TRACE PALEY–WIENER THEOREM FOR REDUCTIVE p-ADIC GROUPS

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§1. Statement of the theorem

1.1. Let G be a reductive p-adic group. A smooth representation (π, E) of the group G on a complex vector space E is called a G-*module*. Usually we shorten the notation and write π or E.

Let $\mathcal{M}(G)$ be the category of G-modules, $\operatorname{Irr} G$ the set of equivalence classes of irreducible G-modules, and $R(G)$ the Grothendieck group of G-modules of finite length; $R(G)$ is a free abelian group with basis $\operatorname{Irr} G$.

We fix a minimal parabolic subgroup $P_0 \subset G$ and its Levi decomposition $P_0 = M_0 \cdot U_0$. By a standard Levi subgroup we mean a subgroup $M \supseteq M_0$ which is a Levi component of the parabolic subgroup $P = M \cdot P_0$ (notation $M < G$). For any standard Levi subgroup $M < G$ the functors $i_{GM} : \mathcal{M}(M) \to \mathcal{M}(G)$ and $r_{MG}: \mathcal{M}(G) \to \mathcal{M}(M)$ define morphisms $i_{GM}: R(M) \to R(G)$, $r_{MG}: R(G) \to R(M)$ (see [2, §2] or [1, 2.5]).

Let $\Psi(G) \subset \{\psi: G \to \mathbf{C}^*\}$ be the group of unramified characters of G. It acts naturally on $\operatorname{Irr} G$ and $R(G)$ by $\psi: \pi \mapsto \psi\pi$. This group has a natural structure of *complex algebraic group* (isomorphic to $(\mathbf{C}^*)^d$).

1.2. Let $\mathcal{H}(G)$ be the Hecke algebra of G (algebra of locally constant complex valued measures on G with compact support). Each measure $h \in \mathcal{H}(G)$ defines a linear form $f_h : R(G) \to \mathbf{C}$ by $f_h(\pi) = \operatorname{tr} \pi(h)$.

It is easy to see that the form $f = f_h$ satisfies the following conditions:

(i) For any standard Levi subgroup $M < G$ and $\sigma \in \operatorname{Irr} M$, the function $\psi \mapsto f(i_{GM}(\psi\sigma))$ is a regular function on the complex algebraic variety $\Psi(M)$.

(ii) There exists an open compact subgroup $K \subset G$ which dominates f, i.e., f is nonzero only on the G-modules E, which have a nontrivial space E^K of K-invariant vectors.

We want to prove that the conditions (i)–(ii) characterize the trace forms $\{f_h\}$. Namely, let $R^*(G) = \operatorname{Hom}_{\mathbf{Z}}(R(G), \mathbf{C}) = \operatorname{Map}(\operatorname{Irr} G, \mathbf{C})$ be the space of all linear forms on $R(G)$. We call a form $f: R(G) \to \mathbf{C}$ *good* if it satisfies conditions (i)–(ii)

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and *trace form* if $f = f_h$ for some $h \in \mathscr{H}(G)$. We denote the subspaces of trace and good forms by $F_{tr} = F_{tr}(G) \subset F_{good} = F_{good}(G)$.

Theorem (Trace Paley-Wiener Theorem). $F_{tr} = F_{good}$.

Remark. This theorem describes the image of the natural morphisms tr: $\mathscr{H}(G) \to R^*(G)$. One can show that $\operatorname{Ker} \mathrm{tr} = [\mathscr{H}(G), \mathscr{H}(G)]$. See Theorem B in [8].

1.3. The paper has the following structure. In Section 2 we recall the basic properties of the central algebra of a p-adic group G. In Section 3 we formulate three technical results, which are proven in Section 5. In Section 4 we prove the theorem, using these results. Section 6 contains some remarks on the proof.

§2. The central algebra $\mathfrak{Z}(G)$

In this section we recall some definitions and results about the central algebra $\mathfrak{Z}(G)$ of a p-adic group G. This algebra is analogous to the center of the universal enveloping algebra for a real group. We follow [1] but we use a slightly different terminology in order to emphasize this analogy.

2.1. *Infinitesimal Characters*

We call a *cuspidal pair* a pair (M, ρ) where $M < G$ is a standard Levi subgroup, $\rho \in \operatorname{Irr} M$ is a cuspidal irreducible M-module. We denote by $\Theta(G)$ the set of all cuspidal pairs up to conjugation by G. A point $\theta \in \Theta(G)$ is called an *infinitesimal* character of G.

For any cuspidal pair (M, ρ) the image of the map ν'_ρ: $\Psi(M) \to \Theta(G)$, given by $\psi \mapsto (M, \psi\rho)$, is called a *connected component* of $\Theta(G)$. This component has the natural structure of a complex affine algebraic variety as a quotient of $\Psi(M)$ by a finite subgroup. We consider $\Theta(G)$ as a complex "algebraic variety", consisting of an infinite number of connected components Θ, and we will write $\Theta(G) = \bigcup \Theta$.

For each $\omega \in \operatorname{Irr} G$ there exists a cuspidal pair (M, ρ), such that ω is a sub-quotient of $i_{GM}(\rho)$. This pair is uniquely defined up to conjugation by G and hence defines a point $\theta \in \Theta(G)$, which is called the *infinitesimal character* of ω (notation: $\theta = \text{inf. ch.}(\omega)$). The map inf. ch.: $\operatorname{Irr} G \to \Theta(G)$ is onto and finite-to-one.

