AVERAGES IN THE PLANE OVER CONVEX CURVES AND MAXIMAL OPERATORS

By

J. BOURGAIN

1. Introduction

The main goal of this paper is to give a somewhat simpler exposition of the result on circular means proved in [1] and extensions to more general curves. The simplifications with respect to [1] consists in avoiding the more combinatorial part and most of the interpolation. The basic approach however is the same and also here information coming from both harmonic analysis and geometry will be combined.

Let Γ be the boundary of a compact convex centrally symmetric body in $\mathbf{R}^2$. Thus also $\Gamma = \{x \in \mathbf{R}^2, \|x\| = 1\}$, where $\| \ \|$ is the norm on $\mathbf{R}^2$ induced by this convex symmetric body. We assume Γ sufficiently smooth and of non-vanishing curvature (extensions will be discussed later). Let σ denote the arc length measure of Γ.

Theorem 1. *Let f be a bounded measurable function on the plane and define for $0 < t < \infty$ the average*

$$A_t f(x) = \int f(x + ty)\sigma(dy). \tag{1}$$

Denote Mf the corresponding maximal operator

$$Mf = \sup_{t>0} |A_t f|. \tag{2}$$

Then for $2 < p < \infty$, there is an inequality

$$\|Mf\|_p \leqq C(\Gamma, p)\|f\|_p. \tag{3}$$

Here $\| \ \|_p$ denotes the $L^p(\mathbf{R}^2)$-norm.

This result was proved [1] in the special case Γ in the unit circle $\{x \in \mathbf{R}^2; |x| = 1\}$. Theorem 1 has the following consequence on differentiation of functions.

Corollary 2. *Under the hypothesis of Theorem* 1

$$f = \lim_{t \to 0} A_t f \quad a.e. \tag{4}$$

If Γ is the unit circle, Theorem 1 answers a problem posed by E. Stein and S. Wainger in [4]. The restriction $p > 2$ is essential in (3), as is clear from the simple example

$$f(x) = \frac{1}{|x| \log(1/|x|)} \chi_{\{|x| \leq 1\}}$$

considering the operator

$$Tf(x) = \int f(x + |x| y) \sigma(dy) \tag{5}$$

(σ = arc length measure of unit circle). Clearly $|Tf| \leq |Mf|$ and $Tf = \infty$ everywhere while $\|f\|_2 < \infty$.

Related operators, more precisely

$$Tf(x) = \int f(x + (1 + \|x\|) y) \sigma(dy); \qquad \sigma = \sigma_\Gamma,$$

will play a special role in proving Theorem 1.

In terms of Fourier-transforms, the study of (5) reduces to the pseudo-differential operator

$$\int \hat{f}(\xi) \frac{1 - \hat{\varphi}(\xi)}{|\xi|^{1/2}} e^{2\pi i (\langle x, \xi \rangle + |\xi|)} d\xi \tag{6}$$

where $\varphi \in \mathcal{S}(\mathbf{R}^2)$ satisfies $\hat{\varphi} = 1$ on a neighborhood of 0. In this paper, we will in particular show the L^p-boundedness of (5) for $p > 2$ by direct geometrical considerations rather than working with (6).

2. The L^2-estimation

Although (3) fails for $p = 2$, the study of the L^2-behaviour is important in our approach. Denote P_t $(t > 0)$ the Poisson-semigroup on $\mathbf{R}^2$, $\hat{\varphi}_t(\xi) = e^{-t|\xi|}$, and define for $\varepsilon > 0$ small

$$M_\varepsilon f = \sup_{t > 0} \left| \int f(x + ty)(\sigma * P_\varepsilon)(dy) \right|$$

thus replacing σ by an "ε-mollification". The failure of the L^2-bound of M is "logarithmic" in the following sense.

Lemma 1.

$$\|M_\varepsilon f\|_2 \leq C(\Gamma) \left(\log \frac{1}{\varepsilon} \right) \|f\|_2. \tag{7}$$

Relation (7) follows from a general L^2-estimate on convolution maximal operators obtained in [2].

Lemma 2. *Consider a kernel $K \in L^1(\mathbf{R}^2)$ and let $K_t(x) = t^{-2}K(t^{-1}x)$ for $t > 0$. Define for $j \in \mathbf{Z}$ the following quantities:*

$$\alpha_j = \sup_{2^j \leq |\xi| \leq 2^{j+1}} |\hat{K}(\xi)| \quad \textit{and} \quad \beta_j = \sup_{2^j \leq |\xi| \leq 2^{j+1}} |\langle \nabla \hat{K}(\xi), \xi \rangle|. \tag{8}$$

Then

$$\left\| \sup_{t>0} |f * K_t| \right\|_2 \leqq C\Gamma(K) \|f\|_2,$$

denoting

$$\Gamma(K) = \sum_{j \in \mathbf{Z}} \alpha_j^{1/2}(\alpha_j + \beta_j)^{1/2}.$$

To derive (7), we let $K = (\sigma * P_t) - P_1$. Thus $(\sigma * P_t)_t = K_t + P_t$ and the maximal operator associated to P_t is taken care of by the Hardy–Littlewood theorem, while the expression $\Gamma(K)$ reduces to

$$C + \sum_{j \geq 0} a_j^{1/2}(a_j + b_j)^{1/2}$$

where

$$a_j = \sup_{|\xi| \sim 2^j} |\hat{\sigma}(\xi)| \quad \text{and} \quad b_j = \sup_{|\xi| \sim 2^j} |\langle \nabla \hat{\sigma}(\xi), \xi \rangle|.$$

