the higher-dimensional case. We only mention Hinkkanen's result [11] that a uniformly quasisymmetric group of R is a quasisymmetric conjugation of an affine group of R.

Most of our theorems are valid (or even meaningful) only for $n \geqq 2$. The major exception is Section C whose theorems are valid also for $n = 1$.

Finally, we remark that we prove in the course of the paper some auxiliary results on quasiconformal mappings of which some may have independent interest. The most important of these is the generalization of the so-called good approximation theorem for quasiconformal mappings from dimension $n = 2$ to all dimensions $n \geqq 2$ (Corollary D).

Notation and terminology. Quasiconformal maps and Möbius transformations may be here also orientation reversing. The group of Möbius transformations of $\bar{R}^n$ is Möb(n), and a *Möbius group* is a subgroup of Möb(n). A map $g \in$ Möb(n) can be extended in a unique manner to a Möbius transformation of the closed hyperbolic space $\bar{H}^{n+1} = H^{n+1} \cup \bar{R}^n$; we do not distinguish between g and its extension to $\bar{H}^{n+1}$.

Our reference for quasiconformal mappings is Väisälä [23] and we refer to it for terms occurring in connexion with quasiconformal mappings, such as ACL, K-quasiconformal, etc.

The *hyperbolic metric* of $H^{n+1} = R^n \times (0, \infty)$ is given by the element of length $|dx|/x_{n+1}$, $x = (x_1, \ldots, x_{n+1})$, and it is denoted by d. The euclidean distance of two points of R^n is $|x - y|$ and we set $|\infty - x| = \infty$, $x \in R^n$. The *spherical metric* of $\bar{R}^n$ is obtained by means of the stereographic projection and is denoted by q. The distance of a point x from a set X in a given metric d is denoted by $d(x, X)$; this notation is used also for the euclidean metric.

The derivative of a map $f: U \to R^n$, $U \subset R^n$ open, at a point x is $f'(x)$ and we regard it as an $n \times n$ matrix. The operator norm of $|f'(x)|$ is

$$f'(x) = \sup_{|u| \leqq 1} |f'(x)(u)|.$$

The Jacobian at x is $J_f(x)$. If $f_i: S \to \bar{R}^n$ are said to converge uniformly to $f: X \to \bar{R}^n$, we mean uniform convergence with respect to the spherical metric.

We denote by m the (euclidean) Lebesgue measure of measurable subsets of $\bar{R}^n$ and m_q is the measure obtained from the spherical metric q.

A map $f: U \to X$, where $U \subset \bar{R}^n$ is open and X is a metric space with metric d, is *approximately continuous* at $x \in U$ if we have for all $\varepsilon > 0$

$$\text{(A2)} \qquad m_q(\{y \in B_q^n(x, r) \cap U : d(f(x), f(y)) \leqq \varepsilon\}) / m_q(B_q^n(x, r)) \to 1$$

as $r \to 0$; here $B_q^n(x, r)$ is the open ball in the spherical metric with radius r and center x. If $x \neq \infty$, we can clearly use here instead the euclidean metric and measure. If X is a separable metric space and if f is measurable with respect to the Borel sets of X and Lebesgue measurable sets of U, then f is a.e. (with respect to m_q) approximately continuous (Federer [4, 2.9.13]).

If a group G acts in X and $Y \subset X$, then Y is *G-invariant* if $g(Y) = Y$ for every $g \in G$.

The open euclidean ball with center x and radius r is $B^n(x, r)$, $B^n(r) = B^n(r, 0)$, and $B^n = B^n(1)$.

An affine map α of R^n is extended to $\bar{R}^n$ by $\alpha(\infty) = \infty$.

We denote by id the identity map of a set X.

The determinant of a matrix A is $\det A$.

The unit $n \times n$ matrix is I.

The standard basis of R^{n+1} is $e_1, \ldots, e_{n+1}$.

The closure and boundary of a set X are denoted $\mathrm{cl}\, X$ and ∂X, respectively. Usually they are taken in $\bar{R}^n$.

B. Lemmas on quasiconformal mappings

While we assume the usual theory of quasiconformal mappings, we now give some more special lemmas needed in the sequel.

In our first lemma, a family $\mathcal{F}$ of embeddings $U \to \bar{R}^n$, $U \subset \bar{R}^n$ open, is topologized by means of the topology of uniform convergence on compact subsets of U (with respect to the spherical metric). The family $\mathcal{F}$ is *compact* if and only if it is possible to pick from every sequence $f_i \in \mathcal{F}$ a subsequence f_{n_i} such that $f_{n_i} \to f$ uniformly on compact subsets of U for some $f \in \mathcal{F}$.

The next result is an application of Gehring–Kelly [7] (cf. also Gehring [5, Sections 5 and 6]).

Lemma B1. *Let $\mathcal{F}$ be a compact family of K-quasiconformal embeddings $U \to \bar{R}^n$, $U \subset \bar{R}^n$ open. Let $F \subset U$ be compact. Then there are positive $a = a(n, K)$, $a' = a'(n, K)$, $b = b(n, K, \mathcal{F})$ and $b' = b'(n, K, \mathcal{F})$ such that we have in the spherical measure*

$$b' m_q(E)^{a'} \leqq m_q(f(E)) \leqq b m_q(E)^a$$

for all measurable $E \subset F$ and $f \in \mathcal{F}$.

Proof. We can reduce the situation so that U and $f(U)$, $f \in \mathcal{F}$, are contained in some fixed bounded set of R^n and thus we can use the euclidean measure rather than the spherical measure. We can also assume that every f is orientation preserving. It is also clear that it suffices to prove the right-hand inequality.

Fix $r > 0$ such that $B^n(x, r) \subset U$ for all $x \in F$ which is possible by compactness. Let $r'' = r''(x, r, f) > 0$ be the biggest number such that $B^n(f(x), r'') \subset f(B^n(x, r))$. Let then $r' = r'(x, f, r) > 0$ be the biggest number such that $f(B^n(x, r')) \subset B^n(f(x), r'')$. Obviously r' and r'' are continuous functions of x, r and f. By Gehring–Kelly [7, Theorem 2] we can assume that r is so small that if $E \subset B^n(x, r')$ is measurable, then

$$m(fE)/r''^n \leqq b[m(E)/r'^n]^a \tag{B1}$$

for some positive $b = b(n, K)$ and $a = a(n, K)$. By continuity and compactness, $r' \geqq r'_0 > 0$ and $r'' \leqq r''_0 < \infty$ for some $r'_0 = r'_0(F, \mathcal{F})$ and $r''_0 = r''_0(F, \mathcal{F})$. If $E \subset F$ and

$d(E) < r'_0$, (B1) implies that $m(fE) \leqq (r_0'^{n}/r^{an})bm(E)^a$ for all $f \in \mathcal{F}$. We now get the right-hand inequality by compactness of F. The lemma follows.

We next turn to the convergence properties of quasiconformal maps. The dilatation $K(f, x)$ of f at x is the dilatation of the differential $f'(x)$ at x if f is differentiable with a non-vanishing Jacobian at x; if this is not the case, then $K(f, x)$ is undefined.

In the next lemma I am indebted to T. Kuusalo and J. Väisälä for some helpful comments.

Lemma B2. *Let $f_i : U \to \bar{R}^n$, $U \subset \bar{R}^n$ open, be a family of K-quasiconformal embeddings such that $f_i \to f$ for some embedding $f : U \to \bar{R}^n$. Suppose that there is $K' \geqq 1$ such that for every $\varepsilon > 0$ the spherical measure*

$$m_q(\{x \in U : K(f_i, x) \geqq K' + \varepsilon\}) \to 0$$

as $i \to \infty$. Then f is K'-quasiconformal.

Proof. A ring A is an open set $A \subset \bar{R}^n$ such that $\bar{R}^n \setminus A$ has two components. Let us recall that the conformal capacity of a ring A is

$$\mathscr{C}(A) = \inf_u \int_A |\nabla u|^n dm$$

where dm is the Lebesgue measure and u ranges over all continuous functions $u : \mathrm{cl}\, A \to R$ taking the value 0 on one component of ∂A and the value 1 on the other and such that u is sufficiently regular. Either one may take u to be C^1 in A or a so-called admissible function (Mostow [16, p. 64]).

We prove the lemma using this notion. It is clear that we can assume that all the sets U, $f(U)$, $f_i(U)$ are in some bounded set of R^n. Let A be a ring such that $\mathrm{cl}\, A \subset U$ and set for $r > 0$

$$A'_r = \{z \in f(A) : B^n(z, r) \subset f(A)\}$$

which is, at least for small r, a ring such that $\mathrm{cl}\, A'_r \subset f(U)$. Suppose that $r > 0$ is given such that this is true. Since $f_i \to f$ uniformly on compact subsets of U (Väisälä [23, 21.1]), there is i_r such that $f_i(A) \supset A'_r$ if $i \geqq i_r$.

Let u'_r be a differentiable admissible function for A'_r. Extend u'_r to $\bar{R}^n$ by the requirements that u'_r is continuous in $\bar{R}^n$ and constant on each of the two components of $\bar{R}^n \setminus A$. Then Mostow [16, Lemma (3.4)] implies that u'_r is ACL and that $\nabla u'_r = 0$ a.e. in $R^n \setminus A'_r$. Hence

$$\text{(B2)} \qquad \int_{A'_r} |\nabla u'_r|^n dm = \int_{A'} |\nabla u'_r|^n dm$$

whenever $A' \supset A'_r$ is measurable.

