# A COUNTEREXAMPLE CONCERNING INTEGRABILITY OF DERIVATIVES OF CONFORMAL MAPPINGS†

*By*

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## 1. Introduction

Let $\Omega$ be a simply connected plane domain and $f$ be a conformal mapping from $\Omega$ onto the unit disk $\Delta$. We shall be concerned with integrability properties of $f'$ on lines which intersect $\Omega$. Without loss of generality the line may be taken to be the real axis. The basic result is due to Hayman and Wu [HW].

**Hayman–Wu Theorem.** $\int_{\mathbf{R}\cap\Omega} |f'(x)|\,dx \leqq 10^{35}$.

Thus, $f' \in L^1(\mathbf{R} \cap \Omega)$ for all such $\Omega$ and $f$. Simpler proofs have been given in [GGJ] and [FHM]. In the latter paper the upper bound $10^{35}$ is reduced to $4\pi^2$, and a conjecture is offered for the best constant.

For which exponents $p > 1$ is it true that $f' \in L^p(\mathbf{R} \cap \Omega)$ for all $f$ and $\Omega$? Taking $\Omega$ to be $\Delta \backslash (-1, 0]$, one sees that $f' \in L^2(\mathbf{R} \cap \Omega)$ can fail. In [Ha, p. 638], I conjectured that $f' \in L^p(\mathbf{R} \cap \Omega)$ would be true for $p \in (1, 2)$. A result in this direction appears in [FHM], where it is shown that there is an absolute constant $\varepsilon > 0$ such that $f'$ always belongs to $L^{1+\varepsilon}(\mathbf{R} \cap \Omega)$.

It turns out, though, that my conjecture is false.

**Theorem 1.** *There exists a simply connected domain $\Omega$ and a number $p \in (1, 2)$ such that*

$$\int_{\mathbf{R}\cap\Omega} |f'(x)|^p dx = \infty$$

*for every conformal mapping $f$ of $\Omega$ onto the unit disk.*

One can also consider a two-dimensional analogue of this problem and ask for which $p$ is it true that $f' \in L^p(\Omega, dxdy)$ for every $\Omega$ and $f$. For $p = 2$ this is clearly the case, and for $p \in (2, 3)$ it follows from elementary distortion theorems. Brennan [Br] adapted a difficult technique of Carleson's to prove that $f' \in L^{3+\varepsilon}(\Omega)$ for some absolute $\varepsilon > 0$. He conjectured that $f' \in L^p(\Omega)$ should hold

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for $p \in (2, 4)$. The slit disk again shows that $f' \in L^4(\Omega)$ can fail. The best partial result in the direction of Brennan's conjecture is due to Pommerenke [P1], [P2]: $f' \in L^p(\Omega)$ for $p \in (2, 3.399)$.

To prove Theorem 1 we show that $p$-integrability fails for the complement of a rather simple "tree". Recently various authors, including Carleson, Makarov, Volberg, Pryztycki, Zdundik, and Urbański, see [M] for references, have used concepts from dynamics to study harmonic measure in domains with fractal boundaries. Wolff's example [W] of a domain in $\mathbf{R}^3$ whose harmonic measure is not supported on a set of dimension 2 is inspired by these ideas. It seems plausible that a scale invariant construction more complicated than our tree might provide a counterexample to the Brennan conjecture.

During the course of this research I benefitted greatly from conversations with C. Bishop, J. Fernández, and J. Manfredi. My thanks go also to E. Villamor, whose careful reading of the manuscript uncovered several errors. I am especially grateful to D. Marshall, who with his computer established the inequality stated as Theorem 2 in the next section. Without his help it is unlikely that I ever would have known that my counterexample really was one.

## 2. The example

For $z_1, z_2 \in \mathbf{C}$ let $[z_1, z_2]$ denote the straight line segment connecting them. Write $a = e^{i\pi/3}$, and take $\alpha \in (0, 1/2)$. Eventually $\alpha$ will be quite small. Define $S = [0, (1-\alpha)\bar{a}] \cup [0, (1-\alpha)\bar{a}^2]$ and define inductively sets $T_k$ and $\mathscr{B}(k)$, $k \geqq 0$, by

$$T_0 = S + i = \{z + i : z \in S\}, \qquad \mathscr{B}(0) = \{i\},$$

$$\mathscr{B}(1) = \text{the two endpoints of } T_0 \text{ on } \operatorname{Im} z = \alpha,$$

$$T_k = T_{k-1} \cup \left[ \bigcup_{b \in \mathscr{B}(k)} (\alpha^k S + b) \right], \qquad k \geqq 1,$$

$$\mathscr{B}(k+1) = \text{the endpoints of } T_k \text{ on } \operatorname{Im} z = \alpha^{k+1}.$$

Thus, there are $2^k$ points in $\mathscr{B}(k)$. (See Fig. 1, where $T_2$ is shown.)

Let $T_{-1} = \{iy : y \in [1, \infty]\}$, and define $T$ to be the closure of $\bigcup_{k=-1}^{\infty} T_k$. Then $T = T(\alpha)$ is a continuum on the sphere containing $\infty$. Let $\Omega = \mathbf{C} \setminus T$, and let $f$ be a conformal mapping from $\Omega$ onto the unit disk with $f(-i) = 0$.

We shall denote by $C$ an absolute positive constant whose value can change from line to line, and use the notation $A \approx B$ to mean that $C^{-1}A \leqq B \leqq CA$ for some $C$. For $b \in \mathscr{B}(k)$, $k \geqq 0$, define

$$m = \operatorname{Re} b, \qquad I(b) = [m - 10^{-2}\alpha^k, m + 10^{-2}\alpha^k].$$

