Appl. Math. J. Chinese Univ. Ser. B
2006, 21(2): 191-202

RESEARCH ON THE EXISTENCE OF SOLUTION OF EQUATION INVOLVING p-LAPLACIAN OPERATOR

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Abstract. By using the perturbation theories on sums of ranges for nonlinear accretive mappings of Calvert and Gupta (1978), the abstract result on the existence of a solution $u \in L^p(\Omega)$ to nonlinear equations involving p-Laplacian operator Δ_p, where $\frac{2N}{N+1}<p<+\infty$ and $N(\geqslant 1)$ denotes the dimension of $\mathbf{R}^N$, is studied. The equation discussed and the methods shown in the paper are continuation and complement to the corresponding results of Li and Zhen's previous papers. To obtain the result, some new techniques are used.

§ 1 Introduction and preliminary

Since p-Laplacian operator Δ_p with $p \neq 2$ arises from a variety of physical phenomena such as non-Newtonian fluids, reaction-diffusion problems and petroleum extraction, etc. it becomes a very popular topic in mathematical fields. We have already studied it in different aspects[1-5]. In this paper, the following Eq. (1) will be discussed: for a given $f \in L^p(\Omega)$, find $u \in L^p(\Omega)$, where $\frac{2N}{N+1}<p<+\infty$ and $N>1$ such that

$$\begin{cases} -\Delta_p u + |u(x)|^{p-2}u(x) + g(x,u(x)) = f(x), \text{a. e. on } \Omega, \\ -\langle \nu, |\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x)), \text{a. e. on } \Gamma, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator, $|\nabla u|^{p-2}\nabla u$ is understood to be zero if $\nabla u = 0$ and ν denotes the exterior normal derivative of Γ. More details on Eq. (1) will be given in § 2.

Now let X be a real Banach space with a strictly convex dual space X^*. We use $\to$ and w-lim to denote the strong and weak convergence, respectively. For any subset G of X, we denote by int$G$ its interior and $\overline{G}$ its closure, respectively. Let $X \hookrightarrow\hookrightarrow Y$ denote that space X is embedded compactly in space Y. A mapping $T: D(T)=X \to X^*$ is said to be

Received: 2005-08-28.

MR Subject Classification: 47H05, 47H09, 49M05.

Keywords: maximal monotone operator, accretive mapping, hemi-continuous mapping, p-Laplacian operator.

Supported by the National Natural Science Foundation of China (10471033).

hemi-continuous on X if $w\text{-}\lim_{t\to 0}T(x+ty)=Tx$, for any $x,y\in X$. Let J denote the duality mapping from X into 2^{X^*} defined by

$$J(x)=\{f\in X^*:\langle x,f\rangle=\|x\|\cdot\|f\|,\|f\|=\|x\|\},x\in X,$$

where $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing between X and X^*. It is well-known that J is a single-valued mapping since X^* is strictly convex.

Let $A: X\to 2^X$ be a given multi-valued mapping. We say that A is boundedly-inversely-compact if for any pair of bounded subsets G and G' of X, the subset $G\cap A^{-1}(G')$ is relatively compact in X. The mapping $A: X\to 2^X$ is said to be accretive if $((v_1-v_2),J(u_1-u_2))\geqslant 0$ for any $u_i\in D(A)$ and $v_i\in Au_i$, $i=1,2$. The accretive mapping A is said to be m-accretive if $R(I+\lambda A)=X$ for some $\lambda>0$.

Let $B: X\to 2^{X^*}$ be a given multi-valued mapping. The graph of B, $G(B)$, is defined by $G(B)=\{[u,w]\mid u\in D(B),w\in Bu\}$. Then $B: X\to 2^{X^*}$ is said to be monotone if $G(B)$ is a monotone subset of $X\times X^*$ in the sense that $(u_1-u_2,w_1-w_2)\geqslant 0$ for any $[u_i,w_i]\in G(B)$, $i=1,2$. The monotone operator B is said to be maximal monotone if $G(B)$ is maximal among all monotone subsets of $X\times X^*$ in the sense of inclusion. The mapping B is said to be coercive if $\lim_{n\to+\infty}\frac{(x_n,x_n^*)}{\|x_n\|}=\infty$ for all $[x_n,x_n^*]\in G(B)$ such that

$$\lim_{n\to+\infty}\|x_n\|=+\infty.$$

Next, we give some concepts and well-known results for the needs in the sequel.

Definition 1.1. The duality mapping $J: X\to 2^{X^*}$ is said to satisfy Condition (I) if there exists a function $\eta: X\to[0,+\infty)$ such that

$$\|Ju-Jv\|\leqslant\eta(u-v),\text{for }\forall\, u,v\in X.\tag{I}$$

Definition 1.2. Let $A: X\to 2^X$ be an accretive mapping and $J: X\to X^*$ be a duality mapping. We say that A satisfies Condition (*) if, for any $f\in R(A)$ and $a\in D(A)$, there exists a constant $C(a,f)$ such that

$$(v-f,J(u-a))\geqslant C(a,f),\text{for any }u\in D(A),v\in Au.\tag{*}$$

Lemma 1.1. [6] Let Ω be a bounded conical domain in $\mathbf{R}^N$. If $mp>N$, then $W^{m,p}(\Omega)\hookrightarrow\hookrightarrow C_B(\Omega)$.

Lemma 1.2. [6] Let Ω be a bounded conical domain in $\mathbf{R}^N$. If $0<mp\leqslant N$ and $q_0=\frac{Np}{N-mp}$, then $W^{m,p}(\Omega)\hookrightarrow\hookrightarrow L^q(\Omega)$, where $1\leqslant q<q_0$.

