QUASIEXTREMAL DISTANCE DOMAINS AND EXTENSION OF QUASICONFORMAL MAPPINGS

By

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1. Introduction

Throughout this paper we let R^n denote euclidean n-space and $\bar{R}^n$ its one point compactification $R^n \cup \{\infty\}$. Next for $x \in R^n$ and $0 < r < \infty$ we let $B^n(x,r)$ denote the open n-ball with center x and radius r and $S^{n-1}(x,r)$ its boundary.

A domain D in $\bar{R}^2$ is said to be a K-*quasidisk* if it is the image of an open disk or half plane under a K-quasiconformal self mapping of $\bar{R}^2$. This paper is concerned with the extension to more general classes of domains D in $\bar{R}^n$ of the following two basic properties of quasidisks.

1.1. **Extremal distance property.** *If D is a quasidisk and if F_1 and F_2 are disjoint continua in D, then*

$$\operatorname{mod}\Gamma \leqq M \operatorname{mod}\Gamma_D,$$

where Γ and Γ_D are the families of curves which join F_1 and F_2 in $\bar{R}^2$ and D, respectively, and where M is a constant which depends only on D.

1.2. **Extension property.** *If D is a quasidisk and if f is a quasiconformal mapping of D onto a domain D' in $\bar{R}^2$, then f has a quasiconformal extension to $\bar{R}^2$ if and only if D' is a quasidisk.*

Property 1.1 is a consequence of a simple reflection principle for the moduli of curve families; see 2.12. Property 1.2 follows from the work of Beurling and Ahlfors [3].

For domains in $\bar{R}^2$ it turns out that these properties are related in the following sense. If D has the extremal distance property, then D and D' have the extension property if and only if D' has the extremal distance property. This is Corollary 3.16 in section 3.

We begin in section 2 by deriving several geometric properties of domains D in $\bar{R}^n$ which have the extremal distance property. We call such domains D *quasiextremal distance* or QED domains. It turns out that a simply connected plane domain of hyperbolic type is QED if and only if it is a quasidisk. We then obtain in

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section 3 a number of extension theorems for QED domains including several generalizations of the above-mentioned result of Beurling and Ahlfors.

2. QED exceptional sets and QED domains

A closed set E in $\bar{R}^n$ is said to be an *M-quasiextremal distance* or *M-QED exceptional set*, $1 \leqq M < \infty$, if for each pair of disjoint continua $F_1, F_2 \subset \bar{R}^n \setminus E$

$$\text{(2.1)} \qquad \operatorname{mod} \Gamma \leqq M \operatorname{mod} \Gamma_E,$$

where Γ and Γ_E are families of curves joining F_1 and F_2 in $\bar{R}^n$ and in $\bar{R}^n \setminus E$, respectively, and mod is the n-modulus. The class of QED exceptional sets contains the class of NED or nullsets for extremal distances; these are the sets E in $\bar{R}^n$ for which (2.1) holds with $M = 1$ for all choices of F_1, F_2. See [1], [2], [21] and Remark 2.4 below.

The conformal or n-capacity can also be used to characterize QED exceptional sets. Let D be an open set in $\bar{R}^n$ and C_1, C_2 compact disjoint sets in D. Set

$$\text{(2.2)} \qquad \operatorname{cap}(C_1, C_2; D) = \inf_{u \in W} \int_{D \cap R^n} |\nabla u|^n dm$$

where $W = W(C_1, C_2; D)$ is the family of all functions u continuous and ACL in D with $u(x) \leqq 0$ for $x \in C_1$ and $u(x) \geqq 1$ for $x \in C_2$. Since a point has zero n-capacity, the point ∞ can be deleted in the definition for W and thus W in (2.2) can be replaced by the family $\tilde{W}$ of functions u which are continuous and ACL in $D \cap R^n$ and satisfy $u(x) \leqq 0$ for $x \in C_1 \cap R^n$ and $u(x) \geqq 1$ for $x \in C_2 \cap R^n$. The classes W and $\tilde{W}$ differ only if $\infty \in D$. It is well-known, see [14], that $\operatorname{cap}(C_1, C_2; D) = \operatorname{mod}(\Gamma)$ where Γ is the family of curves joining C_1 and C_2 in D. Hence (2.1) can be written as

$$\text{(2.3)} \qquad \operatorname{cap}(F_1, F_2; \bar{R}^n) \leqq M \operatorname{cap}(F_1, F_2; \bar{R}^n \setminus E).$$

2.4. **Remark.** If E is an M-QED exceptional set in $\bar{R}^n$ with $m(E) = 0$, then E is NED. This follows from the methods and results in [2] although it is not explicitly mentioned there. To see this let E be an M-QED exceptional set in $\bar{R}^n$ with $m(E) = 0$ and let F_1, F_2 be two continua in $\bar{R}^n \setminus E$. Then for each $u \in W(F_1, F_2; \bar{R}^n \setminus E)$, it follows from Lemmas 3 and 4 and the considerations on pp. 1220–1221 in [2] that there is a function $u^* \in W(F_1, F_2; \bar{R}^n)$ with

$$\int_{R^n} |\nabla u^*|^n dm = \int_{R^n \setminus E} |\nabla u^*|^n dm \leqq \int_{R^n \setminus E} |\nabla u|^n dm.$$

Hence (2.3) holds with $M = 1$ and thus E is NED. This observation together with Corollary 2.16 below yields: *For M-QED exceptional sets E in $\bar{R}^n$ the following conditions are equivalent.*

(i) $m(E)=0$.
(ii) $\operatorname{int} E=\emptyset$.
(iii) E *is* NED.

We shall derive some properties of QED exceptional sets. The first is an immediate consequence of the quasi-invariance of the modulus under quasiconformal mappings.

2.5. **Lemma.** *Suppose that E is an M-QED exceptional set and that $f:\bar{R}^n\to\bar{R}^n$ is a quasiconformal mapping. Then $f(E)$ is an M'-QED exceptional set where*

$$M'=K_I(f)K_O(f)M.$$

Here $K_I(f)$ and $K_O(f)$ denote the inner and outer dilatations of f [22, p. 42].

We shall need the following estimate to establish several metric properties of QED sets.

2.6. **Lemma.** *Suppose that F_1 and F_2 are disjoint continua in $\bar{R}^n$ and that*

$$\min_{j=1,2}\operatorname{dia}(F_j)\geqq a d(F_1,F_2)$$

where a is a positive constant and $d(F_1,F_2)$ denotes the distance between F_1 and F_2. If Γ is the family of curves which join F_1 and F_2 in $\bar{R}^n$, then

$$\operatorname{mod}\Gamma\geqq c>0$$

where c is a constant which depends only on n and a.

Proof. Choose $x_1\in F_1$ and $x_2\in F_2$ so that

$$|x_1-x_2|=d(F_1,F_2).$$

By hypothesis we can choose a point $y_j\in F_j$ such that

$$|y_j-x_j|\geqq\frac{1}{2}\operatorname{dia}(F_j)\geqq\frac{a}{2}|x_1-x_2|$$

for $j=1,2$; by relabeling if necessary we may also assume that $|y_1-x_1|\leqq|y_2-x_2|$. Let $f:\bar{R}^n\to\bar{R}^n$ be a Möbius transformation for which $f(y_2)=\infty$. Then

$$\begin{aligned}\frac{|f(x_2)-f(x_1)|}{|f(y_1)-f(x_1)|}&=\frac{|x_2-x_1|}{|y_1-x_1|}\frac{|y_1-y_2|}{|x_2-y_2|}\\&\leqq\frac{2}{a}\frac{|y_1-y_2|}{|x_2-y_2|}\leqq\frac{2}{a}\frac{|x_2-y_2|+|x_1-x_2|+|y_1-x_1|}{|x_2-y_2|}\\&\leqq\frac{2}{a}\left(1+\frac{2}{a}+1\right)=\frac{4(a+1)}{a^2}=b>0\end{aligned}$$

and hence by [22, Theorem 11.9]

$$\operatorname{mod}\Gamma = \operatorname{mod} f(\Gamma) \geqq \varphi_n(b) = c > 0$$

where $\varphi_n : (0,\infty) \to (0,\infty)$ is a decreasing function depending only on n. (See also Theorem 4 in [4].)

A set $A \subset \bar{R}^n$ is said to be *a-quasiconvex*, $1 \leqq a < \infty$, if each pair of points $x_1, x_2 \in A \setminus \{\infty\}$ can be joined in A by a rectifiable curve γ whose length does not exceed $a\,|x_1 - x_2|$. If $A \subset R^n$, then A is 1-quasiconvex if and only if A is convex in the usual sense.