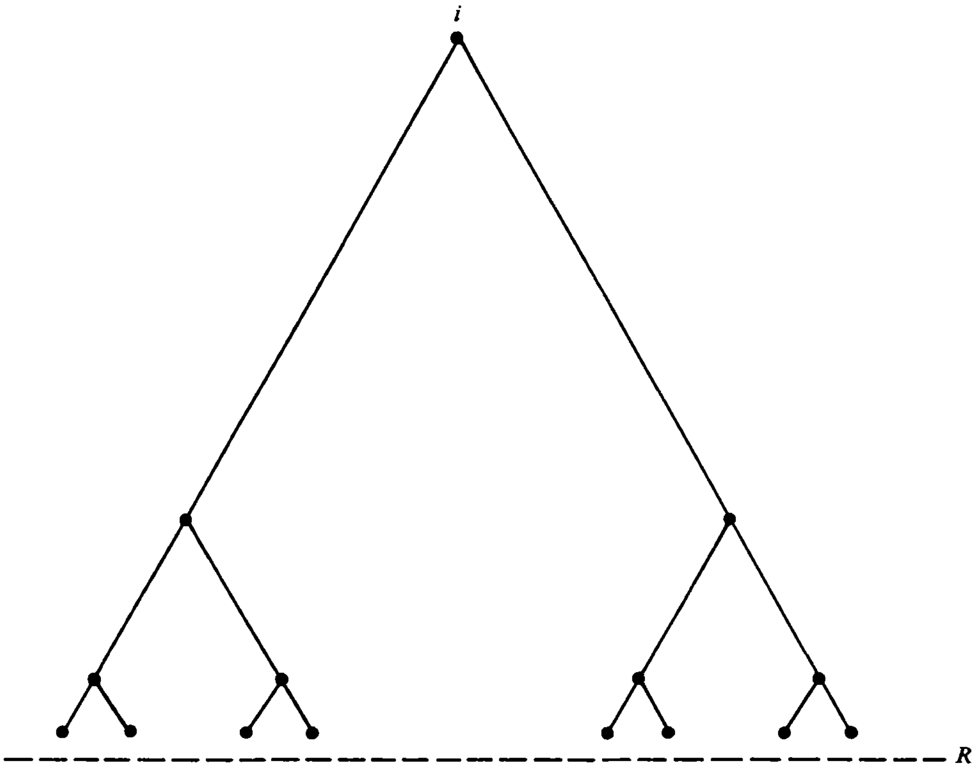


Fig. 1.

The $I(b)$ are pairwise disjoint, and $|I(b)| \approx \operatorname{dist}(I(b), T)$. (We use $|E|$ to denote the Lebesgue measure of $E \subset \mathbf{R}$.) Write

$$g(z) = g(z, -i, \Omega)$$

for the Green's function. Using Schwarz's lemma, the one-quarter theorem, and conformal invariance of Green functions, one sees that $g(z) \approx |f'(z)| \operatorname{dist}(z, T)$ for $z \in \Omega$, $|z+i| > 1/2$. Thus, for $x \in I(b)$,

$$g(x) \approx |f'(x)|\,|I(b)|.$$

Moreover, $g(x) \approx g(x')$ for $x, x' \in I(b)$, by Harnack's inequality. Hence

$$\int_{I(b)} |f'(x)|\,dx \approx g(m), \qquad b \in \mathscr{B}(k).$$

Define $E(k) = \bigcup\{I(b) : b \in \mathscr{B}(k)\}$. Then

$$\int_{E(k)} |f'(x)|\,dx \approx \sum_{b \in \mathscr{B}(k)} g(m). \tag{2.1}$$

We will show that if $\alpha$ is chosen small enough, then a number $q > 2$ exists such that $\forall k \geqq 0$,

$$\sum_{b \in \mathscr{B}(k)} g(m) \geqq C(q\alpha)^{k/2}. \tag{2.2}$$

Take $p \in (1, 2)$, and write $A = \int_{\mathbf{R} \cap \Omega} |f'|^p dx$. Let $p'$ denote the conjugate index of $p$. By Hölder's inequality, for $E \subset \mathbf{R} \cap \Omega$,

$$\int_E |f'| dx \leqq A^{1/p} |E|^{1/p'}.$$

Now $|E(k)| = C(2\alpha)^k$. From (2.1), (2.2), we deduce

$$(q\alpha)^{k/2} \leqq CA^{1/p}(2\alpha)^{k/p'}, \qquad k \geqq 0. \tag{2.3}$$

Let $p_0$ be the exponent conjugate to $2(\log 2\alpha)(\log q\alpha)^{-1}$. Then $p_0 \in (1, 2)$. Taking $k$th roots in (2.3) and letting $k \to \infty$, we see that if $A < \infty$ then $p' \geqq p_0'$. Hence $f' \notin L^p(\mathbf{R} \cap \Omega)$, when $p_0 < p < 2$.

To prove Theorem 1 it thus suffices to prove (2.2). This estimate will be derived from properties of a pair of simple conformal maps. Recall that $a = e^{i\pi/3}$. Let $D$ be the "fork domain"

$$D = \mathbf{C} \setminus [[0, a] \cup \{x \in \mathbf{R} : x \in (-\infty, 1]\}].$$

Let $F_1$ and $F_2$ conformally map $D$ onto the slit plane $\mathbf{C} \setminus \{x \in \mathbf{R} : -\infty < x \leqq 0\}$, with

$$F_1(1) = 0, \quad F_2(a) = 0, \quad |F_i(z)| \sim |z| \quad \text{as } z \to \infty, \quad i = 1, 2.$$

Define $\beta, \gamma \in (0, \infty)$ by

$$\beta = \lim_{z \to a} \left| \frac{F_2(z)}{z - a} \right|, \qquad \gamma = \lim_{z \to 1} \left| \frac{F_1(z)}{z - 1} \right|.$$

**Theorem 2.** $\beta^{1/2} + \gamma^{1/2} > 2^{1/2}$.

The values of $\beta$ and $\gamma$ were computed for me by Donald Marshall, who used Trefethen's program [T], [He, p. 422] for finding parameters for Schwarz–Christoffel transformations. According to the computer

$$\beta = 0.49824727, \qquad \gamma = 0.75253266,$$

which gives $\beta^{1/2} + \gamma^{1/2} > 1.57$. As a check on the computation, in §5 we start with the 4-place decimal approximations to the parameters given by the computer and confirm by calculus that the true values of $\beta$ and $\gamma$ satisfy $\beta^{1/2} + \gamma^{1/2} > 1.56$. It would be desirable to have a conceptual proof of Theorem 2.

