# ON GROWTH OF SOBOLEV NORMS IN LINEAR SCHRÖDINGER EQUATIONS WITH SMOOTH TIME DEPENDENT POTENTIAL

*By*

J. BOURGAIN

## 0 Introduction

In this paper, we consider a linear Schrödinger equation of the form

$$iu_t + \Delta u + V(x,t)u = 0 \tag{0.1}$$

with periodic boundary conditions (i.e. $x \in \mathbb{T}^d$). Here $V$ is a real potential, smooth in $x$ and $t$ and periodic in $x$. Thus we do not specify any further structure with respect to the time dependence of $V$. Denote by $S(t)$ the flowmap of (0.1). Clearly, there is $L^2$-conservation

$$\|S(t)\phi\|_2 = \|\phi\|_2. \tag{0.2}$$

If $\phi \in H^s(\mathbb{T}^d)$, $s > 0$, then $S(t)\phi \in H^s$ for all time. The problem we are concerned with is the growth (if any) of $\|S(t)\phi\|_{H^s}$ for $t \to \infty$. Simple considerations permit us to bound $\|S(t)\phi\|_{H^s}$ by a power of $t$; thus

$$\|S(t)\phi\|_{H^s} \leq Ct^s \quad \text{for } t \to \infty \tag{0.3}$$

(see Lemma 6.2 below). T. Spencer [S] observed that if we assume further that $V$ is periodic in time, a much better bound holds. More precisely, let $V$ be real analytic and periodic in $x$ and $t$ (with arbitrary, fixed periods). Then, for $\phi \in H^s(\mathbb{T}^d)$, $\|S(t)\phi\|_{H^s}$ grows at most like a power of $\log t$ for $t \to \infty$. This fact holds in any dimension. In [B1], we essentially extended the preceding to the case of a potential $V = V(x,t)$ with quasi-periodic time dependence. We also produced examples showing that those logarithmic estimates are necessary, even in the time periodic case. Let us mention that the problem of growth of higher Sobolev norms has also been considered in equations (0.1) where $V = V(x,t)$ is assumed to have certain random behaviour in time.

The main result of this paper is the following rather general



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**Proposition 1.** *Consider* (0.1) *in arbitrary dimension* $d$ *(periodic b.c.), assuming* $V$ *is bounded, smooth in* $x$ *and* $t$*, and periodic in* $x$*. Then, for all* $s < \infty$, $\varepsilon > 0$

$$\|S(t)\phi\|_{H^s} \le C_{\varepsilon,s} t^{\varepsilon} \|\phi\|_{H^s} \quad \text{for } t \to \infty. \tag{0.4}$$

This statement is surprising, since we do not make any specification on the time dependence of $V$ except smoothness. In fact, the proof is relatively simple and based on ideas very similar to the time-periodic case. Observe that if we restrict $t$ to a large time interval $[0, T]$, one may always replace $V$ by a potential $V_1 = V_1(x,t)$ which is periodic in $t$, say with period $4T$. This permits us to consider specific solutions of (0.1) (Bloch waves) of the form

$$u(x,t) = e^{iEt}\psi(x,t), \tag{0.5}$$

where $e^{4iTE}$ is an eigenvalue for the unitary operator $S_1(4T)$ and $\psi$ is periodic in $x$, $4T$-periodic in $t$. Thus one may write

$$\psi(x,t) = \sum_{n,k\in\mathbb{Z}} \widehat{\psi}(n,k)\, e^{i(nx + \frac{\pi k}{2T} t)}. \tag{0.6}$$

The main part of the argument then consists (as in [B1]) in estimating $\widehat{\psi}(n,k)$ and establishing a certain localization. It turns out that the methods as used in [B1] are considerably less restrictive regarding certain specific properties of the potential assumed in that paper. Proposition 1 leads to simpler proofs of the results from [B1] for quasi-periodic potentials in time, although the conclusion is a bit weaker. The proof of Proposition 1 for $d = 1$ occupies Sections 1–7. In Section 8, we sketch the argument in arbitrary dimension $d > 1$. It turns out that the only basic ingredients are the separation properties of the sequence of squares $\{n^2\}$ and, in higher dimension, the "separated-cluster" structure of the set $\{(n, |n|^2) | n \in \mathbb{Z}^d\}$.

The second part of this work consists in obtaining estimates from below on $\|S(t)\phi\|_{H^1}$ for $t \to \infty$ in a model (0.1) where $V = V(x,t)$ is analytic in $x$ and has a random behaviour, with restricted smoothness, in time. More precisely, we take $V$ of the form

$$V(x,t) = \sum_j [g_j(\omega)e^{ix} + \overline{g_j\omega)}\, e^{-ix}]\gamma_j(t) \tag{0.7}$$

where $\{g_j\}$ are independent, complex, normalized Gaussians and $\{\gamma_j\}$ are disjointly supported bump functions, satisfying

$$\sup_j \|\gamma_j\|_{H^s} < \infty. \tag{0.8}$$

We prove the following.

**Proposition 2.** *For an appropriate choice of* $\{\gamma_j\}$ *in* (0.7), *satisfying* (0.8), *one may ensure that*

$$\overline{\lim_{t\to\infty}} \frac{\|S(t)\phi\|_{H^1}}{t^{1/2(1+s)}} > 0, \quad \textit{almost surely} \tag{0.9}$$

*for any data* $\phi \in H^1$, $\phi \neq 0$.

Again, the argument is simple. Letting $u = S(t)\phi$, one has that

$$\frac{d}{dt}\left(\int |u_x(t)|^2 dx\right) = 2 \ \mathrm{Im} \int V_x \bar{u} u_x. \tag{0.10}$$

Restricting $t$ to a small interval $[0, T]$, we may expand (0.10) as a power series in $V$. The key point in the analysis is then the structure of the quadratic term.

## Acknowledgment

As for [B1], these investigations were motivated by discussions with T. Spencer.

### I. Proof of Proposition 1

## 1 Reduction to the periodic case with large period

Consider the equation

$$iu_t + u_{xx} + V(x,t)u = 0 \tag{1.0}$$

with periodic bc.

Assume that $V$ is real and smooth in $x$ and $t$, and that

$$\|V\|_\infty < 1. \tag{1.1}$$

Fix $T$ large and consider the evolution for $0 \leq t \leq T$.

Take smooth $\varphi$ such that

$$\begin{cases} 0 \leq \varphi \leq 1 \\ \varphi(t) = 1 & \text{for } |t| \leq T \\ \varphi(t) = 0 & \text{for } |t| > 2T \end{cases} \tag{1.2}$$

and let

$$V_1(x,t) = \sum_{j\in\mathbb{Z}} V(x, t + 4Tj)\varphi(t + 4Tj). \tag{1.3}$$

Thus

$$V_1(x,t) = V(x,t) \quad \text{for } 0 \le t \le T, \tag{1.4}$$

$$V_1(x,t) = V_1(x, t+4T). \tag{1.5}$$

Since $V_1$ is $2\pi$-periodic in $x$ and $4T$-periodic in $t$

$$V_1(x,t) = \sum_{n\in\mathbb{Z},k\in\mathbb{Z}} \widehat{V}_1(n,k)\, e^{i(nx+\frac{\pi k}{2T}t)}. \tag{1.6}$$

Also

$$|\partial_x^\alpha \partial_t^\beta V_1| < C_{|\alpha|+|\beta|}. \tag{1.7}$$

Fix a large number $M_1 = M_1(T) > \log T$ and let

$$V_2(x,t) = \sum_{\substack{|n|<M_1\\|k|<M_1T}} \widehat{V}_1(n,k)\, e^{i(nx+\frac{\pi k}{2T}t)}. \tag{1.8}$$

From (1.7),

$$\|V_1 - V_2\|_\infty < C_\alpha M_1^{-\alpha}(\log T) < C_\alpha M_1^{-\alpha} \quad \text{for all } \alpha. \tag{1.9}$$

Observe that from (1.4), (1.1) is equivalent to

$$iu_t + u_{xx} + V_1(x,t)u = 0 \tag{1.10}$$

for $0 \le t \le T$.