If Γ is smooth (up to order 5 say) and has everywhere positive gaussian curvature, we have $a_j < C2^{-j/2}$, $b_j < C2^{j/2}$ as in the circle case. This can be shown by a simple direct computation. In fact, C. Herz has shown in [3] that the essential part of $\hat{\sigma}$ is given by the expression ($|\xi| > 1$)

$$\rho(\xi) + \overline{\rho(\xi)} \qquad \text{where } \rho(\xi) = |\xi|^{-1/2} K^{-1/2}(\zeta) e^{2\pi i (\|\xi\|_* - 1/8)}$$

where $K(\zeta)$ is the curvature of Γ at the point with unit outer normal vector ζ and $\| \ \|_*$ is the norm on $\mathbf{R}^2$ dual to $\| \ \|$,

$$\|x\|_* = \sup_{\|y\| \leq 1} |\langle x, y \rangle|.$$

3. Reduction of the problem

We restrict ourselves in the definition (2) of M to values $t \leqq 1$ (by rescaling, this is no loss of generality).

For $k = 1, 2, \ldots$, denote $\mathcal{D}_k$ the σ-algebras generated by the 2^{-k}-size squares in the plane obtained by successive diadic refinement of the unit square partition. Let $\mathbf{E}_k$ be the corresponding expectation operators, thus

$$\mathbf{E}_k[f] = \mathbf{E}[f \mid \mathcal{D}_k].$$

If f is a bounded measurable function on $\mathbf{R}^2$, we have

$$f=\sum_{k=1}^{\infty}\Delta_k f \quad \text{defining} \quad \Delta_k f=\mathbf{E}_k f-\mathbf{E}_{k-1}f.$$

For $\nu\in\mathbf{Z}_+$ and $2^{-\nu}\leqq t\leqq 2^{-\nu+1}$, write

$$f=\mathbf{E}_\nu[f]+\sum_{k\geqq\nu}\Delta_k f,$$

$$A_t f=A_t(\mathbf{E}_\nu[f])+\sum_{k\geqq\nu}A_t(\Delta_k f).$$

A moment's reflection shows that

$$|A_t(\mathbf{E}_\nu[f])|\leqq Cf^*$$

where f^* stands for the Hardy–Littlewood function of f.

Therefore, we are reduced to estimate

$$\sup_{\nu\geqq 0}\ \sup_{2^{-\nu}\leqq t\leqq 2^{-\nu+1}}\left|\sum_{k\geqq\nu}A_t(\Delta_k f)\right|$$

which is dominated by (fixing some $p>2$)

$$\left\{\sum_{\nu\geqq 0}\left[\sum_{s=0}^{\infty}\sup_{t\sim 2^{-\nu}}|A_t(\Delta_{\nu+s}f)|\right]^p\right\}^{1/p}\leqq\sum_{s=0}^{\infty}\left\{\sum_{\nu\geqq 0}\left[\sup_{t\sim 2^{-\nu}}|A_t(\Delta_{\nu+s}f)|^{\;p}\right]\right\}^{1/p}.$$

The $L^p(\mathbf{R}^2)$-norm of latter expression is dominated by

$$\sum_{s=0}^{\infty}\left\{\sum_{\nu\geqq 0}\left\|\sup_{t\sim 2^{-\nu}}|A_t(\Delta_{\nu+s}f)|\right\|_p^p\right\}^{1/p}. \tag{9}$$

Our main goal will be to prove the following estimate:

$$\left\|\sup_{t\sim 2^{-\nu}}|A_t(g)|\right\|_p\leqq C2^{-\alpha(p)s}\|g\|_p,\qquad \text{provided } \mathbf{E}_{\nu+s}[g]=0. \tag{10}$$

Once (10) is obtained, (9) may be majorized by (since $p>2$)

$$\sum_{s=0}^{\infty}2^{-\alpha(p)s}\left(\sum_{\gamma\geqq 0}\|\Delta_{\gamma+s}f\|_p^p\right)^{1/p}\leqq\sum_{s=0}^{\infty}2^{-\alpha(p)s}\|f\|_p\leqq C\|f\|_p.$$

Notice that again by rescaling, it is enough to prove (10) with $\nu=0$. Also, in this situation, the problem trivially localizes to the case g is supported by the unit square $[0,1]^2$.

Consider the following decomposition for the arc length measure σ of Γ:

$$\sigma=\sigma_0+\sum_{k=1}^{\infty}2^{k-1}\sigma_k$$

where

$$\sigma_0 = \chi_{[1 \leq \|y\| \leq 2]}$$

and

$$\sigma_k = \chi_{[1 \leq \|y\| \leq 1+2^{-k}]} - \chi_{[1+2^{-k} < \|y\| \leq 1+2^{-k+1}]}$$

(analogous to the expansion of the Dirac measure δ_0 on $\mathbf{R}$ in the Haar system). Thus

(11) $$|A_t f(x)| \leq \left| \int f(x+ty)\sigma_0(y)dy \right| + \sum_{k=1}^{\infty} 2^{k-1} \left| \int f(x+ty)\sigma_k(y)dy \right|.$$