Define now

$$u_i = u'_r \circ f_i.$$

Clearly, $u_i \,|\, \mathrm{cl}\, f_i^{-1}(A'_r)$ is admissible for $f_i^{-1}(A'_r)$ and hence [16, Lemma (3.4)] again implies that u_i is admissible for A. Furthermore, one easily sees that

$$|\nabla u_i(x)|^n \leqq |\nabla u'_r(f(x))|^n K(f_i, x)\,|J_{f_i}(x)|$$

a.e. in A when J_{f_i} is the Jacobian (cf. [16, p. 92] and [23, 34.6]).

Choose $\varepsilon > 0$ and set

$$E_i = \{x \in A : K(f_i, x) \geqq K' + \varepsilon\}.$$

We now estimate

$$\mathscr{C}(A) \leqq \int_A |\nabla u_i|^n dm \leqq \int_A |\nabla u'_r(f(x))|^n K(f_i, x)\,|J_{f_i}(x)|\, dm(x)$$

$$\text{(B3)} \qquad \leqq (K' + \varepsilon) \int_{f_i(A)} |\nabla u'_r|^n dm + K \int_{f_i(E_i)} |\nabla u'_r|^n dm.$$

Since $m(E_i) \to 0$ as $i \to \infty$ by assumption, the uniform absolute continuity of f_i (Lemma B1) implies that also $m(f_i(E_i)) \to 0$ as $i \to \infty$. Hence the second term in (B3) tends to zero as $i \to \infty$. Suppose that u'_r has been chosen in such a way that

$$\int_{A'_r} |\nabla u'_r|^n dm \leqq \mathscr{C}(A'_r) + \varepsilon.$$

Letting $i \to \infty$ and remembering (B2), we get now

$$\mathscr{C}(A) \leqq (K' + \varepsilon)(\mathscr{C}(A'_r) + \varepsilon).$$

This is true for every $\varepsilon > 0$. Hence $\mathscr{C}(A) \leqq K'\mathscr{C}(A'_r)$ for all small r. If we let $r \to 0$, $\mathscr{C}(A'_r) \to \mathscr{C}(f(A))$ ([16, Theorem 6.1]) and we finally get the conclusion that

$$\mathscr{C}(A) \leqq K'\mathscr{C}(f(A))$$

for all rings A such that $\mathrm{cl}\, A \subset U$.

One can also obtain the conformal capacity of A by the modulus of a path family

$$\mathscr{C}(A) = M(\Gamma_A)$$

when $M(\Gamma_A)$ is the modulus of the path family Γ_A whose elements are paths in A joining the components of $\bar{R}^n \setminus A$ (Gehring [6], see also Väisälä [23, 11.11]). Hence [23, 36.1] implies that the outer dilatation of f is K'.

Using Lemma B1, we see that the assumptions of the lemma are valid also for f_i^{-1} and f^{-1} (or, more precisely, for their restrictions to some suitable subsets of $\bar{R}^n$). Hence the outer dilatation of f^{-1} is also K'. It follows that f is K-quasiconformal [23, Section 13].

Finally, we record the following well-known

Lemma B3. *Let $f : \bar{R}^n \to \bar{R}^n$, $n \geqq 2$, be a conformal homeomorphism, that is, f is 1-quasiconformal. Then f is a Möbius transformation.*

For a proof, see Mostow [16, Lemma (12.2)]. Later Gehring gave a simpler proof of it and [22] gives a half-page proof of it.

For $n \geqq 3$, this follows also from the much deeper Gehring–Resetnjak theorem which asserts that 1-quasiconformal embeddings of a domain of $\bar{R}^n$ into $\bar{R}^n$ are restrictions of Möbius transformations.

C. The space of triples T^n

A Möbius group of $\bar{R}^n$ has the most useful property that the action of such a group can be extended to the hyperbolic space H^{n+1}. This seems to be difficult for general quasiconformal groups but we now introduce a substitute for the hyperbolic space to which the action of any group of homeomorphisms of $\bar{R}^n$ naturally extends.

We remark that in this section, unlike in other sections, we can also have $n = 1$. Of course, we must then define K-quasiconformal maps of $\bar{R}^1$ in a suitable manner and here we follow the convention of [20, 1F].

The substitute for the hyperbolic space is the *triple space* T^n which is a $3n$-manifold defined by

$$T^n = \{(u, v, w) \in (\bar{R}^n)^3 : u, v, w \text{ distinct}\}.$$

There is a natural projection $p : T^n \to H^{n+1}$ defined by

$$p(u, v, w) = \text{the orthogonal projection of } w \text{ (in hyperbolic geometry) onto the hyperbolic line joining } u \text{ and } v.$$

If f is a homeomorphism of $\bar{R}^n$, it induces a homeomorphism of T^n, also denoted by f, by the formula $(u, v, w) \mapsto (f(u), f(v), f(w))$. Note that if f is a Möbius transformation, p commutes with f, that is, the following maps $T^n \to H^{n+1}$ are equal:

$$fp = pf. \tag{C0}$$

This space was studied in [20]. We now give the properties of T^n needed in this paper. If $z \in H^{n+1}$, then $p^{-1}(z)$ is homeomorphic to the set of 2-frames of R^{n+1} which is a compact space. It follows that if $C \subset H^{n+1}$ is compact, then also

(C1) $$p^{-1}(C) \subset T^n$$

is compact. Hence, in a sense, T^n and H^{n+1} are not very much different. It is also easy to see that if $u = (u_1, u_2, u_3) \in T^n$, $p(u)$ is near $x \in \bar{R}^n$ if and only if at least two u_i are near x. More precisely, simple calculations [20, Lemma 3.2] show that if $|u_i - x| \leqq r$ for at least two i, then

(C2) $$|p(u) - x| \leqq (\sqrt{2} + 1)r$$

and if $|u_i - x| \leqq r$ for at most one i, we have

(C3) $$|p(u) - x| \geqq r/(\sqrt{2} + 1).$$

Möbius transformations of $\bar{R}^n$ define hyperbolic isometries of H^{n+1}. This fact generalizes as follows. Given n and $K \geqq 1$, there are $m = m(n, K) > 0$ and $c = c(n, K) > 1$ such that if $u, v \in T^n$ and $f : \bar{R}^n \to \bar{R}^n$ is K-quasiconformal, then

(C4) $$d(p(u), p(v))/c - m \leqq d(pf(u), pf(v)) \leqq cd(p(u), p(v)) + m,$$

as follows by [20, Theorems 3.4 and 3.6]. We need also a similar condition for the distance from a hyperbolic line. If $q = (x, y) \in \bar{R}^n \times \bar{R}^n$ is a pair of distinct points, let $L(q) = L(x, y)$ denote the hyperbolic line joining x and y. Then c and m in (C4) can be so chosen that also

(C5) $$d(p(u), L(q))/c - m \leqq d(pf(u), L(f(q)) \leqq cd(p(u), L(q)) + m.$$

This follows from [20, Theorem 3.8 and (3.16)].

Lemma C1. *Let f_i be a sequence of K-quasiconformal homeomorphisms of $\bar{R}^n$ such that $\{pf_i(u) : i > 0\} \subset H^{n+1}$ is bounded in the hyperbolic metric for some $u \in T^n$. Then, by passing to a subsequence, we can obtain that $f_i \to f$ uniformly in the spherical metric for some K-quasiconformal homeomorphism f of $\bar{R}^n$.*

Proof. Let $u = (u_1, u_2, u_3)$. Pass to a subsequence in such a way that all the limits $\lim_{i\to\infty} f_i(u_j) = x_j \in \bar{R}^n$, $j \leqq 3$, exist. We can assume that $x_j \subset R^n$. All x_j's must be distinct, otherwise $\{pf_i(u) : i > 0\}$ cannot be bounded by (C2). Then the conclusion follows by Väisälä [23, 20.5, 21.1 and 27.3] if $n \geqq 2$; if $n = 1$, this follows since quasisymmetric mappings have the same kind of compactness properties as quasiconformal maps [2, 2.2].

A Möbius group of $\bar{R}^n$ is discrete if and only if it acts discontinuously in H^{n+1}. We get as a consequence an analogue of this for a quasiconformal group G of $\bar{R}^n$. Such a group is *discrete* if we can find no sequence of distinct $g_i \in G$ such that $g_i \to f$ for some homeomorphism f of $\bar{R}^n$. If G acts in X and $x \in X$, G acts *discontinuously* at x if x has a neighbourhood U such that $g(U) \cap U \neq \emptyset$ for only finitely many $g \in G$.

Lemma C1 has as an immediate

Corollary C. *Let G be a quasiconformal group of $\bar{R}^n$. Then the following conditions are equivalent.*

(a) *G is discrete.*

(b) *G acts discontinuously in T^n.*

(c) *For all $u \in T^n$, the set $\{pg(u): g \in G\}$ is a discrete subset of H^{n+1}.*

Let f be a quasiconformal homeomorphism of $\bar{H}^{n+1}$. It is of interest to compare the action of f in T^n and H^{n+1}. We have the following generalization of (C0).

Lemma C2. *Let f be a K-quasiconformal homeomorphism of $\bar{H}^{n+1}$. If $u \in T^n$, then*

$$d(fp(u), pf(u)) \leqq \bar{m} \tag{C6}$$

for some $\bar{m} = \bar{m}(n, K)$.

Proof. Extend f by reflection to a K-quasiconformal homeomorphism of $\bar{R}^{n+1}$ ([23, 35.2]). By (C0), we can compose with Möbius transformations in such a way that $u = (0, e_1, \infty)$ and that $f(u) = u$. The family of K-quasiconformal maps of $\bar{R}^{n+1}$ fixing 0, e_1, and ∞ is compact, as we already observed in the proof of Lemma C1, and (C6) follows.

Finally, we compare a quasiconformal group of H^{n+1} (which defines also a group of $\bar{H}^{n+1}$ by extension to the boundary) with the group it induces in T^n.