## 2 Bloch waves

Consider the equation

$$iu_t + u_{xx} + V_2(x,t)u = 0. \tag{2.1}$$

Fix $n_0 \in \mathbb{Z}$,

$$|n_0| > M_2 > (\log T)^{10} \tag{2.2}$$

with $M_2$ to be specified.

Fix $M_3$ and define

$$\mathcal{B} = \{n \in \mathbb{Z} |\ |n - n_0| < M_3\}, \tag{2.3}$$

$$P_{\mathcal{B}}\phi = \sum_{n\in\mathcal{B}} \widehat{\phi}(n)e^{inx}. \tag{2.4}$$

Consider the IVP (ODE)

$$\begin{cases} iw_t + w_{xx} + P_{\mathcal{B}}[V_2.w] = 0, \\ w(0) = e^{in_0x}. \end{cases} \tag{2.5}$$

Thus $w = P_{\mathcal{B}}w$. Also, since

$$\begin{aligned} \frac{d}{dt}\Big[\int |w(x,t)|^2 dx\Big] &= 2 \text{ Re } \langle w_t, w\rangle \\ &\overset{(2.5)}{=} 2 \text{ Im } \langle -w_{xx} - P_{\mathcal{B}}[V_2w], w\rangle \\ &= 2 \text{ Im } \{\langle w_x, w_x\rangle - \langle V_2w, w\rangle\} = 0, \end{aligned} \tag{2.6}$$

there is conservation of $L^2$-norm

$$\|w(t)\|_2 = \Big(\sum_{n\in\mathcal{B}} |\widehat{w}(n)(t)|^2\Big)^{1/2} = 1. \tag{2.7}$$

Denote by $S(t)$ the flowmap corresponding to

$$iw_t + w_{xx} + P_{\mathcal{B}}[V_2.w] = 0, \tag{2.8}$$

acting on

$$[e^{in.x} | n \in \mathcal{B}] = \mathbb{C}^{|\mathcal{B}|}. \tag{2.9}$$

Since $V_2$ is $4T$-periodic, we have

$$S(t) = S(t-4T)S(4T) \quad \text{for } t \geq 4T. \tag{2.10}$$

Thus, denoting $(\xi_\alpha, E_\alpha)$ an orthonormal basis of eigenvectors for the unitary map $S(4T)$

$$S(4T)\xi_\alpha = e^{iE_\alpha}\xi_\alpha, \tag{2.11}$$

we have

$$S(t)\xi_\alpha = e^{i\frac{E_\alpha}{4T}t}\psi_\alpha(x,t), \tag{2.12}$$

where $\psi_\alpha$ is $2\pi$-periodic in $x$ and $4T$-periodic in $t$. ($E_\alpha$ is specified up to a multiple of $2\pi$.)

Hence, writing in particular $e_{n_0} = e^{in_0x}$,

$$e_{n_0} = \sum_\alpha \langle e_{n_0}, \xi_\alpha \rangle \xi_\alpha, \tag{2.13}$$

(2.12), (2.13) permit us to write

$$S(t)e_{n_0} = \sum_{|E_\alpha + 4n_0^2T| < \pi} e^{i\frac{E_\alpha}{4T}t}\, \psi'_\alpha(x,t) \tag{2.14}$$

$$= e^{-in_0^2t} \sum_\alpha e^{i\tilde{E}_\alpha t}\, \tilde{\psi}_\alpha(x,t), \tag{2.15}$$

where $\psi'_\alpha, \tilde{\psi}_\alpha$ are $2\pi$-periodic in $x$, $4T$-periodic in $t$,

$$|\tilde{E}_\alpha| = \left| \frac{E_\alpha}{4T} + n_0^2 \right| < \frac{\pi}{4T} \tag{2.16}$$

and, by (2.7), for all $t$

$$\|\tilde{\psi}_\alpha(t)\|_2 = |\langle e_{n_0}, \xi_\alpha \rangle|. \tag{2.17}$$

## 3 Fourier transform estimates

Fix $\alpha$ and denote $\tilde{\psi}_\alpha$ by $\psi$. Thus $e^{i(\tilde{E}_\alpha - n_0^2)t}\psi(x,t)$ satisfies (2.8).

Our next purpose is to get more information on $\widehat{\psi}$,

$$\psi(x,t) = \sum_{\substack{n \in \mathcal{B} \\ k \in \mathbb{Z}}} \widehat{\psi}(n,k) e^{i(nx + \frac{\pi k}{2T}t)}. \tag{3.1}$$

Substituting (3.1) in (2.8) thus gives for $n \in \mathcal{B},\ k \in \mathbb{Z}$

$$\Big(n^2 - n_0^2 + \frac{\pi}{2T}k + \tilde{E}_\alpha\Big)\widehat{\psi}(n,k) - \widehat{V_2\psi}(n,k) = 0. \tag{3.2}$$

Let $A, B$ satisfy (cf. (2.2))

$$2M_1 < A < B < \tfrac{1}{2}|n_0|. \tag{3.3}$$

Then, by (2.16), it follows that for

$$AT < |k| < BT, \tag{3.4}$$

$$\left| n^2 - n_0^2 + \frac{\pi}{2T}k + \tilde{E}_\alpha \right| = \left| \frac{\pi}{2T}k + \tilde{E}_\alpha \right| > \frac{\pi}{2}A - 1 > A \quad \text{if } n = n_0 \tag{3.5}$$

and

$$|n^2 - n_0^2 + \frac{\pi}{2T}k + \tilde{E}_\alpha| > 2|n_0| - \frac{\pi}{2}B - 2 > |n_0| > 2B \quad \text{if } n \neq n_0. \tag{3.6}$$

Also, by (1.8)

$$\Big(\sum_{\substack{n\in\mathcal{B}\\ AT<|k|<BT}} |\widehat{V_2\psi}(n,k)|^2\Big)^{1/2}$$
$$\leq \left[\frac{1}{8T\pi}\int_0^{4T}\int_0^{2\pi}|V_2|^2\Big|\sum_{\substack{n\in\mathcal{B}\\ (A-M_1)T<|k|<(B+M_1)T}}\widehat{\psi}(n,k)e^{i(nx+\frac{\pi k}{2T}t)}\Big|^2 dxdt\right]^{1/2}$$
$$\leq \|V_2\|_\infty\Big[\sum_{\substack{n\in\mathcal{B}\\ (A-M_1)T<|k|<(B+M_1)T}}|\widehat{\psi}(n,k)|^2\Big]^{1/2}$$
$$\overset{(1.1)+(1.9)}{<} C\Big(\sum_{n\in\mathcal{B},(A-M_1)T<|k|<(B+M_1)T}|\widehat{\psi}(n,k)|^2\Big)^{1/2}. \tag{3.7}$$

From (3.2), (3.5)+(3.6), (3.7), it follows that

$$\Big[\sum_{\substack{n\in\mathcal{B}\\ AT<|k|<BT}}|\widehat{\psi}(n,k)|^2\Big]^{1/2} < \frac{C}{A}\Big[\sum_{\substack{n\in\mathcal{B}\\ (A-M_1)T<|k|<(B+M_1)T}}|\widehat{\psi}(n,k)|^2\Big]^{1/2}. \tag{3.8}$$

By $q$-fold iteration of (3.8), assuming say

$$10qM_1 < |n_0|, \tag{3.9}$$

we get

$$\Big[\sum_{\substack{n\in\mathcal{B}\\ 2qM_1T<|k|<\frac{1}{4}|n_0|T}}|\widehat{\psi}(n,k)|^2\Big]^{1/2} < M_1^{-q}\Big(\sum_{n\in\mathcal{B},k}|\widehat{\psi}(n,k)|^2\Big)^{1/2}$$
$$= M_1^{-q}\Big[\frac{1}{8T\pi}\int_0^{4T}\int_0^{2\pi}|\psi(x,t)|^2 dxdt\Big]^{1/2}$$
$$< M_1^{-q}\|\psi(0)\|_2 \tag{3.10}$$

by $L^2$-conservation (2.17).