It will thus suffice to have an estimate

(12) $$\left\| \sup_{1<t<2} \left| \int f(x+ty)\sigma_k(y)dy \right| \right\|_p \leq C2^{-k(1+\alpha)} \|f\|_p$$

for some constant $\alpha = \alpha(p) > 0$. Summation over k yields then indeed the bound

$$\left\| \sup_{t\sim 1} |A_t f| \right\|_p \leq C\left(\sum_{k=0}^{\infty} 2^{-k\alpha} \right) \|f\|_p \leq C(p)\|f\|_p.$$

In fact, as is easily seen, an estimate

$$\left\| \sup_{t\sim 1} |A_t f| \right\|_p \leq C2^{-\alpha' s} \|f\|_p$$

is obtained provided $\mathbf{E}_s f = 0$, since the initial terms in (11) can be estimated with the inequality

$$\left| \int f(x+ty)\sigma_k(y)dy \right| \leq C\|f\|_p 2^{-s/p'}.$$

Let thus k be fixed and denote $n = 2^k$. Consider a radius function $t(x)$ ranging in $[1,2]$ and satisfying at the point x

(13) $$\sup_{1<t<2} \left| \int f(x+ty)\sigma_k(y)dy \right| = \left| \int f(x+t(x)y)\sigma_k(y)dy \right|.$$

Denoting

$$\tilde{V}_x = \sigma_k\left(\frac{y-x}{t(x)} \right) \quad \text{and} \quad V_x = |\tilde{V}_x| \quad \text{(absolute value)},$$

expression (13) becomes

$$\left| \int f(y)\tilde{V}_x(y)dy \right|$$

and we estimate its L^p-norm by dualization. Consider thus a function g on $\{|x| \leq c\}$, $\|g\|_q = 1$, $q = p/(p-1)$ and write

$$\left|\left\langle \int f(y)\tilde{V}_x(y)dy, g\right\rangle\right| = \left|\left\langle \int g(x)\tilde{V}_x dx, f\right\rangle\right|$$

$$\leqq \|f\|_p \left\| \int g(x)\tilde{V}_x dx \right\|_{L^q([0,1]^2)}.$$

Thus the question reduces to proving an inequality

$$\left\| \int g(x)\tilde{V}_x dx \right\|_q \leqq C2^{-(1+\alpha)k} \|g\|_q.$$

Let $1 < q_0 < q < q_1 < 2$. By the Marcinkicwicz interpolation theorem and the fact that

$$[L^{q_0,1}, L^{q_1,1}]_{\theta,q} = L^q = [L^{q_0}, L^{q_1}]_{\theta,q} \qquad \text{if } \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1},$$

it suffices to consider a region $\Omega \subset [|x| \leqq c]$ and estimate $(n = 2^k)$

$$\left\| \int_\Omega \tilde{V}_x dx \right\|_q \leqq Cn^{-1-\alpha} |\Omega|^{1/q}. \tag{14}$$

Writing

$$\frac{1}{q} = \frac{1-\theta}{1} + \frac{\theta}{2},$$

it follows from Hölder's inequality that

$$\left\| \int_\Omega \tilde{V}_x dx \right\|_q \leqq \left\| \int_\Omega \tilde{V}_x dx \right\|_1^{1-\theta} \left\| \int_\Omega \tilde{V}_x dx \right\|_2^{\theta}. \tag{15}$$

Of course, there is the trivial inequality

$$\left\| \int_\Omega \tilde{V}_x dx \right\|_1 \leqq \left\| \int_\Omega V_x dx \right\|_1 = \int_\Omega \|V_x\|_1 dx \sim \frac{1}{n}|\Omega|. \tag{16}$$

The L^2-norm may be estimated using Lemma 1. One clearly has

$$\left\langle \int_\Omega V_x dx, f\right\rangle \leqq \frac{C}{n} \int_\Omega \left\{ \sup_{1 \leqq t \leqq 2} [f * (\sigma * P_{1/n})_t](x) \right\} dx$$

and hence from Lemma 1

$$\left\| \int_\Omega \tilde{V}_x dx \right\|_2 \leqq \left\| \int_\Omega V_x dx \right\|_2 \leqq C \frac{\log n}{n} |\Omega|^{1/2}. \tag{17}$$

Substitution of (16), (17) in the right member of (15) shows that in order to get (14), it suffices for given Ω to improve a bit on (16) or on (17), i.e.

(18) $$\left\| \int_{\Omega} \tilde{V}_x dx \right\|_1 \leqq C n^{-1-\alpha} |\Omega|$$

or

(19) $$\left\| \int_{\Omega} \tilde{V}_x dx \right\|_2 \leqq C n^{-1-\alpha} |\Omega|^{1/2}.$$

In fact, we will split $\Omega = \Omega_0 + \Omega_1$ where

$$\Omega_0 = \bigcup_{s=1}^{S} \Omega_{0,s},$$

each $\Omega_{0,s}$ satisfying

(20) $$\left\| \int_{\Omega_{0,s}} \tilde{V}_x dx \right\|_1 \leqq C n^{-1-\alpha} |\Omega_{0,s}|$$

and

(21) $$\left\| \int_{\Omega_1} \tilde{V}_x dx \right\|_2 \leqq C n^{-1-\alpha} |\Omega|^{1/2}.$$

Then

$$\left\| \int_{\Omega} \tilde{V}_x dx \right\|_q \leqq \left\| \int_{\Omega_0} \tilde{V}_x dx \right\|_q + \left\| \int_{\Omega_1} \tilde{V}_x dx \right\|_q$$

where the first term satisfies (18) (by adding up (20)) and the second term satisfies (19).