Let us return now to the domain $\Omega$. For $b \in \mathscr{B}(k)$, there is a unique path in $T$ from $b$ to $i$. This path contains exactly one point of $\mathscr{B}(j)$, $0 \leqq j \leqq k$. Denote it by $b_j$. Thus $b_k = b$, $b_0 = i$, and $b_j$ is the "ancestor" of $b_k$ in the $j$th generation. For $1 \leqq j \leqq k-1$, when the path passes through $b_j$ it either continues in a straight line or makes a 120° turn. Define

$$\nu(b) = \text{number of times the path makes a turn.} \tag{2.4}$$

Thus $0 \leqq \nu(b) \leqq k-1$, and for $0 \leqq j \leqq k-1$, $\nu(b)$ equals $j$ for exactly $2\binom{k-1}{j}$ of the $b$'s in $\mathscr{B}(k)$. Here now is the main estimate in the proof of Theorem 1.

Given $\varepsilon > 0$, $\exists\, \alpha$ such that for $T = T(\alpha)$, all $b \in \mathscr{B}(k)$, and $k \geqq 0$,

$$g(m) \geqq C(e^{-\varepsilon k}\alpha^k \beta^\nu \gamma^{k-\nu})^{1/2}. \tag{2.5}$$

This will be proved in §3, except for some lemmas which will be proved in §4. Accepting it for now and using the binomial theorem, we find that

$$\sum_{b \in \mathscr{B}(k)} g(m) \geqq C[e^{-\varepsilon/2}\alpha^{1/2}(\beta^{1/2} + \gamma^{1/2})]^k.$$

Define $q = q(\varepsilon)$ by $q^{1/2} = (\beta^{1/2} + \gamma^{1/2})e^{-\varepsilon/2}$. By Theorem 2, $q > 2$ when $\varepsilon > 0$ is sufficiently small. Thus (2.2), and hence Theorem 1, follows from (2.5).

## 3. Proof of (2.5)

For $0 < \kappa < 1/4$ and $D$ the fork domain of §2 define

$$D_1(\kappa) = D \setminus \Delta(a, \kappa), \qquad D_2(\kappa) = D \setminus \Delta(1, \kappa).$$

Here $\Delta(z_0, \rho) = \{z \in \mathbf{C} : |z - z_0| < \rho\}$. For $i = 1, 2$ let $F_{i,\kappa}$ be a conformal mapping from $D_i(\kappa)$ onto $\mathbf{C} \setminus \{x \in \mathbf{R} : -\infty < x \leqq 0\}$ with $F_{1,\kappa}(1) = 0$, $F_{2,\kappa}(a) = 0$, $|F_{i,\kappa}(z)| \sim |z|$ as $z \to \infty$. By Theorem 2, when $\kappa$ is sufficiently small, $\beta(\kappa) = |F'_{1,\kappa}(1)|$ and $\gamma(\kappa) = |F'_{2,\kappa}(a)|$ satisfy $\beta(\kappa)^{1/2} + \gamma(\kappa)^{1/2} > 2^{1/2}$. Fix such a $\kappa$ once and for all, and then suppress the dependence on $\kappa$. Thus, we will write $D_1$, $D_2$, $F_1$, $F_2$, $\beta$, $\gamma$, instead of $D_1(\kappa), \ldots, \gamma(\kappa)$.

Define also another fork-type domain $D_3$ by

$$D_3 = \mathbf{C} \setminus [(-\infty, 0] \cup [0, e^{i\pi/6}] \cup [0, e^{-i\pi/6}] \cup \Delta(e^{-i\pi/6}, \kappa)],$$

and let $F_3$ conformally map $D_3$ onto $\mathbf{C} \setminus (-\infty, 0]$ with $|F_3(z)| \sim |z|$ for $z \to \infty$, $F_3(e^{i\pi/6}) = 0$.

Let $\rho \in (0, 1/4)$, and suppose that $z_1$, $z_2$ satisfy

$$|z_1 - 1| < \rho^{3/2}, \qquad |z_2| < \rho^{-1/2}. \tag{3.1}$$

Define

$$D_1(z_1, z_2, \rho) = [D_1 \cap \Delta(z_2, \rho^{-1})] \setminus \bar{\Delta}(z_1, \rho).$$

Here $\bar{\Delta}$ denotes the closure of a disk $\Delta$. The circles $\partial\Delta(z_1, \rho)$, $\partial\Delta(z_2, \rho^{-1})$ intersect (0, 1), resp. the negative real axis, in exactly one point. Denote the first one by $z_1 + \rho e^{i\phi_1}$ and the second by $z_2 + \rho^{-1} e^{i\phi_2}$, with $\phi_1, \phi_2, \in (\pi/2, 3\pi/2)$. Similarly, if

$$|z_1 - a| < \rho^{3/2}, \qquad |z_2| < \rho^{-1/2}, \tag{3.2}$$

or if

$$|z_1 - e^{i\pi/6}| < \rho^{3/2}, \qquad |z_2| < \rho^{-1/2}, \tag{3.3}$$

define for $i = 2, 3$,

$$D_i(z_1, z_2, \rho) = [D_i \cap \Delta(z_2, \rho^{-1})] \setminus \bar{\Delta}(z_1, \rho).$$

Let $z_2 + \rho^{-1} e^{i\phi_2}$ be the same as above, and $z_1 + \rho e^{i\phi_1}$ be, in the case of (3.2), the point where $\partial\Delta(z_1, \rho)$ meets $[0, a]$, and in the case of (3.3) the point where $\partial\Delta(z_1, \rho)$ meets $[0, e^{i\pi/6}]$. See Fig. 2. One should think of $\rho$ as being very small. The points $A$ and $B$ are, resp. $z_2 + \rho^{-1} e^{i\phi_2}$ and $z_1 + \rho e^{i\phi_1}$.

