A CLASS OF ISOPERIMETRIC INEQUALITIES

By

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1. Introduction

The work presented here is a sequel to my paper [4] in which I considered the class Φ of non-decreasing functions $\phi : [0, 2\pi) \to \mathbf{R}$ such that $\phi(2\pi - 0) - \phi(0) \leqq 2\pi$, and the Fourier series

$$e^{i\phi(\theta)} \sim \sum_{n=-\infty}^{\infty} c_n e^{in\theta}. \tag{1}$$

However, my primary aim now is to establish certain isomprimetric inequalities for the class of closed convex curves Γ with perimeter 2π, similar to those of H. Sachs [5] and R. Wegmann [7]. Let

$$\Phi_0 := \{\phi \in \Phi : c_0 = 0\}. \tag{2}$$

Each $\phi \in \Phi_0$ is associated with one of the curves Γ through the relation $(dX/ds, dY/ds) = (\cos\phi(s), \sin\phi(s))$ where (X, Y), (s, ϕ) denote Cartesian, intrinsic coordinates, respectively. We write

$$\begin{aligned} X(s) &= \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} (A_n \cos ns + B_n \sin ns), \\ Y(s) &= \tfrac{1}{2}C_0 + \sum_{n=1}^{\infty} (C_n \cos ns + D_n \sin ns), \end{aligned} \tag{3}$$

and define the invariants

$$I_n(\Gamma) := A_n^2 + B_n^2 + C_n^2 + D_n^2, \tag{4}$$

$$\Delta_n(\Gamma) = A_n D_n - B_n C_n. \tag{5}$$

With $\phi \in \Phi_0$ and Γ related as above, we have

$$I_n(\Gamma) = \frac{2}{n^2}(|c_n|^2 + |c_{-n}|^2), \qquad \Delta_n(\Gamma) = \frac{1}{n^2}(|c_n|^2 - |c_{-n}|^2). \tag{6}$$

It is sometimes easier to study the larger class Φ even in problems arising specifically in Φ_0. In my previous work [4] I evaluated the Heinz constant

$$\mu_1^{(0)} = \inf_{\Gamma} I_1(\Gamma) = 27/2\pi^2 \tag{7}$$

by showing that

$$\inf\{\tfrac{1}{2}|c_{-1}|^2+|c_0|^2+\tfrac{1}{2}|c_1|^2:\phi\in\Phi\}=27/8\pi^2, \tag{8}$$

and in §6 below I give a general result of this type concerning sums of the form $\Sigma a_n I_n(\Gamma)$. This includes Heinz' constant, and a theorem quoted by Wegmann [7] and derived from the work of Sachs [5] that the moment of inertia of Γ, regarded as a wire of unit density, satisfies

$$\sum_{n=1}^{\infty} I_n(\Gamma)\geqq 4\pi^2/27. \tag{9}$$

The extremal Γ in these problems is the equilateral triangle.

An important theorem of Choquet [1] implies that in (1), $|c_1|\geqq|c_{-1}|$, or $\inf\Delta_1(\Gamma)=0$. In §3 I show that in fact

$$|c_1|^2-|c_{-1}|^2\geqq n^{-1}\big||c_n|^2-|c_{-n}|^2\big| \qquad \text{for all } n,$$

which implies $\Delta_1(\Gamma)\geqq n|\Delta_n(\Gamma)|$. The area inside Γ is given by the formula

$$\text{Area}(\Gamma)=\pi\sum_{n=1}^{\infty} n\Delta_n(\Gamma) \tag{10}$$

and in §2 I prove an area theorem,

$$\text{Area}(\Gamma)\leqq(2\pi^5/81)\Delta_1(\Gamma). \tag{11}$$

The constant on the right is 7.5560... and it would be interesting to know its best possible value. The circle shows that this must be at least π.