It should be pointed out that both (18) and (19) considered separately may fail for suitably chosen Ω and radius function $t(x)$.

4. Geometrical estimates

In this section, we first prove some estimates on $\langle V_x, V_y \rangle$ and $\langle \tilde{V}_x, \tilde{V}_y \rangle$ which will be used later on. Again c, C will stand for constants (possibly depending on Γ). By the curvature hypothesis on Γ, one has the following property:

(22) $$\|p\| = 1 = \|q\| \Rightarrow \left\| \frac{p+q}{2} \right\| \leqq 1 - c\,\|p-q\|^2.$$

We will use the following corollary. Suppose $\|p\| \sim 1 \sim \|q\|$ and $\|p-q\| < c\|p\|$, then

$$(23) \qquad \operatorname{dist}(q, \mathbf{R}p) \sim [\|p-q\| - |\,\|p\| - \|q\|\,|]^{1/2} \|p-q\|^{1/2}.$$

To see this, assume $\|p\| > \|q\|$ and let $\|\theta\|_* = 1$ satisfy $\langle p, \theta \rangle = \|p\|$. Denote

$$\delta = \|q\| + \|p-q\| - \|p\|.$$

Writing $p = q + (p-q)$, we get

$$\langle q, \theta \rangle \geqq \|q\| - \delta \quad \text{and} \quad \langle p-q, \theta \rangle \geqq \|p-q\| - \delta.$$

Therefore, by (22)

$$|\sin \operatorname{Angle}(p-q, q)| \sim \left\| \frac{q}{\|q\|} - \frac{p-q}{\|p-q\|} \right\| \sim \left(\frac{\delta}{\|p-q\|} \right)^{1/2},$$

$$\operatorname{dist}(q, \mathbf{R}p) \sim \|p-q\| \, |\sin \operatorname{Angle}(p-q, q)| \sim \delta^{1/2} \|p-q\|^{1/2},$$

proving (23).

The following lemma generalizes the estimates in the circle case (see [1]).

Lemma 3. *Denote r_x the radius $t(x)$ introduced in previous sections when defining V_x. Then*

(1) *The diameter of each of the components of $V_x \cap V_y$ (resp. of the unique component in case of coincidence) is*

$$(24) \qquad \leqq \frac{C}{n} \left(|x-y| + \frac{1}{n} \right)^{-1/2} \left(|\,\|x-y\| - |r_x - r_y|\,| + \frac{1}{n} \right)^{-1/2}$$

(2)

$$(25) \qquad |\langle \tilde{V}_x, \tilde{V}_y \rangle| \leqq C \frac{\langle V_x, V_y \rangle}{1 + n\,|\,\|x-y\| - |r_x - r_y|\,|}.$$

(*We assume $|x-y| < c < \frac{1}{2}(r_x + r_y)$*).

Proof. Let $x = (0,1)$, $y = (a,b)$, $r_x = \|x\|$, $r_y = \|y\|$. Assume

$$\|x-y\| \sim |x-y| \sim |a| + |1-b| < \tfrac{1}{2}.$$

Let $\phi(p)$ stand for $\|p\|$. Notice that by (23)

$$(26) \qquad |a| \sim \operatorname{dist}(y, \mathbf{R}x) \sim \|x-y\|^{1/2} [\|x-y\| - |r_x - r_y|]^{1/2}.$$

Consider the component of $V_x \cap V_y$ (V_x, V_y referring to the support of the respective functions) containing 0.

For $|t| < 1/n$, let $p(t) = (\lambda(t), \mu(t))$ be the solution of the equations

$$(27) \qquad \begin{cases} \phi(-\lambda(t), \quad 1-\mu(t)) - \phi(0,1) = t, \\ \phi(a-\lambda(t), \quad b-\mu(t)) - \phi(a,b) = 0, \end{cases}$$

belonging to the 0-component. Then

(28) $$\operatorname{diam}\operatorname{component}(V_x \cap V_y) \sim \int_{-1/n}^{1/n} (|\dot{\lambda}(t)| + |\dot{\mu}(t)|)dt,$$

(29) $$\langle V_x, V_y \rangle \sim \frac{1}{n} \int_{-1/n}^{1/n} (|\dot{\lambda}(t)| + |\dot{\mu}(t)|)dt,$$

(30) $$|\langle \tilde{V}_x, \tilde{V}_y \rangle| \leqq \frac{C}{n^2} \int_{-1/n}^{1/n} (|\ddot{\lambda}(t)| + |\ddot{\mu}(t)|)dt.$$

Taking t-derivative in equations (27) yields

(31) $$\begin{cases} -\partial_x\phi(-\lambda, 1-\mu)\dot{\lambda} - \partial_y\phi(-\lambda, 1-\mu)\dot{\mu} = 1, \\ -\partial_x\phi(a-\lambda, b-\mu)\dot{\lambda} - \partial_y\phi(a-\lambda, b-\mu)\dot{\mu} = 0; \end{cases}$$