Lemma C3. *Let G be a quasiconformal group of $\bar{H}^{n+1}$ and denote by G also the quasiconformal group of $\bar{R}^n$ which it induces on the boundary of the hyperbolic space. Then H^{n+1}/G is compact if and only if T^n/G is compact.*

Proof. Let G be a K-quasiconformal group of $\bar{H}^{n+1}$.

Suppose first that H^{n+1}/G is compact and let $C \subset H^{n+1}$ be a compact set such that $GC = H^{n+1}$. Let $C' = \{z \in H^{n+1}: d(z, C) \leqq \bar{m}\}$ where $\bar{m} = \bar{m}(n, K)$ is as in (C6). Then C' is also compact and so is $C'' = p^{-1}(C') \subset T^n$ by (C1).

We claim that $GC'' = T^n$. Let $u \in T^n$. Then $p(u) = g(z)$ for some $z \in C$ and $g \in G$. Thus $d(pg^{-1}(u), z) \leqq \bar{m}$ and hence $g^{-1}(u) \in C''$. Our claim is proved and T^n/G is compact.

Suppose then that T^n/G is compact. Let $C \subset T^n$ be a compact set such that $GC = T^n$. If we set $C' = \{z \in H^{n+1}: d(z, p(C)) \leqq \bar{m}\}$, one sees as above that C' is a compact set such that $GC' = H^{n+1}$. The lemma is proved.

D. Conformal structures

A measurable Riemannian structure of $U \subset \bar{R}^n$ would be a measurable map μ which assigns to a.e. $x \in U$ a positive definite bilinear form $\mu(x)$. From the point of view of quasiconformal maps, two such structures μ and μ' are equivalent if they differ a.e. by a constant: $\mu(x) = \lambda_x \mu'(x)$ a.e. in U. This leads us to consider Riemannian structures which are normalized by the requirement that the determinant of μ is a.e. 1. In addition, it is reasonable to require that μ satisfies a boundedness condition. Such structures are to be called conformal structures and we will now define them more formally.

The space S. For $n \geqq 2$ let

$$S = \{A : A \text{ is a real positive definite } n \times n \text{ matrix with determinant } 1\}.$$

Then S can be identified with $\mathrm{SL}(n, R)/\mathrm{SO}(n)$ where $\mathrm{SL}(n, R)$ is the set of $n \times n$ matrices with determinant 1 and $\mathrm{SO}(n)$ is the orthogonal subgroup of it. The general linear group $\mathrm{GL}(n, R)$ acts on S by the rule

(D1) $$X[A] = |\det X|^{-2/n} X^{\mathrm{T}} A X$$

where T is the transpose and det the determinant. Note that the action is actually a right action since $XY[A] = Y[X[A]]$ but we prefer to write it in this manner.

The element of length

$$ds^2 = \frac{\sqrt{n}}{2} \operatorname{tr}(Y^{-1} dY Y^{-1} dY) = \frac{\sqrt{n}}{2} \operatorname{tr}(Y^{-1} dY)^2,$$

where tr is the trace, makes S into a Riemannian space with metric d invariant under the action of $\mathrm{GL}(n, R)$; the factor $\sqrt{n}/2$ is included in order to have equality in (D6). If I is the unit matrix, then

(D2) $$d(I, A) = d(A) = \frac{\sqrt{n}}{2}((\log \lambda_1)^2 + \cdots + (\log \lambda_n)^2)^{1/2}$$

when $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, see [14, p. 27] (note that by this same reference S is a geodesic subspace of $\mathrm{GL}(n, R)/\mathrm{SO}(n)$). Other distances $d(A, B)$ can be calculated by (D2) and the invariance of the action of $\mathrm{GL}(n, R)$.

The space $S = \mathrm{SL}(n, R)/\mathrm{SO}(n)$ is a globally symmetric Riemannian space (see table V in Helgason [10, p. 518]). It is of non-compact type, that is, the orthogonal symmetric Lie algebra associated to S ([10, p. 213]) is of non-compact type ([10, p. 230]). (For the calculation of this algebra see [10, p. 451].) Hence S has negative curvature by [10, Theorem V. 3.1], that is, the sectional curvature is $\leqq 0$ at all points along all plane sections. Hence S is a *complete, simply connected Riemannian manifold of negative curvature.* We will make use of this fact in Theorem F.

We will use on S also another metric, denoted k. It is defined by the requirement that it is also invariant under the action of $\mathrm{GL}(n, R)$ and that

(D3) $$k(A) = k(I, A) = \frac{n}{2} \max(\log \lambda^A_{\max}, \log 1/\lambda^A_{\min})$$

when $\lambda^A_{\max}$ is the biggest eigenvalue of A and $\lambda^A_{\min}$ the smallest. Note that these requirements make indeed k well-defined.

To see that k is a metric, note that it is obviously positive definite. Since obviously $k(I,A)=k(A,I)$, it follows by the $\mathrm{GL}(n,R)$ invariance that k is symmetric. To prove the triangle inequality, it suffices to show that

$$k(I,B)\leqq k(I,A)+k(A,B) \tag{D4}$$

for all $A,B\in S$. Let $C=A^{-1/2}BA^{-1/2}$. Then $k(A,B)=k(I,C)$. Since $B=A^{1/2}CA^{1/2}$, we have

$$\lambda^B_{\max}\leqq\lambda^A_{\max}\lambda^C_{\max}\quad\text{and}\quad\lambda^B_{\min}\geqq\lambda^A_{\min}\lambda^C_{\min}$$

which easily imply (D4).

We get by (D2) and (D3) the following relation between k and d:

$$k/\sqrt{n-1}\leqq d\leqq\begin{cases}k & \text{if } n \text{ is even,}\\ k\sqrt{n-1/n} & \text{if } n \text{ is odd.}\end{cases} \tag{D5}$$

These inequalities are sharp. To see this, let $A\in S$ have eigenvalues $\lambda_1,\ldots,\lambda_n$ and suppose that $k(A)=|\log\lambda_1^{n/2}|$. Then one easily sees that the lower bound occurs when the eigenvalues satisfy (up to permutation) $\lambda_1^{1/(n-1)}=\lambda_2^{-1}=\cdots=\lambda_n^{-1}$. Similarly, if n is even, the upper bound occurs when $\lambda_1=\cdots=\lambda_{n/2}=\lambda^{-1}_{n/2+1}=\cdots=\lambda_n^{-1}$; if n is odd, then $\lambda_1=\cdots=\lambda_{(n-1)/2}=\lambda^{-1}_{(n+1)/2}=\cdots=\lambda^{-1}_{n-1}$ and $\lambda_n=1$.

Let us note the consequence that if $n=2$, then

$$k=d. \tag{D6}$$

Conformal structures. Now we can define that a *conformal structure* of an open subset U of $\bar{R}^n$, $n\geqq 2$, is a measurable map μ which assigns to a.e. $x\in U\cap R^n$ an element $\mu(x)\in S$ such that $k(\mu(x))$ is essentially bounded. The *standard conformal structure* ι of U assigns to every $x\in U\cap R^n$ the unit matrix I and it is meant if no other structure is specified.

Thus here (and also often in the following, even if we do not always mention it) everything is modulo null-sets: $\mu=\nu$ if $\mu(x)=\nu(x)$ a.e. In particular, we need not care about the value of a conformal structure at ∞ which would cause some problems, see Remark D1. Therefore we prefer to regard μ undefined at ∞.

Now we can define when an embedding $f:U\to U'$ is a quasiconformal map in structures μ and μ' of U and U', respectively. If f is differentiable with a non-vanishing Jacobian $J_f(x)$ at $x\in U$, we denote

$$\begin{aligned}&K(f,\mu,\mu',x)=\exp k(\mu(x),f'(x)[\mu'(f(x))]),\quad\text{and}\\ &K(f,\mu,\mu')=\operatorname{sup\,ess}\{K(f,\mu,\mu',x):x\in U \text{ and } f \text{ is differentiable with } J_f(x)\neq 0 \text{ at } x\}.\end{aligned} \tag{D7}$$

We say that f is a K-quasiconformal map $(U,\mu)\to(U',\mu')$ if

1° f is ACL and a.e. differentiable, and

2° $K(f,\mu,\mu')\leqq K<\infty$.

In view of the essential boundedness of $k\circ\mu$ and $k\circ\mu'$, the property of quasiconformality does not depend on the structures but K-quasiconformality may. In the case of the standard structure, we get the usual K-quasiconformal maps $U\to U'$ ([23, 34.6]). The map $f:(U,\mu)\to(U',\mu')$ is *conformal* if it is 1-quasiconformal, that is, if it is quasiconformal and if

(D8) $$\mu(x)=f'(x)[\mu'(f(x))]$$

a.e. in U.

If we mean here the standard structure ι, we omit it from the notations and simply denote $K(f)$ and $K(f,x)$. Since the standard structure is our reference point, it is natural to define the dilatation of a conformal structure μ of U by

$$K(\mu)=K(\mathrm{Id},\iota,\mu)\quad\text{and}\quad K(\mu,x)=K(\mathrm{Id},\iota,\mu,x),$$

Id being the identity map $U\to U$.