In §5 I prove the conjecture of J. L. Ullman and C. J. Titus [6] that $I_1(\Gamma)+2\Delta_1(\Gamma)\geqq 16/\pi^2$. In the notation of (1) this is simply

$$\inf\{|c_1|:\phi\in\Phi_0\}=2/\pi. \tag{12}$$

Equality is attained when Γ collapses to a (double) line segment of length π. This has an application to the theory of harmonic univalent mappings developed by J. Clunie and T. Sheil-Small [2]: let $h(z)$ be a harmonic, univalent, sense-preserving map from the unit disc onto a domain $\mathcal{D}$. Let h be normalized by the conditions

$$h(0)=0, \qquad \left.\frac{\partial h}{\partial z}\right|_{z=0}=1.$$

Then $\mathcal{D}$ has a boundary point w such that $|w|<\pi/2$. The constant $\pi/2$ is best possible.

2. An area theorem

Theorem 1. *Let Γ be a closed convex curve with perimeter 2π. Then*

$$\text{Area}(\Gamma)\leqq(2\pi^5/81)\Delta_1(\Gamma) \tag{13}$$

where the left-hand side denotes the area within Γ *and* $\Delta_1(\Gamma)$ *is defined by* (3) *and* (5).

Proof. Given $\phi \in \Phi$ the function $e^{i\phi}$ may be continued into the unit disc to give a harmonic function $h(z) = f(z) + \bar{g}(z)$ where

$$f(z) = \tfrac{1}{2}c_0 + c_1 z + c_2 z^2 + \cdots \tag{14}$$
$$g(z) = \tfrac{1}{2}\bar{c}_0 + \bar{c}_{-1} z + \bar{c}_{-2} z^2 + \cdots$$

are analytic. For $|\zeta| < 1$ let $\theta_1 \in \Phi$ be such that

$$e^{i\theta_1} = \frac{e^{i\theta} + \zeta}{1 + \bar{\zeta}e^{i\theta}}$$

and $\phi_1(\theta) = \phi(\theta_1(\theta))$. The harmonic continuation $h_1(z)$ of $e^{i\phi_1}$ is $f_1(z) + \bar{g}_1(z)$ where

$$f_1(z) = f\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) - \tfrac{1}{2}(f(\zeta) - \bar{g}(\zeta)), \tag{15}$$
$$g_1(z) = g\left(\frac{z+\zeta}{1+\bar{\zeta}z}\right) + \tfrac{1}{2}(\bar{f}(\zeta) - g(\zeta)).$$

Hence $f_1'(0) = (1 - |\zeta|^2)f'(\zeta)$, $g_1'(0) = (1 - |\zeta|^2)g'(\zeta)$. For any f and g as in (14) we have $|f'(0)|^2 + |g'(0)|^2 = |c_1|^2 + |c_{-1}|^2 \leqq 1$ (by Parseval's identity applied to (1)). Applying this to $|f_1'(0)|^2 + |g_1'(0)|^2$ yields

$$|f'(\zeta)|^2 + |g'(\zeta)|^2 \leqq (1 - |\zeta|^2)^{-2}, \qquad |\zeta| < 1. \tag{16}$$

We recall from [4] that there exists $w(z)$, analytic for $|z| < 1$ and such that $|w(z)| \leqq 1$, $g'(z) \equiv w(z)f'(z)$, $(|z| < 1)$. This follows from the fundamental inequality $|c_{-1}| \leqq |c_1|$ via the Möbius transformation above. We now have

$$\begin{aligned} \text{Area}(\Gamma) &= \pi \sum_{n=1}^{\infty} n\Delta_n(\Gamma) \\ &= \pi \sum_{n=1}^{\infty} \frac{1}{n}(|c_n|^2 - |c_{-n}|^2) \\ &= 2\int_0^1 r \log^2 \frac{1}{r}\, dr \int_0^{2\pi} (|f'(re^{i\theta})|^2 - |g'(re^{i\theta})|^2)d\theta \end{aligned}$$

after differentiating (14) and applying Parseval's identity to the inner integral. But from (16)

$$|f'(re^{i\theta})|^2 - |g'(re^{i\theta})|^2 \leqq \frac{1 - |w(re^{i\theta})|^2}{1 + |w(re^{i\theta})|^2} \frac{1}{(1-r^2)^2}$$

and so

$$\text{Area}(\Gamma) \leqq \frac{\pi^2}{12} \sup_r \int_0^{2\pi} \frac{1-|w(re^{i\theta})|^2}{1+|w(re^{i\theta})|^2}\, d\theta.$$