(32) $$\begin{cases} -\partial_x\phi(-\lambda, 1-\mu)\ddot{\lambda} - \partial_y\phi(-\lambda, 1-\mu)\ddot{\mu} = \\ \qquad -\partial^2_{xx}\phi(-\lambda, 1-\mu)(\dot{\lambda})^2 \\ \qquad -2\partial^2_{xy}\phi(-\lambda, 1-\mu)\dot{\lambda}\dot{\mu} - \partial^2_{yy}\phi(-\lambda, 1-\mu)(\dot{\mu})^2, \\ -\partial_x\phi(a-\lambda, b-\mu)\ddot{\lambda} - \partial_y\phi(a-\lambda, b-\mu)\ddot{\mu} = \\ \qquad -\partial^2_{xx}\phi(a-\lambda, b-\mu)(\dot{\lambda})^2 \\ \qquad -2\partial^2_{xy}\phi(a-\lambda, b-\mu)\dot{\lambda}\dot{\mu} - \partial^2_{yy}\phi(a-\lambda, b-\mu)(\dot{\mu})^2. \end{cases}$$

Notice $(\partial_x\phi(-\lambda, 1-\mu), \partial_y\phi(-\lambda, 1-\mu))$ and $(\partial_x\phi(a-\lambda, b-\mu), \partial_y\phi(a-\lambda, b-\mu))$ give the normal directions at the respective points $(-\lambda, 1-\mu)$ and $(a-\lambda, b-\mu)$. Hence, by the curvature hypothesis

(33) $$\left|\det\begin{bmatrix} \partial_x\phi(-\lambda,1-\mu) & \partial_y\phi(-\lambda,1-\mu) \\ \partial_x\phi(a-\lambda, b-\mu) & \partial_y\phi(a-\lambda,b-\mu) \end{bmatrix}\right| \sim \left|\det\begin{bmatrix} -\lambda & 1-\mu \\ a-\lambda & b-\mu \end{bmatrix}\right| = |(1-b)\lambda + a\mu - a|.$$

Clearly we may suppose $|x-y| \sim |a| + |1-b| \gg n^{-1}$ and in proving (25) that $\|x-y\| - |r_x - r_y| \gg n^{-1}$ or equivalently, by (26), that

(34) $$na^2 \gg |x-y|.$$

If (34) does not hold, i.e. if

(35) $$na^2 < C|x-y|,$$

we only have to prove

(36) $$\operatorname{diam}\operatorname{comp}(V_x \cap V_y) < Cn^{-1/2}(|x-y| + 1/n)^{-1/2}.$$

If (34) holds, then, by (26), (28), (29), (30), we need to prove

$$\int_{-1/n}^{1/n} (|\dot{\lambda}|+|\dot{\mu}|)dt \sim \frac{1}{na} \tag{37}$$

and

$$\int_{-1/n}^{1/n} (|\ddot{\lambda}|+|\ddot{\mu}|)dt \leqq C\frac{|x-y|}{na^3}. \tag{38}$$

Assume first (35). We may suppose $|a|<|1-b|$ since otherwise $|x-y|<cn^{-1}$ and there is nothing to prove. It follows from (31) and (33) that

$$|\dot{\lambda}|+|\dot{\mu}| \leqq C|(1-b)\lambda + a\mu - a|^{-1} \tag{39}$$

hence

$$|1-b||\lambda\dot{\lambda}| \leqq 2|a||\dot{\lambda}|+C.$$

Integrating, for $|t| \leqq 1/n$, we get

$$|1-b|\lambda(t)^2 \leqq C(1/n+|a||\lambda(t)|)$$

thus

$$|\lambda(t)| \leqq Cn^{-1/2}|1-b|^{-1/2}$$

or

$$|\lambda(t)| \leqq C|a||1-b|^{-1} \leqq Cn^{-1/2}|x-y|^{-1/2}$$

by (35). It is geometrically clear that $|\mu(t)| \ll |\lambda(t)|$. Hence (36) is proved, under the assumption

$$n[\|x-y\|-|r_x-r_y|] < C.$$

Assume next (34).

Then the determinant value

$$|a-(1-b)\lambda - a\mu| \sim |a| \tag{40}$$

as long as

$$|\lambda| < \tfrac{1}{10}|a||1-b|^{-1}. \tag{41}$$

By (39), it follows that $|\dot{\lambda}|<C|a|^{-1}$, thus $|\lambda|<C/na$ for $|t|<1/n$, on the component of $V_x \cap V_y$ containing 0.

It follows from (34) that $n^{-1}|a|^{-1} \ll |a||1-b|^{-1}$. By previous reasoning (40) holds on the 0 containing component of $V_x \cap V_y$.

It follows from (31) and (40) that

$$|\dot{\lambda}|+|\dot{\mu}|\sim|a|^{-1}\Rightarrow(37).$$

It remains to prove (38).

From (32) and (40), we clearly get the following bound on $|\ddot{\lambda}|+|\ddot{\mu}|$,

$$\frac{C}{|a|}\begin{bmatrix} -\partial_y\phi(a-\lambda,b-\mu) & \partial_y\phi(-\lambda,1-\mu) \\ \partial_x\phi(a-\lambda,b-\mu) & -\partial_x\phi(-\lambda,1-\mu)\end{bmatrix}\begin{pmatrix}\xi \\ \eta\end{pmatrix}$$

$$\leqq C|a|^{-1}\{|\xi-\eta|+|x-y|(|\xi|+|\eta|)\},$$

where ξ,η are the right members in equations (32). Hence

$$|\ddot{\lambda}|+|\ddot{\mu}|\leqq C|a|^{-1}|x-y|(|\dot{\lambda}|^2+|\dot{\mu}|^2)\leqq C|a|^{-3}|x-y|\Rightarrow(38).$$

This completes the proof. ■

We will also use the following geometrical fact.