Define an algebraic action of $\Psi(G)$ on $\Theta(G)$ by ψ: $(M, \rho) \to (M, \psi|_M \cdot \rho)$. Then inf. ch. is a $\Psi(G)$-equivariant map. We can decompose $R(G) = \bigoplus_\theta R(\theta)$, $\theta \in \Theta(G)$, where $R(\theta)$ is the subgroup, generated by irreducible G-modules with infinitesimal character θ. More generally, for each subset $S \subset \Theta(G)$ we put $R(S) = \bigoplus_\theta R(\theta)$, $\theta \in S$.

2.2. *The Central Algebra $\mathfrak{Z}(G)$*

Definition. *The central algebra $\mathfrak{Z}(G)$ is the algebra of regular functions on the* complex algebraic variety $\Theta(G)$.

The decomposition $\Theta(G)=\bigcup\Theta$ shows that $\mathfrak{Z}(G)=\prod_\Theta \mathfrak{Z}(\Theta)$, where $\mathfrak{Z}(\Theta)$ is the algebra of regular functions on the component Θ. We denote by $\mathfrak{Z}^0(G)$ the ideal $\bigoplus_\Theta \mathfrak{Z}(\Theta)\subset\mathfrak{Z}(G)$, consisting of functions supported on a finite number of components. We can (and will) interpret infinitesimal characters $\theta\in\Theta(G)$ as algebra homomorphisms $\theta\colon \mathfrak{Z}(G)\to\mathbf{C}$, nontrivial on $\mathfrak{Z}^0(G)$.

Theorem (see [1, 2.13]). *On each G-module E there exists a natural action of $\mathfrak{Z}(G)$ such that*

(i) *$z\colon E\to E$ is a morphism of G-modules for each $z\in\mathfrak{Z}(G)$.*

(ii) *Each G-module morphism $\alpha\colon E\to E'$ is a $\mathfrak{Z}(G)$-morphism.*

(iii) *On each irreducible G-module E the action of $z\in\mathfrak{Z}(G)$ is given by* $z=\mathrm{inf.ch.}(E)(z)\cdot 1_E$.

Remarks. (1) The system of actions of $\mathfrak{Z}(G)$ on G-modules is uniquely determined by properties (i)–(iii).

(2) Consider $\mathscr{H}(G)$ as a G-module with respect to the left action of G. Then the corresponding action of $\mathfrak{Z}(G)$ on $\mathscr{H}(G)$ identifies $\mathfrak{Z}(G)$ with the algebra of all endomorphisms of $\mathscr{H}(G)$, invariant with respect to left and right actions of G (see [1, §1]).

For each G-module π, $h\in\mathscr{H}(G)$, $z\in\mathfrak{Z}(G)$ we have $\pi(zh)=z\circ\pi(h)$. Hence the action of $\mathfrak{Z}(G)$ on π can be described in terms of its action on $\mathscr{H}(G)$.

2.3. *Decomposition Theorem*

Let E be a G-module. Then for each connected component $\Theta\subset\Theta(G)$ its characteristic function $1_\Theta\in\mathfrak{Z}(G)$ acts on E as a projector on a G-submodule E_Θ, and $E=\bigoplus_\Theta E_\Theta$. Moreover, for each open compact subgroup $K\subset G$ there exists an open subset $\Theta_K(G)\subset\Theta(G)$, which is a union of a finite number of components, such that for $\Theta\not\subset\Theta^K(G)$, $E_\Theta^K=0$ for all G-modules E (see [1, 2.10]).

This theorem implies that the Hecke algebra $\mathscr{H}(G)$ can be decomposed as a direct sum of two-sided ideals $\mathscr{H}(G)_\Theta$ and for each open compact subgroup $K\subset G$ the corresponding decomposition $\mathscr{H}_K(G)=\bigoplus_\Theta\mathscr{H}_K(G)_\Theta$ has only a finite number of nonzero terms (here $\mathscr{H}_K(G)$ is the algebra of K-bi-invariant measures).

2.4. *Harish-Chandra Homomorphism*

Let $M<G$ be a standard Levi subgroup. Define the morphism $i_{GM}\colon\Theta(M)\to\Theta(G)$ by $(L,\rho)\to(L,\rho)$. This is a finite morphism of algebraic varieties (it is not an inclusion, since cuspidal pairs conjugate under G may be non-conjugate under M). We call the corresponding morphism $i^*_{GM}\colon\mathfrak{Z}(G)\to\mathfrak{Z}(M)$ the *Harish-Chandra homomorphism.* As for real groups $\mathfrak{Z}(M)$ is a finitely generated $\mathfrak{Z}(G)$-module.

It is easy to check that the map $i_{GM}\colon\Theta(M)\to\Theta(G)$ is compatible with the functors i_{GM} and r_{MG}, i.e., for each subset $S_M \subset \Theta(M)$, $i_{GM}R(S_M) \subset R(i_{GM}(S_M))$ and for each subset $S \subset \Theta(G)$, $r_{MG}R(S) \subset R(i_{GM}^{-1}(S))$. In fact, a more precise statement is true.

Proposition. *Let $z \in \mathfrak{Z}(G)$, $z_M = i_{GM}^*(z) \in \mathfrak{Z}(M)$. Then for each M-module σ, $i_{GM}(z_M) = z$ on $i_{GM}(\sigma)$ and for each G-module π, $r_{MG}(z) = z_M$ on $r_{MG}(\pi)$.*

The proof can be found in [1, 2.13–2.16].

2.5. *Finiteness Theorem*

Theorem (see [1, §3]). *Let $K \subset G$ be an open compact subgroup. Then for any finitely generated G-module E the space E^K of K-invariant vectors is a finitely generated $\mathfrak{Z}^0(G)$-module. In particular, the algebra $\mathcal{H}_K(G)$ is a finitely generated $\mathfrak{Z}^0(G)$-module.*

This is an analogue of Harish–Chandra's finiteness theorem for real groups (see [5, 9.5]).