The integral does not exceed $2\pi(1-|w(0)|^2)/(1+|w(0)|^2)$. To prove this, we may assume $w(0)$ real. Then Cauchy's theorem gives (with C the circle $z = re^{i\theta}$)

$$\frac{1-w(0)^2}{1+w(0)^2} = \frac{1}{2\pi i} \int_C \frac{1-w(z)^2}{1+w(z)^2} \cdot \frac{dz}{z}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \frac{1-w(re^{i\theta})^2}{1+w(re^{i\theta})^2}\, d\theta.$$

The inequality follows on taking real parts. We now have

$$\text{Area}(\Gamma) \leqq \frac{\pi^3}{6} \cdot \frac{|c_1|^2-|c_{-1}|^2}{|c_1|^2+|c_{-1}|^2} = \frac{\pi^3}{3} \cdot \frac{\Delta_1(\Gamma)}{I_1(\Gamma)}\,;$$

and apply (7) to the denominator. This gives the result stated.

3. The function $K(\tau)$

Given $\phi \in \Phi$ we define

$$K(\tau) := \frac{1}{2\pi} \int_0^{2\pi} \sin(\phi(x+\tau)-\phi(x))dx = \sum_{n=-\infty}^{\infty} |c_n|^2 \sin n\tau, \tag{17}$$

where the range of definition of ϕ is extended to $\mathbf{R}$ by setting $\phi(\theta+2\pi) \equiv \phi(\theta)+2\pi$. We note that K is sub-additive in the sense that

$$K(\tau_1+\tau_2) \leqq K(\tau_1)+K(\tau_2) \qquad (0 \leqq \tau_1 \leqq 2\pi,\ 0 \leqq \tau_2 \leqq 2\pi-\tau_1). \tag{18}$$

For the function $\sin\theta$ is sub-additive in the same sense and so

$$K(\tau_1+\tau_2) = \frac{1}{2\pi} \int_0^{2\pi} \sin(\phi(x+\tau_1+\tau_2)-\phi(x))dx$$

$$\leqq \frac{1}{2\pi} \int_0^{2\pi} \sin(\phi(x+\tau_1+\tau_2)-\phi(x+\tau_2)) + \sin(\phi(x+\tau_2)-\phi(x))dx$$

$$\leqq K(\tau_1)+K(\tau_2).$$

Theorem 2. *The function K defined by* (17) *satisfies* $K(\tau) \geqq 0$ *for* $0 \leqq \tau \leqq \pi$ *and every* $\phi \in \Phi$.

In fact $K(\tau)\sin\tau \geqq 0$ for every real τ.

Corollary. *For every $\phi \in \Phi$ we have*

$$||c_n|^2 - |c_{-n}|^2| \leqq n(|c_1|^2 - |c_{-1}|^2), \qquad n \geqq 1, \tag{19}$$

$$n|\Delta_n(\Gamma)| \leqq \Delta_1(\Gamma), \qquad n \geqq 1, \quad \phi \in \Phi_0, \tag{20}$$

$$\Delta_1(\Gamma) \leqq \frac{2}{\pi} \operatorname{Area}(\Gamma), \qquad \phi \in \Phi_0. \tag{21}$$

We begin with the corollary. Since

$$|c_n|^2 - |c_{-n}|^2 = \frac{2}{\pi} \int_0^\pi K(\tau) \sin n\tau d\tau \tag{22}$$

and $|\sin n\tau| \leqq n \sin \tau$ for $0 \leqq \tau \leqq \pi$, (19) follows. For $\phi \in \Phi_0$, (20) is equivalent, by (6). Next we have

$$\begin{aligned} \int_0^\pi (\pi - \tau) K(\tau) d\tau &= \pi \sum_{n=1}^\infty \frac{|c_n|^2 - |c_{-n}|^2}{n} \\ &= \pi \sum_{n=1}^\infty n \Delta_n(\Gamma) = \operatorname{Area}(\Gamma) \end{aligned} \tag{23}$$

for $\phi \in \Phi_0$. We deduce (21) from the inequality $\sin \tau \leqq \pi - \tau$ $(0 \leqq \tau \leqq \pi)$, and (23).