Lemma 1.3. [7] Let $X=L^p(\Omega)$ and Ω be a bounded domain in $\mathbf{R}^N$. For $2\leqslant p<+\infty$, the duality mapping $J_p: L^p(\Omega)\to L^{p'}(\Omega)$ defined by $J_pu=|u|^{p-1}\mathrm{sgn}u\,\|u\|_p^{2-p}$, for $u\in L^p(\Omega)$, satisfies Condition (I); for $\frac{2N}{N+1}<p\leqslant 2$ and $N\geqslant 1$, the duality mapping $J_p: L^p(\Omega)\to L^{p'}(\Omega)$ defined by $J_pu=|u|^{p-1}\mathrm{sgn}u$, for $u\in L^p(\Omega)$, satisfies Condition (I), where $\frac{1}{p}+\frac{1}{p'}=1$.

Lemma 1.4. [7] Let Ω be a bounded domain in $\mathbf{R}^N$ and $g:\Omega\times\mathbf{R}\to\mathbf{R}$ be a function satisfying

Caratheodory's conditions such that

(i) $g(x, \cdot)$ is monotonically increasing on $\mathbf{R}$;

(ii) the mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^p(\Omega)$ is well defined, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$.

Let $J_p : L^p(\Omega) \to L^{p'}(\Omega), \frac{1}{p} + \frac{1}{p'} = 1$, be the duality mapping defined by

$$J_p u = \begin{cases} |u|^{p-1}\operatorname{sgn} u, & \text{if } \frac{2N}{N+1} < p \leqslant 2, \\ |u|^{p-1}\operatorname{sgn} u \ \|u\|_p^{2-p}, & \text{if } 2 \leqslant p < +\infty \end{cases}$$

for $u \in L^p(\Omega)$. Then the mapping $B : L^p(\Omega) \to L^p(\Omega)$ defined by $(Bu)(x) = g(x, u(x))$ for any $x \in \Omega$ satisfies Condition(*).

Theorem 1.1. [7]Let X be a real Banach space with a strictly convex dual X^*. Let $J : X \to X^*$ be a duality mapping on X satisfying Condition (I). Let $A, B_1 : X \to 2^X$ be accretive mappings such that

(i) both A, B_1 satisfy Condition (*), or $D(A) \subset D(B_1)$ and B_1 satisfy Condition (*),

(ii) $A + B_1$ is m-accretive and boundedly-inversely-compact.

If $B_2 : X \to X$ be a bounded continuous mapping such that, for any $y \in X$, there is a constant $C(y)$ satisfying $(B_2(u+y), Ju) \geqslant -C(y)$ for any $u \in X$. Then

(a) $\overline{[R(A)+R(B_1)]} \subset \overline{R(A+B_1+B_2)}$.

(b) $\operatorname{int}[R(A)+R(B_1)] \subset \operatorname{int} R(A+B_1+B_2)$.

§ 2 Main results

Let Ω be a bounded conical domain of a Euclidean space $\mathbf{R}^N (N \geqslant 1)$ with its boundary $\Gamma \in C^1$ [8], and suppose that Green's Formula is available. Let $|\cdot|$ denote the Euclidean norm in $\mathbf{R}^N$, $\langle \cdot, \cdot \rangle$ the Euclidean inner-product and ν the exterior normal derivative of Γ.

Let $\varphi : \Gamma \times \mathbf{R} \to \mathbf{R}$ be a given function such that, for each $x \in \Gamma$,

(i) $\varphi_x = \varphi(x, \cdot) : \mathbf{R} \to \mathbf{R}$ is a proper, convex, lower-semicontinuous function with $\varphi_x(0) = 0$.

(ii) $\beta_x = \partial\varphi_x$ (: subdifferential of φ_x) is maximal monotone mapping on $\mathbf{R}$ with $0 \in \beta_x(0)$ and for each $t \in \mathbf{R}$, the function $x \in \Gamma \to (I + \lambda\beta_x)^{-1}(t) \in \mathbf{R}$ is measurable for $\lambda > 0$.

Let $g : \Omega \times \mathbf{R} \to \mathbf{R}$ be a given function satisfying Caratheodory's conditions such that for $\frac{2N}{N+1} < p < +\infty$ and $N \geqslant 1$, the mapping $u \in L^p(\Omega) \to g(x, u(x)) \in L^p(\Omega)$ is defined. Further, suppose that there is a function $T(x) \in L^p(\Omega)$ such that $g(x,t)t \geqslant 0$ for $|t| \geqslant T(x), x \in \Omega$.

In this section we study the following nonlinear boundary value problem: given $f \in L^p(\Omega)$, find $u \in L^p(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geqslant 1$, such that

$$\begin{cases} -\Delta_p u + |u(x)|^{p-2}u(x) + g(x,u(x)) = f(x), & \text{a.e. on } \Omega \\ -\langle \nu, |\nabla u|^{p-2}\nabla u\rangle \in \beta_x(u(x)), & \text{a.e. on } \Gamma. \end{cases} \tag{1}$$

Definition 2. 1. [7] Define $g_+(x)=\lim\inf_{t\to+\infty}g(x,t)$ and $g_-(x)=\lim\sup_{t\to-\infty}g(x,t)$. Further, define a function $g_1: \Omega\times\mathbf{R}\to\mathbf{R}$ by

$$g_1(x,t)=\begin{cases}(\inf_{s\geqslant t}g(x,s))\wedge(t-T(x)), & \forall\, t\geqslant T(x),\\ 0, & \forall\, t\in[-T(x),T(x)],\\ (\sup_{s\leqslant t}g(x,s))\vee(t+T(x)), & \forall\, t\leqslant -T(x).\end{cases}$$

Then $\forall\, x\in\Omega, g_1(x,t)$ is increasing in t and $\lim_{t\to\pm\infty}g_1(x,t)=g_\pm(x)$. Moreover, $g_1: \Omega\times\mathbf{R}\to\mathbf{R}$ satisfies Caratheodory's conditions and the functions $g_\pm(x)$ are measurable on Ω. And, if $g_2(x,t)=g(x,t)-g_1(x,t)$, then $g_2(x,t)t\geqslant 0$ for $|t|\geqslant T(x), x\in\Omega$.

