SOME INTEGRAL INEQUALITIES RELATED TO HARDY'S INEQUALITY

By

R. P. Boas, Jr.

in Evanston, Illinois, U.S.A.

The inequality that goes by the name of Hardy's inequality states, in its integral version, that if $p > 1$, $f(x) \geqq 0$, and

$$F_1(x) = \int_0^x f(t)dt$$

then

$$\int_0^\infty x^{-p} F_1(x)^p dx \leqq \{p/(p-1)\}^p \int_0^\infty f(x)^p dx \,. \tag{1}$$

(See, for example, [4], Theorem 327.) Here, as in all such inequalities, we are to understand (1) as saying that if the right-hand side is finite, so is the left-hand side, and the inequality holds; the "constant" $\{p/(p-1)\}^p$ in (1) is best possible. Another inequality, obtained by Hardy from (1) by using the converse of Hölder's inequality ([4], Theorem 328), states that if $p > 1$, $f(x) \geqq 0$, and

$$F_2(x) = \int_x^\infty t^{-1} f(t)dt$$

then

$$\int_0^\infty F_2(x)^p dx \leqq p^p \int_0^\infty f(x)^p dx \,. \tag{2}$$

The right-hand sides of (1) and (2) are multiples of the same integral, and this fact suggests that the left-hand sides themselves might be comparable. This is, in fact, the case: there are constants c_1 and c_2 (actually functions of p) such that

$$(3) \qquad \int_0^\infty x^{-p}F_1(x)^p dx \leqq c_1 \int_0^\infty F_2(x)^p dx \leqq c_2 \int_0^\infty x^{-p}F_1(x)^p dx .$$

There are various generalizations of Hardy's inequality: see, for example, [4], Theorem 330 and 346. Inequality (3) can be generalized in the same way: if $p > 1$, $s > 0$, and $-1 < c < sp - 1$ we have

$$(4) \qquad \int_0^\infty x^{c-sp} \left\{ \int_0^x f(u)u^s du \right\}^p \leqq C(c, s, p) \int_0^\infty x^c \left\{ \int_x^\infty f(u)du \right\}^p dx;$$

the same inequality holds in the opposite sense with a different C (corresponding to the other part of (3)). For a more symmetric formulation see (15) and (16) below; I state (4) in its present form because it is then the integral analogue of a series inequality that has applications to trigonometric series ([2], Lemma 6.18; special cases appear in [6] and [7]). Now inequalities of this kind tend to be reversed when $0 < p < 1$, and it was observed by R. Askey that both versions of (4) do hold, reversed, for $0 < p < 1$; but this means that (4) holds in both senses for $0 < p < 1$ as well as for $p > 1$. (For the series analogue, and an application, see [1].) It will also appear that (4) holds, in both senses, when $p < 0$, but with $sp - 1 < c < -1$. Inequalities with $p < 0$ have been little investigated; the series analogue of (1) is discussed in [5].

To get the left-hand side of (3) from (4) when $p > 1$, replace $f(u)$ by $u^{-1}f(u)$ and take $c = 0$; then the left-hand side of (4) is

$$\int_0^\infty x^{-sp} dx \left\{ \int_0^x f(u)u^{s-1} du \right\}^p = \int_0^\infty x^{-p} dx \left\{ \int_0^x f(u)du \right\}^p$$

if we take $s = 1$; these choices are possible since $-1 < 0 < p-1$. We are going, in particular, to establish (3) with explicit values for the constants:

$$\int_0^{\infty} x^{-p} F_1(x)^p dx \leqq \left\{ \left(\frac{p}{p-1} \right)^p - 1 \right\} \int_0^{\infty} F_2(x)^p dx ,$$

$$\int_0^{\infty} F_2(x)^p dx \leqq (p^p - 1) \int_0^{\infty} x^{-p} F_1(x)^p dx, \qquad p > 1 .$$

Note that although (4) holds for $0 < p < 1$ also, we cannot specialize it to get (3) when $0 < p < 1$ since the conditions on c become incompatible in this case.

In the series version, (4) has been proved in different ways for $p > 1$ and $0 < p < 1$. I shall give a unified proof of both cases of the integral version, and in fact of a more general set of inequalities; but the constants obtained are not best possible.

We shall need the following lemma, which contains (1) and (2) as special cases. When the convex function in the lemma is a power, it is a restatement of a well-known theorem ([4], Theorem 319). A similar approach to inequalities of this kind has been given by Godunova [3].