§3. Preparation for induction: discrete G-modules and discrete forms

3.1. *Discrete G-modules*

Let $R^I(G) \subset R(G)$ be the subgroup of "strictly induced" G-modules, defined by $R^I(G) = \Sigma_M i_{GM}(R(M))$ with $M \lneqq G$. An irreducible G-module ω is called *discrete* if $\omega \notin R^I(G)$. An infinitesimal character θ is called *discrete* if it is the infinitesimal character of a discrete G-module (or, equivalently, if $R(\theta) \not\subset R^I(G)$). We denote the set of all discrete infinitesimal characters by $\Theta_{\text{disc}}(G)$.

Proposition (see 5.1–5.2). *For each connected component $\Theta \subset \Theta(G)$ the set $\Theta_{\text{disc}} = \Theta \cap \Theta_{\text{disc}}(G)$ consists of a finite number of $\Psi(G)$-orbits.*

Let $M < G$, $\sigma \in \operatorname{Irr} M$. The family $\{i_{GM}(\psi\sigma)\}$ parametrized by $\Psi(M)$ is called a *standard family* of G-modules.

Corollary. *For each connected component $\Theta \subset \Theta(G)$ the group $R(\Theta)$ is generated by a finite number of standard families $\{i_{GM}(\psi\sigma)\}$, corresponding to discrete M-modules σ.*

Indeed, it is clear that $R(\Theta)$ is generated by standard families, corresponding to discrete M-modules σ with infinitesimal characters in $i_{GM}^{-1}(\Theta) \subset \Theta(M)$. Since $i_{GM}^{-1}(\Theta)$ consists of a finite number of components and the map inf. ch.: $\operatorname{Irr} M \to \Theta(M)$ is finite-to-one, the proposition shows that there are only a finite number of such families.

3.2. Let $i_{GM}^*: R^*(G) \to R^*(M)$ and $r_{MG}^*: R^*(M) \to R^*(G)$ be morphisms adjoint to morphisms i_{GM} and r_{MG}.

Proposition (see 5.3). (i) $i^*_{GM}(F_{\mathrm{good}}(G)) \subset F_{\mathrm{good}}(M)$;
(ii) $r^*_{MG}(F_{\mathrm{tr}}(M)) \subset F_{\mathrm{tr}}(G)$.

3.3. A linear form $f \in R^*(G)$ is called *discrete* if $f(R^I(G)) = 0$, i.e.,

$$i^*_{MG}(f) = 0 \qquad \text{for all } M \subsetneqq G.$$

Combinatorial Lemma (see 5.4–5.5). *There are constants $c_M \in \mathbf{Q}$ for each $M \subsetneqq G$ such that for each linear form $f \in R^*(G)$ the form $f^d = f - \Sigma_M c_M r^*_{MG} \circ i^*_{GM}(f)$, $M \subsetneqq G$, is discrete.*

§4. Proof of Theorem 1.2

4.1. The central algebra $\mathfrak{Z}(G)$ naturally acts on $R^*(G)$ by $zf(\omega) = \mathrm{inf.ch.}(\omega)(z) \cdot f(\omega)$, $z \in \mathfrak{Z}(G)$, $f \in R^*(G)$, $\omega \in \mathrm{Irr}\, G$. Using Remark 2 in 2.2 we see that tr: $\mathscr{H}(G) \to R^*(G)$ is a morphism of $\mathfrak{Z}(G)$-modules, so its image F_{tr} is a $\mathfrak{Z}(G)$-submodule. It is also easy to check that the subspace $F_{\mathrm{good}} \subset R^*(G)$ is a $\mathfrak{Z}(G)$-submodule (one should only notice that for each standard family $\{i_{GM}(\psi\sigma)\}$ the corresponding map ν'_σ: $\Psi(M) \to \Theta(G)$ is a morphism of algebraic varieties).

The group $\Psi(G)$ naturally acts on $R^*(G)$ by $\psi(f)(\omega) = f(\psi\omega)$. Clearly the subspace F_{good} is $\Psi(G)$-invariant. In order to prove that F_{tr} is $\Psi(G)$-invariant it is sufficient to observe that the morphism tr: $\mathscr{H}(G) \to R^*(G)$ is $\Psi(G)$-equivariant, where $\Psi(G)$ acts on $\mathscr{H}$ by ψ: $h \mapsto \psi h$.

4.2. Consider any subset $S \subset R(G)$. We call the restrictions to S of good and trace linear forms good and trace functions on S. We denote the corresponding spaces of functions by $F_{\mathrm{good}}[S]$, $F_{\mathrm{tr}}[S]$.

Lemma. *For a finite set S the restriction to S of any linear form f: $R(G) \to \mathbf{C}$ is a trace function. In particular, $F_{\mathrm{tr}}[S] = F_{\mathrm{good}}[S]$.*

This is just another way of saying that characters of irreducible G-modules are linearly independent functionals on $\mathscr{H}(G)$.

Proposition. *Let $S \subset \mathrm{Irr}\, G$ be a union of a finite number of $\Psi(G)$-orbits. Then $F_{\mathrm{tr}}[S] = F_{\mathrm{good}}[S]$.*

Proof. The set S has the natural structure of algebraic variety, as a union of a finite number of $\Psi(G)$-orbits. By definition of good forms $F_{\mathrm{tr}}[S] \subset F_{\mathrm{good}}[S] \subset \mathscr{R}(S)$, where $\mathscr{R}(S)$ is the algebra of regular functions on S.