Lemma 4. *Let* $\rho<\frac{1}{10}$ *and* $x,y,z\in\mathbf{R}^2$ *satisfy*

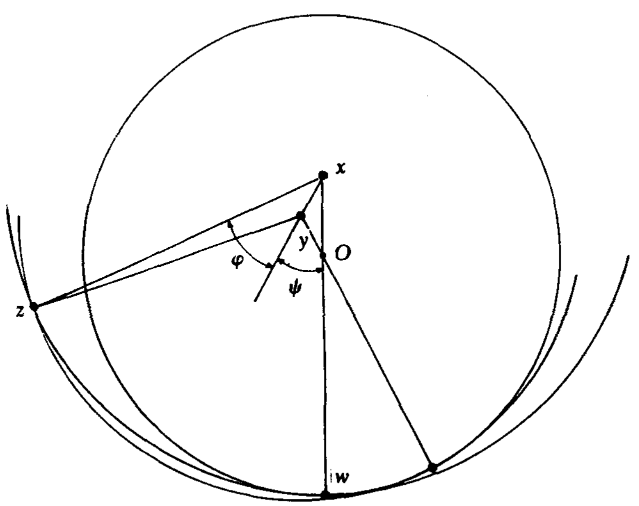
$$\frac{\rho}{10}<\|x\|,\|y\|<\rho, \tag{42}$$

$$\|z\|>1+\kappa\rho,\quad \kappa\ll 1, \tag{43}$$

$$|\|z-x\|-1-\|x\||<\tau \quad \text{and} \quad |\|z-y\|-1-\|y\||<\tau, \tag{44}$$

where $\tau\ll\rho$. *Then*

$$\|x-y\|-|\|x\|-\|y\||\geqq c\kappa|x-y|-C\tau. \tag{45}$$



Proof of Lemma 2. Assume $\|x\| \geqq \|y\|$. Let $w = -x/\|x\|$ and φ, ψ the angles indicated above. It follows from the discussion at the beginning of this section that

(46) $$\|x-y\|+\|y-z\| \geqq \|x-z\|+c\|x-y\|\sin^2\varphi.$$

From (44)

(47) $$\|x-y\|+\|y\|+2\tau \geqq \|x\|+c\|x-y\|\sin^2\varphi.$$

Similarly

(48) $$\|x-y\|+\|y-w\| \leqq 1+\|x\|+C\|x-y\|\sin^2\psi$$

and subtraction from (47) yields

(49) $$2\tau+\|y\|+1-\|y-w\| \geqq c\|x-y\|\sin^2\varphi - C\|x-y\|\sin^2\psi.$$

Also

$$\|x\|+\|z\| \leqq \|x-z\|+C(\sin^2\varphi+\sin^2\psi)\|x\|$$

and again from the hypothesis on z

(50) $$\sin^2\varphi+\sin^2\psi \geqq c\kappa - C\frac{\tau}{\rho}.$$

From (49), (50)

(51) $$C\|x-y\|\sin^2\psi \geqq -2\tau+\|y-w\|-1-\|y\|+c\chi\|x-y\|-C\tau.$$

Assume $\|y-w\|>1$. Then again

$$1+\|y\|-\|y-w\| \leqq C\frac{\operatorname{dist}(y,\mathbf{R}x)^2}{\|y\|} \leqq C\frac{|x-y|^2\sin^2\psi}{\rho}$$

and substituting in (51),

$$\|x-y\|-\|x\|+\|y\| \sim C|x-y|\sin^2\psi \geqq c\kappa|x-y|-C\tau.$$

If $1 \geqq \|y-w\| = \|y+x/\|x\|\|$, clearly $\|x-y\| \geqq \|x\|$ and (45) is trivial. ∎

5. The basic construction

Let now $\Omega \subset [0,1]^2$ (actually by further splitting, we may assume that Ω has an arbitrary small diameter; in particular, $\|x-y\| < \frac{1}{2}(r_x+r_y)$ for $x, y \in \Omega$). For $x \in \Omega$, define the set

$$\Omega_x = \{y \in \Omega;\, |\,\|x-y\| - |r_x - r_y|\,| < n^{-1+\varepsilon}\}.$$

An easy inductive argument allows us to obtain points $x_1, x_2, \ldots, x_J$ in Ω and a complementary set $\Omega_1 \subset \Omega$ such that

$$|\Omega_{x_1}| > n^{-1+\delta}$$

$$|\Omega_{x_2} \backslash \Omega_{x_1}| > n^{-1+\delta}$$

$$\vdots$$

$$\Big|\Omega_{x_J} \backslash \bigcup_{j<J} \Omega_{x_j}\Big| > n^{-1+\delta}$$

and $\Omega_1 = \Omega \backslash \bigcup_{j \leqq J} \Omega_{x_j}$ with the property that

$$|\Omega_1 \cap \Omega_x| \leqq n^{-1+\delta} \qquad \text{for } x \in \Omega. \tag{52}$$