Next, we establish some decay estimates for $n$ away from $n_0$. One has again from (1.8)

(3.11)

$$\Big(\sum_{\substack{n\in\mathcal{B},\,|n-n_0|>qM_1\\ |k|<\frac{1}{4}|n_0|T}}|\widehat{V_2\psi}(n,k)|^2\Big)^{1/2} \leq C\Big(\sum_{\substack{n\in\mathcal{B},|n-n_0|>(q-1)M_1\\ |k|<(\frac{1}{4}|n_0|+M_1)T}}|\widehat{\psi}(n,k)|^2\Big)^{1/2};$$

hence, by (3.6), (3.11),

(3.12)
$$\Big(\sum_{\substack{n\in\mathcal{B},|n-n_0|>qM_1\\ |k|<\frac14|n_0|T}}|\widehat{\psi}(n,k)|^2\Big)^{1/2}\le\frac{C}{|n_0|}\Big(\sum_{\substack{n\in\mathcal{B},|n-n_0|>(q-1)M_1\\ |k|<(\frac14|n_0|+M_1)T}}|\widehat{\psi}(n,k)|^2\Big)^{1/2}.$$

Iteration of (3.12), taking (3.9) into account, then implies

$$\Big(\sum_{\substack{n\in\mathcal{B},|n-n_0|>qM_1\\ |k|<\frac14|n_0|T}}|\widehat{\psi}(n,k)|^2\Big)^{1/2}<\Big(\frac{C}{|n_0|}\Big)^q\Big(\sum_{n\in\mathcal{B},k}|\widehat{\psi}(n,k)|^2\Big)^{1/2}$$
$$<|n_0|^{-q/2}\|\psi(0)\|_2. \tag{3.13}$$

Also, by (3.6), for $n\neq n_0$

$$\Big(\sum_{|k|<\frac14|n_0|T}|\widehat{\psi}(n,k)|^2\Big)^{1/2}<\frac{1}{|n_0|}\Big(\frac{1}{4T}\int_0^{4T}|\widehat{V_2\psi}(n)(t)|^2dt\Big)^{1/2}$$
$$\le\frac{C}{|n_0|}\|\psi(0)\|_2$$
$$<|n_0|^{-1/2}\|\psi(0)\|_2. \tag{3.14}$$

Assume

$$100M_3<|n_0|^{\frac{1}{10}}. \tag{3.15}$$

Then, for $n\in\mathcal{B}$, by (2.3),

$$|n-n_0|<M_3<\frac{1}{100}|n_0|^{\frac{1}{10}} \tag{3.16}$$

and, if $q=[\frac{|n-n_0|}{M_1}]$, necessarily (3.9) holds. Thus, from (3.13), (3.14), for $n\neq n_0$

$$\Big(\sum_{|k|<\frac14|n_0|T}|\widehat{\psi}(n,k)|^2\Big)^{1/2}<|n_0|^{-\frac14(1+\frac{|n-n_0|}{M_1})}\|\psi(0)\|_2. \tag{3.17}$$

It will also be useful to have a similar bound with respect to the $L^\infty$-norm. For $n\neq n_0$ fixed and $|k|<\frac12|n_0|T$, the sequence

$$\frac{1}{n^2-n_0^2+\frac{\pi}{2T}k+\tilde{E}_\alpha} \tag{3.18}$$

has multiplier norm on $L^\infty(dt)$ bounded by $\frac{C}{|n_0|}$.

Also, for $|n-n_0|>qM_1$

$$\Big\| \sum_{|k|<\frac{1}{4}|n_0|T} \widehat{V_2\psi}(n,k)e^{\frac{i\pi k}{2T}t}\Big\|_{L_t^\infty} \leq \tag{3.19}$$

$$(\log|n_0|T)\|V_2\|_\infty . M_1\Big[\max_{|n'-n_0|>(q-1)M_1}\Big\| \sum_{|k|<(\frac{1}{4}|n_0|+M_1)T} \widehat{\psi}(n',k)e^{i\frac{\pi k}{2T}t}\Big\|_\infty\Big]. \tag{3.20}$$

Thus from (3.2), (3.20), we deduce that for $|n-n_0|>qM_1$

$$\Big\| \sum_{|k|<\frac{1}{4}|n_0|_T} \widehat{\psi}(n,k)e^{i\frac{\pi k}{2T}t}\Big\|_{L_t^\infty} \leq \frac{C}{|n_0|}(3.19)$$

$$< |n_0|^{-1/2}\Big[\max_{|n-n_0|>(q-1)M_1}\Big\| \sum_{|k|<(\frac{|n_0|}{4}+M_1)T} \widehat{\psi}(n,k)e^{i\frac{\pi k}{2T}t}\Big\|_\infty\Big]. \tag{3.21}$$

Iteration then again gives the bound

$$\Big\| \sum_{|k|<\frac{1}{4}|n_0|T} \widehat{\psi}(n,k)e^{i\frac{\pi k}{2T}t}\Big\|_{L_t^\infty} < |n_0|^{-\frac{1}{4}(1+\frac{|n-n_0|}{M_1})}\|\psi(0)\|_2 \quad \text{for } n\neq n_0. \tag{3.22}$$

## 4 Localization

With $\psi=\tilde{\psi}_\alpha$ as above, define (cf. (3.1))

$$\Psi_\alpha(x,t) = \sum_{n\in\mathcal{B},|k|<\frac{1}{8}|n_0|T} \widehat{\psi}(n,k)e^{i(nx+\frac{\pi k}{2T}t)}. \tag{4.1}$$

Recall that $\psi$ satisfies the equation

$$i\psi_t+\psi_{xx}+(n_0^2-\tilde{E}_\alpha)\psi+P_{\mathcal{B}}[V_2\psi]=0. \tag{4.2}$$

Write by (1.8)

$$\begin{aligned} &\Big|P_{\mathcal{B}}[V_2\Psi_\alpha]-\Big[\sum_{n\in\mathcal{B},|k|<\frac{1}{8}|n_0|T} \widehat{V_2\psi}(n,k)\ e^{i(nx+\frac{\pi k}{2T}t)}\Big]\Big| \\ &\leq |P_{\mathcal{B}}[V_2\Psi'_\alpha]| + \Big|\sum_{n\in\mathcal{B},|k|<\frac{1}{8}|n_0|T} \widehat{V_2\Psi''_\alpha}(n,k)\ e^{i(nx+\frac{\pi k}{2T}t)}\Big| \end{aligned} \tag{4.3}$$

with

$$\Psi'_\alpha(x,t) = \sum_{n\in\mathcal{B},(\frac{1}{8}|n_0|-M_1)T<|k|<\frac{1}{8}|n_0|T} \widehat{\psi}(n,k)e^{i(nx+\frac{\pi k}{2T}t)}, \tag{4.4}$$

$$\Psi''_\alpha(x,t) = \sum_{n\in\mathcal{B},(\frac{1}{8}|n_0|-M_1)T<|k|<(\frac{1}{8}|n_0|+M_1)T} \widehat{\psi}(n,k)e^{i(nx+\frac{\pi k}{2T}t)}. \tag{4.5}$$