Lemma 2. 1. [7] For $\frac{2N}{N+1}<p<+\infty$ and $N\geqslant 1$, define the mapping $B_1: L^p(\Omega)\to L^p(\Omega)$ by $(B_1u)(x)=g_1(x,u(x))$, for $\forall\, u\in L^p(\Omega)$ and $x\in\Omega$, then B_1 is a bounded, continuous and m-accretive mapping. Also define $B_2: L^p(\Omega)\to L^p(\Omega)$ by $(B_2u)(x)=g_2(x,u(x))$, where $g_2(x,t)=g(x,t)-g_1(x,t)$, then B_2 satisfies the inequality

$$(B_2(u+y), J_pu)\geqslant -C(y) \tag{2}$$

for any $u,y\in L^p(\Omega)$, where $C(y)$ is a constant depending on y and $J_p: L^p(\Omega)\to L^{p'}(\Omega)$ denotes the duality mapping, where $\frac{1}{p}+\frac{1}{p'}=1$.

Lemma 2. 2. [4] Define the mapping $\Phi_p: W^{1,p}(\Omega)\to\mathbf{R}$ by $\Phi_p(u)=\int_\Gamma\varphi_x(u|_\Gamma(x))\mathrm{d}\Gamma(x)$, for any $u\in W^{1,p}(\Omega)$, where $\frac{2N}{N+1}<p<+\infty$ for $N\geqslant 1$. Then Φ_p is a proper, convex and lower-semi-continuous mapping on $W^{1,p}(\Omega)$.

Lemma 2. 3. Define the mapping $B_p: W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$, for $\frac{2N}{N+1}<p<+\infty$ and $N\geqslant 1$, by

$$(v,B_pu)=\int_\Omega\langle|\nabla u|^{p-2}\nabla u,\nabla v\rangle\mathrm{d}x+\int_\Omega|u(x)|^{p-2}u(x)v(x)\mathrm{d}x$$

for any $u,v\in W^{1,p}(\Omega)$. Then B_p is everywhere defined, monotone, hemi-continuous and coercive.

Proof. This result is a special case of Lemma 2. 1 in [4].

Definition 2. 2. Define a mapping $A_p: L^p(\Omega)\to 2^{L^p(\Omega)}$, where $\frac{2N}{N+1}<p<+\infty$ for $N\geqslant 1$ as follows:

$D(A_p)=\{u\in L^p(\Omega)\,|\,\text{there exists an } f\in L^p(\Omega) \text{ such that } f\in B_pu+\partial\Phi_p(u)\}$.

For $u\in D(A_p)$, let $A_pu=\{f\in L^p(\Omega)\,|\,f\in B_pu+\partial\Phi_p(u)\}$.

Lemma 2. 4. The mapping $A_p: L^p(\Omega)\to 2^{L^p(\Omega)}$ for $\frac{2N}{N+1}<p<+\infty$ and $N\geqslant 1$, is accretive.

Proof. We will prove this lemma in two cases.

Case 1. If $p\geqslant 2$, the duality mapping $J_p : L^p(\Omega)\to L^{p'}(\Omega)$ is defined by $J_p u=|u|^{p-1}\cdot \mathrm{sgn}u\,\|u\|_p^{2-p}$ for $u\in L^p(\Omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$. It suffices to prove that for any $u_i\in D(A_p)$ and $v_i\in A_p u_i, i=1,2$,

$$(v_1 - v_2, J_p(u_1 - u_2)) \geqslant 0.$$

Then it leaves us to prove that both

$$(|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2)\,\|u_1 - u_2\|_p^{2-p}, B_p u_1 - B_p u_2) \geqslant 0$$

and

$$(|u_1-u_2|^{p-1}\mathrm{sgn}(u_1-u_2)\,\|u_1-u_2\|_p^{2-p}, \partial\Phi_p(u_1)-\partial\Phi_p(u_2))\geqslant 0$$ are available.

Now take for a constant $k>0$, $\chi_k : \mathbf{R}\to\mathbf{R}$ is defined by $\chi_k(t)=|(t\wedge k)\vee(-k)|^{p-1}\mathrm{sgn}t$. Then χ_k is monotone, Lipschitz with $\chi_k(0)=0$ and χ'_k is continuous except at finitely many points on $\mathbf{R}$. This gives

$$(|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2)\,\|u_1 - u_2\|_p^{2-p}, \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) = \lim_{k\to+\infty}\|u_1 - u_2\|_p^{2-p}(\chi_k(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geqslant 0$$

and

$$(|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2)\,\|u_1 - u_2\|_p^{2-p}, B_p u_1 - B_p u_2) =$$
$$\|u_1 - u_2\|_p^{2-p}\lim_{k\to+\infty}\int_\Omega \langle |\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2, \nabla u_1 - \nabla u_2\rangle \chi'_k(u_1 - u_2)\mathrm{d}x +$$
$$\|u_1 - u_2\|_p^{2-p}\int_\Omega (|u_1|^{p-2}u_1 - |u_2|^{p-2}u_2)|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2)\mathrm{d}x \geqslant 0.$$

The last inequality is available since χ_k is monotone and $\chi_k(0)=0$.