2.7. **Lemma.** *Suppose that E is an M-QED exceptional set in $\bar{R}^n$. Then $D = \bar{R}^n \setminus E$ is a domain which is a-quasiconvex with*

$$a \leqq \exp(bM^{1/(n-1)})$$

where b depends only on n.

Proof. Since E is closed, D is open. Suppose that D is not connected. Let D_1, D_2 be two disjoint components of D. Choose non-degenerate continua $F_j \subset D_j$, $j = 1,2$, and let Γ and Γ_E denote the families of curves joining F_1 and F_2 in $\bar{R}^n$ and D, respectively. Lemma 2.6 implies $\operatorname{mod}\Gamma > 0$. On the other hand $\Gamma_E = \varnothing$ and hence $\operatorname{mod}\Gamma_E = 0$. These two conclusions contradict (2.1) and thus D must be connected.

We show next that D is a-quasiconvex. Fix $x_1, x_2 \in D \setminus \{\infty\}$ and set $r = |x_1 - x_2|$. Since $D \setminus \{\infty\}$ is a domain, there is a curve α joining x_1 to x_2 in $D \setminus \{\infty\}$. Let F_j denote the component of $\alpha \cap \bar{B}^n(x_j, r/4)$ which contains x_j, $j = 1,2$, and let Γ and Γ_E denote the families of curves joining F_1 and F_2 in $\bar{R}^n$ and D, respectively. Then

$$\min_{j=1,2} \operatorname{dia}(F_j) \geqq r/4 \geqq d(F_1, F_2)/4$$

and Lemma 2.6 yields

$$\operatorname{mod}\Gamma \geqq c_0 > 0$$

where c_0 depends only on n. Since E is an M-QED exceptional set,

$$\operatorname{mod}\Gamma_E \geqq \frac{1}{M}\operatorname{mod}\Gamma \geqq \frac{c_0}{M}. \tag{2.8}$$

Let Γ_1 consist of those curves in Γ_E which lie in $B^n(x_2, s)$,

$$s = \frac{r}{4}\exp\left(\left(\frac{c_0}{2M\omega_{n-1}}\right)^{1/(1-n)}\right) = rc_1,$$

and let $\Gamma_2 = \Gamma_E \setminus \Gamma_1$. Suppose that each curve γ in Γ_E has length $l(\gamma) \geqq L > 0$. Then, cf. [22, Theorem 7.1],

$$\operatorname{mod}\Gamma_1 \leqq \frac{\Omega_n s^n}{L^n} = \frac{\Omega_n c_1^n r^n}{L^n},$$

where Ω_n is the n-measure of B^n. On the other hand, each $\gamma \in \Gamma_2$ meets $S^{n-1}(x_2, s)$ and hence

$$\operatorname{mod}\Gamma_2 \leqq \omega_{n-1}\left(\log\frac{4s}{r}\right)^{1-n} = \frac{c_0}{2M}.$$

Cf. [22, 7.5]. These inequalities yield

$$\operatorname{mod}\Gamma_E \leqq \operatorname{mod}\Gamma_1 + \operatorname{mod}\Gamma_2 \leqq \frac{\Omega_n c_1^n r^n}{L^n} + \frac{c_0}{2M}$$

and thus by (2.8)

$$L \leqq r c_1 \left(\frac{2\Omega_n M}{c_0}\right)^{1/n} < r \exp(c M^{1/(n-1)})$$

where

$$c = 2(2\omega_{n-1}/c_0)^{1/(n-1)}$$

depends only on n. Set $c_2 = \exp(cM^{1/(n-1)})$. Then there is a rectifiable curve $\gamma_0 \in \Gamma_E$ with

$$l(\gamma_0) \leqq r c_2 = c_2 |x_1 - x_2|$$

and with endpoints $y_1, y_2 \in \alpha$ such that

$$|x_j - y_j| \leqq |x_1 - x_2|/4$$

for $j = 1,2$.

Next set $r_1 = |x_1 - y_1|$ and let F_1 and F_2 denote the components of $\alpha \cap \bar{B}^n(x_1, r_1/4)$ and $\gamma_0 \cap \bar{B}^n(y_1, r_1/4)$ which contain x_1 and y_1, respectively. Then

$$\min_{j=1,2} \operatorname{dia}(F_j) \geqq \frac{r_1}{4} \geqq \frac{1}{4} d(F_1, F_2),$$

and arguing as above we obtain a curve γ_1 in D such that

$$l(\gamma_1) \leqq c_2 |x_1 - y_1| \leqq c_2 \frac{|x_1 - x_2|}{4}$$

and such that γ_1 joins γ_0 to a point $z_1 \in \alpha$ with

$$|x_1 - z_1| \leqq \frac{1}{4^2} |x_1 - x_2|.$$

Clearly the curves γ_0 and γ_1 contain a rectifiable subcurve joining z_1 to y_2. Now a continuation of this process and a similar construction starting from y_2 towards x_2

leads to two sequences of curves $\gamma_1, \gamma_2, \ldots$ and $\tilde{\gamma}_1, \tilde{\gamma}_2, \ldots$ whose union together with γ_0 contains a rectifiable curve γ in D from x_1 to x_2 with

$$\begin{aligned} l(\gamma) &\leqq l(\gamma_0) + \sum_{i=1}^{\infty} l(\gamma_i) + \sum_{i=1}^{\infty} l(\tilde{\gamma}_i) \\ &\leqq c_2\left(|x_1 - x_2| + \sum_{i=1}^{\infty} \frac{|x_1 - x_2|}{4^i} + \sum_{i=1}^{\infty} \frac{|x_1 - x_2|}{4^i} \right) \\ &= \frac{5}{3} c_2 |x_1 - x_2|. \end{aligned}$$

Thus D is a-quasiconvex with

$$a = \frac{5}{3} c_2 \leqq \exp((c+1)M^{1/(n-1)})$$

as desired.

2.9. **Remark.** Lemma 2.7 is an extension of the following result due to Ahlfors and Beurling [1, Theorem 10]. *If E is an* NED *set in $\bar{R}^2$, then $D = \bar{R}^2 \backslash E$ is a-quasiconvex for each $a > 1$.*

If E is an M-QED exceptional set, then by Lemma 2.7 $\bar{R}^n \backslash E$ is a domain; we call any such domain an *M-quasiextremal distance* or *M-QED domain.*

A set A in $\bar{R}^n$ is *c-locally connected*, cf. [6], if for each $x_0 \in R^n$ and $r > 0$,

(i) points in $A \cap \bar{B}^n(x_0, r)$ can be joined in $A \cap \bar{B}^n(x_0, cr)$,

(ii) points in $A \backslash B^n(x_0, r)$ can be joined in $A \backslash B^n(x_0, r/c)$.

The set A is *linearly locally connected* if it is c-locally connected for some c.

2.10. **Remarks.** When A is open, it is easy to see that the condition (i) holds for a given $x_0 \in R^n$ and $r > 0$ if and only if

(i)′ points in $A \cap B^n(x_0, r)$ can be joined in $A \cap B^n(x_0, cr)$ and similarly for condition (ii). Moreover if condition (i) holds for A and its image under each Möbius transformation $f: \bar{R}^n \to \bar{R}^n$, then condition (ii) holds. For let $x_1, x_2 \in A \backslash B^n(x_0, r)$ and let

$$f(x) = r^2 \frac{x - x_0}{|x - x_0|^2} + x_0.$$

Then $f(x_1), f(x_2) \in f(A) \cap \bar{B}^n(x_0, r)$ and, by hypothesis, these points can be joined by a curve γ in $f(A) \cap \bar{B}^n(x_0, cr)$. Hence $f^{-1}(\gamma)$ joins x_1, x_2 in $A \backslash B^n(x_0, r/c)$.

Finally it is not difficult to show that the property of being linearly locally connected is invariant under quasiconformal self mappings of $\bar{R}^n$. In particular if A is c-locally connected and if $f: \bar{R}^n \to \bar{R}^n$ is K-quasiconformal, then $f(A)$ is c'-locally connected where c' depends only on n, c and K. See [23, Theorem 5.6].

2.11. **Lemma.** *Suppose that D is an M-QED domain. Then D is c-locally connected with*

$$c \leqq 1 + \exp(bM^{1/(n-1)})$$

where b is the constant of Lemma 2.7.