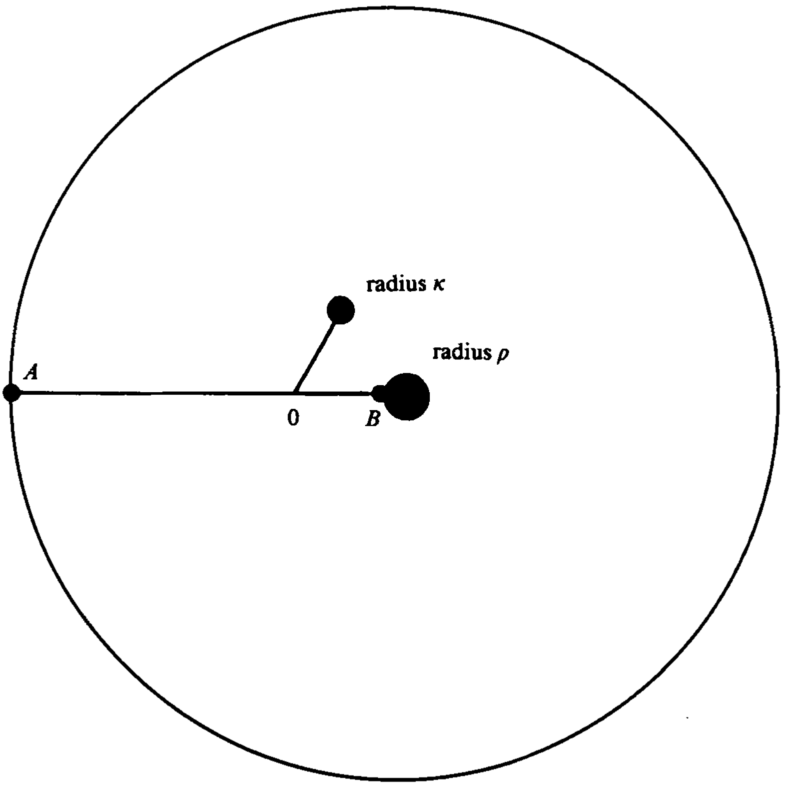


Fig. 2.

Let $R(L)$ denote the open rectangle $(0, L) \times (-\pi, \pi)$. Define, for $i = 1, 2, 3$,

$$L(i) = 2 \log \rho^{-1} - \lambda(i),$$

where $\lambda(1) = \log \gamma$, $\lambda(2) = \log \beta$, $\lambda(3) = \log |F_3'(e^{i\pi/6})|$.

The following lemma, which will be proved in §4, states that for small $\rho$ the $D_i$ are conformally close to $R[L(i)]$.

**Lemma 1.** *There exist absolute constants $\rho_0$ and $C$ such that if (3.$i$) holds for $i = 1, 2$, or 3 and if $\rho \in (0, \rho_0]$, then there exists a homeomorphism $H: D_i(z_1, z_2, \rho) \to R[L(i)]$ whose boundary values satisfy*

(a) $H(z_1 + \rho e^{i\phi}) = i[(\pi - \phi_1) + \phi]$, $-2\pi + \phi_1 \leqq \phi \leqq \phi_1$,

(b) $H(z_2 + \rho^{-1} e^{i\phi}) = L(i) + i[(\pi - \phi_2) + \phi]$, $-2\pi + \phi_2 \leqq \phi \leqq \phi_2$.

*Moreover, $H$ is $(1 + C\rho^{1/2})$-quasiconformal.*

Return now to the situation of (2.5), and recall the notation introduced before (2.4): $m = \operatorname{Re} b$, $b_k = b$, $b_0 = i$, and $b_j$ is the ancestor of $b_k$ lying in $\mathscr{B}(j)$. For $0 \leqq j \leqq k-1$ let $b_j'$ be the point of intersection of the line through $[b_j, b_{j+1}]$ with $\mathbf{R}$. Set $b_k' = b_{k-1}'$, and for $0 \leqq j \leqq k$ let $b_j''$ be the reflection of $b_j'$ in the line $\operatorname{Re} z = \operatorname{Re} b_j$. Write $s = 2 \cdot 3^{-1/2}$. Then $s$ is the side length of an equilateral triangle of altitude 1.

Next, define disks $\Delta_j$, $\tilde{\Delta}_j$, and domains $\Omega_j$, $\Omega'$, by

$$\Delta_j = \Delta(m, s\alpha^{j+1/2}), \qquad j = -1, 0, \ldots, k-1,$$

$$\Delta_k = \Delta(b_k', s\alpha^{k+1/2}),$$

$$\tilde{\Delta}_j = \Delta(b_j'', \kappa s\alpha^j), \qquad j = 0, \ldots, k,$$

$$\Omega' = [(\Omega \cap \Delta_{-1}) \setminus \bar{\Delta}_k] \setminus \left( \bigcup_{j=0}^{k} \bar{\tilde{\Delta}}_j \right),$$

$$\Omega_j = \Omega' \cap (\Delta_{j-1} \setminus \bar{\Delta}_j), \qquad j = 0, \ldots, k.$$

The number $\kappa$ was introduced at the beginning of this section. Recall that if we climb $T$ from $b_k$ back to $b_0$, then at each $b_j$ we either continue in a straight line, denote the set of these $j$ by I, or turn by 120°; denote these by II. Since $b_k' = b_{k-1}'$, we declare that $k \in \mathrm{I}$. Define affine maps $A_1, \ldots, A_k$ by

$$A_j(z) = (b_j' - b_j)^{-1}(z - b_j), \qquad j \in \mathrm{I}, \qquad 1 \leqq j \leqq k,$$

$$A_j(z) = (b_j'' - b_j)^{-1}(z - b_j), \qquad j \in \mathrm{II}, \qquad 1 \leqq j \leqq k.$$

Let $z_j = A_j(m)$, $1 \leqq j \leqq k$. Since $s\alpha^j = |b_j - b_j'| = |b_j - b_j''|$, for $1 \leqq j \leqq k-1$, $A_j$ maps $\Omega_j$ onto one of the four domains $D_i(z_j, z_j, \alpha^{1/2})$, or $D_i^*(z_j, z_j, \alpha^{1/2})$,

$i = 1, 2$, where $*$ denotes reflection in $\mathbf{R}$. Also, $A_k$ maps $\Omega_k$ onto $D_1(1, z_k, \alpha^{1/2})$. Define

$$A_0(z) = e^{\pm(i\pi/6)}(z - b_0), \qquad z_0 = A_0(m),$$

where the sign in $\pm$ is chosen so that $A_0$ maps $\Omega_0$ onto $D_3(z_0, z_0, \alpha^{1/2})$.