Case 2. If $\frac{2N}{N+1}<p<2$ and $N\geqslant 1$, the duality mapping $J_p : L^p(\Omega)\to L^{p'}(\Omega)$ is defined by $J_p u=|u|^{p-1}\mathrm{sgn}u$ for $u\in L^p(\Omega)$ and $\frac{1}{p}+\frac{1}{p'}=1$. It suffices to prove that for any $u_i\in D(A_p)$ and $v_i\in A_p u_i, i=1,2$,

$$(v_1 - v_2, J_p(u_1 - u_2)) \geqslant 0.$$

Now, we define the function $\chi_n : \mathbf{R}\to\mathbf{R}$ by

$$\chi_n(t) = \begin{cases} |t|^{p-1}\mathrm{sgn}t, & \text{if } |t| \geqslant \frac{1}{n}, \\ \left(\frac{1}{n}\right)^{p-2} t, & \text{if } |t| \leqslant \frac{1}{n}. \end{cases}$$

Then χ_n is monotone, Lipschitz with $\chi_n(0)=0$ and χ'_n is continuous except at finitely many points on $\mathbf{R}$. So $(\chi_n(u_1-u_2), \partial\Phi_p(u_1)-\partial\Phi_p(u_2))\geqslant 0$.

Then, for $u_i\in D(A_p)$, $v_i\in A_p u_i$, $i=1,2$, we have

$$(v_1 - v_2, J_p(u_1 - u_1)) = (|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2), B_p u_1 - B_p u_2) + (|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) =$$
$$(|u_1 - u_2|^{p-1}\mathrm{sgn}(u_1 - u_2), B_p u_1 - B_p u_2) + \lim_{n\to\infty}(\chi_n(u_1 - u_2), \partial\Phi_p(u_1) - \partial\Phi_p(u_2)) \geqslant 0.$$

This completes the proof.

Remark 2. 1. Since the methods in [1-5] cannot be used to show that $A_p: L^p(\Omega)\to 2^{L^p(\Omega)}$ is accretive, a new method and some new techniques are applied in Lemma 2. 4.

Lemma 2. 5. The mapping $A_p: L^p(\Omega)\to 2^{L^p(\Omega)}$ for $\frac{2N}{N+1}<p<+\infty$ and $N\geq 1$, satisfies $R(I+\lambda A_p)=L^p(\Omega)$, for any $\lambda>0$.

Proof. First, define the mapping $I_p: W^{1,p}(\Omega)\to (W^{1,p}(\Omega))^*$ by $I_p u=u$ and $(v, I_p u)_{(W^{1,p}(\Omega))^*\times W^{1,p}(\Omega)}=(v,u)_{L^2(\Omega)}$ for $u,v\in W^{1,p}(\Omega)$, where $\frac{2N}{N+1}<p<+\infty$ for $N\geq 1$ and $(\cdot,\cdot)_{L^2(\Omega)}$ denotes the inner product of $L^2(\Omega)$. Then from Lemma 2. 3 in [4], we know that I_p is everywhere defined, monotone and hemi-continuous.

Secondly, for any $\lambda>0$, define the mapping $T_\lambda: W^{1,p}(\Omega)\to 2^{(W^{1,p}(\Omega))^*}$ by $T_\lambda u=I_p u+\lambda B_p u+\lambda\partial\Phi_p(u)$, for $u\in W^{1,p}(\Omega)$, where $\frac{2N}{N+1}<p<+\infty$ and $N\geq 1$. Then T_λ is maximal, monotone and coercive, so it follows that $R(T_\lambda)=(W^{1,p}(\Omega))^*$ for any $\lambda>0$.

Next, noticing the following facts:

if $p\geq 2$, then $W^{1,p}(\Omega)\subset L^p(\Omega)\subset L^{p'}(\Omega)\subset (W^{1,p}(\Omega))^*$, where $\frac{1}{p}+\frac{1}{p'}=1$;

if $\frac{2N}{N+1}<p\leq 2$, since $W^{1,p}(\Omega)\hookrightarrow\hookrightarrow L^{p'}(\Omega)$ when $N\geq 2$ by Lemma 1. 2, and $W^{1,p}(\Omega)\hookrightarrow\hookrightarrow C_B(\Omega)$ when $N=1$ by Lemma 1. 1, we have

$$W^{1,p}(\Omega)\subset L^{p'}(\Omega)\subset L^p(\Omega)\subset (W^{1,p}(\Omega))^*,\text{where } \frac{1}{p}+\frac{1}{p'}=1.$$

So for any $f\in L^p(\Omega)$, there exists $u\in W^{1,p}(\Omega)$ such that

$$f=T_\lambda u=I_p u+\lambda B_p u+\lambda\partial\Phi_p(u)=u+\lambda B_p u+\lambda\partial\Phi_p(u).$$

From the definition of A_p, it follows that $R(I+\lambda A_p)=L^p(\Omega)$, for $\forall\ \lambda>0$.

Proposition 2. 1. The mapping $A_p: L^p(\Omega)\to 2^{L^p(\Omega)}$, for $\frac{2N}{N+1}<p<+\infty$ and $N\geq 1$ is m-accretive.

Proof. In view of Lemmas 2. 4 and 2. 5, the result is available.

Lemma 2. 6. The mapping $A_p: L^p(\Omega)\to 2^{L^p(\Omega)}$ has a compact resolvent for $\frac{2N}{N+1}<p<2$ and $N\geq 1$.