Lemma 1. *If ϕ is convex and continuous, f is non-negative, λ is non-decreasing, and $L = \lambda(\infty) - \lambda(0)$, then*

(5) $$\int_0^{\infty} x^{-1} \phi \left\{ L^{-1} \int_0^{\infty} f(xu) d\lambda(u) \right\} dx \leqq \int_0^{\infty} x^{-1} \phi\{f(x)\} dx;$$

the inequality is reversed when ϕ is concave. In particular,

(6) $$\int_0^{\infty} x^{-1} \left\{ \int_0^{\infty} f(xu) d\lambda(u) \right\}^p dx \leqq L^p \int_0^{\infty} x^{-1} f(x)^p dx, \ p > 1 \text{ or } p < 0;$$

(7) $$\int_0^{\infty} x^{-1} \left\{ \int_0^{\infty} f(xu) d\lambda(u) \right\}^p dx \geqq L^p \int_0^{\infty} x^{-1} f(x)^p dx, \quad 0 < p < 1 .$$

By Jensen's inequality ([4], Theorem 206), if ϕ is convex

$$\phi\left\{L^{-1}\int_0^\infty f(xu)d\lambda(u)\right\} \leqq L^{-1}\int_0^\infty \phi(f(xu))d\lambda(u),$$

so that

$$\int_0^\infty x^{-1}\phi\left\{L^{-1}\int_0^\infty f(xu)d\lambda(u)\right\}dx \leqq L^{-1}\int_0^\infty x^{-1}dx\int_0^\infty \phi(f(xu))d\lambda(u)$$

$$= L^{-1}\int_0^\infty d\lambda(u)\int_0^\infty x^{-1}\phi(f(xu))dx$$

$$= L^{-1}\int_0^\infty d\lambda(u)\int_0^\infty x^{-1}\phi(f(x))dx$$

$$= \int_0^\infty x^{-1}\phi(f(x))dx.$$

This proves (5) when ϕ is convex; when ϕ is concave, Jensen's inequality is reversed and hence so is (5).

If $d\lambda(u) = u^{\alpha-1}du$, $0 < u < 1$; $d\lambda(u) = 0$, $u > 1$, and $\alpha > 0$, (6) becomes

$$\int_0^\infty x^{-1}\left\{\int_0^1 f(xu)u^{\alpha-1}du\right\}^p dx \leqq \alpha^{-p}\int_0^\infty x^{-1}f(x)^p dx,$$

i.e.

$$\int_0^\infty x^{-1-\alpha p}\left\{\int_0^x f(u)u^{\alpha-1}du\right\}^p dx \leqq \alpha^{-p}\int_0^\infty x^{-1}f(x)^p dx.$$

If we take $\alpha = 1 - 1/p$ and replace $f(x)$ by $f(x)x^{1/p}$ we get (1). Similarly we get (2) by letting $d\lambda(u) = u^{-\beta-1}du$ for $u > 1$, and 0 otherwise.

It will be convenient to use the notation

$$K_1 * K_2 = \int_0^\infty K_1(t)K_2(x/t)t^{-1}dt;$$

that is, $K_1 * K_2$ is the convolution of K_1 and K_2 over the multiplicative group of positive real numbers (with its Haar measure). We generalize the functions F_1 and F_2 of (1) and (2) by putting

$$\int_0^\infty f(xu)K_j(u)u^{-1}du = F_j(x). \tag{8}$$

Then Lemma 1 says in particular that

$$\int_0^\infty x^{-1}F_j(x)^p dx \leqq \left\{\int_0^\infty u^{-1}K_j(u)du\right\}^p \int_0^\infty x^{-1}f(x)^p dx, \quad p > 1,$$

with inequality reversed when $0 < p < 1$.

Lemma 2. *Let K_1, K_2 and K_3 be nonnegative and put*

$$m_j = \int_0^\infty K_j(u)u^{-1}du. \tag{9}$$

If $K_1 \leqq K_2 * K_3$ then

$$\int_0^\infty x^{-1}F_1(x)^p dx \leqq m_2{}^p \int_0^\infty x^{-1}F_3(x)^p dx, \quad p > 1 \text{ or } p < 0. \tag{10}$$

The inequality is reversed if $0 < p < 1$ and $K_1 \geqq K_2 * K_3$.