Notice that in this construction $J \leqq n^{1-\delta}$. Define for $j = 1, \ldots, J$

$$\Omega_1' = \Omega_{x_1} \quad \text{and} \quad \Omega_j' = \Omega_{x_j} \backslash (\Omega_{x_1} \cup \cdots \cup \Omega_{x_{j-1}}),$$

also

$$\Omega_0 = \bigcup_{j=1}^{J} \Omega_j'.$$

It will suffice from previous discussion to show that the Ω_j' satisfy (20), thus

$$\Big\| \int_{\Omega_j'} \tilde{V}_x dx \Big\|_1 < C n^{-1-\varepsilon'} |\Omega_j'| \tag{53}$$

and also

$$\Big\| \int_{\Omega_1} \tilde{V}_x dx \Big\|_2 \leqq C n^{-1-\varepsilon'} |\Omega|^{1/2}. \tag{54}$$

We start with (54), writing by Lemma 3(ii)

$$\Big\| \int_{\Omega_1} \tilde{V}_x dx \Big\|_2^2 \leqq C \iint_{\Omega_1 \times \Omega_1} \frac{\langle V_x, V_y \rangle}{1 + n |\, \|x - y\| - |r_x - r_y| \,|} dx dy. \tag{55}$$

Defining

$$\mathscr{D} = \{(x, y) \in \Omega_1 \times \Omega_1 \,|\, y \in \Omega_x\}$$

and using Lemma 3(i), the right member of (55) may be estimated by

$$n^{-\varepsilon} \iint_{\Omega_1^2 \backslash \mathscr{D}} \langle V_x, V_y \rangle dx dy + C \iint_{\mathscr{D}} \frac{1}{n^{3/2} |x - y|^{1/2}} dx dy. \tag{56}$$

From the L^2-theory, the first term in (56) is dominated by

$$n^{-\varepsilon} \Big\| \int_{\Omega_1} V_x dx \Big\|_2^2 \leqq n^{-\varepsilon} \frac{|\Omega_1|}{n^2} (\log n)^2. \tag{57}$$

It follows from (52) that the sections $\mathscr{D}(x)$ of $\mathscr{D} \subset \Omega_1 \times \Omega_1$ satisfy

$$|\mathscr{D}(x)| < n^{-1+\delta}.$$

It easily follows from a rearrangement argument that given $a \in \mathbf{R}^2$, $D \subset \mathbf{R}^2$,

$$\int_D |a-y|^{-1/2} dy \leqq C|D|^{3/4}. \tag{58}$$

Hence, the second term in (56) is dominated by

$$\begin{aligned} Cn^{-3/2}|\Omega_1| \int_{D(x)} |x-y|^{-1/2} dy &\leqq Cn^{-3/2} n^{-3/4+(3/4)\delta} |\Omega_1| \\ &= Cn^{-(9/4)+\delta} |\Omega_1|. \end{aligned} \tag{59}$$

Taking $\delta < \frac{1}{4}$, the proof of the L^2-estimate (54) follows from (57) and (59).

6. End of the proof

It remains to prove (53) for the sets Ω'_j. Here we use Lemma 4. By hypothesis, $|\Omega'_j| > n^{-1+\delta}$ and there is a point x_j with $\Omega'_j \subset \Omega_{x_j}$. Of course, we may assume $x_j = 0$, $r_{x_j} = 1$. Denoting $\Omega' = \Omega'_j$, we then have the property

$$|\,\|x\| - |r_x - 1|\,| < n^{-1+\varepsilon} \qquad \text{for } x \in \Omega'. \tag{60}$$

Divide Ω' in the respective regions $[r_x \leqq 1]$ and $[r_x > 1]$. The computations for both are analogous and we therefore only treat the second case, i.e.

$$|1 + \|x\| - r_x| < n^{-1+\varepsilon} \qquad \text{for } x \in \Omega'. \tag{61}$$

(To handle the first case, Lemma 4 has to be suitably restated.) Since clearly

$$\left\| \int_{\|x\| \leqq n^{-1/2}} \tilde{V}_x dx \right\|_1 \leqq n^{-1-\delta} |\Omega'|$$

we may also assume $\|x\| \geqq n^{1/2}$ for $x \in \Omega'$.

For $l = 0, 1, 2, \ldots, \log n$, define

$$\Omega_l = \{x \in \Omega'; 2^l n^{-1/2} \leqq \|x\| \leqq 2^{l+1} n^{-1/2}\}.$$

It will suffice to prove for fixed l an inequality

$$\left\| \int_{\Omega_l} \tilde{V}_x dx \right\|_1 \leqq Cn^{-1-\varepsilon'} |\Omega'| \tag{62}$$

since (53) then simply follows from addition over l.

Let thus l be fixed and denote $\rho = 2^{l+1} n^{-1/2}$. Let $\kappa = n^{-\varepsilon}$. In order to use L^2-estimates, we need to restrict the functions $\tilde{V}_x$ to the set $[\|x\| > 1 + \kappa\rho]$. Let thus φ be a $[0,1]$-valued function on $\mathbf{R}^2$ satisfying the conditions

$$\varphi(z) = 0 \quad \text{if } \|z\| \leqq 1 + \kappa\rho, \qquad \varphi(z) = 1 \quad \text{if } \|z\| > 1 + 2\kappa\rho; \tag{63}$$
$$|\nabla\varphi| < C/\kappa\rho$$

and define for $x \in \Omega_l$

$$\hat{V}_x = \tilde{V}_x \cdot \varphi.$$

Thus

$$\hat{V}_x(z) \neq 0 \Rightarrow \|z\| > 1 + \kappa\rho.$$

Let $x, y \in \Omega_l$. Clearly $\langle \tilde{V}_x, \tilde{V}_y \rangle = 0$ unless there is a point $z \in V_x \cap V_y$ satisfying $\|z\| \geqq 1 + \kappa\rho$. From (61), we then have

$$|1 + \|x\| - \|x - z\|| < n^{-1+\varepsilon} + \frac{2}{n} \leqq 2n^{-1+\varepsilon}.$$