Proof. Since A_p is m-accretive by Proposition 2. 1, it suffices to prove that if $u+\lambda A_p u=f(\lambda>0)$ and $\{f\}$ is bounded in $L^p(\Omega)$, then $\{u\}$ is relatively compact in $L^p(\Omega)$. Now define functions $\chi_n,\xi_n: \mathbf{R}\to\mathbf{R}$ by

$$\chi_n(t)=\begin{cases}|t|^{p-1}\mathrm{sgn}t, \text{if } |t|\geq \frac{1}{n},\\ \left(\frac{1}{n}\right)^{p-2}t, \text{if } |t|\leq\frac{1}{n},\end{cases}$$

and

$$\xi_n(t) = \begin{cases} |t|^{2-\frac{2}{p}}\mathrm{sgn}t, \text{if } |t| \geq \frac{1}{n}, \\ \left(\frac{1}{n}\right)^{1-\frac{2}{p}} t, \text{if } |t| \leq \frac{1}{n}. \end{cases}$$

Noticing that $\chi'_n(t) = (p-1) \times \left(\frac{p'}{2}\right)^p \times (\xi'_n(t))^p$, for $|t| \geq \frac{1}{n}$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\chi'_n(t) = (\xi'_n(t))^p$, for $|t| \leq \frac{1}{n}$, we can know $(\chi_n(u), \partial\Phi_p(u)) \geq 0$ for $u \in W^{1,p}(\Omega)$ since χ_n is monotone, Lipschitz with $\chi_n(0) = 0$ and χ'_n is continuous except at finitely many points on **R**. Then

$$(|u|^{p-1}\mathrm{sgn}u, A_pu) = \lim_{n\to\infty}(\chi_n(u), A_pu) \geq \lim_{n\to\infty}(\chi_n(u), B_pu) =$$
$$\lim_{n\to\infty}\int_\Omega |\nabla u|^p \chi'_n(u)\mathrm{d}x + \lim_{n\to\infty}\int_\Omega |u|^{p-2}u\chi_n(u)\mathrm{d}x \geq$$
$$\mathrm{const}\lim_{n\to\infty}\int_\Omega |\mathrm{grad}(\xi_n(u))|^p \mathrm{d}x \geq$$
$$\mathrm{const}\int_\Omega |\mathrm{grad}(|u|^{2-\frac{2}{p}}\mathrm{sgn}u)|^p \mathrm{d}x.$$

We now have from $f = u + \lambda A_pu$ that

$$\begin{aligned} \|f\|_p \| |u|^{2-\frac{2}{p}}\mathrm{sgn}u \|_{\frac{\frac{p^2}{2(p-1)p'}}{\frac{p}{2(p-1)}}} &\geq (|u|^{p-1}\mathrm{sgn}u, f) = \\ (|u|^{p-1}\mathrm{sgn}u, u) &+ \lambda(|u|^{p-1}\mathrm{sgn}u, A_pu) \geq \qquad (3) \\ \| |u|^{2-\frac{2}{p}}\mathrm{sgn}u \|_{\frac{\frac{p^2}{2(p-1)}}{\frac{p}{2(p-1)}}} &+ \lambda\mathrm{const} \| \mathrm{grad}(|u|^{2-\frac{2}{p}}\mathrm{sgn}u) \|_p^p, \end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. This gives

$$\| |u|^{2-\frac{2}{p}}\mathrm{sgn}u \|_p^{\frac{p}{2(p-1)}} \leq \| |u|^{2-\frac{2}{p}}\mathrm{sgn}u \|_{\frac{\frac{p^2}{2(p-1)}}{\frac{p}{2(p-1)}}} \leq \|f\|_p \leq \mathrm{const}$$

in view of the fact that $p < \frac{p^2}{2(p-1)}$ when $\frac{2N}{N+1} < p < 2$ for $N \geq 1$. Again from (3), we have $\| \mathrm{grad}(|u|^{2-\frac{2}{p}}\mathrm{sgn}u) \|_p \leq \mathrm{const}$. Hence $\{f\}$ bounded in $L^p(\Omega)$ implies that $\{|u|^{2-\frac{2}{p}}\mathrm{sgn}u\}$ is bounded in $W^{1,p}(\Omega)$.

We notice that $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{p^2}{2(p-1)}}(\Omega)$ when $N \geq 2$ by Lemma 1. 2 and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N = 1$ by Lemma 1. 1, hence $\{|u|^{2-\frac{2}{p}}\mathrm{sgn}u\}$ is relatively compact in $L^{\frac{p^2}{2(p-1)}}(\Omega)$. This gives that $\{u\}$ is relatively compact in $L^p(\Omega)$ since the Nemytskii mapping $u \in L^{\frac{p^2}{2(p-1)}}(\Omega) \to |u|^{\frac{p}{2(p-1)}}\mathrm{sgn}u \in L^p(\Omega)$ is continuous.

Remark 2. 2. Lemma 2. 6 is a new result compared with those in [1-5] since the solution of Eq. (1) to be found in this paper is in $L^p(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$.

Remark 2. 3. Since $\Phi_p(u + \alpha) = \Phi_p(u)$ for any $u \in W^{1,p}(\Omega)$ and $\alpha \in C_0^\infty(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$, we have $f \in A_pu$ that implies $f = B_pu$ in the sense of distributions.