We have

$$F_1(x) \leqq \int_0^\infty f(xu)u^{-1}du \int_0^\infty t^{-1}K_2(t)K_3(u/t)dt$$

$$= \int_0^\infty t^{-1}K_2(t)dt \int_0^\infty f(xu)u^{-1}K_3(u/t)du$$

$$= \int_0^\infty t^{-1}K_2(t)dt \int_0^\infty f(xtv)v^{-1}K_3(v)dv.$$

We now apply Lemma 1, with $\phi(x) = x^p$, replacing f by the inner integral in the preceding line, $d\lambda(u)$ by $t^{-1}K_2(t)dt$, and L by m_2 (from (9)). The result is that

$$(11) \qquad \int_0^\infty x^{-1}F_1(x)^p dx \leqq m_2^p \int_0^\infty x^{-1}F_3(x)^p dx, \quad p > 1 \quad \text{or } p < 0,$$

with the inequality reversed if $0 < p < 1$. Since K_2 and K_3 enter symmetrically we can replace m_2 by m_3 and F_3 by F_2 in (10).

In particular, we have inequalities of the form (11) whenever $K_1 = K_2 * K_3$ with all three functions positive. This equation is equivalent to $k_1 = k_2k_3$, where k_j is the Mellin transform of K_j, so a necessary condition is that any zeros of k_2 are also zeros of k_1; but this does not enable us to decide whether a positive K_3 exists for given positive K_1 and K_2. On the other hand, we can generate inequalities by taking given K_2 and K_3 and forming K_1. For a simple example, take $K_2(t) = t^\alpha$ on $(0,1)$ and $K_2(t) = 0$ for $t > 1$, with $\alpha > 0$; take $K_3 = K_2$. Then $K_1(t) = -t^\alpha \log t$ on $(0,1)$, $K_1(t) = 0$ for $t > 1$. Lemma 2 says that

$$\int_0^\infty x^{-1}\left\{\int_0^1 f(xu)u^{\alpha-1}\log u^{-1}du\right\}^p dx \leqq \alpha^{-p}\int_0^\infty x^{-1}\left\{\int_0^1 f(xu)u^{\alpha-1}du\right\}^p dx,$$

i.e.

$$\int_0^\infty x^{-1}\left\{\int_0^x f(v)v^{\alpha-1}x^{-\alpha}\log(x/v)dv\right\}^p dx \leqq \alpha^{-p}\int_0^\infty x^{-1}\left\{\int_0^x f(v)v^{\alpha-1}x^{-\alpha}\right\}^p dx.$$

Replacing $f(x)$ by $x^{1-\alpha}f(x)$ we have

$$\int_0^\infty x^{-1-\alpha p}\left\{\int_0^x f(v)(\log x-\log v)dv\right\}^p dx \leqq \alpha^{-p}\int_0^\infty x^{-1-\alpha p}\left\{\int_0^x f(v)dv\right\}^p dx.$$

In particular, if $p>1$ and $\alpha = 1-1/p$ we get

$$\int_0^\infty \left\{\frac{1}{x}\int_0^x f(v)(\log x-\log v)\right\}^p dx \leqq (p/(p-1))^p\int_0^\infty \left\{\frac{1}{x}\int_0^x f(v)dv\right\}^p dx.$$

To get (4) we modify the idea of Lemma 2.

Lemma 3. *With notation as in Lemma* 2, *suppose that* $K_1+K_2 = MK_2 * K_3$ (*with a constant* M). *Then*

$$\int_0^\infty x^{-1}F_1(x)^p dx \leqq (M^p m_3^p - 1)\int_0^\infty x^{-1}F_2(x)^p dx, \quad p>1, \tag{12}$$

with the inequality reversed if $0<p<1$.

For $p<0$ I do not know how to get any more than follows directly from Lemma 2.