Suppose that $1 \leqq j \leqq k-1$. If $j \in \mathrm{I}$ then $A_j(\Omega_j)$ is a $D_1$ or $D_1^*$, and

$$|z_j - 1| = \frac{|m - b_j'|}{|b_j' - b_j|} \leqq C\alpha.$$

Similarly, $|z_j - a| \leqq C\alpha$ for $j \in \mathrm{II}$, $|z_0 - e^{\pm(i\pi/6)}| \leqq C\alpha$, and $|z_k| \leqq C$. Choose $\alpha$ so small that $C < \alpha^{-1/4}$. Then one of the hypotheses (3.1)–(3.3) is satisfied for $A_j(\Omega_j)$, with $\rho = \alpha^{1/2}$, when $z_1$, $z_2$ there are replaced by $z_j$, $z_j$ $(0 \leqq j \leqq k-1)$ or $1, z_k$. Let $L_j = L(1)$ for $j \in \mathrm{I}$, $L_j = L(2)$ for $j \in \mathrm{II}$, and $L_0 = L(3)$, where the $L(i)$ were defined before the statement of Lemma 1. Let $H_j$ be the QC map of $A_j(\Omega_j)$ onto $R(L_j)$ provided by Lemma 1. (In case $A_j(\Omega_j)$ is a $D_i^*$, we use the map $\bar{H}(\bar{z})$.) Define $h_j = H_j \circ A_j$, $0 \leqq j \leqq k$. Then, by (a) and (b) of Lemma 1, $h_j = h_{j-1}$ on $\Omega' \cap \partial\Delta_{j-1}$, $1 \leqq j \leqq k$. Define $L = \sum_{j=0}^{k} L_j$ and $h : \Omega' \to R(L)$ by

$$h(z) = h_j(z) + \sum_{i=j+1}^{k} L_i, \qquad 0 \leqq j \leqq k, \quad z \in \bar{\Omega}_j.$$

By Lemma 1, $h$ is a $(1 + C\alpha^{1/4})$-QC homeomorphism of $\Omega'$ onto $R(L)$. In §4 we shall see there are absolute constants $C$ and $\eta > 0$ such that

$$\text{(3.4)} \quad \begin{aligned} &|\operatorname{Re} h(m) - (1/2)\log \alpha^{-1}| < C, && |\operatorname{Im} h(m)| < \pi - \eta, \\ &|\operatorname{Re} h(-i) - (L - (1/2)\log \alpha^{-1})| < C, && |\operatorname{Im} h(-i)| < \pi - \eta. \end{aligned}$$

To finish, we need two estimates for Green functions. The second one is certainly known and probably so is the first, but for completeness proofs will be given at the end of §4.

**Lemma 2.** *Let $\Omega_1$, $\Omega_2$ be simply connected plane domains, and $h$ be a $K$-quasiconformal homeomorphism of $\Omega_1$ and $\Omega_2$. Assume that $K \leqq 2$ and that $g(z, z', \Omega) \leqq 1$, where $z, z' \in \Omega$. Then*

$$g(z, z', \Omega_1) \geqq C[g(h(z), h(z'), \Omega_2)]^K.$$

**Lemma 3.** *Suppose that $P_i \in R(L)$, $i = 1, 2$, with $L > 1$, and that $1 \leqq \operatorname{Re} P_1 < \operatorname{Re} P_2 \leqq L - 1$, $|\operatorname{Im} P_i| \leqq \pi - \eta$, $i = 1, 2$. Then there is a constant $C$ depending only on $\eta$ such that*

$$g(P_1, P_2, R(L)) \geqq Ce^{(-1/2)(\operatorname{Re} P_2 - \operatorname{Re} P_1)}.$$

Returning to the proof of (2.5), choose $\alpha$ so small that $\frac{1}{2}\log(\alpha^{-1}) > C + 1$, where $C$ is the constant in (3.4). Then

(3.5) $$\operatorname{Re} h(-i) - \operatorname{Re} h(m) \leqq L - \log(\alpha^{-1}) + C,$$

and also the hypotheses of Lemma 3 are satisfied for $P_1 = h(m)$, $P_2 = h(-i)$.

Using Lemmas 2, 3, (3.5), and the fact that $h$ is $(1 + C\alpha^{1/4})$-QC, we obtain

$$g(m, -i, \Omega') \geqq C \exp[(-1/2)(L + \log\alpha + C)(1 + C\alpha^{1/4})]$$
$$\geqq C \exp[-(1/2)(L + \log\alpha)]\exp[-C\alpha^{1/4}(L + \log\alpha)].$$

Now $L = \Sigma_{j=0}^{k} L_j$, so

(3.6) $$L + \log\alpha = -k\log\alpha - (v - k)\log\gamma - v\log\beta - \log|F_3'(e^{i\pi/6})|,$$

where $v$ is the number of $j \in \mathrm{II}$. Thus

$$g(m, -i, \Omega') \geqq C(\alpha^k\beta^v\gamma^{k-v})^{1/2}E,$$

where, using (3.6),

(3.7) $$\log(E^{-1}) = C\alpha^{1/4}(L + \log\alpha) \leqq C\alpha^{1/4}[\log(\alpha^{-1}) + C]k.$$

For given $\varepsilon > 0$, choose $\alpha$ so small that

$$C\alpha^{1/4}[\log(\alpha^{-1}) + C] < \varepsilon/2.$$

Then $E \geqq e^{-\varepsilon k/2}$, and hence

$$g(m, -i, \Omega') \geqq C(e^{-\varepsilon k}\alpha^k\beta^v\gamma^{k-v})^{1/2}.$$

Since $\Omega' \subset \Omega$, the proof of (2.5) is complete, except for the proof of the lemmas.

## 4. Proof of the lemmas

We shall prove Lemma 1 for $D_1(z_1, z_2, \rho)$, and shall write $F$ instead of $F_1$. The proofs for $D_2$ and $D_3$ require only obvious changes. Square root transformations show that as $z \to 1$ we have

(4.1) $$F(z) = \gamma(z - 1) + O(|z - 1|^{3/2}),$$

(4.2) $$(z - 1)F'(z)/F(z) = 1 + O(|z - 1|^{1/2}),$$

and as $z \to \infty$

(4.3) $$F(z) = z + O(|z|^{1/2}),$$

(4.4) $$zF'(z)/F(z) = 1 + O(|z|^{-1/2}).$$

Let $Q = \log F(D_1(z_1, z_2, \rho))$. Then (3.1), (4.1), (4.3) show that $Q$ is a quadrilateral with a pair of opposite sides on $\operatorname{Im} z = \pm\pi$. (Breaking tradition, we denote points in the image plane by $z$ also.) The other sides of $Q$ are the image curves of $\log F(\partial\Delta(z_1, \rho))$ and $\log F(\partial\Delta(z_2, \rho^{-1}))$, which we denote by $\Gamma_1$ and $\Gamma_2$, respectively. When $\rho$ is small the $\Gamma_i$ are essentially vertical segments lying on the lines $\operatorname{Re} z = \log \gamma + \log \rho$, $\operatorname{Re} z = -\log \rho$. More precisely, define $A_i$ and $B_i$, $i = 1, 2$, by

$$(A_1 + iB_1)(\phi) = \log F(z_1 + \rho e^{i\phi}), \qquad \phi_1 - 2\pi \leqq \phi \leqq \phi_1,$$

$$(A_2 + iB_2)(\phi) = \log F(z_2 + \rho^{-1} e^{i\phi}), \qquad \phi_2 - 2\pi \leqq \phi \leqq \phi_2.$$