Proof. Fix $x_0 \in R^n$ and $r > 0$. By Lemma 2.7, D is a-quasiconvex with

$$a \leqq \exp(bM^{1/(n-1)}).$$

Hence each pair of points $x_1, x_2 \in D \cap \bar{B}^n(x_0, r)$ can be joined in $D \cap \bar{B}^n(x_0, s)$ where

$$s \leqq r + a\,|x_1 - x_2|/2 \leqq r + ar = (1+a)r.$$

Since

$$1 + a \leqq 1 + \exp(bM^{1/(n-1)})$$

the points x_1, x_2 can be joined in $D \cap \bar{B}^n(x_0, cr)$ and c has the desired upper bound.

Next if D' is the image of D under a Möbius transformation of $\bar{R}^n$, then D' is M-QED by Lemma 2.5 and points in $D' \cap \bar{B}^n(x_0, r)$ can be joined in $D' \cap \bar{B}^n(x_0, cr)$ by what was proved above. Thus D is c-locally connected by the remarks in 2.10.

2.12. **Remarks.** Suppose that D is a ball or a half space, that F_1, F_2 are disjoint continua in D and that Γ and Γ_D are the families of curves joining F_1 and F_2 in $\bar{R}^n$ in D, respectively. Let Γ^* denote the family of curves joining F_1^* and F_2^* in $\bar{R}^n$ where $F_j^* = F_j \cup \varphi(F_j)$ and φ denotes reflection in ∂D. Then

$$\operatorname{mod}\Gamma \leqq \operatorname{mod}\Gamma^* = 2\operatorname{mod}\Gamma_D$$

and hence D is a 2-QED domain. It is easy to see that the constant 2 is best possible.

Next if

$$0 < \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n$$

and if D is the image of the exterior of a ball under the affine mapping

$$f(x_1, \ldots, x_n) = (\lambda_1 x_1, \ldots, \lambda_n x_n),$$

then Lemma 2.5 implies that D is M-QED where

$$M = 2(\lambda_n/\lambda_1)^n.$$

If, in particular, $\lambda_1 = 1$ and $\lambda_2 = \cdots = \lambda_n = t > 1$, then D is a-quasiconvex only if

$$a > t = (M/2)^{1/n}.$$

This observation yields lower bounds for the constants a and c in Lemmas 2.7 and 2.11.

Lemmas 2.7 and 2.11 give quantitative information about the connectivity of a QED domain. The following result yields a density condition for this class of domains.

2.13. **Lemma.** *Suppose that D is an M-QED domain in $\bar{R}^n$. Then for each $x_0 \in \bar{D} \cap R^n$ and $0 < r \leqq \operatorname{dia}(D)$*

$$\frac{m(D \cap B^n(x_0, r))}{m(B^n(x_0, r))} \geqq \frac{c}{M} \tag{2.14}$$

where $c > 0$ depends only on n.

Proof. Fix $x_0 \in \bar{D} \cap R^n$. Since $r \leqq \operatorname{dia}(D)$ we can choose $x_3 \in \bar{D}$ so that $|x_3 - x_0| = r/2$. Set $s = r/10$, choose $x_1, x_2 \in D$ such that $|x_0 - x_1| < s$, $|x_2 - x_3| < s$, and let α be a curve joining x_1 and x_2 in D. Let F_1 be the x_1-component of $\alpha \cap \bar{B}^n(x_0, 2s)$ and F_2 the x_2-component of $\alpha \setminus B^n(x_0, 3s)$. Next denote by Γ and Γ_D the families of curves which join F_1 and F_2 in $\bar{R}^n$ and in D, respectively. Set

$$\rho(x) = \begin{cases} \dfrac{1}{s} & \text{in } D \cap B^n(x_0, r), \\ 0 & \text{elsewhere.} \end{cases}$$

Since each $\gamma \in \Gamma_D$ contains a subcurve β which joins $S^{n-1}(x_0, 2s)$ and $S^{n-1}(x_0, 3s)$ in D,

$$\int_\gamma \rho\, ds \geqq \int_\beta \rho\, ds = \frac{1}{s} l(\beta) \geqq 1,$$

ρ is admissible for Γ_D and

$$\operatorname{mod} \Gamma_D \leqq \int_{R^n} \rho^n\, dm = \frac{1}{s^n} \int_{D \cap B^n(x_0, r)} dm$$

$$= 10^n \Omega_n \frac{m(D \cap B^n(x_0, r))}{m(B^n(x_0, r))}.$$

Next

$$\min_{j=1,2} \operatorname{dia}(F_j) \geqq s > \frac{1}{4} d(F_1, F_2)$$

and thus Lemma 2.6 implies that

$$\operatorname{mod} \Gamma \geqq c_0 > 0$$

where c_0 depends only on n. Since D is M-QED, we obtain

$$\frac{m(D \cap B^n(x_0,r))}{m(B^n(x_0,r))} \geq \frac{c}{M}$$

where $c = c_0/(10^n \Omega_n)$.

2.15. **Remark.** Suppose that $t > 1$ and that D is the image of the unit ball $B^n(0,1)$ under

$$f(x_1,\ldots,x_{n-1},x_n) = (x_1,\ldots,x_{n-1},tx_n).$$

Then as in Remark 2.12, D is M-QED where $M = 2t^n$ while

$$\frac{m(D \cap B^n(0,t))}{m(B^n(0,t))} = \frac{1}{t^{n-1}} = \left(\frac{2}{M}\right)^{(n-1)/n}.$$

Hence the exponent of M in (2.14) is asymptotically sharp for large n.

2.16. **Corollary.** *The boundary ∂D of a* QED *domain D has n-dimensional measure zero.*

Proof. By Lemma 2.13 a point $x_0 \in \partial D \backslash \{\infty\}$ cannot be a point of density for $E = \bar{R}^n \backslash D$ and hence not for ∂D.

A domain D in R^n is said to be *uniform* if there exist constants a, b such that each $x_1, x_2 \in D$ can be joined by a rectifiable curve γ in D with

$$\begin{aligned} &l(\gamma) \leqq a\,|x_1 - x_2|, \\ &\min(s, l(\gamma) - s) \leqq b d(\gamma(s), \partial D). \end{aligned} \tag{2.17}$$

Here γ is parametrized by arc length s.

The next lemma is essentially due to P. Jones [15].

2.18. **Lemma.** *A uniform domain D is an M-*QED *domain where M depends only on n and the constants for D.*

Proof. Let F_1 and F_2 be two disjoint continua in D. Let $\varepsilon > 0$ and choose $u \in W(F_1, F_2; D)$ such that

$$\int_D |\nabla u|^n dm \leqq \operatorname{cap}(F_1, F_2; D) + \varepsilon/2.$$

Then for small $t > 0$ the function $v = (1+t)(1-t)^{-1}(u - t)$ satisfies

$$\int_D |\nabla v|^n dm \leqq \operatorname{cap}(F_1, F_2; D) + \varepsilon$$

and $v(x) \leqq -t$ for $x \in F_1$, $v(x) \geqq 1+t$ for $x \in F_2$. By [15, Theorem 2] there exists an ACL-function $v^*: R^n \to R$ such that $v^* = v$ in D and

$$M \int_D |\nabla v|^n dm \geqq \int_{R^n} |\nabla v^*|^n dm,$$

where the constant M depends only on n and the constants for D. Choose a smooth convolution approximation φ of v^* with $\varphi \leqq 0$ on F_1, $\varphi \geqq 1$ on F_2 and

$$\int_{R^n} |\nabla v^*|^n dm \geqq \int_{R^n} |\nabla \varphi|^n dm - \varepsilon.$$

Then $\varphi \in \tilde{W}(F_1, F_2; \bar{R}^n)$ and the last three inequalities yield

$$\operatorname{cap}(F_1, F_2; \bar{R}^n) \leqq \int_{R^n} |\nabla \varphi|^n dm \leqq M \operatorname{cap}(F_1, F_2; D) + \varepsilon(M+1).$$

Letting $\varepsilon \to 0$ yields the desired result.

Although the classes of QED, linearly locally connected and uniform domains do not coincide, it is possible to obtain more precise relations between them when $n = 2$. In particular, we shall show that for finitely connected plane domains these classes are the same.

We say that $D \subset \bar{R}^n$ is a *K-quasiball* if D is the image of an open ball or half space under a K-quasiconformal self mapping of $\bar{R}^n$ and that $S \subset \bar{R}^n$ is a *K-quasisphere* if it is a boundary of a K-quasiball. Next a domain $D \subset \bar{R}^n$ is said to be a *K-quasisphere domain* if each component of ∂D is either a point or a K-quasisphere. We use the more standard terms quasidisk and quasicircle when $n = 2$.

We shall show that every quasisphere domain is linearly locally connected and that this property characterizes this class of domains when $n = 2$. We require first the following result.