Thus

$$\|(4.3)\|_\infty \le |\mathcal{B}|^{1/2}(|n_0|T)^{1/2}\Big[\sum_{n\in\mathcal{B},k}\big(|\widehat{V_2\Psi'_\alpha}(n,k)|^2+|\widehat{V_2\Psi''_\alpha}(n,k)|^2\big)\Big]^{1/2}$$

$$\overset{(2.3)}{\le} C(M_3|n_0|T)^{1/2}\Big(\sum_{n\in\mathcal{B},k}\big(|\widehat{\Psi}'_\alpha(n,k)|^2+|\widehat{\Psi}''_\alpha(n,k)|^2\big)\Big)^{1/2}$$

$$\overset{(4.4)+(4.5)}{\le} C(M_3|n_0|T)^{1/2}\Big\{\sum_{\substack{n\in\mathcal{B}\\ (\frac{1}{8}|n_0|-M_1)T<|k|<(\frac{1}{8}|n_0|+M_1)T}}|\widehat{\psi}(n,k)|^2\Big\}^{1/2}$$

$$\overset{(3.10)}{<} C(M_3|n_0|T)^{1/2}\, M_1^{-\frac{1}{20}\frac{|n_0|}{M_1}}\,\|\psi(0)\|_2. \tag{4.6}$$

Take in (2.2)

$$M_2 = M_1^2 + (\log T)^{10}. \tag{4.7}$$

Thus the factor in (4.6) is

$$< |n_0|T^{1/2}e^{-9|n_0|^{1/2}} < e^{-8|n_0|^{1/2}} \tag{4.8}$$

and

$$\|(4.3)\|_\infty < e^{-8|n_0|^{1/2}}\|\psi(0)\|_2. \tag{4.9}$$

From (4.2), (4.9), we see that $\Psi_\alpha$ satisfies (4.2) approximately, in the sense that

$$i\dot{\Psi}_\alpha + \Delta\Psi_\alpha + (n_0^2-\tilde{E}_\alpha)\Psi_\alpha + P_{\mathcal{B}}[V_2\Psi_\alpha] = O(e^{-8|n_0|^{1/2}}\|\psi(0)\|) \tag{4.10}$$

where $O(\ )$ refers to the $L^\infty$-norm. Recall (2.17)

$$\|\psi(0)\|_2 = |\langle e_{n_0},\xi_\alpha\rangle|. \tag{4.11}$$

Define, cf. (2.14), (2.15),

$$\tilde{f}_{n_0}(x,t) = \sum_\alpha e^{i\tilde{E}_\alpha t}\Psi_\alpha(x,t), \tag{4.12}$$

$$f_{n_0}(x,t) = e^{-in_0^2 t}\tilde{f}_{n_0}(x,t). \tag{4.13}$$

Since

$$\sum|\langle e_{n_0},\xi_\alpha\rangle| \le |\mathcal{B}|^{1/2}\|e_{n_0}\| < 2M_3^{1/2}, \tag{4.14}$$

it follows from (4.10), (4.11) and (4.12) that

$$\begin{aligned} i\partial_t\tilde{f}_{n_0} + \Delta\tilde{f}_{n_0} + n_0^2\tilde{f}_{n_0} + P_{\mathcal{B}}[V_2\tilde{f}_{n_0}] &= O(e^{-8|n_0|^{1/2}}M_3^{1/2})\\ &< O(e^{-7|n_0|^{1/2}}). \end{aligned} \tag{4.15}$$

Similarly

$$i\partial_t f_{n_0} + \Delta f_{n_0} + P_{\mathcal{B}}[V_2 f_{n_0}] = O(e^{-7|n_0|^{1/2}}). \tag{4.16}$$

Since, by (2.8) and the integral equation

$$e^{in_0^2 t}\big(S(t)e_{n_0}\big) = e_{n_0} + ie^{in_0^2 t}\int_0^t e^{i(t-\tau)\Delta} P_{\mathcal{B}}[V_2(\tau)S(\tau)e_{n_0}]d\tau, \tag{4.17}$$

we have for $|t| < c$, $c$ a sufficiently small constant,

$$|1 - e^{in_0^2 t}\widehat{S(t)e_{n_0}}(n_0)| \le \|e_{n_0} - e^{in_0^2 t}(S(t)e_{n_0})\|_2 \le C|t| < \frac{1}{10}. \tag{4.18}$$

Consider a smooth bump function on $\mathbb{R}$ satisfying

$$0 \le \varphi \le 1, \text{ supp } \varphi \subset [-c, c], \tag{4.19}$$

$$\int \varphi = 1, \tag{4.20}$$

$$|\widehat{\varphi}(\lambda)| < Ce^{-\lambda^{1/2}} \tag{4.21}$$

From (4.18), (4.20),

$$\left|\int e^{in_0^2 t}.\widehat{S(t)e_{n_0}}(n_0).\varphi(-t)dt\right| > \frac{1}{2}. \tag{4.22}$$

From (2.15), (4.1), (4.12), (4.21), the left side of (4.22) gives

$$\begin{aligned}
&\left|\sum_\alpha \int e^{i\tilde{E}_\alpha t}\,\widehat{\tilde{\psi}}_\alpha(n_0).\varphi(-t)dt\right| \\
=&\left|\sum_\alpha\sum_k \widehat{\tilde{\psi}}_\alpha(n_0,k)\widehat{\varphi}\Big(\frac{k}{4T} + \frac{1}{2\pi}\tilde{E}_\alpha\Big)\right| \\
\overset{(4.21)}{=}&\left|\int \widehat{\tilde{f}_{n_0}}(n_0)\varphi(-t)dt\right| + O(M_3 T e^{-\frac{1}{10}|n_0|^{1/2}}) && (4.23)\\
=&\left|\int \widehat{f}_{n_0}(n_0)e^{in_0^2 t}\varphi(-t)dt\right| + O(e^{-|n_0|^{1/3}}) && (4.24)\\
\le&\|\widehat{f}_{n_0}(n_0)\|_{L^\infty_{|t|<c}} + O(e^{-|n_0|^{1/3}}). && (4.25)
\end{aligned}$$

Hence, from (4.22), (4.25),

$$\|\widehat{f}_{n_0}(n_0)\|_{L^\infty_{|t|<c}} > \frac{1}{3}. \tag{4.26}$$

Next, take $n \neq n_0$. From (4.12), (4.13), (4.1), (3.22), (4.11), (3.15),

$$\begin{aligned}\|\widehat{f_{n_0}}(n)\|_\infty &\leq \sum_\alpha \|\widehat{\Psi_\alpha}(n)\|_\infty \\ &= \sum_\alpha \Big\| \sum_{|k|<\frac{1}{8}|n_0|T} \widehat{\widetilde{\psi}}_\alpha(n,k)\, e^{i\frac{\pi k}{2T}t} \Big\|_\infty \\ &\overset{(3.22)}{<} |n_0|^{-\frac{1}{4}(1+\frac{|n-n_0|}{M_1})} \Big( \sum_\alpha \|\psi_\alpha(0)\|_2 \Big) \\ &\overset{(4.11)}{<} |\mathcal{B}|^{1/2} |n_0|^{-\frac{1}{4}(1+\frac{|n-n_0|}{M_1})} \\ &< |n_0|^{-\frac{1}{5}(1+\frac{|n-n_0|}{M_1})}. \end{aligned} \tag{4.27}$$

By (4.16), (4.26), (4.27) and normalizing, we may then clearly get from $f_{n_0}$ an approximate solution $f'_{n_0}$ of (2.8), i.e.,

$$i\partial_t f'_{n_0} + \Delta f'_{n_0} + P_{\mathcal{B}}[V_2 f'_{n_0}] = O(e^{-7|n_0|^{1/2}}) \tag{4.28}$$

satisfying, for some $|t_0| < c$,

$$\widehat{f'_{n_0}}(n_0)(t_0) = 1, \tag{4.29}$$

$$\|f'_{n_0}(t_0)\|_\infty < 2, \tag{4.30}$$

$$\widehat{f'_{n_0}}(n) = 0 \quad \text{if } |n-n_0| > M_3, \tag{4.31}$$

$$\|\widehat{f'_{n_0}}(n)\|_\infty < 3|n_0|^{-\frac{1}{5}(1+\frac{|n-n_0|}{M_1})} \quad \text{for } n \neq n_0. \tag{4.32}$$