The familiar rules for the dilatations of compositions and inverses of quasiconformal mappings generalize straightforwardly using the fact that k is a $\mathrm{GL}(n,R)$-invariant metric of S. We record them here for later reference:

(D9) $$K(f,\mu,\mu',x)=K(f^{-1},\mu',\mu,f(x)),\quad\text{and}$$
$$K(gf,\mu,\mu'',x)\leqq K(f,\mu,\mu',x)K(g,\mu',\mu'',f(x))$$

a.e. in U for quasiconformal maps $f:(U,\mu)\to(U',\mu')$ and $g:(U',\mu')\to(U'',\mu'')$. It follows that

$$K(f,\mu,\mu')=K(f^{-1},\mu',\mu)\quad\text{and}\quad K(gf,\mu,\mu'')\leqq K(f,\mu,\mu')K(g,\mu',\mu'').$$

There are also other measures for the dilatation of a map although this is, perhaps, the most widely used. In particular, it is used by Väisälä [23, 34.6] which is our standard reference. We will use it whenever convenient but one of our theorems (Theorem F) is most naturally expressed by means of the dilatation obtained using the metric d of S instead of k. Thus we define the D-dilatation of a quasiconformal map $f:(U,\mu)\to(U',\mu')$ at a point x where f is differentiable by

(D10) $$D(f,\mu,\mu',x)=\exp d(\mu(x),f'(x)[\mu'(f(x))])$$

and set $D(f,\mu,\mu')=\sup\mathrm{ess}_{x\in U}\,D(f,\mu,\mu',x)$; $D(f)$, $D(\mu)$, $D(f,x)$, and $D(\mu,x)$ are defined similarly. The properties (D9) are valid also for D and (D5) gives a relation between K and D.

The D-dilatation was considered by Ahlfors [1] and was suggested by Earle (unpublished).

If $f: U \to U'$ is quasiconformal (in the standard structures), and if μ and μ' are conformal structures of U and U', then one can define the image $f_*\mu$ of μ and pre-image $f^*\mu'$ of μ' by the formulae

(D11) $$f_*\mu(f(x)) = f'(x)^{-1}[\mu(x)] \quad \text{and} \quad f^*\mu'(x) = f'(x)[\mu'(f(x))];$$

in order that $f_*\mu$ be defined at all points, we must require that $f: U \to U'$ is a homeomorphism (note that $f_*\mu = (f^{-1})^*\mu$). Thus $f_*\mu$ and $f^*\mu'$ are uniquely determined by the requirement that $f:(U,\mu)\to(U',f_*\mu)$ and $f:(U,f^*\mu')\to(U',\mu')$ are conformal (see (D8)). This fact and (D9) now imply that if $f: U\to U'$ and $g:(U,\mu)\to(U,\nu)$ are quasiconformal, then

(D12) $$K(fgf^{-1}, f_*\mu, f_*\nu, f(x)) = K(g,\mu,\nu,x)$$

a.e. in U. Finally we note that

(D13) $$(gf)_*\mu = g_*f_*\mu \quad \text{and} \quad (gf)^*\mu'' = f^*g^*\mu''$$

when $f: U\to U'$ and $g: U'\to U''$ are quasiconformal and μ and μ'' are conformal structures on U and U'', respectively.

Finally, we define the generalization of the complex dilatation for $n \geqq 2$. It is called the *matrix dilatation* and is defined for the quasiconformal map $f: U\to U'$ by

(D14) $$\mu_f(x) = |\det f'(x)|^{-2/n} f'(x)^{\mathrm{T}} f'(x) = f'(x)[I]$$

from which expression we see that μ_f is a.e. defined and that $\mu_f = f^*\iota$ if we regard it as a conformal structure and hence $f:(U,\mu_f)\to(U',\iota)$ is conformal. We record the following composition rule for the matrix dilatation

(D15) $$\mu_{fg}(x) = g'(x)[\mu_f(g(x))]$$

a.e. in U when $f: U\to U'$ and $g: U'\to U''$ are quasiconformal.

Convergence theorems. As is well-known, a limit of K-quasiconformal embeddings is also K-quasiconformal provided that it is an embedding. We now give a generalization of this. It is also a generalization of Lemma B2 on which it is based. Actually, Lemma B2 almost suffices for us but in Theorem F we need the strengthened form for uncountable quasiconformal groups.

Theorem D. *Let $f_i:(U,\mu)\to(V,\nu)$ be K-quasiconformal embeddings and suppose that $f_i\to f$ for some embedding $U\to V$. Suppose that for some $K'\geqq 1$ and all $\varepsilon>0$*

(D16) $$m_q(\{y\in U: K(f_i,\mu,\nu,y)\geqq K'+\varepsilon\})\to 0$$

as $i\to\infty$. Then f is a K'-quasiconformal embedding $(U,\mu)\to(V,\nu)$ and $f_i\to f$ uniformly on compact subsets of U.

Proof. By (D9), each f_i is $K(\mu)K(\nu)K$-quasiconformal (in the ordinary structures). Hence f is in any case quasiconformal and the convergence is uniform on compact subsets of U by Väisälä [23, 21.1 and 37.3]. Hence it suffices to prove that

$$K(f, \mu, \nu, x) \leqq K' \tag{D17}$$

for all points $x \in U$ such that $x \neq \infty \neq f(x)$, that f is differentiable at x with a non-vanishing Jacobian, and that μ is approximately continuous at x (see (A2)) and ν at $f(x)$, these points having full measure in U. See [23, 33.3] and [4, 2.9.13].

Pick then affine maps α and β of R^n such that $\beta(0) = x$, $\alpha f \beta(0) = 0$ and that $\beta^*\mu(0) = \beta[\mu(x)] = I = \alpha_*\nu(0) = \alpha^{-1}[\nu(f(x))]$. Replace now f, f_i, μ and ν by $\alpha f \beta$, $\alpha f_i \beta$, $\beta^*\mu$ and $\alpha_*\nu$. By (D9) and (D11), f_i still satisfy (D16), and $K(f, \mu, \nu)$ does not change.

Hence, when proving (D17), we can assume in addition that $x = 0 = f(0)$ and that $\mu(0) = I = \nu(0)$.

Define the maps $B^n \to R^n$ by

$$g_i(y) = 2^i f(2^{-i}y) \quad \text{and} \quad g_{ij}(y) = 2^i f_j(2^{-i}y);$$

by passing to a subsequence we can indeed assume that g_i and g_{ij} are well-defined (and that the image is a subset of R^n). Then $g_i \to f'(0) \mid B^n$ as $i \to \infty$ and, for fixed i, $g_{ij} \to g_i$ as $j \to \infty$ and the convergences are uniform. Hence we can find a sequence $j_1 < j_2 < \cdots$ such that setting

$$h_i = g_{ij_i},$$

then

$$h_i \to f'(0) \mid B^n$$

uniformly on B^n.

Define conformal structures μ_i and ν_i of B^n by

$$\mu_i(y) = \mu(2^{-i}y) \quad \text{and} \quad \nu_i(y) = \nu(2^{-i}y);$$

these are the pre-images of μ and ν under the map $y \mapsto 2^{-i}y$. It is clear by (D16) that the sequence j_i can be so chosen that if

$$F_i = \{y \in B^n : K(h_i, \mu_i, \nu_i, y) \geqq K' + 2^{-i}\}, \quad \text{then } m(F_i) \leqq 2^{-i}. \tag{D18}$$

Next pick $\varepsilon > 0$ and consider the sets

$$\begin{aligned} &E_i = \{y \in B^n : K(\mu_i, y) = K(\mu, 2^{-i}y) \geqq 1 + \varepsilon\}, \quad \text{and} \\ &E'_i = \{y \in B^n : K(\nu_i, h_i(y)) = K(\nu, 2^{-i}h_i(y)) \geqq 1 + \varepsilon\}. \end{aligned} \tag{D19}$$

We claim that both $m(E_i)\to 0$ and $m(E'_i)\to 0$ as $i\to\infty$. For $m(E_i)$ this is clear by the approximate continuity of μ at 0. To see it for $m(E'_i)$, let

$$E''_i = \{y \in B^n(2|f'(0)|) : K(\nu, 2^{-i}y) \geqq 1+\varepsilon\}$$

and note that $h_i(B^n)\subset B^n(2|f'(0)|)$ for big i. By the approximate continuity of ν at 0, $m(E''_i)\to 0$ as $i\to\infty$. Since $h_i(E'_i)\subset E''_i$, the claimed result now follows by Lemma B1. It can be applied since h_i and $f'(0)$ are $KK(\mu)K(\nu)$-quasiconformal and since they form a compact family of mappings (to apply Lemma B1, we can clearly assume that they are maps $2B^n\to R^n$).

As we observed above, h_i are $KK(\mu)K(\nu)$-quasiconformal (in the ordinary structures). Furthermore, by (D9), (D18) and (D19),

$$K(h_i, y)\leqq (1+\varepsilon)^2(K'+2^{-i})$$

if $y\in B^n\setminus(E_i\cup E'_i\cup F_i)$. Since $m(E_i\cup E'_i\cup F_i)\to 0$ as $i\to\infty$ and ε was arbitrary, we can apply Lemma B2 to obtain that $f'(0)\,|\,B^n = \lim_{i\to\infty} h_i$ is K'-quasiconformal. Thus $K(f,x) = K(f'(0))\leqq K'$ and the theorem is proved.

We have the following corollary which generalizes for $n\geqq 2$ the so-called good approximation ([12, IV. 5.6]) of planar quasiconformal maps.

Corollary D. *Let $f_i : U\to \bar{R}^n$ be K-quasiconformal embeddings. Suppose that $f_i\to f$ for some embedding $f: U\to\bar{R}^n$ and that $\mu_{f_i}\to\mu$ in measure for some measurable map $\mu : U\to S$. Then f is K-quasiconformal and*

$$\mu_f = \mu$$

a.e. in U.

Proof. Since $\mu_{f_i}\to\mu$ in measure and f_i are K-quasiconformal, $K(\mu)\leqq K$. Hence μ defines a conformal structure of U. Regard f_i as maps $(U,\mu)\to(\bar{R}^n,\iota)$. Then $K(f_i,\mu,\iota)\leqq K(\mu)K\leqq K^2$ and our assumptions imply that, given $\varepsilon>0$,

$$m_q(\{y\in U : k(\mu(y),\mu_{f_i}(y)) = \log K(f_i,\mu,\iota,x)\geqq\varepsilon\})\to 0$$

as $i\to\infty$. Hence Theorem D implies that $f:(U,\mu)\to(\bar{R}^n,\iota)$ is conformal. That is $\mu_f=\mu$ a.e. in U.