We may assume in the proof of the theorem that $\tau > \pi/2$. Else there exists $m \in \mathbf{Z}^+$ such that $\tau' = m\tau \in (\pi/2, \pi]$ and, by the sub-additivity, we shall have $K(\tau) \geqq m^{-1}K(m\tau) = m^{-1}K(\tau')$, so we only need to prove that $K(\tau') \geqq 0$. Now let $a \in [0, 2\pi)$ be given. Since $\phi(x + \tau) - \phi(x)$ is periodic we can change the range of integration in (18) to $[a - \tau, a - \tau + 2\pi]$. We write this as $[a - \tau, a - \tau + \pi/2) \cup [a, a + \pi/2) \cup E$. These ranges are disjoint $(\tau > \pi/2)$ and we make the substitution $x \to x - \tau$ in the first of them. We then have

$$K(\tau) = \int_a^{a+\pi/2} 2 \sin\left(\frac{\phi(x+\tau) - \phi(x-\tau)}{2}\right) \cos\left|\frac{\phi(x+\tau) + \phi(x-\tau)}{2} - \phi(x)\right| dx$$
$$+ \int_E \sin(\phi(x+\tau) - \phi(x)) dx.$$

We now vary the function ϕ on the range $[a, a + \pi/2)$. This does not affect the integral over E and we show that there exists a suitable c, $\phi(a) \leqq c \leqq \phi(a + \pi/2)$ such that if we make $\phi(x) = c$ for $x \in [a, a + \pi/2)$ then the value of the first integral above is not increased.

Lemma 1. *Let $F: \mathbf{R}^+ \to \mathbf{R}$ be continuous and strictly decreasing, $x_1 \leqq x_2 \leqq \cdots \leqq x_N$ be given and the weights $w_i \geqq 0$, $1 \leqq i \leqq N$. Then the minimum of*

$$S := \sum_{i=1}^{N} w_i F(|x_i - y_i|)$$

obtained by varying the numbers y_i *subject to the constraint* $u \leqq y_1 \leqq y_2 \leqq \cdots \leqq y_N \leqq v$ *is attained when* $y_1 = y_2 = \cdots = y_N = y$, *for some* $y \in [u, v]$.

Proof. We begin by assuming $x_1 < x_2 < \cdots < x_N$ and $w_i > 0$. The minimum is clearly attained. Suppose that for some $i < N$, we have $y_i < y_{i+1}$. Then $F(|x_i - y_i|) \leqq F(|x_i - y_{i+1}|)$ else we should replace y_i by y_{i+1}. As F is strictly decreasing, we deduce that $|x_i - y_i| \geqq |x_i - y_{i+1}|$, that is $x_i \geqq \frac{1}{2}(y_i + y_{i+1})$. Also $F(|x_{i+1} - y_{i+1}|) \leqq F(|x_{i+1} - y_i|)$ with implies

$$|x_{i+1} - y_{i+1}| \geqq |x_{i+1} - y_i| \quad \text{or} \quad x_{i+1} \leqq \tfrac{1}{2}(y_i + y_{i+1}).$$

Since $x_i < x_{i+1}$ this is a contradiction and the result follows, subject to the assumption above about the x_i and w_i. Since F is continuous the general case may be established by a limiting argument. The values y_i for which the minimum is attained may not be unique.

We apply the lemma to the Riemann sums approximating the first integral above. We divide $[a, a + \pi/2)$ into N intervals, with ξ_i a point in the i-th interval, $1 \leqq i \leqq N$. We put

$$x_i = \frac{\phi(\xi_i - \tau) + \phi(\xi_i + \tau)}{2}, \qquad w_i = 2l_i \sin \frac{\phi(\xi_i + \tau) - \phi(\xi_i - \tau)}{2}$$

where l_i is the length of the i-th interval. As ϕ is non-decreasing, we have $x_i \leqq x_{i+1}$, and as $\tau \leqq \pi$, we have $\phi(\xi_i + \tau) - \phi(\xi_i - \tau) \leqq 2\pi$ and so $w_i \geqq 0$. We put $F =$ cosine: since

$$\left| \frac{\phi(x + \tau) - \phi(x)}{2} - \frac{\phi(x) - \phi(x - \tau)}{2} \right| \leqq \pi$$

we have F strictly decreasing. Finally $u = \phi(a)$, $v = \phi(a + \pi/2)$, $y_i = \phi(\xi_i)$. By the lemma, each Riemann sum attains its minimum when $\phi(x)$ takes a suitable constant value on $[a, a + \pi/2)$: hence this is true of the integral itself.