Estimate

$$\begin{aligned} &\|P_{\mathcal{B}}[V_2 f'_{n_0}] - V_2 f'_{n_0}\|_\infty \\ &\leq (\log M_3) \Big\| V_2 \Big[ \sum_{|n-n_0|>M_3-M_1} \widehat{f'_{n_0}}(n) e^{inx} \Big] \Big\|_\infty \\ &\leq (\log M_3) \Big[ \sum_{|n-n_0|>M_3-M_1} \|\widehat{f'_{n_0}}(n)\|_\infty \Big] \overset{(4.32)}{<} |n_0|^{-\frac{1}{6}\frac{M_3}{M_1}}. \end{aligned} \tag{4.33}$$

Taking also (1.9) into account, we see that (4.28) implies

$$i\partial_t f'_{n_0} + \Delta f'_{n_0} + V_1 f'_{n_0} = 0(e^{-7|n_0|^{1/2}} + |n_0|^{-\frac{1}{6}\frac{M_3}{M_1}} + C_\alpha M_1^{-\alpha}), \tag{4.34}$$

where $\alpha > 0$ is arbitrary.

Recall (2.2), (3.15), (4.7):

$$\begin{cases} M_1 > \log T, \\ |n_0| > M_2 > (\log T)^{10}, \\ 100M_3 < |n_0|^{1/10}, \\ M_2 = M_1^2 + (\log T)^{10}. \end{cases}$$

Hence, letting

$$M_1 = T^\varepsilon, \quad M_3 = M_1^2, \quad \varepsilon = \varepsilon(s), \tag{4.35}$$

we get

$$i\partial_t f'_{n_0} + \Delta f'_{n_0} + V_1 f'_{n_0} = O(T^{-\alpha}) \quad \text{for all } \alpha. \tag{4.36}$$

Also $n_0$ is assumed to satisfy

$$|n_0| > T^\delta, \tag{4.37}$$

where $\delta = 100\varepsilon$. The "$O$" in (4.36) depends on $\varepsilon, \alpha$

Let $S(t)$ denote the flowmap for the equation

$$iu_t + \Delta u + V_1 u = 0.$$

Then (4.35) and the integral equation imply for $0 < t < T$

$$\|f'_{n_0}(t) - S(t)\xi_{n_0}\|_{L_t^\infty L_x^2} \le \left\| \int_0^t S(t)S(\tau)^{-1}[0(T^{-\alpha})]d\tau \right\|_{L_t^\infty L_x^2} < C_\alpha T^{-\alpha}, \tag{4.38}$$

where

$$\xi_{n_0}(x) \equiv f'_{n_0}(x, 0). \tag{4.39}$$

It follows in particular from (4.18), (4.29), (4.30) and (4.38) that

$$1 = |\langle f'_{n_0}(t_0), e_{n_0}\rangle| > |\langle f'_{n_0}(t_0), S(t_0)e_{n_0}\rangle| - \frac{2}{10} > |\langle S(t_0)\xi_{n_0}, S(t_0)e_{n_0}\rangle| - \frac{3}{10}$$
$$= |\langle \xi_{n_0}, e_{n_0}\rangle| - \frac{3}{10}.$$

Thus, taking (4.31), (4.32) into account, we may ensure

$$\widehat{\xi}_{n_0}(n_0) = 1, \tag{4.40}$$

$$\|\xi_{n_0}\|_\infty < 2, \tag{4.41}$$

$$\widehat{\xi}_{n_0}(n) = 0 \quad \text{for } |n - n_0| > M_3, \tag{4.42}$$

$$|\widehat{\xi}_{n_0}(n)| < 5|n_0|^{-\frac{1}{5}(1+\frac{|n-n_0|}{M_1})} \quad \text{for } n \ne n_0. \tag{4.43}$$

## 5 $H^s$-bounds (1)

Take $\phi$ such that

$$\widehat{\phi}(n) = 0 \quad \text{if } |n| < T^{2\delta} \text{ or } |n| > T^A, \tag{5.1}$$

where $A$ is a fixed large constant.

Define

$$\phi_1 = \sum_n \widehat{\phi}(n)\xi_n, \tag{5.2}$$

where $\{\xi_n \,|\, |n| > T^\delta\}$ are obtained above.

From (4.40), (4.43), we have the approximation

$$\begin{aligned}
\|\phi - \phi_1\|_{H^s} &= \Big(\sum_m |m|^{2s} \Big|\langle \sum_n \widehat{\phi}(n)(\xi_n - e_n), e_m\rangle\Big|^2\Big)^{1/2} \\
&= \Big(\sum_m |m|^{2s} \Big(\sum_{n \neq m} |\widehat{\phi}(n)|\, |\widehat{\xi}_n(m)|\Big)^2\Big)^{1/2} \\
&< \Big[\sum_m |m|^{2s} \Big(\sum_n |\widehat{\phi}(n)|\, |n|^{-\frac{1}{5}(1+\frac{|n-m|}{M_1})}\Big)^2\Big]^{1/2} \\
&\overset{(5.1)}{<} C_{s,\varepsilon} T^{-\frac{\varepsilon}{4}} \|\phi\|_{H^s}.
\end{aligned} \tag{5.3}$$

Also, from (5.1), (4.42), (4.35),

$$\widehat{\phi}_1(n) = 0 \quad \text{if } |n| < T^{2\delta} - T^{2\varepsilon} \text{ or } |n| > T^A + T^{2\varepsilon}, \tag{5.4}$$

which permits us to iterate approximation (5.2), (5.3).

Assume $T$ sufficiently large (depending on $\varepsilon, s, A$).

Fix $N_0$ to be specified.

Estimate for $0 \le t \le T$

$$\begin{aligned}
&\Big(\sum_{|m|<N_0} |m|^{2s} |\widehat{S(t)\phi_1}(m)|^2\Big)^{1/2} \overset{(5.2)}{=} \\
&\Big(\sum_{|m|<N_0} |m|^{2s} \Big|\sum_n \widehat{\phi}(n)\widehat{S(t)\xi_n}(m)|^2\Big)^{1/2} \overset{(4.38),(5.1)}{<} \\
&\Big(\sum_{|m|<N_0} |m|^{2s} \Big|\sum_n \widehat{\phi}(n)\widehat{f_n'(t)}(m)\Big|^2\Big)^{1/2} + O(N_0^s T^{A/2-\alpha} \|\phi\|_2).
\end{aligned} \tag{5.5}$$

Recalling (4.31), (4.35), we have

$$\widehat{f_n'(t)}(m) = 0 \quad \text{if } |m-n| > M_3 = T^{2\varepsilon}. \tag{5.6}$$

Thus, by (4.35), (5.4),

$$\phi_0 = \sum_{|n|<4M_3} \widehat{\phi}(n)e^{inx} = 0. \tag{5.7}$$

Define for $\ell \geq 1$

$$\phi_\ell = \sum_{2^{\ell-1}M_3\leq|n|<2^{\ell+2}M_3} \widehat{\phi}(n)e^{inx}. \tag{5.8}$$