The example in Lehto–Virtanen [12, IV. 5.4] shows that without the assumption that $\mu_{f_i}\to\mu$ in measure, it is not in general possible to pass to a subsequence in such a way that $\mu_{f_i}\to\mu_f$ a.e. in U.

Remarks. **D1.** Actually, a conformal structure is a field of bilinear forms on the tangent space of U. If $x\in U\cap R^n$, it is natural to identify the tangent space at x with R^n. However, if $x=\infty$, no such natural identification exists but we would have to use auxiliary Möbius transformations to define $\mu(x)$ as an element of S, that is, we take $g \in \text{Möb}(n)$, $g(\infty) \neq \infty$ and consider $g^*\mu(g^{-1}(\infty))$ (or $g_*\mu(g(\infty))$). But an examination of the transformation formula (D11) shows that this depends on g. Since we usually need not consider a conformal structure at ∞, we have not introduced this complication, to define a conformal structure as a field of bilinear forms of the tangent space.

Note, however, that $K(g^*\mu, g^{-1}(x)) = K(\mu, x)$ is independent of g. Hence its definition for $x = \infty$ does not pose difficulties. Similarly, such properties of μ like continuity and approximate continuity are independent of g and hence well-defined also for $x = \infty$. Understood this way, the point x in Theorems F and G can be also ∞. Observe that even if we can define $\mu(\infty) \in S$ in such a way that μ becomes continuous (or approximately continuous) at ∞, this may not be true of $g^*\mu$ at $g^{-1}(\infty)$.

The same remark applies also to the matrix dilatation which is also a field of bilinear forms in the tangent space. However, if $f(x) = \infty$, $x \neq \infty$, then $\mu_{gf}(x)$ is independent of $g \in \text{Möb}(n)$, $g(\infty) \neq \infty$, as follows by (D15), and is thus well-defined. On the other hand, $\mu_{fg}(g^{-1}(\infty))$ depends on g and this $\mu_f(\infty)$ cannot be defined in this manner.

Again, continuity of μ_f is well-defined for $x = \infty$ and so is $K(f, x)$ if f is regular enough at x.

D2. If $n = 2$, then the matrix dilatation of f at a point x determines the complex dilatation of f at x and conversely; this is due to the fact that both give essentially the ratio of eigenvalues of $f'(x)^T f'(x)$ as well as their eigenspaces. Then the solvability of the Beltrami equation implies that given a conformal structure μ on $U \subset \bar{R}^2$, there is a quasiconformal map $f: U \to U'$ such that $\mu = \mu_f$ a.e. in U.

E. The center of the smallest disk

In the next section we will show that any quasiconformal group of $\bar{R}^n$ acts as a group of conformal homeomorphisms with respect to some conformal structure of $\bar{R}^n$. This result is based on a lemma on the uniqueness of centers of smallest disks which we now present.

A smooth Riemannian manifold M has negative curvature if the sectional curvature of M is ≤ 0 along each planar section at every point of M. We denote by d the geodesic metric of M and let

$$D(x, r) = \{z \in M : d(z, x) \leq r\}.$$

We assume now that M is complete, simply connected and has negative curvature. Then all geodesics of M are non-intersecting and homeomorphic to the real line. Any two distinct points $x, y \in M$ can be joined by a unique geodesic and we denote by xy the closed geodesic segment joining x and y; then $d(x, y) =$ the length of xy.

If $z \in M$, $x \neq z \neq y$, we let $\operatorname{ang}(z, x, y) \in [0, \pi]$ be the angle between zx and zy. Using this notation we have

$$\text{(E1)} \qquad d(x, y)^2 \geqq d(z, x)^2 + d(z, y)^2 - 2d(z, x)d(z, y)\cos\operatorname{ang}(z, x, y),$$

see Corollary I. 13.2 of Helgason [10]; also the other results mentioned above can be found in this book.

Formula (E1) makes it possible to prove

Lemma E. *Let M be a simply connected, complete Riemannian manifold of negative curvature. Let $X \subset M$ be non-empty and bounded. Then there is a uniquely determined disk $D(P_X, r_X)$ with smallest radius r_X containing X. Furthermore, if $X \subset D(x, r)$ for some $x \in M$ and $r \geqq 0$, then*

$$\text{(E2)} \qquad d(x, P_X) \leqq r.$$

Proof. We can assume that $X \neq$ a point. Then

$$r = \inf\{r' : D(y, r') \supset X \text{ for some } y \in M\} > 0.$$

It is easy to see that there is at least one $y \in M$ such that $D(y, r) \supset X$. Suppose that also $D(z, r) \supset X$. If $z \neq y$, we get a contradiction as follows.

Let w be the midpoint of the segment yz and let $D = D(y, r) \cap D(z, r)$. Pick $u \in D$, $u \neq w$. Then either $\operatorname{ang}(w, u, y) \geqq \pi/2$ or $\operatorname{ang}(w, u, z) \geqq \pi/2$. Suppose, for instance, that the first case occurs. Then (E1) implies that

$$r^2 \geqq d(y, u)^2 \geqq d(w, y)^2 + d(w, u)^2 - 2d(w, y)d(w, u)\cos\operatorname{ang}(w, u, y) > d(w, u)^2.$$

Hence $d(w, u) < r$ if $u \in D$. Since D is compact, it follows that $X \subset D \subset D(w, r')$ for some $r' < r$, a contradiction. Hence P_X is well-defined.

Suppose then that $X \subset D(x, r)$. Let $D(y, \rho)$ be the smallest disk containing X. If $y \notin D(x, r)$, let $\{w\} = xy \cap \partial D(x, r)$. Then, using (E1), one sees that $\operatorname{ang}(w, x, u) < \pi/2$ for all $u \in D(x, r) \cap D(y, \rho)$, $u \neq w$. Hence $\operatorname{ang}(w, y, u) \geqq \pi/2$ and it would follow as above that $D(w, r') \supset D(x, r) \cap D(y, \rho)$ for some $r' < \rho$. This contradiction proves (E2).

Remarks E1. Let $\mathscr{C}(M)$ be the family of all non-empty, bounded subsets of M. Define the Hausdorff metric ρ in $\mathscr{C}(M)$ by

$$\text{(E3)} \qquad \rho(X, Y) = \sup\{d(x, Y), d(y, X) : x \in X, y \in Y\}.$$

This makes $\mathscr{C}(M)$ a pseudo-metric space (ρ is a metric in the subfamily of non-empty, *closed* and bounded subsets of M).

We will need the fact that $X \mapsto P_X$ *is a continuous map* $\mathscr{C}(M) \to M$. To see this, suppose that it is not continuous at X. Then there is a sequence $X_i \in \mathscr{C}(M)$ such that

$$\text{(E4)} \qquad \rho(X_i, X) \to 0$$

but that

$$d(P_X, P_{X_i}) > \varepsilon \tag{E5}$$

for some $\varepsilon > 0$. Then, for big i, $X_i \subset \{z \in M : d(z, X) \leqq 1\}$ which is a compact subset of M. Hence, by passing to a subsequence, we can assume that the limits

$$x = \lim_{i\to\infty} P_{X_i} \quad \text{and} \quad r = \lim_{i\to\infty} r_{X_i} \tag{E6}$$

exist. If $\varepsilon' > 0$, then $D(x, r+\varepsilon') \supset X_i$ for big i and hence $D(x, r+\varepsilon') \supset X$ by (E4). It follows that $D(x, r) \supset X$. Suppose that also $D(y, r') \supset X$. Then $D(y, r'') \supset X_i$ if $r'' > r'$ for big i and it follows that $r' \geqq \lim_{i\to\infty} r_i = r$. Thus $r = r_X$ and consequently $x = P_X$. This contradicts (E5) and (E6) and the continuity is proved.

E2. If $X \subset D(x, r)$, $r > 0$, and if $x \in X$, then actually slightly more than (E2) is true. To see this, observe that in any case $r_X \leqq r$. If $r_X < r$, then $d(x, P_X) \leqq r_X < r$. If $r_X = r$, then $D(x, r)$ is the smallest disk containing X and hence $P_X = x$ and $d(x, P_X) = 0 < r$. Thus always $d(x, P_X) < r$.

Now $P_X = P_{\mathrm{cl}X}$ and the family of all closed subsets of $D(x, r)$ containing x is compact in the ρ-metric. This follows since, as is well known, the family of all non-empty and closed subsets of $D(x, r)$ is compact ([3, pp. 58–59]). As we saw, the map $X \mapsto P_X$ is continuous and it follows that, if $r > 0$, there is $r' < r$ (r' may depend on x) such that

$$d(x, P_X) < r' \tag{E7}$$

whenever $X \subset D(x, r)$ and $x \in X$.

If M is the hyperbolic space, one can make more precise calculations ([18]) and show that in this situation in fact

$$d(x, P_X) \leqq \operatorname{ar\,cosh}(\cosh r)^{1/2} \leqq \min(r/\sqrt{2}, r/2 + \log\sqrt{2}) \tag{E8}$$

and in the right-hand inequality the equality holds only for $r = 0$.

E3. Lemma E gives a simple proof of the following theorem.

Let F be a group of isometries of a simply connected, complete Riemannian manifold M of negative curvature. Suppose that Gx is bounded for some $x \in M$. Then there is $x_0 \in M$ fixed by every $g \in G$.

By Lemma E, we can take for x_0 the center of the smallest disk containing Gx.