Choose a cocompact lattice Λ in the center $Z(G)$ of the group G and put $Y = \Psi(\Lambda) = \mathrm{Hom}(\Lambda, \mathbf{C}^*)$. Then the restriction map r: $\Psi(G) \to Y$ is a finite epimorphism of algebraic groups. Define a map c': $\mathrm{Irr}\, G \to Y$ by c': $\omega \mapsto (\text{central char.}(\omega))|_\Lambda$. Its restriction to S, c: $S \to Y$ is a finite $\Psi(G)$-equivariant submersive morphism of algebraic varieties. In particular, $\mathscr{R}(S)$ is a finitely generated $\mathscr{R}(Y)$-module. We claim that $F_{tr}[S]$, $F_{good}[S]$ are $\mathscr{R}(Y)$-submodules of $\mathscr{R}(S)$. For $F_{good}[S]$ it is clear. For $F_{tr}[S]$ one can either directly describe the action of $\mathscr{R}(Y)$, which is after all just the group algebra of Λ, on $F_{tr}[S]$, or use the homomorphism $\mathscr{R}(Y)\to \mathfrak{Z}(G)$, corresponding to $c''\colon \Theta(G)\to Y$.

Let $y\in Y$, $\mathcal{M}_y\subset\mathscr{R}(Y)$ the corresponding maximal ideal. For each $\mathscr{R}(Y)$-module $\mathscr{F}$ we put $\mathscr{F}_y=\mathscr{F}/\mathcal{M}_y\mathscr{F}$ — the fiber of $\mathscr{F}$ at y. Since $c\colon S\to Y$ is finite and submersive, the set $S_y=c^{-1}(y)$ is finite and the fiber $\mathscr{R}(S)_y$ coincides with $\mathscr{R}(S_y)$.

By the lemma above $F_{tr}[S_y]=F_{good}[S_y]$, which gives

$$F_{good}[S]\subset F_{tr}[S]+\mathcal{M}_y\mathscr{R}(S).$$

Put $\mathscr{F}=\mathscr{R}(S)/F_{tr}[S]$, $\mathscr{F}'=F_{good}[S]/F_{tr}[S]\subset\mathscr{F}$. Then the inclusion above can be written as

$$(*)\qquad \mathscr{F}'\subset\mathcal{M}_y\mathscr{F}\qquad\text{for each } y\in Y.$$

Since $\mathscr{F}$ is a finitely generated $\mathscr{R}(Y)$-module it is locally free at almost every point $y\in Y$. Since it is $\Psi(G)$-equivariant, it is locally free everywhere. Hence $(*)$ implies that $\mathscr{F}'=0$, i.e., $F_{tr}[S]=F_{good}[S]$.

4.3. Proof of Theorem 1.2. Let $f\in F_{good}(G)$. By definition f is dominated by an open compact subgroup $K\subset G$ (see condition (ii) in 1.2). Using 2.3 we can write $f=\Sigma 1_\Theta\cdot f$, where $1_\Theta\cdot f=0$ for all but a finite number of components Θ. It is sufficient to prove that $1_\Theta\cdot f\in F_{tr}(G)$ for each component Θ, so we can fix Θ and assume that $1_\Theta\cdot f=f$.

By Proposition 3.1 the group $R(\Theta_{disc})$ is generated by a subset $S\subset \operatorname{Irr} G$, which is a union of a finite number of $\Psi(G)$-orbits. Using Proposition 4.2, we can find a trace form f_h such that $f=f_h$ on $R(\Theta_{disc})$. Replacing f by $f-1_\Theta\cdot f_h$ we can assume that f is zero on the subgroup $R_{disc}=R(\Theta_{disc}(G))$.

Consider the form $f^d=f-\Sigma_M c_M r^*_{MG} i^*_{GM} f$, defined in 3.3. This form is discrete, i.e., it equals 0 on $R^I(G)$. It also equals 0 on R_{disc}, since $f(R_{disc})=0$ and the operators $i_{GM}\circ r_{MG}$ preserve R_{disc} because they preserve $R(\theta)$ for each $\theta\in\Theta(G)$. Since $R(G)=R_{disc}+R^I(G)$, $f^d\equiv 0$, i.e., $f=\Sigma_M c_M r^*_{MG}\circ i^*_{GM} f$, $M\subsetneqq G$.

Let $M\subsetneqq G$. By induction in $\dim M$ we can assume that $F_{tr}(M)=F_{good}(M)$. Then $i^*_{GM}f\in F_{good}(M)=F_{tr}(M)$ and hence $r^*_{MG}\circ i^*_{GM}f\in r^*_{MG}F_{tr}(M)\subset F_{tr}(G)$ (see 3.2(i), (ii)). This shows that $f\in F_{tr}(G)$.

§5. Proofs of Propositions 3.1, 3.2 and 3.3

5.1. Fix a connected component $\Theta\subset\Theta(G)$. We want to prove that the subset $\Theta_{disc}\subset\Theta$ of discrete infinitesimal characters consists of a finite number of $\Psi(G)$-orbits.

Lemma. *Θ_{disc} is a constructible subset of Θ, i.e., a union of a finite number of subsets, locally closed in Zariski topology.*

This lemma is "morally obvious", since the whole situation is algebraic. But, as usual in such cases, the proof is a little tiresome.