Similarly,

$$|1 + \|y\| - \|y - z\|| < 2n^{-1+\varepsilon}$$

in which case, by Lemma 4 ($\tau = 2n^{-1+\varepsilon}$) and (61),

$$|\|x - y\| - |r_x - r_y|| \geqq c\kappa^2 |x - y| - Cn^{-1+\varepsilon},$$

thus

$$1/n + |\|x - y\| - |r_x - r_y|| \geqq n^{-4\varepsilon}(|x - y| + 1/n). \tag{64}$$

If $\langle \hat{V}_x, \hat{V}_y \rangle \neq 0$, then $V_x \cap V_y$ consists of 2 (possibly 1) components of diameter (by Lemma 3,(i)) at most

$$Cn^{-1+2\varepsilon}(|x - y| + 1/n)^{-1}$$

on each of which, by (63), φ oscillates at most

$$\frac{C}{\kappa\rho} n^{-1+2\varepsilon}\left(|x - y| + \frac{1}{n}\right)^{-1}.$$

Hence

$$|\langle \hat{V}_x, \hat{V}_y \rangle| \leqq |\langle \tilde{V}_x, \tilde{V}_y \rangle| + \langle V_x, V_y \rangle \frac{C}{\kappa\rho} n^{-1+2\varepsilon}\left(|x - y| + \frac{1}{n}\right)^{-1}$$

and by Lemma 3 and (64)

$$|\langle \hat{V}_x, \hat{V}_y \rangle| \leqq \frac{C}{\rho} n^{-3+7\varepsilon}\left(|x - y| + \frac{1}{n}\right)^{-2}. \tag{65}$$

Estimate the left member of (62) by

$$\int_{\Omega_l} \| V_x(1-\varphi)\|_1 dx + \left\| \int_{\Omega_l} \hat{V}_x dx \right\|_1 . \tag{66}$$

Since $\|x\| > \rho/2$ for $x \in \Omega_l$, $r_x > 1 + \rho/4$ by (61) and it follows from construction of φ that for each $x \in \Omega'$

$$\| V_x(1-\varphi)\|_1 \leqq \frac{1}{n} \operatorname{length}(\Gamma_x \cap [1 \leqq \|z\| \leqq 1 + 2\kappa\rho])$$

and thus (cf. proof of Lemma 4)

$$\| V_x(1-\varphi)\|_1 \leqq C\frac{\sqrt{\kappa}}{n}.$$

Hence, the first term in (66) is dominated by $Cn^{-\varepsilon/2-1}|\Omega_l|$. Again since $\|x\| \leqq \rho$ for $x \in \Omega_l$, the function $\int_{\Omega_l} V_x dx$ is supported by a set of measure at most $C\rho$, and thus by the Cauchy–Schwartz inequality and (65)

$$\left\| \int_{\Omega_l} \hat{V}_x dx \right\|_1 \leqq C\rho^{1/2} \left\{ \iint_{\Omega_l \times \Omega_l} \rho^{-1} n^{-3+7\varepsilon} \left(|x-y| + \frac{1}{n} \right)^{-2} dx dy \right\}^{1/2}$$

$$\leqq C \frac{|\Omega'|}{n} \frac{n^{5\varepsilon}}{(n|\Omega'|)^{1/2}}$$

and hence (62), with $\varepsilon' = \delta/2 - 5\varepsilon$, since $|\Omega'| > n^{-1+\delta}$.

Thus the estimates (53) are obtained and the proof of Theorem 1 is completed.

7. Extension to vanishing curvature

The differentiation result (4) may be generalized to curves Γ that are a bit more general than those covered by Theorem 1. Let Γ be the smooth boundary of a compact convex body in the plane containing 0 as inner point. Assume that the curvature of Γ only vanishes at finitely many points. We may then approximate the arc length measure σ of Γ by a sum $\sigma_1 + \sigma_2 + \cdots + \sigma_S$,

$$\|\sigma - (\sigma_1 + \cdots + \sigma_S)\|_{M(\mathbf{R}^2)} < \varepsilon$$

where the σ_s $(1 \leqq s \leqq S)$ are the arc length measures of disjoint curves $\Gamma_s \subset \Gamma$ (for which the curvature stays away from 0). Each of the Γ_s can be embedded in a Γ_s' satisfying the conditions of Theorem 1 and hence we have a maximal inequality

$$\left\| \sup_{t>0} \left| \int f(x+ty)\sigma_s(dy) \right| \right\|_p \leqq C(p,\Gamma)\|f\|_p \qquad (p > 2).$$

This reasoning yields

Corollary 3. *Let σ be the arc length measure of a curve Γ as described above and $A_t f$ defined by* (1), *for f a bounded measurable function. Then*

$$f = \lim_{t \to 0} A_t f \qquad a.e.$$

It is easily seen that Corollary 3 fails when, for instance,

$$\Gamma = \{\max(|x_1|, |x_2|) = 1\}$$

is boundary of the square.

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I.H.E.S.
BURES-SUR-YVETTE, FRANCE
AND
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS, USA

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