F. Invariant conformal structures

We will now show that a quasiconformal group of $U \subset \bar{R}^n$ admits a *G-invariant conformal* structure μ. That is, every $g \in G$ is a conformal map $(U, \mu) \to (U, \mu)$, or, equivalently, $g_*\mu = \mu = g^*\mu$. Explicitly, this means (see (D8)) that

(F0) $$g'(x)[\mu(g(x))] = \mu(x)$$

a.e. in U when $g \in G$ is fixed. Here it is most convenient to work using the Earle–Ahlfors dilatation $D(f)$ of a quasiconformal map (see (D10)). Sullivan [17] gives a different proof of it for countable G.

Theorem F. *Let G be a quasiconformal group of U, $U \subset \bar{R}^n$ open. Then U has a G-invariant conformal structure μ. Furthermore, if $D(g) \leqq D$ for every $g \in G$, then*

(F1) $$D(\mu) \leqq D.$$

Proof. We assume first that G is countable. In this case U has a measurable G-invariant subset $U' \subset R^n$ of full measure (that is, $U \setminus U'$ is a null set) such that every $g \in G$ is differentiable with a non-vanishing Jacobian with $D(g, x) \leqq D$ at every $x \in U'$. Then $\mu_g(x)$ is defined for every $x \in U'$ and we set

(F2) $$M_x = \{\mu_g(x) : g \in G\}$$

for $x \in U'$. If g, $f \in G$, then $\mu_{fg}(x) = g'(x)[\mu_f(g(x))]$ (see (D15)) and hence, for every $x \in U'$ and every $g \in G$,

(F3) $$g'(x)[M_{g(x)}] = \{g'(x)[\mu_f(g(x))] : f \in G\} = \{\mu_{fg}(x) : f \in G\} = M_x.$$

Thus the *set* M_x satisfies the transformation rule (F0) that a G-invariant conformal structure should satisfy.

Now $\mu_g(x) \in S$ which is a simply connected, complete Riemannian space of negative curvature, as we observed in Section D. Furthermore, $d(\mu_g(x), I) \leqq \log D$ for all $g \in G$ and $x \in U'$. Hence we can apply Lemma E and set for $x \in U'$

(F4) $$\mu(x) = P_{M_x}.$$

Then μ satisfies (F1) by (E2). The map $A \mapsto h[A]$, $h \in \mathrm{GL}(n, R)$, is an isometry of S. Thus, by (F3), $g'(x)[\mu(g(x)] = \mu(x)$ for every $g \in G$ and $x \in U'$.

Hence μ is a G-invariant conformal structure of U if it is measurable. To see this, let $G = \{g_0, g_1, \ldots\}$ and set $M(x, j) = \{\mu_{g_i} : i \leqq j\}$ and $\mu_j(x) = P_{M(x,j)}$. The map $X \mapsto P_X$ is continuous in the Hausdorff metric (E3) (see Remark E1). This implies first that each μ_j is measurable and then that $\mu_j(x) \to \mu(x)$ as $j \to \infty$ $(x \in U')$. Hence μ is measurable.

This proves the theorem in the countable case. We now suppose that G is uncountable. Regard U as a PL manifold. The space of all (not necessarily injective) PL maps $U \to U$ is clearly separable in the topology of uniform convergence on compact subsets. Since every $g \in G$ can be approximated arbitrarily closely by such maps, it follows that also G is separable. Hence there is a countable subgroup $G_0 \subset G$ which is dense in the topology of uniform convergence on compact subsets. We have shown that there is a G_0-invariant conformal structure μ on U. By Theorem D, μ is also G-invariant. The theorem is proved.

Remarks. **F1.** Actually, since always $I = \mu_{id}(x) \in M_x$, there is by (E7) $D' = D'(D, n)$ with the property that $D' < D$ if $D > 1$ and that the D-dilatation of μ satisfies

(F5) $$D(\mu) \leqq D' \leqq D$$

if $D(g) \leqq D$ for all $g \in G$. In view of (D5), it now follows that if $K(g) \leqq K$ for all $g \in G$, then

(F6) $$K(\mu) \leqq K' \leqq K^{\sqrt{n-1}}$$

for some $K' = K'(K, n)$ such that $K' < K^{\sqrt{n-1}}$ if $K > 1$; if n is odd, then one has the slightly better estimate $K' < K^{(n-1)/\sqrt{n}}$ if $K > 1$.

If $n = 2$, then one can give an even better estimate, cf. [18, p. 77] and (E8). We have

(F7) $$K(\mu) \leqq (\sqrt{K + 1/K} + \sqrt{K} - 1/\sqrt{K})/\sqrt{2} \leqq \min(K^{1/\sqrt{2}}, \sqrt{2K}),$$

and on the right-hand inequality the equality is true only if $K = 1$.

F2. If $n = 2$, then there is always a quasiconformal map f of $\bar{R}^n$ such that $\mu(x) = \mu_f(x)$ a.e. in U (see Remark D2). Hence $fG(f^{-1} \mid f(U))$ is a conformal group of U (with respect to the ordinary structure).

Note that here we need not assume that G contains only orientation preserving quasiconformal maps. We assumed this in [18] since one customarily considers the complex dilatation only for orientation preserving quasiconformal maps.

F3. Suppose that μ_g, $g \in G$, are defined at a neighbourhood of a point $x \in U$ and that they are equicontinuous at x (see Remark D1 for the case $x = \infty = g(x)$). Then μ is continuous at x, as follows from the fact that the map $X \mapsto P_X$ is continuous (Remark E1). In particular, if μ_g, $g \in G$, are defined and equicontinuous everywhere, then μ is continuous.

F4. Let d be a smooth Riemannian metric of $U \subset \bar{R}^n$ and suppose that G is a group of homeomorphisms of U such that every $x \in U$ has a neighbourhood W such that every $g \in G$ satisfies a bilipschitz condition on W with a bilipschitz constant not depending on x nor on g. Then one could show as above that U has a measurable Riemannian structure preserved by $g \in G$.

G. Conformal structures at radial points

We will now show that if a quasiconformal group G of $\bar{R}^n$ admits an invariant conformal structure which is approximately continuous (see (A2)) at a radial point of G, then G is a quasiconformal conjugation of a Möbius group.

Radial points for Möbius groups are defined using the fact that the action of the group extends to the hyperbolic space $\bar{H}^{n+1}$. Such an extension is not known to exist for a general quasiconformal group G but we circumvent the difficulty using the triple space T^n and the projection $p : T^n \to H^{n+1}$ defined in Section C. Let $x \in \bar{R}^n$. Suppose that there is a sequence $g_i \in G$ such that given a triple $z = (u, v, w) \in T^n$ (i.e. $u, v, w \in \bar{R}^n$ are distinct) and a hyperbolic line $L \subset H^{n+1}$ with endpoint x, then, as $i \to \infty$,

$$\text{(G0)} \qquad \begin{aligned} &z_i = p(g_i(z)) = p(g_i(u), g_i(v), g_i(w)) \to x \text{ in } \bar{H}^{n+1}, \quad \text{and} \\ &d(z_i, L) \leqq M \end{aligned}$$

for some $M \geqq 0$ when $d(z_i, L)$ is the hyperbolic distance of z_i and L. If this is true, we say that x is a *radial point* of G.

If (G0) is true for one particular triple z and one hyperbolic line L, then it is true for all triples and hyperbolic lines, with M depending on z and L, as follows by (C4).

We denote by $R(G)$ the radial point set of G. If G is a Möbius group, then this is the usual radial point set of G (these points are also called radial limit points, conical limit points or non-tangential limit points). Furthermore, radial points are preserved under quasiconformal mappings: If $f : \bar{R}^n \to \bar{R}^n$ is quasiconformal and G a quasiconformal group of $\bar{R}^n$, then

$$\text{(G1)} \qquad R(fGf^{-1}) = f(R(G))$$

in view of (C5), (C2), and (C3). Thus our definition seems reasonable.

Theorem G. *Let G be a quasiconformal group of $\bar{R}^n$ and let μ be a G-invariant conformal structure of $\bar{R}^n$. Suppose that μ is approximately continuous at a radial point x of G. Then there is a quasiconformal homeomorphism f of $\bar{R}^n$ such that fGf^{-1} is a Möbius group of $\bar{R}^n$ and that*

$$\text{(G2)} \qquad K(f) \leqq K(\mu)^2 K(\mu, x).$$

Remark. Since a conformal structure μ is actually a field of bilinear forms on the tangent space, we have not defined μ at ∞. In order to consider Theorem G for $x = \infty$, we must replace G, μ, and ∞ by gGg^{-1}, $g_*\mu$, and $g(\infty)$ for some $g \in \text{Möb}(n)$ such that $g(\infty) \neq \infty$. See Remark D1.

Proof. By (D12), we can replace G by hGh^{-1} and μ by $h_*\mu$, if h is a Möbius transformation; inequality (G2) is not affected. Hence we can assume that $x = 0$. Let $g_i \in G$ and $z = (u, v, w) \in T^n$ be points that satisfy (G0) for some $M > 0$ with respect to the hyperbolic line L with endpoint $x = 0$ and ∞.

Choose a linear map α of R^n such that $\alpha_*\mu(0) = I$; then

$$\text{(G3)} \qquad K(\alpha) = K(\mu, x).$$

Pick then positive numbers $\lambda_i > 0$ such that

(G4) $$d(e_{n+1}, \lambda_i p g_i(z)) = d(e_{n+1}, p(\lambda_i g_i(z))) \leqq M$$

which is possible by (G0). In view of (G0),

(G5) $$\lambda_i \to \infty$$

as $i \to \infty$. Define now maps $f_i : \bar{R}^n \to \bar{R}^n$ by

(G6) $$f_i(u) = \lambda_i \alpha g_i(u) = \alpha(\lambda_i g_i(u)).$$

By (D9), each g_i is $K(\mu)^2$-quasiconformal and thus, in view of (G3), every f_i is $K(\mu)^2 K(\mu, x)$-quasiconformal.