Estimate, taking (5.6) and (5.7) into account,

$$(5.5) < (2M_3)^s\Big(\sum_{|m|<2M_3}\Big|\sum_n \widehat{\phi}_0(n)\widehat{f_n'(t)}(m)\Big|^2\Big)^{1/2} \tag{5.9}$$

$$+\sum_{\ell\geq1}(2^{\ell+1}M_3)^s\Big(\sum_{2^\ell M_3\leq|m|<2^{\ell+1}M_3}\Big|\sum_n \widehat{\phi}_\ell(n)\widehat{f_n'(t)}(m)\Big|^2\Big)^{1/2} \tag{5.10}$$

$$+\,O(N_0^sT^{-\alpha}\|\phi\|_2) \tag{5.11}$$

$$\leq$$

$$\sum_{\ell\geq1}(2^{\ell+1}M_3)^s\Big\|\sum_n \widehat{\phi}_\ell(n)f_n'(t)\Big\|_2 \tag{5.12}$$

$$+\,O(N_0^sT^{-\alpha}\|\phi\|_2) \tag{5.13}$$

$$\overset{(4.38)}{<}$$

$$\sum_{\ell\geq1}(2^{\ell+1}M_3)^s\Big\|\sum_n \widehat{\phi}_\ell(n)S(t)\xi_n\Big\|_2 \tag{5.14}$$

$$+\,O(N_0^sT^{-\alpha}\|\phi\|_2) \tag{5.15}$$

$$<$$

$$\sum_{\ell\geq1}(2^{\ell+1}M_3)^s\Big\|\sum_n \widehat{\phi}_\ell(n)\xi_n\Big\|_2 \tag{5.16}$$

$$+\,O(N_o^sT^{-\alpha}\|\phi\|_2). \tag{5.17}$$

Taking in (5.3) $s=0$, it follows in particular that

$$\Big\|\sum_n \widehat{\phi}(n)\xi_n\Big\|_2 \leq 2\|\phi\|_2. \tag{5.18}$$

Thus

$$(5.16) < \sum_{\ell\geq1}(2^{\ell+1}M_3)^s\,\|\phi_\ell\|_2$$

$$\overset{(5.8)}{<} 4^s\sum_{\ell\geq1}\|\phi_\ell\|_{H^s}. \tag{5.19}$$

Since the range of $\ell < A \log T$, by (5.1), it follows from the preceding that

$$(5.5) < (C_{A,s} \log T)\|\phi\|_{H^s} + 0(N_0^s T^{-\alpha}\|\phi\|_2) \tag{5.20}$$

$$\overset{(5.3)}{<} (C_{A,s} \log T + N_0^s T^{-\alpha})\|\phi_1\|_{H^s}. \tag{5.21}$$

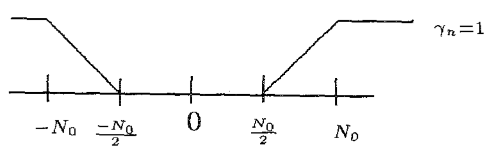
Next estimate

$$\left(\sum_{|m|>N_0} |m|^{2s} |\widehat{S(t)\phi_1}(m)|^2\right)^{1/2}. \tag{5.22}$$

Denote by $Q$ the (Fourier multiplier) operator

$$Q\psi = \sum \gamma_n \widehat{\psi}(n) e^{inx}, \tag{5.23}$$

where $\gamma_n$ is the multiplier

(5.24) 


Coming back to the equation

$$iu_t + \Delta u + V_1 u = 0 \tag{5.25}$$

and denoting

$$I_s(t) = \|(Qu)(t)\|_{H^s}^2, \tag{5.26}$$

we get (assuming $s \geq 2$ even)

$$\begin{aligned}\dot{I}_s = \frac{d}{dt}\|(Qu)(t)\|_{H^s}^2 &= 2\,\mathrm{Re}\,\langle(-\Delta)^s Qu, Qu_t\rangle \\ &= -2\,\mathrm{Im}\,\langle(-\Delta)^s Qu, Q(V_1 u)\rangle \\ &= 2\,\mathrm{Im}\,\langle(-\Delta)^{s/2} Qu, (-\Delta)^{s/2} Q(V_1 u)\rangle \\ &\leq \|Qu(t)\|_{H^s}\, \|(-\Delta)^{s/2} Q(V_1 u) - V_1(-\Delta)^{s/2} Qu\|_2. \end{aligned} \tag{5.27}$$

From definition (5.24) of $\gamma$, one clearly has

$$\begin{aligned}\|(-\Delta)^{s/2} Q(V_1 u) - V_1(-\Delta)^{s/2} Qu\|_2 &< \|[Q, V_1]\|\, \|u(t)\|_{H^s} + \frac{C_s}{N_0}\|u(t)\|_{H^s} \\ &< \frac{C_s}{N_0}\|u(t)\|_{H^s}. \end{aligned} \tag{5.28}$$

Substitute (5.28) in (5.27); it follows that

$$|\dot{I}_s| \le \frac{C_s}{N_0} I_s^{1/2} \, \|u(t)\|_{H^s}. \tag{5.29}$$

With $u = S(t)\phi_1$, estimate (5.21) and the definition of $\gamma$ in (5.24) show that, for $0 \le t \le T$,

$$\|(u - Qu)(t)\|_{H^s} < [C \log T + N_0^s T^{-\alpha}] \|\phi_1\|_{H^s}. \tag{5.30}$$

Hence

$$\|u(t)\|_{H^s} \le I_s(t)^{1/2} + [C \log T + N_0^s T^{-\alpha}] \|\phi_1\|_{H^s}, \tag{5.31}$$

and from (5.29)

$$|\dot{I}_s| \le \frac{C_s}{N_0} [I_s + (T^\epsilon + N_0^{2s} T^{-\alpha}) \|\phi_1\|_{H^s}^2]. \tag{5.32}$$

Thus, for $0 \le t \le T$,

$$I_s(t) < \frac{C_s T}{N_0} (T^{6\varepsilon s} + N_0^{2s} T^{-\alpha}) \|\phi_1\|_{H^s}^2 \tag{5.33}$$

provided $N_0 > C_s T$. Take

$$N_0 = T^2. \tag{5.34}$$

From (5.33), we get in particular

$$\|Qu)(t)\|_{H^s} < \|\phi_1\|_{H^s} \quad \text{for } 0 \le t \le T, \tag{5.35}$$

which, combined with (5.30), yields

$$\|S(t)\phi_1\|_{H^s} \le C(\log T) \|\phi_1\|_{H^s} \quad \text{for } 0 \le t \le T. \tag{5.36}$$

Invoking the approximation procedure (5.2), (5.3), (5.4), one may also derive (using an iterative argument) that

$$\|S(t)\phi\|_{H^s} \le C(\log T) \|\phi\|_{H^s} \quad \text{for } 0 \le t \le T, \tag{5.37}$$

provided $\phi$ satisfies (5.1).

## 6 $H^s$-bounds (2)

In this section, we estimate

$$\|S(t)\phi\|_{H^s} \quad (0 \le t \le T),$$

assuming that $\phi \in H^s$ satisfies

$$\widehat{\phi}(n) = 0 \quad \text{for } |n| < T^A. \tag{6.1}$$

The following bound holds for $\phi \in H^s$ without further assumptions.

**Lemma 6.2.**

$$\|S(t)\phi\|_{H^s} < C_s(1+|t|)^s \, \|\phi\|_{H^s}. \tag{6.2}$$

**Proof.** Assume that $s$ is an even integer and define

$$I_s(t) = \|S(t)\phi\|_{H^s}^2. \tag{6.3}$$

Then, again writing $u(t) = S(t)\phi$, we have from the equation

$$\begin{aligned} |\dot{I}_s| &= 2|\operatorname{Im} \langle(-\Delta)^s u, V_1 u\rangle| \\ &= 2|\operatorname{Im} \langle(-\Delta)^{s/2} u, (-\Delta)^{s/2} V_1 u\rangle| \\ &\le C_s \|u(t)\|_{H^s} \, \|u(t)\|_{H^{s-1}}. \end{aligned} \tag{6.4}$$

By interpolation, we have

$$\|u(t)\|_{H^{s-1}} \le \|u(t)\|_{H^s}^{(s-1)/s} \, \|u(t)\|_{H^0}^{1/s} = \|\phi\|_2^{1/s} \, \|u(t)\|_{H^s}^{1-1/s}. \tag{6.5}$$

Substituting (6.5) in (6.4), we get

$$|\dot{I}_s| < C_s \|\phi\|_2^{1/s} I_s^{1-1/2s}, \tag{6.6}$$

$$I_s(t)^{1/2s} \le \|\phi\|_{H^s}^{1/s} + C_s \|\phi\|_2^{1/s} t, \tag{6.7}$$

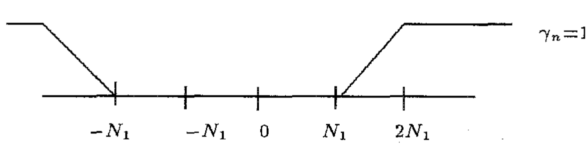
and hence (6.2).