Proof. *Step 1.* Let $\mathcal{B}$ be a commutative algebra over $\mathbf{C}$, E a G-$\mathcal{B}$-module, i.e., a G-module together with a homomorphism $\mathcal{B} \to \mathrm{End}_G(E)$. Suppose E is finitely generated as a G-$\mathcal{B}$-module and for each open compact subgroup $K \subset G$ $\mathcal{B}$-module E^K is projective and finitely generated. In this situation we call E a $\mathcal{B}$-family of G-modules. For any homomorphism of algebras $\mathcal{B} \to \mathcal{B}'$ we denote by $E_{\mathcal{B}'}$ the induced $\mathcal{B}'$-family of G-modules $E_{\mathcal{B}'} = \mathcal{B}' \otimes_{\mathcal{B}} E$.

Usually we consider $\mathcal{B}$ to be the algebra of regular functions on an algebraic variety S, and then call E an S-family of G-modules. For each morphism $\phi\colon S' \to S$ we denote by $E_{S'}$ an induced S'-family of G-modules. For instance, for any point $s \in S$ the corresponding G-module E_s is the specialization of the family E at s.

For an S-family of G-modules E we define a function $\nu_E\colon S \to R(G)$ by $\nu_E(s) = E_s$. We call functions of this type *regular.* A regular function $\nu\colon S \to R(G)$ is called irreducible if $\nu(S) \subset \mathrm{Irr}\, G$. Two irreducible functions ν, ν' are called *disjoint* if for each $s \in S$, $\nu(s) \neq \nu'(s)$.

Step 2. Let $\nu\colon S \to R(G)$ be a regular function. Then there exists a dominant étale morphism $\phi\colon V \to S$, irreducible regular functions $\{\lambda_j\colon V \to R(G)\}$ and $n_j \in \mathbf{Z}^+$ such that $\phi^*(\nu) = \Sigma\, n_j \lambda_j$.

Let E be an S-family of G-modules, representing ν. We can find an open compact subgroup $K \subset G$ such that E is generated by E^K as a G-module. If we choose K to be good, then any nonzero subquotient E' of E has a nonzero space E'^K (see [1, §3]). This implies that we can everywhere replace G-modules E' by $\mathcal{H}_K$-modules E'^K, where $\mathcal{H}_K = \mathcal{H}_K(G)$, i.e., essentially we study families of finite-dimensional $\mathcal{H}_K$-modules.

We can assume that S is irreducible. Let $\mathcal{B} = \mathcal{R}(S)$, L be the field of fractions of $\mathcal{B}$. There are two possibilities.

(i) $\mathcal{H}_K \subset \mathrm{End}_L(E_L^K)$ generates the whole space as L-module. Then, after replacing S by an open subset, we can assume that $\mathcal{H}_K$ generates $\mathrm{End}_{\mathcal{B}}(E^K)$ as a $\mathcal{B}$-module, i.e., E_s is irreducible for each $s \in S$.

(ii) $\mathcal{H}_K$ does not generate $\mathrm{End}_L(E_L^K)$ as an L-module. Then for some finite extension L' of L the representation of $\mathcal{H}_K$ in $E_{L'}^K$ is reducible. Replacing S by some V, étale over S, we may assume that $L' = L$.

Let $E_1 \subset E_L^K$ be a nontrivial $\mathcal{H}_K \otimes L$ submodule. Replacing S by an open subset we can assume that $E_1 = (E_0)_L$, where $E_0 \subset E^K$ is an $\mathcal{H}_K \otimes \mathcal{B}$ submodule, such that E_0 and E^K/E_0 are projective $\mathcal{B}$-modules. Induction on $\dim_{\mathcal{B}} E^K$ finishes the proof.

Step 3 (Decomposition in irreducible functions). Let $\nu_1, \ldots, \nu_r\colon S \to R(G)$ be regular functions. Then there exist a dominant étale morphism $\phi\colon V \to S$, irreducible regular functions $\lambda_j\colon V \to R(G)$ and integers $n_{ij} \in \mathbf{Z}^+$ such that $\phi^*(\nu_i) = \Sigma\, n_{ij}\lambda_j$. Moreover, we can choose λ_j to be disjoint.

Indeed, repeating Step 2 we can find λ_j. If λ, λ' are two irreducible functions, then the set of points $v \in V$ such that $\lambda(v) = \lambda'(v)$ is closed in Zariski topology. Hence, replacing V by an open subset, we can assume that each two functions λ_i, λ_j either coincide, or are disjoint.

Step 4. Let $\nu_1, \ldots, \nu_n, \mu_1, \ldots, \mu_m: S \to R(G)$ be regular functions. Consider the subset $S_0 \subset S$ consisting of all points s, such that $\{\nu_1(s), \ldots, \nu_n(s)\}$ lie in a subgroup of $R(G)$, generated by $\{\mu_1(s), \ldots, \mu_m(s)\}$. Then there exists a dominant étale morphism $\phi: V \to S$ such that $\phi^{-1}(S_0)$ is either empty or the whole V.

It follows immediately from Step 3.

Step 5 Proof of the Lemma. In order to prove that the subset $\Theta_{\text{disc}} \subset \Theta$ is constructible it is sufficient to check the following condition.

(*) Let $S \subset \Theta$ be a locally closed subvariety, $S_0 = S \setminus (S \cap \Theta_{\text{disc}})$. Then there exists a dominant étale morphism $\phi: V \to S$ such that $\phi^{-1}(S_0)$ is either empty or the whole V.

Let (N, ρ) be a cuspidal pair, corresponding to Θ, $\nu'_\rho: \Psi(N) \to \Theta$ the corresponding morphism. For each standard Levi subgroup $M < G$, which contains N, consider the regular function $\nu_M: \Psi(N) \to R(M)$ given by $\psi \mapsto i_{MN}(\psi\rho)$.