Proposition 2. 2. Let $f\in L^p(\Omega)$ and $u\in L^p(\Omega)$ be such that $f\in A_p u$, where $\frac{2N}{N+1}<p<+\infty$ for $N\geqslant 1$. Then

(a) $-\Delta_p u+|u(x)|^{p-2}u(x)=f(x)$, a. e. $x\in\Omega$,

(b) $-\langle\nu, |\nabla u|^{p-2}\nabla u\rangle\in\beta_x(u(x))$, a. e. $x\in\Gamma$, for $\frac{2N}{N+1}<p<+\infty$ and $N\geqslant 1$.

Proof. (a) Similar to the proof of Proposition 2. 2 in [5], the result of (a) is true.

(b) We'll prove it under the additional condition $|\beta_x(u)|\leqslant a|u|^{\frac{p}{p'}}+b(x)$, where $b(x)\in L^{p'}(\Gamma)$, $\frac{1}{p}+\frac{1}{p'}=1$ and $a\in\mathbf{R}$. Refer to the result of [9] for the general case.

Now, from (a), $f\in A_p u$ implies that $f(x)=-\Delta_p u+|u(x)|^{p-2}u(x)\in L^p(\Omega)$. By using Green's Formula, we have for any $v\in W^{1,p}(\Omega)$ that

$$\int_\Gamma\langle\nu, |\nabla u|^{p-2}\nabla u\rangle v|_\Gamma \mathrm{d}\Gamma(x) = \int_\Omega \mathrm{div}(|\nabla u|^{p-2}\nabla u)v\mathrm{d}x + \int_\Omega\langle|\nabla u|^{p-2}\nabla u,\nabla v\rangle\mathrm{d}x.$$

Then $-\langle\nu, |\nabla u|^{p-2}\nabla u\rangle\in W^{-\frac{1}{p},p'}(\Gamma)=(W^{\frac{1}{p},p}(\Gamma))^*$, where $W^{\frac{1}{p},p}(\Gamma)$ is the space of traces of $W^{1,p}(\Omega)$.

Now let the mapping $B: L^p(\Gamma)\to L^{p'}(\Gamma)$, $\frac{1}{p}+\frac{1}{p'}=1$, be defined by $Bu=g(x)$, for any $u\in L^p(\Gamma)$, where $g(x)=\beta_x(u(x))$ a. e. on Γ. Clearly, $B=\partial\Psi$ where $\Psi(u)=\int_\Gamma\varphi_x(u(x))\mathrm{d}\Gamma(x)$ is a proper, convex and lower-semi-continuous function on $L^p(\Gamma)$. Now define the mapping $K: W^{1,p}(\Omega)\to L^p(\Gamma)$ by $K(v)=v|_\Gamma$ for any $v\in W^{1,p}(\Omega)$. Then $K^*BK: W^{1,p}(\Omega)\to(W^{1,p}(\Omega))^*$ is maximal monotone since both K, B are continuous. Finally, for any $u, v\in W^{1,p}(\Omega)$, we have

$$\Psi(Kv)-\Psi(Ku)=\int_\Gamma[\varphi_x(v|_\Gamma(x))-\varphi_x(u|_\Gamma(x))]\mathrm{d}\Gamma(x)\geqslant$$

$$\int_\Gamma\beta_x(u|_\Gamma(x))(v|_\Gamma(x)-u|_\Gamma(x))\mathrm{d}\Gamma(x)=$$

$$(BKu,Kv-Ku)=(K^*BKu,v-u).$$

Hence we get $K^*BK\subset\partial\Phi_p$ and so $K^*BK=\partial\Phi_p$. Therefore, we have $-\langle\nu, |\nabla u|^{p-2}\nabla u\rangle\in\beta_x(u(x))$, a. e. on Γ.

Remark 2. 4. If $\beta_x\equiv 0$ for any $x\in\Gamma$, then $\partial\Phi_p(u)\equiv 0$ for $\forall\ u\in W^{1,p}(\Omega)$, where $\frac{2N}{N+1}<p<+\infty$ and $N\geqslant 1$.

Proposition 2. 3. If $\beta_x\equiv 0$ for any $x\in\Gamma$, then $\{f\in L^p(\Omega)\,|\int_\Omega f\mathrm{d}x=0\}\subset R(A_p)$ for $\frac{2N}{N+1}<p<+\infty$ and $N\geqslant 1$.

Proof. We can easily know that $R(B_p)=(W^{1,p}(\Omega))^*$ in view of Lemma 2. 3. Note that for any $f\in L^p(\Omega)$ with $\int_\Omega f\mathrm{d}x=0$, the linear function $u\in W^{1,p}(\Omega)\to\int_\Omega fu\mathrm{d}x$ is an element of

$(W^{1,p}(\Omega))^*$. So there exists $u \in W^{1,p}(\Omega)$ such that

$$\int_\Omega f v \mathrm{d}x = \int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla v \rangle \mathrm{d}x + \int_\Omega |u|^{p-2} u v \mathrm{d}x$$

for any $v \in W^{1,p}(\Omega)$. thus $f = A_p u$ from Remark 2.4.

Definition 2.3. [7] For $t \in \mathbf{R}, x \in \Gamma$, let $\beta_x^0(t) \in \beta_x(t)$ be the element with least absolute value if $\beta_x(t) \neq \Phi$ and $\beta_x^0(t) = \pm\infty$, where $t > 0$ or < 0, respectively, in case $\beta_x(t) = \Phi$. Finally, let $\beta_\pm(x) = \lim_{t \to \pm\infty} \beta_x^0(t)$ (in the extended sense) for $x \in \Gamma$. $\beta_\pm(x)$ defines measurable functions on Γ, in view of our assumptions on β_x.