Suppose first that $p>1$. For positive numbers A and B we have

$$A^p + B^p \leqq (A+B)^p \tag{13}$$

(for example by [4], Theorem 19). Hence (with notation given by (8))

$$F_1(x)^p + F_2(x)^p \leqq \{F_1(x) + F_2(x)\}^p$$

$$= M^p\left\{\int_0^\infty f(xu)u^{-1}du \int_0^\infty K_3(t)K_2(u/t)t^{-1}dt\right\}^p,$$

$$\int_0^\infty x^{-1}F_1(x)^p dx + \int_0^\infty x^{-1}F_2(x)^p dx$$

$$\leqq M^p \int_0^\infty x^{-1}dx \left\{\int_0^\infty K_3(t)t^{-1}dt \int_0^\infty u^{-1}f(xu)K_2(u/t)du\right\}^p$$

$$= M^p \int_0^\infty x^{-1}dx \left\{\int_0^\infty K_3(t)t^{-1}dt \int_0^\infty v^{-1}f(xvt)K_2(v)dv\right\}^p$$

$$= M^p \int_0^\infty x^{-1}dx \left\{\int_0^\infty K_3(t)t^{-1}F_2(xt)dt\right\}^p.$$

Now the integral in $\{\cdots\}$ in the preceding line is formed from F_2 as F_3 is formed from f (in (8)). Hence by Lemma 1, in the formulation given just before Lemma 2,

$$\int_0^\infty x^{-1}F_1(x)^p dx + \int_0^\infty x^{-1}F_2(x)^p dx \leqq M^p m_3{}^p \int_0^\infty x^{-1}F_2(x)^p dx.$$

Since $m_1 + m_2 = Mm_2m_3$ we have $Mm_3 > 1$, and hence

$$\int_0^\infty x^{-1}F_1(x)^p dx \leqq (M^p m_3^p - 1)\int_0^\infty x^{-1}F_2(x)^p dx, \qquad p > 1.$$

If $0 < p < 1$, (13) is reversed and hence all our inequalities are reversed. We also have

$$(14)\qquad \int_0^\infty x^{-1}F_1(x)^p dx + \int_0^\infty x^{-1}F_2(x)^p dx \leqq M^p m_2^p \int_0^\infty x^{-1}F_3(x)^p dx,$$

and so if it happens that $K_3 = K_1$ (as it will in our derivation of (4)), then

$$\int_0^\infty x^{-1}F_2(x)^p dx \leqq (M^p m_1^p - 1) \int_0^\infty x^{-1}F_1(x)^p dx, \quad p > 1,$$

again with inequality reversed if $0 < p < 1$.

We now specialize Lemma 3 by taking

$$K_1(u) = u^\alpha,\ 0 < u < 1;\ 0, u > 1 \qquad (\alpha > 0),$$

$$K_2(u) = u^{-\beta},\ u > 1;\ 0, 0 < u < 1 \qquad (\beta > 0).$$

Then we have $(\alpha+\beta)K_1 * K_2 = K_1 + K_2$. This can be verified by direct calculation, but it is easier to verify the Mellin transform of this equation: we have

$$k_1(z) = \int_0^\infty u^{\alpha+z-1}K_1(u)du = (\alpha+z)^{-1},$$

$$k_2(z) = (\beta - z)^{-1}.$$

$$k_1(z) + k_2(z) = (\alpha+\beta)k_1(z)k_2(z).$$

We also have $M = \alpha+\beta$, $m_1 = 1/\alpha$, $m_2 = 1/\beta$. Hence

$$(15)\qquad \int_0^\infty x^{-\alpha p-1}\left\{\int_0^x f(t)\, t^{\alpha-1}\, dt\right\}^p dx$$

$$\leqq \left\{\left(\frac{\alpha+\beta}{\alpha}\right)^p - 1\right\} \int_0^\infty x^{\beta p-1}\left\{\int_x^\infty f(t)t^{-\beta-1}dt\right\}^p dx, \quad p > 1;$$

$$(16)\qquad \int_0^\infty x^{\beta p-1}\left\{\int_x^\infty f(t)t^{-\beta-1}dt\right\}^p dx$$

$$\leqq \left\{\left(\frac{\alpha+\beta}{\beta}\right)^p - 1\right\}\int_0^\infty x^{-\alpha p-1}\left\{\int_0^x f(t)t^{\alpha-1}dt\right\}^p dx, \quad p>1,$$

with both inequalities reversed when $0<p<1$.