2.19. **Lemma.** *If $G_1, \ldots, G_k$ are pairwise disjoint K-quasiballs all of which meet $S^{n-1}(x_0, r_1)$ and $S^{n-1}(x_0, r_2)$, then*

$$k \leqq a\left(\frac{r_2 + r_1}{|r_2 - r_1|}\right)^{n-1}$$

where a depends only on n and K.

Proof. We may assume $r_2 > r_1$. Set $t = |r_2 - r_1|/2$. For each $i = 1, \ldots, k$ choose $x_i \in G_i$ such that

$$|x_i - x_0| = \frac{r_1 + r_2}{2}.$$

By Lemma 2.5, see also the remarks in 2.12, each G_i is an M-QED domain where M depends only on K. Lemma 2.13 yields for $i = 1, \ldots, k$

$$m(G_i \cap B^n(x_i, t)) \geqq \frac{c}{M} m(B^n(x_i, t)) = \frac{c\,\Omega_n t^n}{M}$$

where $c > 0$ depends only on n. Since the quasiballs G_i are disjoint,

$$\Omega_n (r_2^n - r_1^n) = m(B^n(x_0, r_2) \setminus \bar{B}^n(x_0, r_1))$$

$$\geqq \sum_{i=1}^{k} m(G_i \cap B^n(x_i, t)) \geqq \frac{c\,\Omega_n}{M} k t^n = \frac{c\,\Omega_n}{M 2^n} k (r_2 - r_1)^n$$

and thus

$$k \leqq a \frac{r_2^n - r_1^n}{(r_2 - r_1)^n} = a \frac{1 - s^n}{(1 - s)^n}$$

where $s = r_1 / r_2 < 1$ and $a = M 2^n / c$ depends only on n and K. The elementary inequality

$$1 - s^n \leqq (1 - s)(1 + s)^{n-1}$$

easily follows by induction and hence

$$k \leqq a \left(\frac{r_2 + r_1}{|r_2 - r_1|} \right)^{n-1}$$

as desired.

2.20. **Lemma.** *If D is a K-quasisphere domain, then D is c-locally connected where c depends only on n and K.*

Proof. Let C_0 be a nondegenerate component of ∂D and let D_0 denote the component of $\bar{R}^n \setminus C_0$ which contains D. Then D_0 is a K-quasiball and hence $c = c(n, K)$-locally connected by, for example, the remarks in 2.12 and Lemmas 2.5 and 2.11.

Fix $x_0 \in R^n$, $r > 0$ and $d > c$. We shall show that D is d-locally connected. Since each image of D under a Möbius transformation is again a K-quasisphere domain, it suffices by the remarks in 2.10 to show that each pair of points $x_1, x_2 \in D \cap B^n(x_0, r)$ can be joined in $D \cap B^n(x_0, dr)$. Suppose that this is not true for a given pair x_1, x_2. Then these points are separated by

$$F = \partial D \cup S^{n-1}(x_0, dr).$$

By [19, Theorem V.14.3 and p. 137], there is a component E of F which does this.

Now observe that E meets $S^{n-1}(x_0, dr)$ since otherwise $E \subset \partial D$ and hence could not separate x_1 and x_2. Let

$$E_0 = S^{n-1}(x_0, dr) \cup \left(\bigcup_{\alpha} C_\alpha \right)$$

where $\{C_\alpha\}$ is the collection of all components of ∂D which meet $S^{n-1}(x_0, dr)$. Then E_0 is a connected subset of F,

$$E \cap E_0 \supset E \cap S^{n-1}(x_0, dr) \neq \emptyset$$

and thus $E_0 \subset E$. Suppose that there exists a point $y \in \partial D \setminus E_0$. Then y lies in a component C of ∂D with

$$C \cap S^{n-1}(x_0, dr) = \emptyset.$$

Choose $\varepsilon > 0$ so that

$$\varepsilon < q(C, S^{n-1}(x_0, dr))$$

where q is the chordal metric in $\bar{R}^n$. Then [19, Corollary 1 on p. 83] yields a set $H \subset \partial D$ such that H is both open and closed in ∂D with

$$C \subset H \subset \{x : q(x, C) < \varepsilon\}.$$

Thus

$$H \cap S^{n-1}(x_0, dr) = \emptyset$$

and H is closed in F. On the other hand,

$$F \setminus H = S^{n-1}(x_0, dr) \cup (\partial D \setminus H)$$

is also closed in F. Hence y does not belong to the same component of F as $S^{n-1}(x_0, dr)$, i.e. $y \notin E$. Thus $E = E_0$ or

$$E = S^{n-1}(x_0, dr) \cup \left(\bigcup_\alpha C_\alpha \right).$$

For each non-degenerate component C_α let D_α and G_α denote the components of $\bar{R}^n \setminus C_\alpha$ labeled so that $D \subset D_\alpha$. Then the G_α are pairwise disjoint K-quasiballs and hence by Lemma 2.19 at most k of the C_α meet $S^{n-1}(x_0, cr)$ where

$$k \leqq a \left(\frac{d+c}{d-c} \right)^{n-1}, \qquad a = a(n, K).$$

By relabeling we may assume that these are the components $C_1, \ldots, C_k$. Then for $i = 1, \ldots, k$, x_1 and x_2 lie in $D_i \cap B^n(x_0, r)$ and hence x_1 and x_2 can be joined in $D_i \cap B^n(x_0, cr)$. This says that x_1 and x_2 are not separated by

$$F_i = S^{n-1}(x_0, cr) \cup C_i.$$

For $j = 1, \ldots, k$ let

$$E_j = \bigcup_{i=1}^{j} F_i$$

and suppose that x_1, x_2 are not separated by E_j for some $j < k$. Then since

$$E_j \cap F_{j+1} = S^{n-1}(x_0, cr)$$

we can apply [24, Theorem II.5.18] to conclude that x_1, x_2 are not separated by E_{j+1} and hence not by

$$E_k = S^{n-1}(x_0, cr) \cup \left(\bigcup_{i=1}^{k} C_i \right).$$

In particular, there is an arc γ which joins x_1 and x_2 in $B^n(x_0, cr)$ and does not meet any C_i, $i = 1, \ldots, k$. Choose C_α with $\alpha \notin \{1, \ldots, k\}$. Then C_α meets $S^{n-1}(x_0, dr)$ and not $S^{n-1}(x_0, cr)$. Hence $C_\alpha \cap \gamma = \emptyset$ and we conclude that

$$E \cap \gamma = (S^{n-1}(x_0, dr) \cap \gamma) \cup \left(\bigcup_{\alpha} C_\alpha \cap \gamma \right) = \emptyset.$$

This means that E does not separate x_1 and x_2 and the proof is complete.

2.21. **Theorem.** *A domain D in $\bar{R}^2$ is a quasicircle domain if and only if it is linearly locally connected.*

Proof. Suppose that D is a domain in $\bar{R}^2$. If D is linearly locally connected, then by [6, Lemma 5], D is a quasicircle domain. The converse follows from Lemma 2.20.

2.22. **Theorem.** *If D is a finitely connected domain in R^2, then the following conditions are equivalent.*

(i) *D is a* QED *domain.*
(ii) *D is linearly locally connected.*
(iii) *D is a quasicircle domain.*
(iv) *D is uniform.*

Proof. That (i) implies (ii) follows from Lemma 2.11; that (ii) implies (iii) is a consequence of Theorem 2.21. By [20, Theorem 5] and [10, Theorem 5] a finitely connected quasicircle domain is uniform. Finally (iv) implies (i) by Lemma 2.18.

2.23. **Remark.** Suppose that $D \neq R^2$ is a simply connected domain in R^2. Then Theorem 2.22 implies the well-known equivalence of the following conditions.

(i) D is a QED domain.
(ii) D is linearly locally connected.
(iii) D is a quasidisk.
(iv) D is uniform.

The equivalence of (i) and (iii) was proved by V. Gol'dstein and S. Vodop'janov [12]. For the equivalence of (iii) and (iv) see [18, Corollary 2.33] while the equivalence of (ii) and (iii) follows from [6, Lemmas 4 and 5]. Cf. also [8].

2.24. **Remark.** Finally for an arbitrary domain $D \subset \bar{R}^n$ we have the following relations between the classes of domains considered in this paper.

(i) If D is uniform, then D is QED.

(ii) If D is QED, then D is linearly locally connected.

(iii) If D is a quasisphere domain, then D is linearly locally connected.

(iv) There exists a QED domain D which is not uniform.

(v) There exists a quasisphere domain D which is not QED, and hence not uniform.

(vi) When $n > 2$, there exists a domain D which is uniform, and hence QED and linearly locally connected, but not a quasisphere domain.