Proposition 2.4. Let $f \in L^p(\Omega)$ such that

$$\int_\Gamma \beta_-(x) \mathrm{d}\Gamma(x) < \int_\Omega f \mathrm{d}x < \int_\Gamma \beta_+(x) \mathrm{d}\Gamma(x), \tag{4}$$

where $\frac{2N}{N+1} < p < +\infty$ and $N \geq 1$, then $f \in \operatorname{int} R(A_p)$.

Proof. Let $f \in L^p(\Omega)$ and satisfy (4), by Proposition 2.1, there exists $u_n \in L^p(\Omega)$ such that, for any $n \geq 1$, $f = \frac{1}{n} u_n + A_p u_n$. In the same reason as that of Proposition 3.4 in [7], we only need to prove $\| u_n \|_p \leq \text{const}, \forall\ n \geq 1$.

Indeed, suppose to the contrary that $1 \leq \| u_n \|_p \to \infty$, as $n \to \infty$. with $v_n = \frac{u_n}{\| u_n \|_p}$, let $\psi: \mathbf{R} \to \mathbf{R}$ be defined by $\psi(t) = |t|^p$, $\partial\psi: \mathbf{R} \to \mathbf{R}$ be its subdifferential and for $\mu > 0$, $\partial\psi_\mu: \mathbf{R} \to \mathbf{R}$ denote the Yosida-approximation of $\partial\psi$. Also let $\theta_\mu: \mathbf{R} \to \mathbf{R}$ denote the indefinite integral of $[(\partial\psi_\mu)']^{\frac{1}{p}}$ with $\theta_\mu(0) = 0$, so that $(\theta'_\mu)^p = (\partial\psi_\mu)'$. In view of Calvert and Gupta[7], we have

$$(\partial\psi_\mu(v_n), \partial\Phi_p(u_n)) \geq \int_\Gamma \beta_x((1 + \mu\partial\psi)^{-1})(u_n|_\Gamma(x))) \times \partial\psi_\mu(v_n|_\Gamma(x)) \mathrm{d}\Gamma(x) \geq 0. \tag{5}$$

Now multiplying the equation $f = \frac{1}{n} u_n + A_p u_n$ by $\partial\psi_\mu(v_n)$, we get

$$(\partial\psi_\mu(v_n), f) = (\partial\psi_\mu(v_n), \frac{1}{n} u_n) + (\partial\psi_\mu(v_n), B_p u_n) + (\partial\psi_\mu(v_n), \partial\Phi_p(u_n)).$$

Since $\partial\psi_\mu(0) = 0$, it follows that $(\partial\psi_\mu(v_n), u_n) \geq 0$. Also, we can know

$$(\partial\psi_\mu(v_n), B_p u_n) = \int_\Omega \langle |\nabla u_n|^{p-2} \nabla u_n, \nabla v_n \rangle (\partial\psi_\mu)'(v_n) \mathrm{d}x + \int_\Omega |u_n|^{p-2} u_n \partial\psi_\mu(v_n) \mathrm{d}x \geq$$

$$\| u_n \|_p^{p-1} \int_\Omega |\operatorname{grad}(\theta_\mu(v_n))|^p \mathrm{d}x.$$

Then we get from (5),

$$\| u_n \|_p^{p-1} \int_\Omega |\operatorname{grad}(\theta_\mu(v_n))|^p \mathrm{d}x +$$

$$\int_\Gamma \beta_x((1 + \mu\partial\psi)^{-1}(u_n|_\Gamma(x))) \times \partial\psi_\mu(v_n|_\Gamma(x)) \mathrm{d}\Gamma(x) \leq \tag{6}$$

$$(\partial\psi_\mu(v_n), f).$$

Since $|\partial\psi_\mu(t)| \leq |\partial\psi(t)|$ for any $t \in \mathbf{R}$ and $\mu > 0$, we see from $\| v_n \|_p = 1$ for $n \geq 1$ that

$\| \partial\psi_\mu(v_n) \|_{p'} \leqslant C$ for $\mu > 0$, where C is a constant which does not depend on n or μ and $\frac{1}{p}+\frac{1}{p'}=1$.

From (6) we have

$$\int_\Omega |\mathrm{grad}(\theta_\mu(v_n))|^p \mathrm{d}x \leqslant \frac{C}{\| u_n \|_p^{p-1}} \text{ for } \mu > 0 \text{ and } n \geqslant 1. \tag{7}$$

Now we easily know $(\theta'_\mu)^p = (\partial\psi_\mu)' \to (\partial\psi)'$ as $\mu \to 0$ a.e. on $\mathbf{R}$.

Letting $\mu \to 0$, from Fatou's lemma and (7) we get

$$\int_\Omega |\mathrm{grad}(|v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n)|^p \mathrm{d}x \leqslant \frac{C}{\| u_n \|_p^{p-1}}. \tag{8}$$

From (8) we know $|v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \to k$. (a constant) in $L^p(\Omega)$, as $n \to +\infty$.

Next we'll show that $k \neq 0$ is in $L^p(\Omega)$ in two aspects.