To obtain (4), we replace $f(t)$ by $t^{\beta+1}f(t)$ in (15), and put $c=\beta p-1$; then the left-hand side of (15) is

$$\int_0^\infty x^{-\alpha p-1}\left\{\int_0^x f(t)t^{\alpha+\beta}dt\right\}^p dx;$$

take $\alpha+\beta=s$; then $-\alpha p-1=c-sp$. Since $\beta>0$ we must have $c>-1$; since $\alpha>0$ we must have $sp-c>1$; and $C(c,s,p)$ in (4) is

$$\left(\frac{sp}{sp-c-1}\right)^p - 1.$$

Similarly if we replace $f(t)$ by $t^{s-\alpha+1}f(t)$ in (16) we have

$$\int_0^\infty x^{\beta p-1}\left\{\int_0^\infty f(t)t^{s-\alpha-\beta}dt\right\}^p dx \leqq \left\{\left(\frac{\alpha+\beta}{\beta}\right)^p - 1\right\}\int_0^\infty x^{-\alpha p-1}\left\{\int_0^x f(t)t^s dt\right\}^p dx.$$

Again take $-\alpha p-1=c-sp$, $\alpha+\beta=s$; we obtain

$$\int_0^\infty x^c\left\{\int_x^\infty f(u)du\right\}^p dx \leqq \left\{\left(\frac{sp}{1+c}\right)^p - 1\right\}\int_0^\infty x^{c-sp}\left\{\int_0^x f(u)u^s du\right\}^p dx,$$

and this is (4) reversed, with an explicit value for the constant.

As we remarked above, the preceding argument breaks down when $p<0$ (since the necessary inequalities run in opposite directions). However, we can still use Lemma 2. Since $K_1 \leqq MK_1 * K_2$, we can apply Lemma 2 with K_2 replaced by K_1 and K_3 by K_2. The result is

$$\int_0^\infty x^{-1}F_1(x)^p dx \leqq Mm_1^p\int_0^\infty x^{-1}F_2(x)^p dx,$$

or explicitly, when $\alpha > 0,\ \beta > 0,\ p < 0$,

$$(17)\qquad \int_0^\infty x^{-\alpha p-1}\left\{\int_0^x f(t)t^{\alpha-1}dt\right\}^p dx \leqq \left(\frac{\alpha+\beta}{\alpha}\right)^p \int_0^\infty x^{\beta p-1}\int_x^\infty f(t)t^{-\beta-1}dt\right\}^p dx,$$

and similarly

$$(18)\qquad \int_0^\infty x^{\beta p-1}\left\{\int_x^\infty f(t)t^{-\beta-1}dt\right\}^p dx \leqq \left(\frac{\alpha+\beta}{\beta}\right)^p \int_0^\infty x^{-\alpha p-1}\left\{\int_0^x f(t)t^{\alpha-1}dt\right\}^p dx.$$

To put these in the form of (4), we again replace $f(t)$ by $t^{\beta+1}f(t)$ in (17), and put $c = \beta p - 1$; but since $p < 0$ we have to have $c < -1$; and with $\alpha p = sp - c - 1$ we must have $c > sp - 1$; thus (4), and similarly the reversed inequality, hold for $p < 0$ with $s > 0$ and $sp - 1 < c < -1$.

Since inequalities with $p < 0$ are not frequently met with, it is of some interest to write one out explicitly. For example, take $p = -2,\ s = 1,\ c = -2$; then

$$\int_0^\infty dx\left\{\int_0^x uf(u)du\right\}^{-2} \leqq \frac{1}{4}\int_0^\infty x^{-2}dx\left\{\int_x^\infty f(u)du\right\}^{-2}.$$

REFERENCES

1. R. Askey and R. P. Boas, Jr., Some integrability theorems for power series with positive coefficients, to appear.

2. R. P. Boas, Jr., Integrability theorems for trigonometric transforms, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, Vol. **38** 1967.

3. E. K. Godunova, Integral 'nye neravenstva s vypuklymi funkcijami, *Izv. Vysš. Učebn. Zaved. Matematika* 1968, no. **1** (68), 47–49.

4. G. H. Hardy, J. E. Littlewood, and G. Pólya, Inequalities, Cambridge, 1934.

5. K. Knopp, Neuere Sätze über Reihen mit positiven Gliedern, *Math. Z.*, **30** (1929), 387–413.

6. L. Leindler, Über verschiedene Konvergenzarten trigonometrischer Reihen, *Acta. Sci. Math.* (Szeged), **25** (1964), 233–249.

7. G. Sunouchi, On the convolution algebra of Beurling, *Tôhoku Math. J.*, **19** (1967), 303–310.

NORTHWESTERN UNIVERSITY
EVANSTON, ILLINOIS, U.S.A.