The first three conclusions follow from Lemmas 2.18, 2.11 and 2.20, respectively. For (iv), note that if E is a closed set in R^n with $m_{n-1}(E) = 0$, then E is NED by [21] and hence $D = R^n \setminus E$ is a 1-QED domain. On the other hand if we choose E as the set of points with integer coordinates, then the second condition in (2.17) fails and D will not be uniform. For (v) choose a closed, totally disconnected set in R^n with $m(E) > 0$. Then $D = R^n \setminus E$ is a 1-quasisphere domain with $\partial D = E \cup \{\infty\}$ and hence D is not QED by Corollary 2.16. Finally when $n > 2$, then $D = R^n \setminus R^1$ is a uniform domain while $R^1 \cup \{\infty\}$ is neither a point nor a quasisphere.

3. Extension of quasiconformal and quasi-isometric mappings

We shall show in this section that a quasiconformal mapping between QED domains in $\bar{R}^n$ has a homeomorphic extension to the closures of the domains when $n \geqq 2$ and a quasiconformal extension to $\bar{R}^n$ when $n = 2$. Section 2 then yields several extension theorems for quasiconformal mappings on various subclasses of QED domains. We also prove corresponding results for injective local quasi-isometries.

We begin with the following result.

3.1. **Theorem.** *Suppose that D and D' are domains in $\bar{R}^n$, that D is M-QED and that D' is c'-locally connected. If f is a K-quasiconformal mapping of D onto D', then f has a homeomorphic extension to $\bar{D}$. Moreover, if x_1, x_2, x_3, x_4 are distinct points in $\bar{D}$ with*

$$\frac{|x_1 - x_2|}{|x_3 - x_2|}\frac{|x_3 - x_4|}{|x_1 - x_4|} \leqq a$$

then

$$\frac{|f(x_1) - f(x_2)|}{|f(x_3) - f(x_2)|}\frac{|f(x_3) - f(x_4)|}{|f(x_1) - f(x_4)|} \leqq b \tag{3.2}$$

where b is a constant which depends only on n, K, M, c' and a.

Proof. We begin by deriving (3.2) whenever $x_1, x_2, x_3, x_4 \in D$. By cooposing f with a pair of Möbius transformations and appealing to Lemma 2.5 and to the remarks in 2.10, we see that it is sufficient to consider the case where $x_4 = \infty$ and $f(x_4) = \infty$; then we must show that

$$\frac{|x_1 - x_2|}{|x_3 - x_2|} \leqq a \quad \text{implies} \quad \frac{|y_1 - y_2|}{|y_3 - y_2|} \leqq b \tag{3.3}$$

where $y_j = f(x_j)$, $j = 1,2,3,4$.

First we choose t so that

$$|y_1 - y_2| = c'^2 t |y_3 - y_2| = c'^2 tr$$

and suppose that $t > 1$. Because D' is c'-locally connected there exist continua F'_1 and F'_2 which join y_2 to y_3 in $D' \cap \bar{B}^n(y_2, c'r)$ and y_1 to $y_4 = \infty$ in $D' \setminus B^n(y_2, c'tr)$, respectively. Set $F_j = f^{-1}(F'_j)$ and let Γ and Γ_D denote the families of curves joining F_1 and F_2 in $\bar{R}^n$ and in D, respectively. If $\gamma \in \Gamma_D$, then $f(\gamma)$ joins $S^{n-1}(y_2, c'r)$ to $S^{n-1}(y_2, c'tr)$ and thus

$$\operatorname{mod} \Gamma_D \leqq K \operatorname{mod} f(\Gamma_D) \leqq K\omega_{n-1}(\log t)^{1-n}.$$

Next

$$\min_{j=1,2} \operatorname{dia}(F_j) \geqq |x_3 - x_2| \geqq \frac{1}{a}|x_1 - x_2| \geqq \frac{1}{a} d(F_1, F_2)$$

and by Lemma 2.6

$$\operatorname{mod} \Gamma \geqq c$$

where $c > 0$ depends only on n and a. Since D is an M-QED domain, these inequalities yield

$$c \leqq \operatorname{mod} \Gamma \leqq M \operatorname{mod} \Gamma_D \leqq MK\omega_{n-1}(\log t)^{1-n}$$

or

$$t \leqq \exp\left(\left(\frac{MK\omega_{n-1}}{c}\right)^{1/(n-1)}\right).$$

Now this inequality holds trivially whenever $t \leqq 1$. Hence we obtain (3.3) with

$$b = c'^2 \exp\left(\left(\frac{MK\omega_{n-1}}{c}\right)^{1/(n-1)}\right).$$

Next we use what was proved above to conclude that f has a homeomorphic extension to $\bar{D}$; again it is sufficient to consider the case where $\infty \in D$ and $f(\infty) = \infty$. Fix $x_0 \in \partial D$ and choose points $x_j \in D$ so that $x_j \to x_0$ and $f(x_j) \to y_0$ as $j \to \infty$. Then $y_0 \in \partial D' \subset R^n$. Given $\varepsilon > 0$ fix k such that

$$|f(x_k) - y_0| \leqq \varepsilon.$$

Suppose that $x \in D$ and that

$$|x - x_0| \leqq \tfrac{1}{3}|x_k - x_0| = \delta.$$

For large j, $x_j \neq x$, $|x_j - x_0| \leqq \delta$ and

$$\begin{aligned} |x - x_j| &\leqq |x - x_0| + |x_j - x_0| \\ &\leqq 3\delta - |x_j - x_0| \\ &= |x_k - x_0| - |x_j - x_0| \\ &\leqq |x_k - x_j|. \end{aligned} \tag{3.4}$$

With (3.4) we can apply (3.2) with $x_1 = x$, $x_2 = x_j$, $x_3 = x_k$ and $x_4 = \infty$ to conclude that

$$|f(x) - f(x_j)| \leqq b\,|f(x_k) - f(x_j)|$$

where $b = b(n, K, M, c')$. Letting $j \to \infty$ we obtain

$$|f(x) - y_0| \leqq b\,|f(x_k) - y_0| \leqq b\varepsilon$$

and this shows that $f(x) \to y_0$ as $x \to x_0$ in D. Thus f has a continuous extension to $\bar{D}$ which we again denote by f. By continuity (3.2) holds whenever $x_1, x_2, x_3, x_4 \in \bar{D}$, where b is the original constant corresponding to $a + 1$, and this, in turn, implies that f is injective in $\bar{D}$ and hence a homeomorphism.

Theorem 3.1, Lemma 2.11 and Lemma 2.18 imply the following results.

3.5. **Corollary.** *If D and D' are* QED *domains in $\bar{R}^n$, then each quasiconformal mapping of D onto D' has a homeomorphic extension to $\bar{D}$.*

3.6. **Corollary.** *If D and D' are uniform domains in R^n, then each quasiconformal mapping of D onto D' has a homeomorphic extension to $\bar{D}$.*

3.7. **Remark.** In the case of bounded uniform domains Corollary 3.6 also follows from [9, Corollary 3.30] since then both f and f^{-1} belong to some Lipschitz class Lip_α, $\alpha > 0$.

3.8. **Quasiconformal extension to $\bar{R}^2$.** In the plane Theorem 3.1 can be considerably sharpened. We require first the following results on quasidisks.

3.9. **Lemma.** *Suppose that G is a K-quasidisk in R^2, that $z_0 \in R^2 \setminus G$ and that α is a component of $G \cap S^1(z_0, r)$. Then*

$$\mathrm{dia}(\alpha) \leqq c\,|z_1 - z_2|$$

where z_1, z_2 are the endpoints of α and c depends only on K.

Proof. Let θ denote the angle subtended by α at z_0. If $0 < \theta \leqq \pi$, then

$$\mathrm{dia}(\alpha) = |z_1 - z_2|.$$

If $\theta > \pi$, then consider the ray from z_0 through the point $2z_0 - z_1$ on the opposite side of z_1 in $S^1(z_0, r)$. Since G lies in R^2, this ray meets each of the components γ_1 and γ_2 of $\partial G \backslash \{z_1, z_2\}$; thus

$$\operatorname{dia}(\gamma_j) \geqq |z_1 - z_0| = r$$

for $j = 1,2$. On the other hand, since ∂G is a K-quasicircle,

$$\min_{j=1,2} \operatorname{dia}(\gamma_j) \leqq a\,|z_1 - z_2|,$$

where $a = a(K)$, and hence

$$\operatorname{dia}(\alpha) \leqq 2r \leqq 2a\,|z_1 - z_2|.$$

3.10. **Lemma.** *If (G_j) is an infinite sequence of pairwise disjoint K-quasidisks, then*

$$\lim_{j \to \infty} q(G_j) = 0$$

where $q(G_j)$ is the chordal diameter of G_j.