Now each $\lambda_i g_i$ is a $K(\mu)^2$-quasiconformal map of $\bar{R}^n$. Hence (G4) and Lemma C1 imply that we can pass to a subsequence in such a way that $\lambda_i g_i \to \bar{g}$ for some $K(\mu)^2$-quasiconformal $\bar{g}$ and that the convergence is uniform in the spherical metric. It follows that

(G7) $$f_i \to f$$

where $f = \alpha\bar{g}$ is $K(\mu)^2 K(\mu, x)$-quasiconformal and that the convergence is uniform in the spherical metric.

Thus f satisfies (G2) and we conclude the proof by showing that fgf^{-1} is a conformal map of $\bar{R}^n$ for every $g \in G$. Since conformal maps of $\bar{R}^n$, $n \geqq 2$, are Möbius transformations (Lemma D3) it follows that fGf^{-1} is indeed a Möbius group.

Pick $g \in G$ and let

$$g' = fgf^{-1} \quad \text{and}$$
$$g'_i = f_i g f_i^{-1} = (\lambda_i \alpha) g_i g g_i^{-1} (\lambda_i \alpha)^{-1}.$$

Consider the conformal structure $\mu_i = f_{i*}\mu$ of $\bar{R}^n$. By (D12), each g'_i is conformal in this structure. Since g'_i is conformal in μ_i, we have (see (D11))

$$\mu_i(u) = f_{i*}\mu(u) = (\lambda_i\alpha)_* g_{i*}\mu(u) = (\lambda_i\alpha)_*\mu(u) = \alpha_*\mu(u/\lambda_i).$$

Now, μ is approximatively continuous at 0 and so is $\alpha_*\mu$ and, furthermore, $\alpha_*\mu(0) = I$. Remembering that $\lambda_i \to \infty$, we obtain that, given $\varepsilon > 0$,

$$K(\mu_i, x) \leqq 1 + \varepsilon$$

if $x \in \bar{R}^n \setminus A_i$ where $m_q(A_i) \to 0$ as $i \to \infty$. The maps g'_i and g' form a compact family of $K(\mu)^8$-quasiconformal maps of $\bar{R}^n$. Hence it follows by Lemma B1 that there are measurable sets $B_i \subset \bar{R}^n$ such that $m_q(B_i) \to 0$ as $i \to \infty$ and that

$$K(\mu_i, x) \leqq 1 + \varepsilon \quad \text{and} \quad K(\mu_i, g'_i(x)) \leqq 1 + \varepsilon$$

if $x \in \bar{R}^n \setminus B_i$. Since g_i' is conformal in μ_i, we have by (D12) that $K(g_i', x) \leqq K(\mu_i, x) K(\mu_i, g_i'(x))$ and hence

$$K(g_i', x) \leqq (1+\varepsilon)^2$$

if $x \in \bar{R}^n \setminus B_i$. Thus $K(g_i', x) \to 1$ in measure and consequently Lemma B2 implies that $g' = \lim_{i\to\infty} g_i'$ is conformal. The theorem is proved.

We can now give a topological characterization for a quasiconformal group to be a quasiconformal conjugation of a Möbius group. Gromow [9, p. 209] has indicated a similar theorem.

Corollary G. *Let G be a quasiconformal group of $\bar{R}^n$ and suppose that either*

(a) *T^n/G is compact (T^n as in Section C), or*

(b) *G can be extended to a group of homeomorphisms of the closed hyperbolic space $\bar{H}^{n+1}$ in such a way that H^{n+1}/G is compact and that G is a quasiconformal group of H^{n+1}.*

Then G is a quasiconformal conjugation of a Möbius group and every $x \in \bar{R}^n$ is a radial point of G.

Proof. We will show that in both cases every $x \in \bar{R}^n$ is a radial point of G. Since a conformal structure is a.e. approximately continuous, Theorems F and G imply that $fGf^{-1} \subset \text{Möb}(n)$ for some quasiconformal f.

Suppose that T^n/G is compact and that G is a K-quasiconformal group. Then there is a compact set $C \subset T^n$ such that $GC = T^n$. Hence there is $M > 0$ such that

$$d(p(u), p(v)) < M$$

for all $u, v \in C$. By (C4), there is $M' = M'(M, K, n)$ such that if $g \in G$ and $u, v \in C$, then

$$d(g(u), g(v)) \leqq M'.$$

Fix $z \in C$. Since every $x' \in H^{n+1}$ is of the form $pg(z')$ for some $z' \in C$ and $g \in G$ we have that $d(x', pg(z)) \leqq M'$. Thus every closed hyperbolic disk of radius M' intersects $p(Gz)$. It follows that every $x \in \bar{R}^n$ is a radial point of G.

Case (b) follows now by Lemma C3.

Thus the class of Möbius groups of $\bar{R}^n$ such that H^{n+1}/G is compact and the class of quasiconformal groups of $\bar{R}^n$ for which T^n/G is compact are essentially the same class of groups.

Remarks. G1. If G is a K-quasiconformal group of $\bar{R}^n$ and the G-invariant conformal structure μ is obtained from Theorem F, then the map f of Theorem G for which fGf^{-1} is a Möbius group satisfies

$$K(f) \leqq K'^3 \leqq K^{3\sqrt{n-1}} \tag{G8}$$

where $K' = K'(K, n)$ is as in (F6). Recall that $K' < K^{\sqrt{n-1}}$ if $K > 1$ and that for odd n a slightly better estimate is true.

Note that in Theorem G it would be impractical to use the D-dilatation since if $D(f_i) \leqq D$ and $f_i \to f$ for some quasiconformal f, it is not known whether $D(f) \leqq D$.

G2. Suppose that G is a quasiconformal group of $\bar{R}^n$ such that action of G can be extended to $\bar{H}^{n+1}$ in such a way that G defines a quasiconformal group of H^{n+1}. Then $x \in \bar{R}^n$ is a radial point of G if and only if, given $z \in H^{n+1}$ and a hyperbolic line L with endpoint x, the condition (G0) remains true if we set in it $z_i = g_i(z)$. This is a consequence of Lemma C2.

H. Limit points and conformal structures

We will now prove a similar theorem involving limit points of a quasiconformal group G. This is a set which includes the radial points of the previous section. If there is a G-invariant conformal structure of $\bar{R}^n$ which is continuous at a limit point, then G is again a quasiconformal conjugation of a Möbius group.

It is convenient to define limit points for a general quasiconformal group by means of the space of triples T^n and the projection $p : T^n \to H^{n+1}$ of Section C. We say that $x \in \bar{R}^n$ is a *limit point* of a quasiconformal group G of $\bar{R}^n$ if there are $g_i \in G$ and $z \in T^n$ such that

$$x = \lim_{i \to \infty} pg_i(z). \tag{H1}$$

It follows by (C4) that if (H1) is true for some $z \in T^n$, then it is true for all $z \in T^n$. The limit point set of G is denoted by $L(G)$. Obviously, $L(G)$ is a closed subset of $\bar{R}^n$.

In view of (C2) and (C3), we can give also the following characterization for the limit set: $x \in L(G)$ if and only if there are distinct $u, v \in \bar{R}^n$ and $g_i \in G$ such that

$$x = \lim_{i \to \infty} g_i(u) = \lim_{i \to \infty} g_i(v). \tag{H2}$$

It follows that $L(G)$ is preserved under conjugations

$$L(fGf^{-1}) = f(L(G)) \tag{H3}$$

for any homeomorphism of $\bar{R}^n$, quasiconformal or not.

We will now show that limit sets of discrete quasiconformal groups have the same kind of properties as limit sets of Kleinian groups. For the definitions of the notions "discrete" and "discontinuous", see Section C.

Theorem H1. *Let G be a discrete quasiconformal group of $\bar{R}^n$. Then*

(a) *$x \in L(G)$ if and only if there is a sequence of distinct elements $g_i \in G$ and $y \in \bar{R}^n$ such that*

(H4)
$$\lim_{i\to\infty} g_i(y) = x,$$

(b) *the set of points of* $\bar{R}^n$ *where* G *acts discontinuously is* $\bar{R}^n \setminus L(G)$.

Proof. We will prove (a) which then implies at once (b) by (H2). It is also clear by (H2) that if $x \in L(G)$ then (H4) is true. Thus it suffices to prove that if (H4) is true, then $x \in L(G)$.

If $x \notin L(G)$, then, by (H2), for no $u \in \bar{R}^n \setminus \{y\}$ can there be a subsequence of g_i (denoted in the same manner) such that $x = \lim_{i\to\infty} g_i(u)$. Hence, by [23, 19.2 and 20.3], given $u \in \bar{R}^n \setminus \{y\}$, we can pass to a subsequence in such a way that $g_i(v) \to h(v)$ for $v \in \bar{R}^n \setminus \{u, y\}$ for some function $h : \bar{R}^n \setminus \{u, y\} \to \bar{R}^n$. Furthermore, convergence is uniform on compact subsets of $\bar{R}^n \setminus \{u, y\}$. Since G is discrete, h cannot be an embedding. Hence, by [23, 21.1], h is a constant, $h(v) = c$ for all $v \in \bar{R}^n \setminus \{u, y\}$. Since $u \in \bar{R}^n \setminus \{y\}$ was arbitrary, we have in fact that $g_i(v) \to c$ uniformly on compact subsets of $\bar{R}^n \setminus \{y\}$.

We assumed that $x \notin L(G)$. Hence $c \neq x$. It is not now difficult to see that $g_i^{-1}(v) \to y$ uniformly on compact subsets of $\bar{R}^n \setminus \{c\}$. Hence $y \in L(G)$. By (H3), then also $g_i(y) \in L(G)$ and, $L(G)$ being closed, $x \in L(G)$. This contradiction proves the theorem.