Choose a subvariety $V_1 \subset \Psi(N)$ such that $\nu'_\rho(V_1) \subset S$ and $\nu'_\rho: V_1 \to S$ is dominant étale, and denote by ν_M the restrictions of functions ν_M to V_1. Using Step 1 we can replace V_1 by V_2, étale over V_1, such that on V_2 for each M we have a decomposition $\nu_M = \sum n_i \lambda_{Mi}$ with irreducible λ_{Mi}.

Now consider on V_2 two systems of regular functions: the first is $\{\lambda_{G,i}\}$ and the second is $\{i_{GM}(\lambda_{M,j}) \mid M \subsetneqq G\}$. It is clear that the set $(V_2)_0$ of points v for which the second system generates the first one is exactly the pre-image $\phi_2^{-1}(S_0)$, where $\phi_2: V_2 \to S$ is the natural étale morphism. So Step 4 proves (*) and the lemma.

5.2. Proof of Proposition 3.1. For each G-module π denote by π^+ its Hermitian contragradient G-module. This defines an involutive map $+: R(G) \to R(G)$, which maps Irr G into itself. This involution is compatible with morphisms i_{GM}. For each subgroup $M < G$ the involution $+$ acts on $\Psi(M)$ and on the set of cuspidal pairs (M, ρ). This defines an action of $+$ on $\Theta(G)$, which is compatible with the map inf. ch. It is easy to see that the action of $+$ on the algebraic varieties $\Psi(M)$, $\Theta(G)$ is antiholomorphic (more precisely, antialgebraic).

By Langlands theory (see [3, XI.2]) the group $R(G)$ is generated by standard families $\{i_{GM}(\psi\sigma)\}$ in which σ is a *tempered* M-module. Hence each point $\theta \in \Theta_{\text{disc}}$ can be written as $\theta = \text{inf. ch.}(\psi\pi)$, with $\psi \in \Psi(G)$, and π an irreducible tempered G-module. Since π is tempered, it is unitary, i.e., $\pi^+ = \pi$. This implies

$$\theta^+ \in \Psi(G)\theta. \tag{*}$$

Suppose for a moment that $\Psi(G)=\{1\}$, i.e., G has compact center. Then the subset $\Theta_{\rm disc}\subset\Theta$ is constructible by 5.1 and is pointwise fixed by the antialgebraic involution $+$. This implies that this subset is finite.

For arbitrary $\Psi(G)$ the same proof, applied to the subset $\tilde\Theta_{\rm disc}=\Theta_{\rm disc}/\Psi(G)$ of the algebraic quotient variety $\tilde\Theta=\Theta/\Psi(G)$, shows that $\tilde\Theta_{\rm disc}$ is finite, i.e., $\Theta_{\rm disc}$ consists of a finite number of $\Psi(G)$-orbits.

5.3. Proof of Proposition 3.2. Fix $M<G$. The inclusion $i^*_{GM}(F_{\rm good}(G))\subset F_{\rm good}(M)$ is straightforward. We prove the inclusion $r^*_{MG}(F_{\rm tr}(M))\subset F_{\rm tr}(G)$ using a modification of the method used in [4].

Let $P=MU$ be the standard parabolic subgroup, corresponding to M, $P^-=MU^-$ the opposite parabolic. Choose a small good subgroup $K\subset G$, i.e., an open compact subgroup such that $K=(K\cap U^-)\cdot(K\cap M)\cdot(K\cap U)$. Put $\Gamma=K\cap M$ and consider Hecke algebras $\mathscr{H}_K(G)$ and $\mathscr{H}_\Gamma(M)$. We want to prove that for each $h\in\mathscr{H}_\Gamma(M)$ the form $r^*_{MG}(f_h)\in R^*(G)$ is a trace form. Let J denote the subspace of $h\in\mathscr{H}_\Gamma(M)$ for which it is true. We should prove that $J=\mathscr{H}_\Gamma(M)$.

Step 1. Fix a central element $a\in Z(M)$ such that $\mathrm{Ad}\, a\,|_U$ is a strictly contracting operator. Then for each $h\in\mathscr{H}_\Gamma(M)$, $a^nh\in J$ for large n.

Proof. Let e_Γ, e_K be units in $\mathscr{H}_\Gamma(M)$ and $\mathscr{H}_K(G)$. For each $m\in M$ put $h(m)=e_\Gamma\cdot\delta_m\cdot e_\Gamma$, $h'(m)=\delta_U^{-1/2}(m)\cdot e_K\cdot\delta_m\cdot e_K$, where δ_m is the δ-distribution at m and $\delta_U=\mathrm{mod}_U\in\Psi(M)$ (this factor appears since we use the normalized functor r_{MG}, see [1, 2.5]).

We can assume that a given measure h is of the form $h(m)$ for some $m\in M$. Replacing a by its power and m by a^nm we can assume that a, m lie in

$$M^+=\{m\in M\mid m(K\cap U)m^{-1}\subset K\cap U \text{ and } m^{-1}(K\cap U^-)m\subset K\cap U^-\}.$$

Then (see [4, 2.1])

$$h(a^nm)=h(a)^n\cdot h(m)\quad\text{and}\quad h'(a^nm)=h'(a)^nh'(m)=h'(m)\cdot h'(a)^n.$$