Proof. If not we can, after passing to a subsequence if necessary, choose $z_j, w_j \in G_j$ such that $z_j \to z_0 \neq \infty$ and $w_j \to w_0 \neq z_0$. Fix $0 < r_1 < r_2 < |z_0 - w_0|$. Then there exists a j_0 such that $|z_j - z_0| < r_1$ and $|w_j - z_0| > r_2$ for $j \geqq j_0$. This says that infinitely many G_j meet both $S^1(z_0, r_1)$ and $S^1(z_0, r_2)$ contradicting the conclusion of Lemma 2.19.

3.11. **Theorem.** *Suppose that D and D' are domains in $\bar{R}^2$, that D is M-QED and that D' is c'-locally connected. If f is a K-quasiconformal mapping of D onto D', then f has a K^*-quasiconformal extension to $\bar{R}^2$ where K^* depends only on the constants K, M and c'.*

Proof. By Theorem 3.1, f has a homeomorphic extension, denoted again by f, which maps $\bar{D}$ onto $\bar{D}'$. Next by Lemma 2.11, D is c-locally connected where $c = c(M)$ and it follows from Theorem 2.21 that D and D' are K_1-quasicircle domains where K_1 depends only on M and c'.

Let C be a quasicircle component of ∂D. Then $C' = f(C)$ is also a quasicircle and there exist K_1-quasiconformal mappings g and g' of $\bar{R}^2$ onto itself such that $g(C) = \bar{R}^1$, $g'(C') = \bar{R}^1$ and $g' \circ f \circ g^{-1}(\infty) = \infty$. Moreover, we may assume that g maps the component G of $\bar{R}^2 \backslash \bar{D}$ bounded by C onto the lower half plane H and that g' does the same for the corresponding component G' of $\bar{R}^2 \backslash \bar{D}'$. Then $h = g' \circ f \circ g^{-1}$ is a homeomorphism which maps $\overline{g(D)}$ onto $\overline{g'(D')}$, R^1 onto R^1 and is K_2-quasiconformal in $g(D)$, $K_2 = KK_1^2$. Now $g(D)$ is M_1-QED with $M_1 = K_1^2 M$ and by the remarks in 2.10, $g'(D')$ is c_1'-locally connected where c_1' depends only on c' and K_1. Choose $x \in R^1$, $t > 0$ and let

$$x_1 = x + t, \quad x_2 = x, \quad x_3 = x - t, \quad x_4 = \infty.$$

Then by Theorem 3.1 applied to h,

$$\frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leqq b$$

where b depends only on K_1, M_1 and c_1'. From interchanging the roles of x_1 and x_3 above, we conclude that $h \mid R^1$ is b-quasisymmetric and hence, by a theorem of Beurling and Ahlfors [3], there exists a homeomorphism $h^* : \bar{H} \to \bar{H}$ which agrees with h on ∂H and is K_3-quasiconformal in H, $K_3 = K_3(b)$.

Mapping back we obtain a homeomorphism f_G of $\bar{D} \cup G$ onto $\bar{D}' \cup G'$ which extends f and which is K^*-quasiconformal in D and in G, where K^* depends only on K, M and c'. Define $f^* : \bar{R}^2 \to \bar{R}^2$ as $f^*(z) = f(z)$ when $z \in \bar{D}$ and $f^*(z) = f_G(z)$ when z belongs to a quasidisk component of G of $\bar{R}^2 \backslash \bar{D}$. Next we show that f^* is a homeomorphism. Since f^* is injective, it suffices to show that f^* is continuous and this clearly follows if we establish the continuity of f^* at $z_0 \in \partial D$.

Let $z_j \to z_0$ and suppose that $f^*(z_j) \to w_0$. We want to show that $w_0 = f^*(z_0)$. If infinitely many z_j belong either to $\bar{D}$ or to a single component G of $\bar{R}^2 \backslash \bar{D}$, then this follows from the fact that f is continuous in $\bar{D}$ and f_G in $\bar{G}$, respectively. Suppose that the points z_j lie in infinitely many distinct components G_i of $\bar{R}^2 \backslash \bar{D}$. Passing to a subsequence, if necessary, we may assume that $z_j \in G_j$ where the G_j are pairwise disjoint. For each j choose $w_j \in \partial G_j \subset \partial D$. Since the K_1-quasidisks G_j are pairwise disjoint, Lemma 3.10 implies that $q(G_j) \to 0$ as $j \to \infty$. Thus $w_j \to z_0$ and hence $f^*(w_j) \to f^*(z_0)$ by the continuity of f in $\bar{D}$. Next, because the K_1-quasidisks $f^*(G_j)$ are pairwise disjoint, the same reasoning shows that $f^*(z_j)$ approaches the same limit as $f(w_j)$. Thus $w_0 = f(z_0)$.

It remains to show that f^* is K^*-quasiconformal in $\bar{R}^2$. Suppose first that $\infty \in D$ and that $f^*(\infty) = \infty$. By Corollary 2.16, ∂D has zero planar measure. Hence by a well-known removability theorem it suffices to show that there is a constant c such that

$$L(z_0, r) \leqq c\, l(z_0, r) \tag{3.12}$$

for all $z_0 \in \partial D \backslash \{\infty\}$ and $0 < r < \infty$, where

$$L(z_0, r) = \max_{|z - z_0| = r} |f^*(z) - f^*(z_0)|,$$

$$l(z_0, r) = \min_{|z - z_0| = r} |f^*(z) - f^*(z_0)|.$$

By making a pair of changes of variable we may assume that $z_0 = 0$ and $f^*(z_0) = 0$.

Suppose first that $z_1, z_2 \in \bar{D}$ with $|z_1| = |z_2| = r$. Then by (3.2)

$$|w_2| \leqq b_1 |w_1| \tag{3.13}$$

where $w_j = f^*(z_j)$ for $j = 1,2$ and $b_1 = b_1(K, M, c')$.

Suppose next that $z_3 \in R^2 \setminus \bar{D}$ with $|z_3| = r$. Then $z_3 \in G$ where G is a K_1-quasidisk in R^2 with $0 \notin G$; let z_1, z_2 denote the endpoints of the component α of $G \cap S^1(0, r)$ which contains z_3, labeled so that $|w_1| \leqq |w_2|$. Here again $w_j = f^*(z_j)$ for $j = 1, 2, 3$. We shall show that

$$(3.14) \qquad \frac{1}{b_2}|w_1| \leqq |w_3| \leqq b_2 |w_2|$$

where b_2 depends only on K, M and c'.

Choose $z_4 \in \partial G \subset \bar{D}$ so that $|w_4| = |w_3|$, and suppose first that $|z_3 - z_4| \leqq \frac{1}{3}|z_1 - z_4|$. Then

$$|z_4| \leqq \tfrac{1}{3}|z_1 - z_4| + |z_3| \leqq \tfrac{4}{3}|z_3| + \tfrac{1}{3}|z_4|,$$

$$|z_4| \geqq |z_3| - \tfrac{1}{3}|z_1 - z_4| \geqq \tfrac{2}{3}|z_1| - \tfrac{1}{3}|z_4|.$$

Hence

$$\tfrac{1}{2}|z_1| \leqq |z_4| \leqq 2|z_2|$$

and Theorem 3.1 applied to $f^* | \bar{D}$ yields

$$\frac{1}{b_3}|w_1| \leqq |w_3| = |w_4| \leqq b_3 |w_2|,$$

where $b_3 = b_3(K, M, c')$. Suppose next that $|z_3 - z_4| > \frac{1}{3}|z_1 - z_4|$. Then by Lemma 3.9

$$\frac{|z_1 - z_4|}{|z_3 - z_4|}\frac{|z_3 - z_2|}{|z_1 - z_2|} \leqq 3\frac{\operatorname{dia}(\alpha)}{|z_1 - z_2|} \leqq 3c$$

where $c = c(K_1)$. Since G and $G' = f^*(G)$ are K_1-quasidisks and hence $2K_1^2$-QED domains, we can apply Theorem 3.1 to $f^* | \bar{G}$ with $a = 3c$ to obtain

$$(3.15) \qquad \frac{|w_1 - w_4|}{|w_3 - w_4|}\frac{|w_3 - w_2|}{|w_1 - w_2|} \leqq b_4$$

where $b_4 = b_4(K, M, c')$. If $|w_4| \geqq 2|w_1|$, then

$$|w_3 - w_4| \leqq 2|w_4| \leqq 4|w_1 - w_4|$$

and with (3.15)

$$|w_3| \leqq 4b_4|w_1 - w_2| + |w_2| \leqq 4b_4(|w_1| + |w_2|) + |w_2|$$
$$\leqq (8b_4 + 1)|w_2|$$

where the inequality $|w_1| \leqq |w_2|$ has also been used. Similarly if $|w_3| \leqq |w_1|/2$ and hence $|w_3| \leqq |w_2|/2$, then

$$|w_1 - w_2| \leqq 2|w_2| \leqq 4|w_3 - w_2|$$

and

$$|w_1| \leqq 4b_4|w_3 - w_4| + |w_4| \leqq (8b_4 + 1)|w_3|$$

where (3.15) and the equality $|w_3| = |w_4|$ have been used. Thus we obtain (3.14) with

$$b_2 = \max(b_3, 2, 8b_4 + 1).$$

Finally (3.13) and (3.14) imply (3.12) with $c = b_1 b_2^2$ and $z_0 = 0$ completing the proof for the case where $\infty \in D$ and $f(\infty) = \infty$. The general case can then be reduced to this special case by composing f with two auxiliary Möbius transformations.