Then, using (3.1), (4.2), (4.4), a calculation we leave to the reader shows that for $i = 1, 2$ and $\phi_i - 2\pi \leqq \phi \leqq \phi_i$,

$$|A_i'(\phi)| \leqq C\rho^{1/2}, \qquad |B_i'(\phi) - 1| \leqq C\rho^{1/2}. \tag{4.5}$$

Choose $\rho_0$ small enough so that $\inf_\phi B_i'(\phi) > 0$ when $\rho \leqq \rho_0$. Then each $\Gamma_i$ is intersected exactly once by each line $\operatorname{Im} z = y$, $-\pi \leqq y \leqq \pi$. Thus, $\Gamma_i$ may be described in the form $x = \sigma_i(y)$, $|y| \leqq \pi$. From (4.5) it follows that, with $\phi = \phi_i(y)$,

$$|\sigma_i'(y)| = \left|\frac{A_i'(\phi)}{B_i'(\phi)}\right| \leqq C\rho^{1/2}, \qquad |y| \leqq \pi. \tag{4.6}$$

From (3.1), (4.1), (4.3), it follows that

$$|L - (\sigma_2(y) - \sigma_1(y))| \leqq C\rho^{1/2}, \tag{4.7}$$

where $L = L(1) = 2 \log \rho^{-1} - \log \gamma$.

Thus, $Q$ is nearly a translation of the rectangle $R(L) = (0, L) \times (-\pi, \pi)$. For $z = x + iy \in Q$, define

$$h_1(z) = L \, \frac{x - \sigma_1(y)}{\sigma_2(y) - \sigma_1(y)} + iy.$$

Then $h_1$ maps $Q$ 1–1 onto $R(L)$. Define

$$P_i(y) = B_i^{-1}(y) + (\pi - \phi_i), \qquad |y| \leqq \pi,$$

for $i = 1, 2$, where $B_i^{-1}$ denotes the inverse function of $B_i$. The $P_i$ are increasing homeomorphisms of $[-\pi, \pi]$ onto itself. From (4.5), it follows that for $|y| \leqq \pi$ and $i = 1, 2$,

$$|P_i'(y) - 1| \leqq C\rho^{1/2} \tag{4.8}$$

and, since $P_i(\pi) = \pi$,

$$|P_i(y) - y| \leqq C\rho^{1/2}. \tag{4.9}$$

Define, for $z = x + iy \in R(L)$,

$$h_2(z) = x + i\left[\left(1 - \frac{x}{L}\right)P_1(y) + \frac{x}{L}P_2(y)\right].$$

Then $h_2$ maps $R(L)$ 1–1 onto itself. Define $H : D_1(z_1, z_2, \rho) \to R(L)$ by $H = h_2 \circ h_1 \circ (\log F)$. Then $H$ is a homeomorphism of $D_1(z_1, z_2, \rho)$ onto $R(L)$ which satisfies conclusions (a) and (b) of Lemma 1.

To conclude the proof of Lemma 1 we will show that $h_1$ and $h_2$ are $(1 + C\rho^{1/2})$-QC. Write $h_1(z) = u(z) + iy$, $h_2(z) = x + iv(z)$. Then

$$\text{(4.10)} \qquad \left|\frac{(h_1)_{\bar z}}{(h_1)_z}\right|^2 = \frac{(u_x - 1)^2 + u_y^2}{(u_x + 1)^2 + u_y^2} \quad \text{and} \quad \left|\frac{(h_2)_{\bar z}}{(h_2)_z}\right|^2 = \frac{(v_y - 1)^2 + v_x^2}{(v_y + 1)^2 + v_x^2}.$$

Calculation gives

$$u_x - 1 = \frac{L - (\sigma_2(y) - \sigma_1(y))}{(\sigma_2(y) - \sigma_1(y))},$$

$$u_y = -L\,\frac{(\sigma_2(y) - \sigma_1(y))\sigma_1'(y) + (x - \sigma_1(y))(\sigma_2'(y) - \sigma_1'(y))}{(\sigma_2(y) - \sigma_1(y))^2},$$

$$v_x = \frac{P_2(y) - P_1(y)}{L},$$

$$v_y - 1 = \frac{(P_1'(y) - 1)L + (P_2'(y) - P_1'(y))x}{L},$$

where in the first two equations $z = x + iy \in Q$ and in the second two $z \in R(L)$.

Now $L \geqq 1$, and for $z \in Q$, $0 \leqq x - \sigma_1(y) \leqq \sigma_2(y) - \sigma_1(y)$. Using (4.7), (4.6), (4.9) and (4.8), we deduce that $|u_x - 1|$, $|u_y|$, $|v_x|$ and $|v_y - 1|$ are all bounded above by $C\rho^{1/2}$. From (4.10) it follows that $h_1$ and $h_2$ are $(1 + C\rho^{1/2})$-QC, and the proof of Lemma 1 is complete.

Next, we shall verify (3.4). From the definitions in §3, it follows that $h(m)$ equals either $F_1((1 + a)/2)$ or its complex conjugate. Write

$$\log F_1\left(\frac{1 + a}{2}\right) = C_1 + iC_2.$$

Then $|C_2| < \pi$. The construction of $h_1$, $h_2$ and the various inequalities in this section show that

$$\sup_{z \in Q} |h_2 \circ h_1(z) - (z + \log(\rho\gamma)^{-1})| \leqq C\rho^{1/2}.$$

Thus, when $\rho_0$ is chosen sufficiently small, we have

$$|\operatorname{Im} h_2 \circ h_1(C_1 \pm iC_2)| < \tfrac{1}{2}(\pi + |C_2|) \equiv \pi - \eta,$$

$$|\operatorname{Re} h_2 \circ h_1(C_1 \pm iC_2) + \log \rho| \leqq C.$$

Since $\rho = \alpha^{1/2}$, the inequalities for $h(m)$ in (3.4) are verified. Those for $h(-i) = F_3(2)$ are established in the same way, where if necessary we take smaller values of $\rho_0$ and $\eta$.