Let π be an admissible G-module, $\sigma=r_{MG}(\pi)$. Then, arguing as in [4, §3] we see that for large n

$$\mathrm{tr}\,\pi(h'(a^nm))=\mathrm{tr}\,\sigma(h(a^nm)).$$

Since both functions of n are $\mathbf{Z}$-finite, for $n\geqq 1$, equality holds already for $n=1$ (see [4, §§4,5]). This proves that

$$r^*_{MG}(f_{h(am)})=f_{h'(am)}\text{ is a trace form,}$$

i.e.,

$$h(am)=ah(m)\in J.$$

Step 2. By the finiteness theorem 2.5, $\mathscr{H}_\Gamma(M)$ is finitely genrated as $\mathfrak{Z}(M)$-module and hence is finitely generated as $\mathfrak{Z}(G)$-module. Since $J \subset \mathscr{H}_\Gamma(M)$ is a $\mathfrak{Z}(G)$-submodule, we can deduce from step 1 that for some $n \in \mathbf{Z}$, $a^n\mathscr{H}_\Gamma(M) \subset J$. But $a^n\mathscr{H}_\Gamma(M) = \mathscr{H}_\Gamma(M)$, so $J = \mathscr{H}_\Gamma(M)$.

5.4. *Relations Between Morphisms i and r*

Let $W_G = \mathrm{Nom}(M_0, G)/M_0$ be the Weyl group of G. For $m < G$ we consider W_M as a subgroup of W_G.

We call two standard Levi subgroups $M, N < G$ associate if $N = wMw^{-1}$ for some $w \in W_G$. Each such element w defines an isomorphism $w\colon R(M) \to R(N)$, which depends only on the class $wW_M = W_N w$ of w. If $w'\colon R(L) \to R(M)$ is defined, we have $ww' = w \circ w'\colon R(L) \to R(N)$.

Lemma. *The system of groups $R(M)$ for $M < G$, isomorphisms w and morphisms i_{NM}, r_{MN} for $M < N < G$ satisfies the following relations.*

(i) *For $L < M < N < G$, $i_{NM} \circ i_{ML} = i_{NL}$, $r_{LM} \circ r_{MN} = r_{LN}$.*

(ii) *Let $M, N < G$, W_G^{NM} be the set of representatives of $W_N \backslash W_G / W_M$ of minimal length. Then*

$$r_{NG} \circ i_{GM} = \sum_w i_{NN_w} \circ w \circ r_{M_w M}$$

with $w \in M_G^{NM}$, $M_w = w^{-1}Nw \cap M$, $N_w = wM_w w^{-1} = N \cap wMw^{-1}$.

(iii) *If $N = wMw^{-1}$, then*

$$i_{G,N} \circ w(\sigma) = i_{GM}(\sigma) \qquad \text{for } \sigma \in R(M).$$

Proof. (i) is standard, (ii) is the reformulation of [2, 2.11]. The statement (iii) is well known, but we were unable to find a reference. So we give a proof of it.

It is sufficient to check (iii) on generators of $R(M)$. Hence by Langlands theory (see [3, XI 2]) we can assume that $\sigma = i_{ML}(\psi\omega)$, where $L < M$, $\psi \in \Psi(L)$, ω is a tempered irreducible L-module. Since $i_{GM}(\sigma) = i_{GL}(\psi\omega)$ and $i_{GN} \circ w(\sigma) = i_{G,wLw^{-1}}(w(\psi\omega))$, we can just assume that $L = M$, $\sigma = \psi\omega$.

Since characters of both parts in (iii) are regular functions in ψ it is sufficient to check (iii) for a generic character ψ.

Using the formula

$$\mathrm{Hom}_G(i_{GM}(\psi\omega), i_{GN}(w(\psi\omega))) = \mathrm{Hom}_N(r_{NG} \circ i_{GM}(\psi w), w(\psi\omega))$$

and using (ii) we see that this Hom space is 1-dimensional, since for generic ψ all subquotients of $r_{NG} \circ i_{GM}(\psi\omega)$ except $w(\psi\omega)$ have central characters different from the central character of $w(\psi\omega)$. Hence it is sufficient to check that the G-module $i_{GM}(\psi\omega)$ (and similarly $i_{GN}(w(\psi\omega))$) is irreducible for generic ψ.

The same proof as above shows that for generic ψ, $\mathrm{End}_G(i_{GM}(\psi\omega)) = \mathbf{C}$. If ψ is unitary, the G-module $i_{GM}(\psi\omega)$ is unitary and, hence, irreducible. Since irreducibility is an open condition on ψ, $i_{GM}(\psi\omega)$ is irreducible for generic ψ.

Corollary. *For each $M<G$ denote by T_M the operator $T_M = i_{GM}\circ r_{MG}: R(G)\to R(G)$. Then*

(i) $T_N\circ i_{GM} = \Sigma_w i_{GM_w}\circ r_{M_wM}$, $w\in W_G^{NM}$, $M_w = M\cap w^{-1}Nw$,

(ii) $T_N\circ T_M = \Sigma_w T_{M_w}$, $w\in W_G^{NM}$, $M_w = M\cap w^{-1}Nw$.

Proof. (i) $T_N\circ i_{GM} = i_{GN}\circ r_{NG}\circ i_{GM} = \Sigma_w i_{GN}\circ i_{NN_w}\circ w\circ r_{M_wM} = \Sigma_w i_{GN_w}\circ w\circ r_{M_wM}$ $= \Sigma_w i_{GM_w}\circ r_{M_wM}$.

(ii) $T_N\circ T_M = T_N\circ i_{GM}\circ r_{MG} = \Sigma_w i_{GM_w}\circ r_{M_wM}\circ r_{MG} = \Sigma_w i_{GM_w}\circ r_{M_wG} = \Sigma_w T_{M_w}$.