We now consider G-invariant conformal structures at limit points of G and have the following analogue of Theorem G. If $x = \infty$ in it, the same remark applies as for Theorem G.

Theorem H2. *Let* G *be a quasiconformal group of* $\bar{R}^n$ *and let* μ *be a* G*-invariant conformal structure of* $\bar{R}^n$*. Suppose that* μ *is continuous at a limit point* x *of* G*. Then there is a quasiconformal map* f *of* $\bar{R}^n$ *such that* fGf^{-1} *is a Möbius group and such that*

(H5)
$$K(f) \leq K(\mu)^2 K(\mu, x).$$

Proof. As in the proof of Theorem G, we can assume that $x = 0$. Let $g_i \in G$ and $z \in T^n$ be so chosen that (H1) is true for $x = 0$.

Pick now a linear homeomorphism α of R^n such that $\alpha_* \mu(0) = I$. Choose then $\lambda_i > 0$ and $b_i \in R^n$ such that, setting $\beta_i(u) = \lambda_i u + b_i$, which is a similarity of R^n, we have

$$\beta_i(p(\alpha g_i(z))) = p(\beta_i \alpha g_i(z)) = e_{n+1}.$$

Set $f_i = \beta_i \alpha g_i$ which is $K(\mu)^3$-quasiconformal. By Lemma C1, we can pass to a subsequence in such a way that $f_i \to f$ uniformly for some quasiconformal homeomorphism f of $\bar{R}^n$.

We claim that fGf^{-1} is a Möbius group.

Pick $g \in G$. Set

$$h = fgf^{-f} \quad \text{and} \quad h_i = f_i g f_i^{-1}.$$

Each h_i is conformal in $f_{i*}\mu = (\beta_i \alpha g_i)_* \mu = (\beta_i \alpha)_* \mu$ since g_i is conformal in μ. Since the derivative $\beta_i' = \lambda_i I$, we have

$$f_{i*}\mu(y) = (\beta_i \alpha)_*(y) = \alpha_* \mu(\beta_i^{-1}(y)),$$

$y \in R^n$. Furthermore, $\beta_i^{-1}(e_{n+1}) \to 0$ as $i \to \infty$. Hence $\beta_i^{-1}(y) \to 0$ uniformly on compact subsets of R^n. It follows that

$$f_{i*}\mu(y) \to I = \alpha_* \mu(0)$$

uniformly on compact subsets of R^n. By (D9), $K(h_i, x) \to 1$ uniformly on compact subsets of R^n. It follows that $h = \lim_{i \to \infty} h_i$ is conformal and hence a Möbius transformation by Lemma B3.

Finally, to get (H5), note that, being conformal in μ, each g_i is $K(\mu)^2$-quasiconformal by (D9). Since $\alpha_* \mu(0) = I$, $K(\alpha) = K(\mu, x)$. Since β_i is a similarity, $f_i = \beta_i \alpha g_i$ is $K(\mu)^2 K(\mu, x)$-quasiconformal and (H5) follows. The theorem is proved.

Remarks. H1. If the conformal structure μ in Theorem H2 is obtained from Theorem F, and G is a K-quasiconformal group, the estimate (G8) for the dilatation is valid also for the map f of Theorem H2.

H2. If μ_g, $g \in G$, is an equicontinuous family, then Theorem F gives a continuous G-invariant conformal structure (see Remark F3); for the definition and continuity of $\mu_g(x)$ if $x = \infty$ or $g(x) = \infty$, see Remark D1. Hence fGf^{-1} is a Möbius group for some quasiconformal f if $L(G) \neq \emptyset$. If G is discrete, this occurs by Theorem H1 as soon as G is infinite.

Let us note still the following, somewhat striking consequence of Theorem H2. If $L(G) = \bar{R}^n$ and G is not a quasiconformal conjugate of a Möbius group, then a G-invariant conformal structure μ can be continuous at no $x \in R^n$. And there is such a μ by Theorem F.

I. Radial points

In view of Theorems F and G, it is interesting to know when the radial point set of a quasiconformal group G has positive measure since in this case G is a quasiconformal conjugation of a Möbius group. We will now investigate this matter and prove a theorem which generalizes a corresponding theorem for Möbius groups.

In the following theorem, the action of G on $\bar{R}^n \times \bar{R}^n$ is the *diagonal action* $(x, y) \to (g(x), g(y))$, $g \in G$. A set $F \subset \bar{R}^n \times \bar{R}^n$ is a measurable *fundamental set* of

G if it is measurable and if F contains exactly one point from each orbit Gp, $p \in \bar{R}^n \times \bar{R}^n$; the action of G is *ergodic* in $\bar{R}^n \times \bar{R}^n$ if every G-invariant measurable subset has either zero or full measure in $\bar{R}^n \times \bar{R}^n$ (with respect to the product spherical measure).

Theorem I. *Let G be a quasiconformal group of $\bar{R}^n$, $n \geqq 2$, and let R_G be its radial point set. Then*

(a) *either R_G or $\bar{R}^n \setminus R_G$ is a null-set, and*

(b) *if G is discrete, then either G has a measurable fundamental set on $\bar{R}^n \times \bar{R}^n$ or the action of G on $\bar{R}^n \times \bar{R}^n$ is ergodic and the last case occurs if and only if $m(R_G) > 0$.*

Proof. We start from the fact that this theorem is true for Möbius groups (Sullivan [17, pp. 482–483]).

If $m(R_G) > 0$, then, as we observed above, there is a quasiconformal homeomorphism f of $\bar{R}^n$ such that $H = fGf^{-1}$ is a Möbius group. Since quasiconformal maps are absolutely continuous and since $R_H = f(R_G)$ (see (G1)), we get (a) by [17, Lemma 1, p. 482] (note that in this lemma it is not essential that H is discrete).

Similarly, [17, p. 483] implies that if G is discrete and $m(R_G) > 0$, then G acts ergodically on $\bar{R}^n \times \bar{R}^n$.

We conclude the proof by showing that if $m(R_G) = 0$, then the action of G on $\bar{R}^n \times \bar{R}^n$ has a measurable fundamental set. Our proof generalizes Sullivan's argument in [17, Lemma 2, p. 482].

If $p = (x, y)$ is a pair of distinct points of $\bar{R}^n$, let $L(p) = L(x, y)$ be the hyperbolic line joining x and y. Pick $z \in T^n$ and let $m_0 > 0$. We claim that if $x, y \notin R_G$, then the hyperbolic distance

$$d(p(z), L(g(x), g(y))) \leqq m_0 \tag{I1}$$

for only finitely many $g \in G$. Note that if the inequality (I1) is true, then by (C5)

$$d(pg(z), L(x, y))) \leqq m_1$$

where $m_1 = m_1(m_0, K, n)$ when G is a K-quasiconformal group.

Suppose that (I1) is true for infinitely many $g \in G$. Since the set of accumulation points of $\{u \in H^{n+1} : d(u, L(x, y)) \leqq m_1\}$ is contained in $\{x, y\}$ (see Corollary C), it would follow that either x or y is in R_G contrary to our assumption. This proves our claim.

Pick now $u \in H^{n+1}$. Let $A = (\bar{R}^n \setminus R_G) \times (\bar{R}^n \setminus R_G) \setminus$(diagonal) and set

$$F' = \{p \in A : d(u, L(p)) \leqq d(u, L(g(p))) \text{ for all } g \in G\}.$$

It follows easily by (I1) that F' is a measurable set such that if $p \in A$, then $Gp \cap F'$ contains at least one but at most finitely many points.

We now obtain a measurable fundamental domain for G as follows. Let $p \in A$. Since each $Gp \cap F'$ is finite, it is easy to find a rule that picks a point x_p from each of these sets. For instance, if $Gp \cap F' \subset R^{2n}$, we can use the natural alphabetical order of R^{2n}, and the general case can be treated similarly. It is then clear that

$$F = \{x_p : p \in A\}$$

is a measurable fundamental set for the action of G in A and adding a null-set we obtain a measurable fundamental set for the action of G in $\bar{R}^n \times \bar{R}^n$. The theorem is proved.

Remark. Here it is essential that $n \geqq 2$ but our proof still shows that if G is a discrete quasiconformal group of $\bar{R}$, then the action of G has a measurable fundamental set in $(\bar{R} \setminus R_G) \times (\bar{R} \setminus R_G)$. (In this case it is not known whether R_G has always full or zero measure in $\bar{R}$.)

Addendum. In Theorems G and H2 it would be possible to conclude the proof differently which would have some advantages. First of all the proof would be slightly simpler. More importantly, we would have the additional information that the conformal structure μ in Theorems G and H2 is the pullback of the ordinary conformal structure of $\bar{R}^n$ under the map f constructed in the proofs. This is interesting in itself and would give a better estimate for the dilatation of f than we indicated: dilatations of f and μ are equal, and this regardless whether we use the K-dilatation or the D-dilatation.

We indicate this for Theorem G, the case of Theorem H2 being exactly similar. The last paragraph of the proof of Theorem G should be replaced by the following one:

Since μ is approximately continuous at 0 and $\lambda_i \to \infty$, it follows that $\mu_i \to \iota$ ($=$ the ordinary conformal structure of $\bar{R}^n$) in measure. Hence the assumptions of Theorem D are satisfied if we regard f_i as a map $(\bar{R}^n, \mu) \to (\bar{R}^n, \iota)$ and consequently $f : (\bar{R}^n, \mu) \to (\bar{R}^n, \iota)$ is conformal, or, in other words, μ is the pull-back of the ordinary structure of $\bar{R}^n$ under f.

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