**Proof of Lemma 2.** We may assume that $\Omega_1$ and $\Omega_2$ are the unit disk, that $z = h(z) = 0$, and that $z' = r \in (e^{-1}, 1)$. Then, by reflection, $h$ has a $K$-QC extension to **C** which maps the unit circle onto itself. Recall that $K \leqq 2$. Distortion theorems for QC maps (see e.g. [LV, p. 64, Thm. 3.1]) imply that $|h(r)| \geqq C > 0$ and that

$$g(0, h(r)) \leqq C(1 - |h(r)|) \leqq C|h(1) - h(r)| \leqq C(1-r)^{1/K} \leqq Cg(0,r)^{1/K}.$$

**Proof of Lemma 3.** If $D$ is the half-disk $\{z : |z| < 1, \operatorname{Im} z > 0\}$ and $0 < y_1 < y_2 \leqq 0.99$ then conformal mapping onto a half-plane and calculation show that

$$g(iy_1, iy_2, D) \geqq C \frac{y_1}{y_2}. \tag{4.11}$$

Consider the situation of Lemma 3. By Harnack's inequality we may assume $\operatorname{Im} P_1 = \operatorname{Im} P_2 = 0$. Let $\Phi(z) = \sin(\frac{1}{2}iz)$. Then $\Phi$ maps $R(L)$ 1–1 onto a domain containing the half disk $\{z : |z| < \Phi(L), \operatorname{Im} z > 0\}$. Also $|\Phi(P_2)| < (0.99)|\Phi(L)|$, since $1 \leqq P_1 < P_2 \leqq L - 1$.

Hence, by (4.11)

$$g(P_1, P_2, R(L)) \geqq C \frac{\sinh(\frac{1}{2})P_1}{\sinh(\frac{1}{2})P_2} \geqq Ce^{(P_1 - P_2)/2}.$$

## 5. Proof of Theorem 2

Let $H = \{z : \operatorname{Im} z > 0\}$, and for $l_1, l_2 > 0$ define

$$H(l_1, l_2) = H \setminus [[0, l_1 e^{2\pi i/3}] \cup [0, il_2]].$$

There is a unique conformal map from $H$ onto $H(l_1, l_2)$ with the properties $f(z) \sim z$ at $\infty$, $f(0) = 0$, and $f^{-1}(l_1 e^{2\pi i/3}) < 0 < f^{-1}(il_2)$. There is exactly one solution of $f(x) = 0$ in each of the intervals $(-\infty, f^{-1}(l_1 e^{2\pi i/3}))$, $(f^{-1}(il_2), \infty)$. Denote them by $P$ and $Q$, respectively. Then, taking $0 < \arg z < \pi$,

$$f(z) = (z-P)^{1/3} z^{1/6} (z-Q)^{1/2}, \qquad z \in H, \tag{5.1}$$

since the function displayed and the conformal mapping have the same argument on **R** and are both $\simeq z$ at $\infty$. Thus, there is a 1–1 correspondence between points $(P, Q) \in (-\infty, 0) \times (0, \infty)$ and points $(l_1, l_2) \in (0, \infty) \times (0, \infty)$. Define the homeomorphism $G : (-\infty, 0) \times (0, \infty) \to \mathbf{R}^2$ by

$$(5.2) \qquad G(P, Q) = 6(\log l_1, \log l_2).$$

The "6" will ease the arithmetic later on. I thank D. Marshall for pointing out that maps from $H$ to $H(l_1, l_2)$ have the simple form (5.1).

Define also, when $f$ is given by (5.1),

$$s_1(P, Q) = f^{-1}(l_1, e^{2\pi i/3}), \qquad s_2(P, Q) = f^{-1}(il_2).$$

Then $s_1$ and $s_2$ are the critical points of the polynomial $f^6$ in the intervals $(P, 0)$ and $(0, Q)$. Logarithmic differentiation shows they are the roots of the equation $6z^2 - (4P + 3Q)z + PQ = 0$, and hence, after a small miracle,

$$(5.3) \qquad \begin{aligned} s_1(P, Q) &= \tfrac{1}{12}[(4P + 3Q) - (16P^2 + 9Q^2)^{1/2}], \\ s_2(P, Q) &= \tfrac{1}{12}[(4P + 3Q) + (16P^2 + 9Q^2)^{1/2}]. \end{aligned}$$

Write $f(z) = f(z, P, Q)$ to show the dependence on $P$ and $Q$. Then $f$ is a Schwarz–Christoffel mapping from $H$ onto the degenerate polygonal domain $H(l_1, l_2)$, and the pre-vertices of $f$ corresponding to $\infty, 0, l_1e^{2\pi i/3}, 0, il_2, 0$, are $\infty, P, s_1, 0, s_2, Q$.

The fork domain $D$ of Theorem 2 is mapped onto $H(1, 1)$ by the function $z \to iz^{1/2}$. Let $f = f(\cdot, 1, 1)$. One easily shows that the numbers $\beta$ and $\gamma$ of Theorem 2 are given by

$$(5.4) \qquad \beta = |f''(s_1)|^{-1}, \qquad \gamma = |f''(s_2)|^{-1},$$

where $s_i = s_i(P, Q)$, $(P, Q) = G^{-1}(0, 0)$. ($G$ is defined by (5.2).)