5.5. Proof of the Combinatorial Lemma 3.3. For each $M<G$ put $d(M)=\dim\Psi(M)$ — the "depth" of M in G. Define a decreasing filtration $\{R^i\subset R(G)\}$ by $R^i = \Sigma_{d(M)\geqq i} i_{GM}(R(M))$. By definition $R^i = R(G)$ for $i\leqq d(G)$, $R^{d(G)+1} = R^I(G)$ — the subgroup of strictly induced G-modules and $R^i = 0$ for $i>d(M_0)$.

Corollary 5.4 (i) shows that each operator T_N preserves this filtration and gives an explicit description of the action of T_N on quotients. For instance, if we put $P(N) = \#(\mathrm{Norm}(N, W_G)/W_N)$, then for $d = d(N)$ the action of T_N on R^d/R^{d+1} is given by

$$T_N i_{GM}(\sigma) = P(N)\cdot i_{GM}(\sigma)\quad \text{if } M\sim N,$$

$$T_N i_{GM}(\sigma) = 0 \qquad\qquad \text{if } M\not\sim N.\quad d(M)=d.$$

This implies that the operator $A_d = \Pi_{d(N)=d}(T_N - P(N))$ preserves the filtration $\{R^i\}$ and annihilates R^d/R^{d+1}.

Put $A = A_{d(M_0)}\circ A_{d(M_0)-1}\circ\cdots\circ A_{d(G)+1}$. Then $A(R^I(G))=0$ and by Corollary 5.4(ii) A takes the form $A = P(1-\Sigma_M c_M T_M)$, $M\lneqq G$, with $c_M\in\mathbf{Q}$, $P\in\mathbf{Z}$, $P\neq 0$. Then the adjoint operator $A^*: R^*(G)\to R^*(G)$ maps $R^*(G)$ into the orthogonal complement to $R^I(G)$, i.e., into the subspace of discrete forms, and is of the form $A^* = P(1-\Sigma_M c_M r^*_{MG} i^*_{GM})$, $M\lneqq G$ with $c_M\in\mathbf{Q}$, $P\in\mathbf{Z}$, $P\neq 0$. But this is eaxctly the statement of Lemma 3.3.

§6. Miscellaneous remarks

6.1. Let $M<G$, $\sigma\in\operatorname{Irr} M$. Then the natural morphism $\nu'_\sigma: \Psi(M)\to\Theta(G)$, $\psi\mapsto \text{inf.ch.}\, i_{GM}(\psi\sigma)$ is a finite morphism of algebraic varieties. Indeed, it is a composition of the morphism $\nu'': \Psi(M)\to\Theta(M)$, $\psi\mapsto\text{inf.ch.}(\psi\sigma)$ which is finite by 4.2, and a finite morphism $i_{GM}: \Theta(M)\to\Theta(G)$.

Fix a connected component $\Theta\subset\Theta(G)$ and consider a finite set of standard families, given by $\{(M_\xi,\sigma_\xi)\}$, generating $R(\Theta)$. Put $X = \bigcup_\xi \Psi(M_\xi)$ and consider the natural map $\nu\colon X \to R(\Theta)$. Using ν we realize $R^*(\Theta)$ as some space of functions on X. Clearly, the space $F_{\text{good}}(\Theta)$ lies in the subspace $\mathcal{R}(X)$ of regular functions on X. Since $\mathcal{R}(X)$ is a finitely generated $\mathfrak{Z}(\Theta)$-module, $F_{\text{good}}(\Theta)$ is also a finitely generated $\mathfrak{Z}(\Theta)$-module.

In an earlier version of the paper we used this realization in order to prove the theorem. Let us sketch the proof.

First, using 3.2 and 3.3, we reduced the proof to the statement that any discrete form $f \in F_{\text{good}}(\Theta)$ is a trace form. Since $\mathcal{R}(X)$ is a finitely generated $\mathfrak{Z}(\Theta)$-module it is sufficient to check that for each point $\theta \in \Theta$ the function $f \in \mathcal{R}(X)$ can be approximated by trace functions modulo any power of the maximal ideal $\mathcal{M}_\theta \subset \mathfrak{Z}(\Theta)$.

Assume $\Psi(G) = 1$ (in the general case one should argue as in 4.2). By 4.1 f is equal to a trace function f' on a finite subset $\nu^{-1}(\theta) \subset X$. Now we can apply the following simple statement from linear algebra.

(*) Let $A\colon V \to V$ be an endomorphism of a finite-dimensional vector space. Then for each $n \in \mathbf{Z}^+$ there exists a polynomial $P(t) \in \mathbf{C}[t]$ such that $\operatorname{tr} P(A + X) = \operatorname{tr} A + o(\|X\|^n)$ for $X \in \operatorname{End} V$.

This statement allows us to replace f' by a trace form f'', which coincides with f' and f on $\nu^{-1}(\theta)$ and is a constant modulo a high power of $\mathcal{M}_\theta$. Since f is discrete, it is constant on each component of X, and hence f'' approximates f modulo a high power of $\mathcal{M}_\theta$.

6.2. Discrete G-modules and discrete forms, which appeared in our proof, apparently play a very important role in harmonic analysis on G. Some interesting results can be found in [6].

6.3. It would be interesting to study more thoroughly the combinatorial structures, described in 5.4–5.5. They have some relations with Hopf algebras (see [7]). Here we want only to note that the operators T_M in 5.4 do not coincide (and even do not commute) for associate Levi subgroups.

Example. Let $G = \mathrm{GL}(3)$, $M = \mathrm{GL}(2) \times \mathrm{GL}(1)$, $N = \mathrm{GL}(1) \times \mathrm{GL}(2)$. Then $T_M T_N(\pi) \neq T_N T_M \pi$, if $\pi \in R(G)$ is the trivial G-module.

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