For $N_1$ to be specified, let

$$Q\psi = \sum \gamma_n \widehat{\psi}(n) e^{inx} \tag{6.8}$$

with $\gamma = (\gamma_n)$ the Fourier multiplier (cf. (5.14))

(6.9)




Let

$$I_s(t) = \|(Qu)(t)\|_{H^s}^2. \tag{6.10}$$

From (5.29), (6.2), we get

$$|\dot{I}_s| < \frac{C_s}{N_1}\|\phi\|_{H^s}(1+|t|)^s I_s^{1/2}; \tag{6.11}$$

hence, for $0 \le t \le T$,

$$\|Qu(t)\|_{H^s} = I_s(t)^{1/2} \le \|\phi\|_{H^s} + C_s\|\phi\|_{H^s}\frac{T^{s+1}}{N_1}. \tag{6.12}$$

Choose

$$N_1 > C_s T^{s+1}; \tag{6.13}$$

it follows from (6.12) that, for $0 \le t \le T$,

$$\|QS(t)\phi\|_{H^s} \le 2\|\phi\|_{H^s}. \tag{6.14}$$

Finally, estimate

$$\|[Q, S(t)]\|_{H^s \to H^s}. \tag{6.15}$$

Since

$$(i\partial_t + \partial_x^2 + V_1)Qu = [V_1, Q]u, \tag{6.16}$$

we get from the integral equation

$$Qu(t) = S(t)[Qu(0)] + i\int_0^t S(t)S(\tau)^{-1}[V_1(\tau), Q]u(\tau)d\tau,$$

$$\|[Q, S(t)]\phi\|_{H^s} \le \int_0^t \|S(t)S(\tau)^{-1}[V_1(\tau), Q]u(\tau)\|_{H^s}d\tau. \tag{6.17}$$

Thus, by (6.2) and a commutator estimate,

$$\begin{aligned}(6.17) &< C_s\int_0^t (1+t)^s(1+\tau)^s\frac{1}{N_1}\|u(\tau)\|_{H^s}d\tau \\ &< C_s(1+t)^{3s+1}N_1^{-1}\|\phi\|_{H^s}.\end{aligned} \tag{6.18}$$

Thus

$$(6.15) < C_s\frac{(1+t)^{3s+1}}{N_1}. \tag{6.19}$$

Take

$$A = 10s \tag{6.20}$$

in (6.1) and

$$N_1 = T^{5s}. \tag{6.21}$$

Collecting estimates (6.14), (6.19), we obtain for $0 \le t \le T$

$$\|S(t)\phi\|_{H^s} = \|S(t)Q\phi\|_{H^s} \le 2\|\phi\|_{H^s} + \|[Q, S(t)]\phi\|_{H^s} \le 3\|\phi\|_{H^s} \tag{6.22}$$

provided

$$\phi \in H^s \quad \text{and} \quad \widehat{\phi}(n) = 0 \quad \text{for } |n| < T^{10s}. \tag{6.23}$$

## 7 $H^s$-bounds (3)

From (5.37), (6.22), it follows that, if

$$\phi \in H^s, \quad \widehat{\phi}(n) = 0 \quad \text{for } |n| < T^{2\delta}, \tag{7.1}$$

then, for $0 \le t \le T$,

$$\|S(t)\phi\|_{H^s} \le C(\log T)\|\phi\|_{H^s}. \tag{7.2}$$

We now conclude the proof.

Denote by $S(t_1, t_2)$ the flowmap from $t_1$ to $t_2$.

Let $\overline{N} = [T^{2\delta}]$ and

$$P_{\overline{N}}\psi = \sum_{|n| \in \overline{N}} \widehat{\psi}(n) e^{inx}, \tag{7.3}$$

the restriction operator.

Write, for $0 \le t \le T$,

$$\begin{aligned} \|S(0,t)\phi\|_{H^s} &\le \|S(0,t)(\phi - P_{\overline{N}}\phi)\|_{H^s} + \|S(0,t)(P_{\overline{N}}\phi)\|_{H^s} \\ &\overset{(7.2)}{\le} T^\varepsilon \|\phi\|_{H^s} + \|S(0,t)P_{\overline{N}}\phi\|_{H^s}. \end{aligned} \tag{7.4}$$

$$\|S(0,t)P_{\overline{N}}\phi\|_{H^s} \le \|S(1,t)(I - P_{\overline{N}})S(0,1)P_{\overline{N}}\phi\|_{H^s} \tag{7.5}$$

$$+ \|S(1,t)P_{\overline{N}}S(0,1)P_{\overline{N}}\phi\|_{H^s}. \tag{7.6}$$

Again from (7.2), (6.2),

$$(7.5) < T^\varepsilon \|S(0,1)P_{\overline{N}}\phi\|_{H^s} < C_s T^\varepsilon \overline{N}^s \|\phi\|_2. \tag{7.7}$$

Iterating, we get an expression ($r < t$)

$$\|S((r,t)P_{\overline{N}}S(r-1,r)P_{\overline{N}}S(r-2,r-1)\cdots P_{\overline{N}}S(0,1)P_{\overline{N}}\phi\|_{H^s} \lesssim \tag{7.8}$$

$$\|S(r+1,t)(I-P_{\overline{N}})S(r,r+1)P_{\overline{N}}S(r-1,r)\cdots P_{\overline{N}}\phi\|_{H^s} \tag{7.9}$$

$$+\|S(r+1,t)P_{\overline{N}}S(r,r+1)P_{\overline{N}}\cdots P_{\overline{N}}\phi\|_{H^s}; \tag{7.10}$$

and, by (7.2), (6.2) and $L^2$-conservation,

$$\begin{aligned}(7.9) &\leq T^\varepsilon\|S(r,r+1)P_{\overline{N}}S(r-1,r)\cdots P_{\overline{N}}\phi\|_{H^s}\\ &\leq C_sT^\varepsilon\overline{N}^s\|P_{\overline{N}}S(r-1,r)\cdots P_{\overline{N}}\phi\|_{H^0}\\ &\leq C_sT^\varepsilon\overline{N}^s\|\phi\|_2.\end{aligned} \tag{7.11}$$

Collecting previous estimates, we see that for $0 \leq t \leq T$

$$\begin{aligned}\|S(t)\phi\|_{H^s} &\leq T^\varepsilon\|\phi\|_{H^s} + C_sT^{1+\varepsilon}\overline{N}^s\|\phi\|_2\\ &\leq T^{1+\varepsilon+2\delta s}\|\phi\|_{H^s}\\ &\leq T^2\|\phi\|_{H^s},\end{aligned} \tag{7.12}$$

taking $\varepsilon = \varepsilon(s)$ sufficiently small ($\delta = 100\varepsilon$).

Interpolating with the $L^2$-conservation

$$\|S(t)\phi\|_2 = \|\phi\|_2, \tag{7.13}$$

one concludes that, for

$$0 < s_1 < s, \tag{7.14}$$

$$\|S(t)\phi\|_{H^{s_1}} \leq T^{2\frac{s_1}{s}}\|\phi\|_{H^{s_1}} \tag{7.15}$$

for $0 \leq t \leq T$ and $T$ sufficiently large (depending on $s$).