To prove Theorem 2, we shall find a sufficiently accurate estimate of $G^{-1}(0, 0)$ and then verify $\beta^{1/2} + \gamma^{1/2} > 2^{1/2}$ by (5.3), (5.4), and direct calculation. When Marshall ran Trefethen's program using the data for $H(1, 1)$ the output gave for $(P_1, Q_1) = G^{-1}(0, 0)$,

$$P_1 = -1.01661167, \qquad s_1 = -0.47233507,$$

$$s_2 = 0.52153271, \qquad Q_1 = 1.45387605,$$

and also

$$\beta = 0.49824727, \qquad \gamma = 0.75253266.$$

We shall prove that at least

$$(5.5) \qquad -1.0176 < P_1 < -1.0156, \qquad 1.4529 < Q_1 < 1.4549,$$

$$(5.6) \qquad -0.4739 < s_1 < -0.4709, \qquad 0.5201 < s_2 < 0.5230.$$

Using the facts that $f_z(s_i, P, Q) = 0$ and $|f(s_i, P_1, Q_1)| = 1$, it follows that

$$|f_{zz}(s_i, P_1, Q_1)| = \left|\frac{\partial^2 \log f}{\partial z^2}(s_i, P_1, Q_1)\right|$$
$$= \tfrac{1}{6}[2(s_i - P_1)^{-2} + s_i^{-2} + 3(s_i - Q_1)^{-2}].$$

Then (5.4)–(5.6) give

$$\beta > \frac{6}{12.151} > 0.49, \qquad \gamma > 0.74,$$

so that definitely

$$\beta^{1/2} + \gamma^{1/2} > 1.56.$$

To prove (5.5) and (5.6), let

(5.7) $$P_0 = -1.0166, \qquad Q_0 = 1.4539.$$

Then (5.3) give

(5.8) $$-0.472324 < s_1(P_0, Q_0) < -0.472328,$$
$$0.521541 < s_2(P_0, Q_0) < 0.521545.$$

Define

$$h(t, P, Q) = \log[(t - P)^2 |t|\, |t - Q|^3].$$

Then

$$G(P, Q) = h(s_1(P, Q), P, Q) + ih(s_2(P, Q), P, Q),$$

Let $\Delta_1 = \Delta(P_0 + iQ_0, 0.001)$. I claim that

(5.9) $$\sup_{\Delta_1}\left|\frac{\partial s_i}{\partial P}\right| < 0.57, \quad \sup_{\Delta_1}\left|\frac{\partial s_i}{\partial Q}\right| < 0.57, \qquad i = 1, 2,$$

and that

(5.10) $$(0, 0) \in G(\Delta_1), \qquad \text{so that } (P_1, Q_1) \in \Delta_1.$$

Then (5.5) and (5.6) follow from (5.7)–(5.10).

Now (5.9) follows from differentiation of (5.3) and straightforward estimation. Since $G$ is a homeomorphism of $\mathbf{R}_- \times \mathbf{R}_+$, to prove (5.10) it will suffice to prove

(5.11) $$|G(P_0 + iQ_0)| < \inf_{z \in \partial\Delta_1} |G(z) - G(P_0 + iQ_0)|.$$

Direct calculation gives

(5.12) $$|G(P_0 + iQ_0)| < 0.0002.$$

To obtain a lower bound for $G - G(P_0 + iQ_0)$ on $\partial\Delta_1$ write

$$G(P + iQ) - G(P_0 + iQ_0)$$
$$= a[(P - P_0) + i(Q - Q_0)] + b[(P - P_0) + i(Q - Q_0)] + \Phi(P + iQ)$$
$$= L(P, Q) + \Phi(P, Q),$$

where $a = \frac{1}{2}(G_P - iG_Q)(P_0 + iQ_0)$, $b = \frac{1}{2}(G_P + iG_Q)(P_0 + iQ_0)$. Then

$$\inf_{\partial\Delta_1} |L| = ||a| - |b||(0.001),$$

$$\sup_{\partial\Delta_1} |\Phi| \leqq \sup_{\Delta_1} (|G_{PP}|, |G_{PQ}|, |G_{QQ}|)(0.001)^2.$$

Thus (5.11) follows from (5.12), and the bounds

$$|b| > 3.44846, \qquad |a| < 1.44718, \tag{5.13}$$

so that $\inf_{\partial\Delta_1} |L| > 0.002$, and

$$\sup_{\Delta_1} (|G_{PP}|, |G_{PQ}|, |G_{QQ}|) < 50. \tag{5.14}$$

**Proof of (5.13).** Write $G = G_1 + iG_2$ and

$$dG(P_0 + iQ_0) = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix},$$

where $a_1 = (G_1)_P$, $a_2 = (G_1)_Q$, $a_3 = (G_2)_p$, $a_4 = (G_2)_Q$, all evaluated at $P_0 + iQ_0$. By the chain rule and the fact that $h_t(S_i(P, Q), P, Q) = 0$, we have, with $s_i = s_i(P_0, Q_0)$,

$$a_1 = -2(s_1 - P_0)^{-1}, \qquad a_2 = 3(Q_0 - s_1)^{-1},$$
$$a_3 = -2(s_2 - P_0)^{-1}, \qquad a_4 = 3(Q_0 - s_2)^{-1}.$$

Now use the relations

$$a = \tfrac{1}{2}[(a_1 + a_4) + i(a_3 - a_2)] \quad \text{and} \quad b = \tfrac{1}{2}[(a_1 - a_4) + i(a_3 + a_2)],$$

with (5.7), (5.8), keep track of six decimal places, and (5.13) follows.

**Proof of (5.14).** Consider first the function $h(t, P, Q)$, defined between (5.8) and (5.9). Its second derivatives are all negative, and the one of largest absolute value is $h_{tt}$,

$$|h_{tt}(t, P, Q)| = 2(t - P)^{-2} + t^{-2} + 3(t - Q)^{-2}.$$

From (5.8) and (5.9), we see that if $P+iQ\in\Delta_1$ then $s_i(P,Q)$ satisfy the bounds on $s_1$, $s_2$ in (5.6). From this, one can show that if $P+iQ\in\Delta_1$, then each second partial derivative of $h$ evaluated at $(s_i(P,Q),P,Q)$ has absolute value $<12.2$.

For $G_1=\operatorname{Re} G$ we have, using $H_t(s_i,P,Q)=0$,

$$(G_1)_{PP}(P+iQ)$$
$$=h_{tt}(s_i,P,Q)[(s_i)_P(P,Q)]^2+2h_{tP}(s_i,P,Q)[(s_i)_P(P,Q)]+h_{PP}(s_i,P,Q).$$

Using (5.9) and the preceding paragraph,

$$\sup_{\Delta_1}|(G_1)_{PP}|\leqq(12.2)((0.57)^2+(0.57)+1)<25.$$

Similarly, the other second partial derivatives of $G_1$ and of $G_2$ are majorized by 25, and (5.14) is proved.

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