(i) If $p \geqslant 2$, since $\| |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \|_p = \| v_n \|_{2p-2}^{2-\frac{2}{p}} \geqslant \| v_n \|_p^{2-\frac{2}{p}} = 1$, it follows that $k \neq 0$ in $L^p(\Omega)$;

(ii) if $\frac{2N}{N+1} < p < 2$, $\| |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \|_p = \| v_n \|_{2p-2}^{2-\frac{2}{p}} \leqslant \| v_n \|_p^{2-\frac{2}{p}} = 1$, then $\{ |v_n|^{2-\frac{2}{p}} \cdot \mathrm{sgn}v_n \}$ is bounded in $W^{1,p}(\Omega)$. By Lemma 1.1, $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C_B(\Omega)$ when $N=1$ and $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^{\frac{p^2}{2(p-1)}}(\Omega)$ by Lemma 1.2 when $N \geqslant 2$. So $\{ |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \}$ is relatively compact in $L^{\frac{p^2}{2(p-1)}}(\Omega)$. Then there exists a subsequence of $\{ |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \}$, which for simplicity, we denote by $\{ |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \}$, satifying $|v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \to g$ in $L^{\frac{p^2}{2(p-1)}}(\Omega)$. By noticing that $p \leqslant \frac{p^2}{2(p-1)}$ when $\frac{2N}{N+1} < p < 2$ for $N \geqslant 1$, it follows that $k = g$ a.e. on Ω. Now,

$$1 = \| v_n \|_p^p = \int_\Omega \left| |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n \right|^{\frac{p^2}{2(p-1)}} \mathrm{d}x \leqslant$$

$$\mathrm{const}\int_\Omega \left| |v_n|^{2-\frac{2}{p}}\mathrm{sgn}v_n - g \right|^{\frac{p^2}{2(p-1)}} \mathrm{d}x + \mathrm{const} \| g \|_{\frac{p^2}{2(p-1)}}^{\frac{p^2}{2(p-1)}},$$

and it follows that $g \neq 0$ in $L^p(\Omega)$ and $k \neq 0$ in $L^p(\Omega)$. Assuming now $k > 0$, we see from (6),

$$\int_\Gamma \beta_x((1+\mu\partial\psi)^{-1}(u_n|_\Gamma(x))) \times \partial\psi_\mu(v_n|_\Gamma(x))\mathrm{d}\Gamma(x) \leqslant (\partial\psi_\mu(v_n), f).$$

Choosing a subsequence such that $u_n|_\Gamma(x) \to +\infty$ a.e. on Γ, and letting $n \to +\infty$, we have $\int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x) \leqslant \int_\Omega f(x)\mathrm{d}x$, which is a contradiction with (4). Similarly, if $k < 0$, it also leads to a contradiction. Thus $f \in \mathrm{int}R(A_p)$.

Theorem 2.1. Let $f \in L^p(\Omega)$ be such that

$$\int_\Gamma \beta_-(x)\mathrm{d}\Gamma(x) + \int_\Omega g_-(x)\mathrm{d}x < \int_\Omega f(x)\mathrm{d}x < \int_\Gamma \beta_+(x)\mathrm{d}\Gamma(x) + \int_\Omega g_+(x)\mathrm{d}x,$$

then Eq. (1) has a solution in $L^p(\Omega)$, where $\frac{2N}{N+1} < p < +\infty$ and $N \geqslant 1$.

Proof. Let $A_p : L^p(\Omega) \to 2^{L^p(\Omega)}$ be the m-accretive operator defined in Def. 2. 2 and B_i : $L^p(\Omega) \to L^p(\Omega)$ be given by $(B_i u)(x) = g_i(x, u(x))$ as those in Lemma 2. 1, for $x \in \Omega$ and $i = 1, 2$. By Lemma 1. 4 and Lemma 2. 1, B_1 satisfies Condition (*) and B_2 satisfies (2), respectively. By Lemma 1. 3, the duality mapping $J_p : L^p(\Omega) \to L^{p'}(\Omega)$ satisfies Condition (I).

It then suffices to show that $f \in R(A_p + B_1 + B_2)$ in view of Proposition 2. 2 which would be implied by $f \in \text{int}[R(A_p) + R(B_1)]$ in view of Theorem 1. 1.

Next we will check that the conditions of Theorem1. 1 are verified.

First, we will prove that $A_p + B_1$ is boundedly-inversely-compact.

In fact, we only need to prove that if $w \in A_p u + B_1 u$ with $\{w\}$ and $\{u\}$ being bounded in $L^p(\Omega)$, then $\{u\}$ is relatively compact in $L^p(\Omega)$. Now we discuss it in two aspects:

(i) if $p \geqslant 2$, since

$$\int_\Omega |\nabla u|^p \mathrm{d}x \leqslant (u, B_p u) = (u, A_p u) - (u, \partial\Phi_p(u)) \leqslant$$

$$(u, A_p u) + (u, B_1 u) = (u, w) \leqslant \| u \|_p \| w \|_{p'} \leqslant \text{const},$$

it follows that $\{u\}$ is bounded in $W^{1,p}(\Omega)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. Then $\{u\}$ is relatively compact in $L^p(\Omega)$ since $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$;

(ii) if $\frac{2N}{N+1} < p < 2$, from the above fact that $w \in A_p u + B_1 u$ with $\{w\}$ and $\{u\}$ being bounded in $L^p(\Omega)$, we have $w - B_1 u \in A_p u$ with $\{w - B_1 u\}$ and $\{u\}$ being bounded in $L^p(\Omega)$, which gives that $\{u\}$ is relatively compact in $L^p(\Omega)$ since A_p is m-accretive by Proposition 2. 1 and has a compact resolvent by Lemma2. 6.

By using the similar methods for the proof of the result in [3], the other conditions of Theorem 1. 1 are also satisfied. Furthermore, as in the proof of the result in [3], by dividing it into two cases and using Proposition 2. 3 and 2. 4, we also have $f \in \text{int}[R(A_p) + R(B_1)]$.

Remark 2. 5. From the proof of Theorem 2. 1, we can see that Lemma 2. 6 is the key step to prove that $A_p + B_1$ is boundedly-inversely-compact when $\frac{2N}{N+1} < p < 2$.

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