The following two corollaries are immediate consequences of Theorem 3.11, Lemmas 2.5, 2.11, 2.18 and [18, Corollary 2.33] or [10, Corollary 3].

3.16. **Corollary.** *If D is a* QED *domain in $\bar{R}^2$ and if f is a quasiconformal mapping of D onto D', then f has a quasiconformal extension to $\bar{R}^2$ if and only if D' is* QED.

3.17. **Corollary.** *If D is a uniform domain in R^2 and if f is a quasiconformal mapping of D onto a domain D' in R^2, then f has a quasiconformal extension to $\bar{R}^2$ if and only if D' is uniform.*

For finitely connected domains D in $\bar{R}^2$ we obtain

3.18. **Corollary.** *Suppose that D is a linearly locally and finitely connected domain in $\bar{R}^2$. If f is a quasiconformal mapping of D onto a domain D', then f has a quasiconformal extension to $\bar{R}^2$ if and only if D' is linearly locally connected.*

Proof. If $D = \bar{R}^2$, then there is nothing to prove and in the case $D \neq \bar{R}^2$ we can compose f with two auxiliary Möbius transformations and hence assume $D, D' \subset R^2$. Now Corollary 3.16 or Corollary 3.17 together with Theorem 2.22 yield the result.

3.19. **Remarks.** (a) Since a quasidisk $D \subset R^2$ is uniform, linearly locally connected and QED, all three corollaries are generalizations of the Beurling–Ahlfors extension theorem.

(b) Corollaries 3.16 and 3.17 do not hold for $n \geqq 3$. A counterexample is provided by a quasiconformal mapping of a smooth knotted torus onto one which is not knotted.

(c) If D is a QED domain in $\bar{R}^n$, $n \geqq 2$, with $\bar{D} = \bar{R}^n$, then $E = \bar{R}^n \setminus D$ is NED; see Remark 2.4. In this case it follows from results of Ahlfors and Beurling [1] when $n = 2$ and Aseev and Syčev [2] when $n \geqq 3$ that every K-quasiconformal mapping of D into $\bar{R}^n$ has a K-quasiconformal extension to $\bar{R}^n$.

(d) We give an example in section 4 to show that Corollary 3.18 does not hold when D is infinitely connected.

3.20. **Structure of QED and uniform domains.** Theorem 3.11 can be used to interpret the geometric structure of QED and uniform domains in $\bar{R}^2$.

Suppose that D and D' are domains in $\bar{R}^2$. If there exists a quasiconformal mapping of $\bar{R}^2$ which carries D onto D', then D is QED if and only if D' is. This statement is false if we know only that there exists a quasiconformal mapping of D which carries D onto D'; for an example, let D be the upper half plane and $f(z) = z^2$. On the other hand if we know that D and D' are linearly locally connected and that there exists a quasiconformal mapping of D which carries D onto D', then Theorem 3.11 implies that D is QED if and only if D' is. Thus the collection of QED domains is invariant under quasiconformal mappings in the class of plane domains which are linearly locally connected, i.e. in the class of quasicircle domains.

Alternatively we may think of a domain $D \subset \bar{R}^2$ as being determined by the shape of its boundary components and by their relative position and size as measured by its conformal moduli. Then Theorem 3.11 implies that D is QED if and only if D is a quasicircle domain whose conformal geometry is quasiconformally equivalent to that of another QED domain. In particular, it is natural to ask for geometric conditions on the boundary components of a quasicircle domain D which are necessary and sufficient to guarantee that D is QED.

Obviously the same remarks and questions hold for uniform domains in $\bar{R}^2$.

3.21. **Extension of local quasi-isometries.** Suppose that f is a mapping of $E \subset \bar{R}^n$ into $\bar{R}^n$. We say that f is an *L-quasi-isometry* in E if

$$\frac{1}{L}|x_1 - x_2| \leqq |f(x_1) - f(x_2)| \leqq L|x_1 - x_2|$$

for each pair of points $x_1, x_2 \in E \setminus \{\infty\}$ and if $f(\infty) = \infty$ whenever $\infty \in E$. We say that f is a *local L-quasi-isometry* in E if for each $L' > L$ each $x \in E$ has a neighborhood U such that f is an L'-quasi-isometry in $E \cap U$.

The next theorem is a counterpart of Theorem 3.1 for injective local quasi-isometries.

3.22. **Theorem.** *If f is an injective local L-quasi-isometry of a quasiconvex domain $D \subset \bar{R}^n$ into a domain $D' \subset \bar{R}^n$, then f extends to a quasi-isometry f^* of $\bar{D}$ onto $\bar{D}'$ if and only if D' is quasiconvex. In this case f^* is an L^*-quasi-isometry with $L^* = L\max(a, a')$ where a and a' are the constants for D and D'.*

Proof. Suppose first that f extends to an L^*-quasi-isometry f^* of $\bar{D}$ onto $\bar{D}'$. Let $y_1, y_2 \in D' \setminus \{\infty\}$. Since D is an a-quasiconvex domain, there is a curve γ in D joining $f^{-1}(y_1)$ to $f^{-1}(y_2)$ with

$$l(\gamma) \leqq a|f^{-1}(y_1) - f^{-1}(y_2)|.$$

Now $f(\gamma)$ joins y_1 to y_2 in D' and

$$l(f(\gamma)) \leqq L^* l(\gamma) \leqq L^* a|f^{-1}(y_1) - f^{-1}(y_2)| \leqq L^{*2} a|y_1 - y_2|.$$

Thus D' is a'-quasiconvex with $a' = L^{*2}a$.

Next suppose that D' is a'-quasiconvex and that f is an injective local L-quasi-isometry of an a-quasiconvex domain D onto D'. Fix $x_1, x_2 \in D\backslash\{\infty\}$. There is a rectifiable curve γ joining x_1 and x_2 in D with

$$l(\gamma) \leqq a\,|x_1 - x_2|.$$

Thus

$$|f(x_1) - f(x_2)| \leqq l(f(\gamma)) \leqq Ll(\gamma) \leqq La\,|x_1 - x_2|.$$

Since f is injective, f^{-1} is a local L-quasi-isometry in D' and arguing as above yields

$$|x_1 - x_2| \leqq La'|f(x_1) - f(x_2)|.$$

Hence f is an L^*-quasi-isometry in D where $L^* = L\max(a, a')$ and we can extend f to $\bar{D}$ by continuity.

3.23. **Remark.** Theorem 3.22 together with section 2 yields several extension results for injective local quasi-isometries. For example, *if f is an injective local quasi-isometry of a uniform domain $D \subset R^n$ onto a domain $D' \subset R^n$, then f extends to a quasi-isometry of $\bar{D}$ onto $\bar{D}'$ if and only if D' is uniform.* If D and D' are uniform, then the extension follows from Theorem 3.22 and from the fact that uniform domains are quasiconvex, cf. (2.17). On the other hand, it is easy to see that the image of a uniform domain D under a quasi-isometry $f: D \to R^n$ is again a uniform domain.

We conclude this section with the following analogue of Theorem 3.11 for injective local quasi-isometries.

3.24. **Theorem.** *Suppose that D and D' are domains in $\bar{R}^2$, that D is M-QED and that D' is c'-locally connected. If f is an injective local L-quasi-isometry of D onto D' and, in the case $\infty \notin D$, if the unbounded complementary components of D and D' correspond under f, then f has an L^*-quasi-isometric extension to $\bar{R}^2$ where L^* depends only on the constants L, M and c'.*

The formulation of this result requires a word of explanation. If f is an injective local quasi-isometry, then f defines a homeomorphism of D onto D'. In this case for each component E of $\bar{R}^2\backslash D$ there exists a unique component E' of $\bar{R}^2\backslash D'$ such that $f(x) \to E'$ if and only if $x \to E$ in D. The second hypothesis on f in Theorem 3.24 requires that $\infty \in E'$ whenever $\infty \in E$. This condition is clearly necessary for f to have a quasi-isometric extension to $\bar{R}^2$.