We now split $[0, 2\pi)$ into four ranges of length $\pi/2$, taking $a = 0$, $\pi/2$, π, $3\pi/2$ successively, at each stage replacing ϕ by a suitable constant and not increasing the value of $K(\tau)$. After all four stages, we have replaced ϕ by a step function with equal steps and jumps α_1, α_2, α_3, α_4, say, where $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 2\pi$. But then for $\pi/2 < \tau \leqq \pi$,

$$K(\tau) = \left(\frac{1}{2} - \frac{\tau}{2\pi}\right) \sum_{i=1}^{4} \sin \alpha_i + \left(\frac{\tau}{2\pi} - \frac{1}{4}\right) \sum_{i=1}^{4} \sin(\alpha_i + \alpha_{i+1}) \tag{24}$$

(in which α_5 is to be interpreted as α_1). Since $\alpha_3 + \alpha_4 = 2\pi - \alpha_1 - \alpha_2$, $\alpha_4 + \alpha_1 = 2\pi - \alpha_2 - \alpha_3$ the second sum on the right is zero. The first sum is $\geqq \sin 2\pi = 0$ by the sub-additivity of the sine function. Hence the original $K(\tau)$ was non-negative.

4. Admissible functions

In this section we establish the basic inequalities required for §6 and the proof of the Ullman and Titus conjecture. This latter appears to be harder than, for example, the evaluation of Heinz' constant, because in addition its proof requires, in this treatment, both the Area Theorem and the inequality $n|\Delta_n(\Gamma)| \leqq \Delta_1(\Gamma)$ arising from Theorem 2.

We say that a function $F:[0,\pi]\to\mathbf{R}^+$ is *admissible* if it has a representation

$$F(\tau)=\sum_{n=2}^{\infty} S_n(\tau), \qquad 0<\tau\leqq\pi, \tag{25}$$

where $S_n(\tau)\geqq 0$, is supported on $[0,2\pi/n]$ and is symmetric about π/n, i.e., $S_n(2\pi/n-\tau)\equiv S_n(\tau)$. Any decreasing function is admissible, we put

$$S_2(\tau)=F(\tau), \qquad \pi/2\leqq\tau\leqq\pi,$$

$$S_4(\tau)=F(\tau)-S_2(\tau), \qquad \pi/4\leqq\tau\leqq\pi/2,$$

$$S_8(\tau)=F(\tau)-S_2(\tau)-S_4(\tau), \qquad \pi/8\leqq\tau\leqq\pi/4,$$

and so on. In general (25) will not hold at $\tau=0$.

Lemma 2. *Let F be admissible. For a given $\phi\in\Phi$ let*

$$J(\tau)=\frac{1}{2\pi}\int_0^{2\pi}\sin^2\left(\frac{\phi(x+\tau)-\phi(x)}{2}\right)dx \tag{26}$$

and $K(\tau)$ be as defined by (17). *Then*

$$\int_0^{\pi}\{2J(\tau)-\lambda K(\tau)\}F(\tau)d\tau\leqq\kappa(\lambda)\int_0^{\pi}F(\tau)d\tau \tag{27}$$

where $\kappa(\lambda)=(9-3\lambda\sqrt{3})/4\pi$, $(\lambda<1/3\sqrt{3})$, $\kappa(\lambda)=2/\pi$, $(\lambda\geqq 1/3\sqrt{3})$.

Proof. We need the following elementary inequality: if $\theta_n\geqq 0\ (1\leqq n\leqq N)$ and $\theta_1+\theta_2+\cdots+\theta_N=2\pi$ then

$$\sum_{n=1}^{N}\left\{2\sin^2\frac{\theta_n}{2}-\lambda\sin\theta_n\right\}\leqq 2\pi\kappa(\lambda). \tag{28}$$

We note that when the supremum on the left-hand side is attained, any two non-zero θ_n's must be equal and so there exists $N_1\leqq N$ such that θ_n is either $2\pi/N_1$ or 0 for every $n\leqq N$. When $\lambda<1/3\sqrt{3}$, $N_1=3$. Otherwise $N_1=2$. This proves (28).

Now put $L(\tau)=2J(\tau)-\lambda K(\tau)$. Then if $\tau_n\geqq 0$, $(1\leqq n\leqq N)$ and $\tau_1+\tau_2+\cdots+\tau_N=2\pi$ we have

(29) $$\sum_{n=1}^{N} L(\tau_n) \leqq 2\pi\kappa(\lambda).$$

We use (28) in the same fashion as in the proof of Lemma 2 [4].

We have to show that for an admissible F, (27) holds. It is enough to prove (27) for each S_n. But

$$\int_0^{2\pi/n} S_n(\tau)L(\tau)d\tau = \int_0^{2\pi/n} S_n\left(\frac{2\pi}{n}-\tau\right) L\left(\frac{2\pi}{n}-\tau\right) d\tau$$

$$= \frac{1}{2}\int_0^{2\pi/n} S_n(\tau)\left\{L(\tau)+L\left(\frac{2\pi}{n}-\tau\right)\right\} d\tau.$$

From (29), we have $n\{L(\tau)+L(2\pi/n-\tau)\} \leqq 2\pi\kappa(\lambda)$ and so the integral above does not exceed

$$\kappa(\lambda)\int_0^{2\pi/n} S_n(\tau)(\pi/n)d\tau = \kappa(\lambda)\int_0^{2\pi/n} S_n(\tau)\tau d\tau.$$

This completes the proof.

5. The conjecture of Ullman and Titus

Theorem 3. *We have, for c_1 as in* (1),

(30) $$\inf\{|c_1| : \phi \in \Phi_0\} = 2/\pi.$$

Indeed the stronger result

(31) $$I_1(\Gamma) + \tfrac{3}{2}\Delta_1(\Gamma) \geqq 16/\pi^2$$

holds for every Γ.

Equality holds in (30) when $\phi(x) = 0$ $(0 \leqq x < \pi)$, $= \pi$ $(\pi \leqq x < 2\pi)$ and in (31) when Γ is a double line segment of length π. In the original conjecture the factor before $\Delta_1(\Gamma)$ (which is of course positive) was 2. In this form Ullman and Titus could prove their conjecture in the special case when $\phi(x+\pi) \equiv \phi(x) + \pi$.

We consider the integral

(32) $$I := \frac{2}{\pi}\int_0^{\pi}\left(1 - 2J(\tau) + \frac{1}{3\sqrt{3}}K(\tau)\right)(1+\cos\tau)d\tau.$$

Since

$$1 - 2J(\tau) = \frac{1}{2\pi}\int_0^{2\pi}\cos(\phi(x+\tau)-\phi(x))dx = \sum_{n=-\infty}^{\infty}|c_n|^2\cos n\tau,$$

moreover $K(\tau)$ satisfies (17), we have

$$I = |c_{-1}|^2 + 2|c_0|^2 + |c_1|^2 + \frac{4}{3\pi\sqrt{3}} \sum{}' \frac{d_n}{n} + \frac{4}{3\pi\sqrt{3}} \sum{}'' \frac{nd_n}{n^2-1},$$

where $d_n = |c_n|^2 - |c_{-n}|^2$, $n \geqq 1$ and Σ', Σ'' denote summation over odd, even positive integers. Now let $\phi \in \Phi_0$. Then $c_0 = 0$ and $d_n = n^2\Delta_n(\Gamma)$, hence

$$I = |c_{-1}|^2 + |c_1|^2 + \frac{4}{3\pi^2\sqrt{3}} \operatorname{Area}(\Gamma) + \frac{4}{3\pi\sqrt{3}} \sum{}'' \frac{n\Delta_n(\Gamma)}{n^2-1}.$$