Proof for Theorem 3.24. The hypotheses imply that f is a K-quasiconformal mapping of D onto D' where $K = L^2$. Hence by Theorem 3.11, f has a K^*-quasiconformal extension to $\bar{R}^2$ where K^* depends only on L, M and c'; hence D' is M'-QED where $M' = K^{*2}M$. By Lemma 2.7, D and D' are a-

quasiconvex and a depends only on M and M'. Theorem 3.22 implies that f has an extension, denoted again by f, as an L'-quasi-isometry of $\bar{D}$ onto $\bar{D}'$, where L' depends only on L and a and thus only on L, M and c'.

Next let C be a nondegenerate component of ∂D. Then, cf. the proof of Theorem 3.11, the boundary component C is a K-quasicircle where K depends only on M. Let C' be the boundary component of D' which corresponds to C under f. Again C' is a K'-quasicircle and K' depends only on c'. Let G and G' denote the components of $\bar{R}^2\backslash\bar{D}$ and $\bar{R}^2\backslash\bar{D}'$ bounded by C and C', respectively. Then $G\subset R^2$ if and only if $G'\subset R^2$ and we can apply [7, Theorem 7] to get an L^*-quasi-isometry of $\bar{G}$ onto $\bar{G}'$ which agrees with f on C. Moreover, L^* depends only on L', K and K' and thus only on L, M and c'.

Proceeding in this way we obtain an injective mapping $f^*:\bar{R}^2\to\bar{R}^2$ which extends f, maps ∞ onto ∞ and satisfies the inequality

$$|z_1-z_2|/L^*\leqq|f^*(z_1)-f^*(z_2)|\leqq L^*|z_1-z_2| \tag{3.25}$$

whenever z_1 and z_2 are finite points in the closure of the same component of $\bar{R}^2\backslash\partial D$. A trivial argument then yields (3.25) for all $z_1,z_2\in R^2$ and thus completes the proof.

Finally the following consequences of Theorem 3.24 extend Corollary 1 in [7] in precisely the same way that Corollaries 3.16 and 3.17 extend the aforementioned theorem of Beurling and Ahlfors.

3.26. **Corollary.** *If D is a* QED *domain in $\bar{R}^2$ and if f is an injective local quasi-isometry of D onto D', then f has a quasi-isometric extension to $\bar{R}^2$ if and only if D' is* QED *and the unbounded complementary components of D and D' correspond under f.*

3.27. **Corollary.** *If D is a uniform domain in R^2 and if f is an injective local quasi-isometry of D onto D', then f has a quasi-isometric extension to $\bar{R}^2$ if and only if D' is uniform and the unbounded complementary components of D and D' correspond under f.*

4. Quasicircle domains and conformal mappings

We conclude this paper by exhibiting two infinitely connected domains D, D' in $\bar{R}^2$ and a conformal mapping f of D onto D' which has no quasiconformal extension to $\bar{R}^2$. This example will show that the hypothesis that D be finitely connected is essential in Corollary 3.18.

4.1. **Theorem.** *There exists a compact, totally disconnected set E in R^2 and a conformal mapping f of $D=\bar{R}^2\backslash E$ onto $D'=B^2\backslash F$ where F is a closed, totally disconnected subset of B^2.*

Since D and D' are 1-quasicircle domains, this theorem yields the desired example. The proof of Theorem 4.1 is based on the following results due to Grötzsch [13], see also [16], and to Ahlfors and Beurling [1, Theorem 16], respectively.

4.2. **Lemma.** *Suppose that G is a domain in $\bar{R}^2$ and that $z_0 \in \partial G \setminus \{\infty\}$. Then the following conditions are equivalent.*

(i) $\lim_{z \to z_0} f(z)$ *exists for all conformal mappings f of G into $\bar{R}^2$.*

(ii) *For each $r > 0$, $\operatorname{mod} \Gamma = \infty$ where Γ is the family of all closed curves γ in $G \cap B(z_0, r)$ which have nonzero winding number about z_0.*

4.3. **Lemma.** *There exists a compact, totally disconnected set F in R^2 such that $m(F) > 0$ and such that $\lim_{z \to z_0} f(x)$ exists for each $z_0 \in F$ and each conformal mapping f of $\bar{R}^2 \setminus F$ into $\bar{R}^2$.*

We require the following easy consequence of the above two results.

4.4. **Corollary.** *Suppose that G is a domain in R^2 with $m(G) < \infty$ and that $0 < \varepsilon < 1$. Then there exists a compact set E in G such that $m(G \setminus E) < \varepsilon m(G)$ and such that $\lim_{z \to z_0} f(z)$ exists for each $z_0 \in E$ and each conformal mapping f of $G \setminus E$ into R^2.*

Proof. Let F be the set described in Lemma 4.3. Since $m(F) > 0$, F has a point of density and we can pick an open disk B_0 and a compact set $E_0 \subset F \cap B_0$ such that

$$m(B_0 \setminus E_0) < \frac{\varepsilon}{2} m(B_0). \tag{4.5}$$

Then from Lemmas 4.2 and 4.3 we see that $\lim_{z \to z_0} f(z)$ exists for each $z_0 \in E_0$ and each conformal mapping f of $B_0 \setminus E_0$ into $\bar{R}^2$.

Because $m(G) < \infty$ we can choose disjoint open disks B_j in G, $j = 1, 2, \ldots, n$, such that

$$m\left(G \setminus \bigcup_{j=1}^{n} B_j\right) < \frac{\varepsilon}{2} m(G). \tag{4.6}$$

Let E_j denote the image of E_0 under the similarity mapping which carries B_0 onto B_j. Then

$$E = \bigcup_{j=1}^{n} E_j$$

is a compact subset of G,

$$m(G \setminus E) = m\left(G \setminus \bigcup_{j=1}^{n} B_j\right) + \sum_{j=1}^{n} m(B_j \setminus E_j) < \varepsilon m(G)$$

by (4.5) and (4.6) and $\lim_{z \to z_0} f(z)$ exists for each $z_0 \in E$ and each conformal mapping f of $G \setminus E$ into $\bar{R}^2$.

Proof of Theorem 4.1. For $j = 1,2,\ldots$ let $G_j = \{z : 2^{-(j+1)} < |z| < 2^{-j}\}$ and let E_j denote the compact subset of G_j given in Corollary 4.4 corresponding to $\varepsilon = 2^{-3j}$. Next let $D = \bar{R}^2 \setminus E$ where

$$E = \bigcup_{j=1}^{\infty} E_j \cup \{0\},$$

let Γ denote the family of closed curves in $D \cap B^2$ which have nonzero winding number about 0 and set

$$\rho(z) = \begin{cases} \dfrac{1}{2\pi |z|} & \text{if } z \in D \cap B^2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\int_\gamma \rho ds \geqq \frac{1}{2\pi} \left| \int_\gamma \frac{dz}{z} \right| = |n(\gamma,0)| \geqq 1$$

for each rectifiable curve γ in Γ and

$$\operatorname{mod} \Gamma \leqq \int_{R^2} \rho^2 dm = (2\pi)^{-2} \sum_{j=1}^{\infty} \int_{G_j \setminus E_j} \frac{dm}{|z|^2}$$

$$\leqq (2\pi)^{-2} \sum_{j=1}^{\infty} 2^{2(j+1)} m(G_j \setminus E_j) < \infty.$$

Hence by Lemma 4.2, there exists a conformal mapping g of D into $\bar{R}^2$ such that $\lim_{z \to 0} g(z)$ does not exist; since D is locally connected at 0, this implies that the cluster set $C(g,0)$ of g at 0 is a nondegenerate continuum. Next by Corollary 4.4, $\lim_{z \to z_0} g(z)$ does exist for each $z_0 \in E \setminus \{0\}$ and hence g has a homeomorphic extension to $\bar{R}^2 \setminus \{0\}$. Thus $G = g(\bar{R}^2 \setminus \{0\})$ is a simply connected subdomain of $\bar{R}^2 \setminus C(g,0)$ and the Riemann mapping theorem yields a conformal mapping h of G onto B^2. The conclusion of Theorem 4.1 then follows with $f = h \circ g$ and $F = f(E \setminus \{0\})$.

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