The Area Theorem gives

$$\frac{4}{3\pi^2\sqrt{3}} \operatorname{Area}(\Gamma) \leqq \frac{8\pi^3}{243\sqrt{3}} \Delta_1(\Gamma) \leqq 0.59\Delta_1(\Gamma)$$

and since $n|\Delta_n(\Gamma)| \leqq \Delta_1(\Gamma)$ by (21) we have

$$\frac{4}{3\pi\sqrt{3}} \sum{}'' \frac{n\Delta_n(\Gamma)}{n^2-1} \leqq \frac{2}{3\pi\sqrt{3}} \Delta_1(\Gamma) \leqq 0.13\Delta_1(\Gamma).$$

Hence

$$2I \leqq I_1(\Gamma) + \tfrac{3}{2}\Delta_1(\Gamma). \tag{33}$$

On the other hand $(1 + \cos\tau)$ decreases on $[0, \pi]$ and so is admissible. Lemma 2 yields

$$\frac{2}{\pi} \int_0^\pi \left(2J(\tau) - \frac{1}{3\sqrt{3}} K(\tau)\right)(1 + \cos\tau)d\tau \leqq \frac{4}{\pi^2} \int_0^\pi \tau(1 + \cos\tau)d\tau.$$

The right-hand side is $2 - 8/\pi^2$, hence we have $I \geqq 8/\pi^2$ and we insert this into (33). This completes the proof.

6. On an inequality of H. Sachs

Finally we prove an inequality which includes as special cases my result [4] on the Heinz constant, and that of Sachs concerning the moment of inertia of Γ.

Theorem 4. *Let $a_0, a_1, a_2, \ldots,$ be such that the function*

$$p(\tau) := \tfrac{1}{2}a_0 + a_1 \cos\tau + a_2 \cos 2\tau + \cdots \tag{34}$$

is non-negative on $[-\pi, \pi]$, non-increasing on $[0, \pi]$ and convex on $[2\pi/3, \pi]$. Then the functional

$$\Lambda(\phi, p) := a_0|c_0|^2 + a_1(|c_{-1}|^2 + |c_1|^2) + \cdots \tag{35}$$

is minimized by the function

(36) $$\phi_3(x):=0\ \left(0\leqq x<\frac{2\pi}{3}\right),\quad \frac{2\pi}{3}\left(\frac{2\pi}{3}\leqq x<\frac{4\pi}{3}\right),\quad \frac{4\pi}{3}\left(\frac{4\pi}{3}\leqq x<2\pi\right).$$

That is to say

(37) $$\Lambda(\phi,p)\geqq\frac{27}{4\pi^2}\sum_{\substack{m=1\\ m\not\equiv 0\ (\mathrm{mod}\ 3)}}^{\infty}\frac{a_m}{m^2}.$$

Remark. No doubt the conditions on p can be weakened somewhat. However, they are satisfied by some interesting examples. If we set $a_0=1$, $a_1=\frac{1}{2}$, $a_n=0$ for $n\geqq 2$ we obtain (8). If we set $a_0=2\pi^2/3$, $a_n=2/n^2$ for $n\geqq 1$ (which makes $p(\tau)=(\pi-\tau)^2$) we obtain (9). For $r<1$ we may set $a_n=2r^n$ so that $p(\tau)$ is the Poisson kernel.

Proof. We denote by $q(\tau)$ the even function of period $2\pi/3$ such that

(38) $$q(\tau)=p\left(\tau+\frac{2\pi}{3}\right),\qquad 0\leqq\tau<\pi/3,$$

and write

(39) $$q(\tau)\sim\tfrac{1}{2}b_0+b_3\cos 3\tau+b_6\cos 6\tau+\cdots$$

where

$$b_{3k}=\frac{6}{\pi}\int_0^{\pi/3}q(\tau)\cos 3k\tau d\tau=\frac{6}{\pi}\int_{2\pi/3}^{\pi}p(\tau)\cos 3k\tau d\tau$$

in view of (38). Thus $b_0\geqq 0$ and for $k>0$,

$$b_{3k}=-\frac{2}{\pi k}\int_{2\pi/3}^{\pi}\sin 3k\tau dp(\tau).$$

Since p is non-increasing, and convex on $[2\pi/3,\pi]$, we deduce that $b_{3k}\geqq 0$ by the second mean value theorem.

Next, put $p_1(\tau)=p(\tau)-q(\tau)$, which is non-negative and supported on $[-2\pi/3,2\pi/3]$, and consider the integral

(40) $$I'(\phi):=\frac{2}{\pi}\int_0^{\pi}(1-2J(\tau))p_1(\tau)d\tau$$

where $J(\tau)$ is defined by (26). We have

(41) $$\begin{aligned}I'&=(a_0-b_0)|c_0|^2+(a_1-b_1)(|c_{-1}|^2+|c_1|^2)+\cdots\\&\leqq\Lambda(\phi,p)\end{aligned}$$

because $b_n\geqq 0$ for every n.

We write $p_1(\tau) = p_2(\tau) + p_3(\tau)$ where

$$p_2(\tau) = p(\tau) - p(2\pi/3) \ (0 \leqq \tau < 2\pi/3), \quad 0 \text{ else},$$

$$p_3(\tau) = p(2\pi/3) - q(\tau) \ (0 \leqq \tau < 2\pi/3), \quad 0 \text{ else},$$

so that $p_2(\tau)$ is non-increasing and therefore admissible in the sense explained in §4. Since p_3 is an S_3, we see that p_1 is admissible and Lemma 2, with $\lambda = 0$, yields

$$I' \geqq \frac{2}{\pi} \int_0^{\pi} \left(1 - \frac{9\tau}{4\pi}\right) p_1(\tau) d\tau. \tag{42}$$

The function ϕ_3 has $J(\tau) = 9\tau/8\pi$ $(0 \leqq \tau \leqq 2\pi/3)$ and since $p_1(\tau) = 0$ for $\tau > 2\pi/3$, the right-hand side of (42) is simply $I'(\phi_3)$. When $\phi = \phi_3$, we have $|c_n|^2 = 27/4\pi^2 n^2$ $(n \equiv 1 \pmod 3)$, $= 0$ else. Since $b_n = 0$ unless $n = 3k$, we have

$$I'(\phi_3) = \sum_{\substack{m=1 \\ m \not\equiv 0 \pmod 3}}^{\infty} \frac{27 a_m}{4\pi^2 m^2} = \Lambda(\phi_3, p). \tag{43}$$

Combining (41), (42) and (43) we obtain our result.

Remark. We notice that the proof gives a slightly stronger result than (37), inasmuch as we can replace $\Lambda(\phi, p)$ on the left by $\Lambda_1(\phi, p)$ in which a_n is replaced by $a_n - b_n$. For example, when $a_n = 2r^n$ and $p(\tau)$ is the Poisson kernel,

$$b_0 = \frac{12}{\pi} \arctan \left\{ \frac{1}{\sqrt{3}} \left(\frac{1-r}{1+r} \right) \right\}. \tag{44}$$

If we ignore the other b_n's we get

$$\left(1 - \frac{6}{\pi} \arctan \left\{ \frac{1}{\sqrt{3}} \left(\frac{1-r}{1+r} \right) \right\} \right) |c_0|^2 + \sum_{n=1}^{\infty} r^n (|c_{-n}|^2 + |c_n|^2) \geqq \frac{27}{4\pi^2} \sum_{\substack{m=1 \\ m \not\equiv 0 \pmod 3}}^{\infty} \frac{r^m}{m^2}. \tag{45}$$

The coefficient of $|c_0|^2$ is $O(r)$ as $r \to 0$. Hence we may multiply by $(1/r)(\log 1/r)^{\alpha - 1}$ $(\alpha > 0)$, and integrate over $[0, 1]$. For a suitable $c(\alpha) < \infty$ this yields

$$c(\alpha)|c_0|^2 + \sum_{n=1}^{\infty} \frac{1}{n^{\alpha}} (|c_{-n}|^2 + |c_n|^2) \geqq \frac{27}{4\pi^2} \zeta(2 + \alpha) \left(1 - \frac{1}{3^{2+\alpha}}\right) \tag{46}$$

where ζ denotes the Riemann zeta-function. The inequality is sharp: equality is attained when $\phi = \phi_3$.

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