By (1.4), the same statement holds for the flowmap of the original equation (1.0). Consequently,

$$\|S(t)\phi\|_{H^{s_1}} \leq C_{s_1,\kappa}(1+|t|)^\kappa\|\phi\|_{H^{s_1}} \tag{7.16}$$

for all $s_1$ and $\kappa > 0$.

## 8 Dimension $d > 1$

The same result may be obtained in arbitrary dimension $d \geq 1$ using a similar approach. The main ingredient is the following well-known fact on separations in the set $\{n, |n|^2\}$, due to A. Granville and T. Spencer (cf. [B1]).

**Lemma 8.1.** *Fix*

$$0 < \rho < \tfrac{1}{10}. \tag{8.1}$$

*Then there is a partition of* $\mathbb{Z}^d$

$$\mathbb{Z}^d = \bigcup \Omega_\alpha \tag{8.2}$$

*such that the following properties hold with*

$$\Omega'_\alpha = \{(n, |n|^2) : n \in \Omega_\alpha\}. \tag{8.3}$$

*If* $n \in \Omega_\alpha$, *then*

$$\operatorname{diam} \Omega'_\alpha < |n|^\rho. \tag{8.4}$$

*If* $n_1 \in \Omega_\alpha$, $n_2 \in \Omega_\beta$ *and* $\alpha \neq \beta$, *then*

$$|n_1 - n_2| + \big| |n_1|^2 - |n_2|^2 \big| > |n_1|^{\rho_1} \tag{8.5}$$

*for some* $0 < \rho_1 = \rho_1(\rho, d) < \rho$.

Consider $f = f(x)$ on $\mathbb{T}^d$ such that

$$\operatorname{supp} \widehat{f} \subset B(0, 2N) \backslash B(0, N) \equiv \mathcal{D}; \quad N > \overline{N} \equiv T^{2\delta}. \tag{8.6}$$

Denote

$$\mathcal{B} = \bigcup_{\Omega_\alpha \cap \mathcal{D} \neq \phi} \{n \in \mathbb{Z}^d : \operatorname{dist}(\Omega_\alpha, n) < N^{1/2}\}, \tag{8.7}$$

and let $S(t)$ be the flowmap of

$$iu_t + u_{xx} + P_{\mathcal{B}}[uV_2] = 0, \tag{8.8}$$

where $V_2$ is defined as above, cf. (1.8),

$$V_2(x,t) = \sum_{\substack{|n| < M_1 \\ |k| < M_1 T}} \widehat{V}_1(n,k) e^{i(nx + \frac{\pi k}{2T})t}; \; M_1 \equiv T^\epsilon. \tag{8.9}$$

One may then write

$$f = \sum_\alpha \langle f, \xi_\alpha \rangle \xi_\alpha, \tag{8.10}$$

where $\{\xi_\alpha\}$ is the orthonormal eigenvector basis for $S(4T)$ acting on $\mathbb{C}^{|\mathcal{B}|}$; cf. (2.11), (2.12).

Thus

$$S(t)f = \sum_\alpha e^{i\tilde{E}_\alpha t}\psi'_\alpha(x,t) \tag{8.11}$$

with

$$|\tilde{E}_\alpha| < \frac{\pi}{4T}; \tag{8.12}$$

$$e^{i\tilde{E}_\alpha t}\psi'_\alpha \text{ satisfies (8.8);} \tag{8.13}$$

hence, from $L^2$-conservation,

$$\|\psi'_\alpha(t)\|_2 = |\langle f, \xi_\alpha\rangle| \quad \text{for all } t; \tag{8.14}$$

$$\psi'_\alpha \text{ is } 2\pi\text{-periodic in } x \text{ and } 4T\text{-periodic in } t. \tag{8.15}$$

Fix $\alpha$ and denote $\psi'_\alpha$ by $\psi$. Thus, cf. (3.1),

$$\psi(x,t) = \sum_{\substack{n\in\mathcal{B}\\ k\in\mathbb{Z}}} \widehat{\psi}(n,k)e^{i(nx+\frac{\pi k}{2T}t)} \tag{8.16}$$

satisfies, by (8.8),

$$\left(|n|^2 + \frac{\pi}{2T}k + \tilde{E}_\alpha\right)\widehat{\psi}(n,k) - \widehat{V_2\psi}(n,k) = 0. \tag{8.17}$$

Define

$$\tilde{\Omega}_\alpha = \left\{(n,k) \in \mathbb{Z}^{d+1} : n \in \Omega_\alpha \text{and } |n^2 + \frac{\pi}{2T}k| < \frac{|n|^{\rho_1}}{100}\right\}. \tag{8.18}$$

By (8.4), (8.5),

$$\operatorname{diam}\left\{\left(n, \frac{k}{T}\right) : (n,k) \in \tilde{\Omega}_\alpha\right\} < 2|n|^\rho; \tag{8.19}$$

and for $(n_1,k_1) \in \tilde{\Omega}_\alpha$, $(n_2,k_2) \in \tilde{\Omega}_\beta$, $\alpha \neq \beta$,

$$\Delta\big((n_1,k_1),(n_2.k_2)\big) \equiv |n_1 - n_2| + \left|\frac{k_1}{T} - \frac{k_2}{T}\right| > \frac{1}{2}|n_1|^{\rho_1}. \tag{8.20}$$

Denote

$$\mathcal{D}_1 = \bigcup_{\Omega_\alpha\cap\mathcal{D}\neq\phi} \tilde{\Omega}_\alpha \tag{8.21}$$

and

$$\mathcal{D}_2 = \{(n,k) : \Delta - \text{dist}\big((n,k), \mathcal{D}_1\big) < \tfrac{1}{20} N^{\rho_1}\} \tag{8.22}$$

so that

$$\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{B} \times \mathbb{Z}. \tag{8.23}$$

From (8.18), (8.20), (8.21), it follows that if $(n,k) \notin \mathcal{D}_1$ and

$$\Delta - \text{dist}\big((n,k), \mathcal{D}_1\big) < \tfrac{1}{4} N^{\rho_1}, \tag{8.24}$$

then

$$\left| |n|^2 + \frac{\pi}{2T} k \right| > \frac{1}{200} N^{\rho_1} \tag{8.25}$$

and thus, cf. (8.12),

$$\left| |n|^2 + \frac{\pi}{2T} k + \tilde{E}_\alpha \right| > \frac{1}{200} N^{\rho_1}. \tag{8.26}$$

From (8.17), (8.26), one deduces easily that

$$\begin{aligned}
\sum_{10^{-2} N^{\rho_1} < \Delta - \text{dist}((n,k), \mathcal{D}_1) < 10^{-1} N^{\rho_1}} |\widehat{\psi}(n,k)|^2 &< \left(\frac{C}{N^{\rho_1}}\right)^{\frac{10^{-3} N^{\rho_1}}{M_1}} \left(\sum_{n,k} |\widehat{\psi}(n,k)|^2\right)^{1/2} \\
&< e^{-10 \frac{N^{\rho_1}}{M_1}} \|\psi(0)\|_2. \qquad (8.27)
\end{aligned}$$

Define

$$\Psi(x,t) = \sum_{(n,k) \in \mathcal{D}_2} \widehat{\psi}(n,k) e^{i(nx + \frac{\pi k}{2T} t)}. \tag{8.28}$$

Taking (8.22), (8.27) into account, we see that each $\Psi = \Psi_\alpha$ satisfies

$$i\Psi_t + \Psi_{xx} - \tilde{E}_\alpha \Psi + P_{\mathcal{B}}[\Psi V_2] = O(e^{-10 \frac{N^{\rho_1}}{M_1}} \|\psi(0)\|_2). \tag{8.29}$$

Thus, returning to (8.11) and denoting

$$\Phi(x,t) = \sum_\alpha e^{i\tilde{E}_\alpha t} \Psi_\alpha(x,t), \tag{8.30}$$

we obtain an approximate solution of (8.8), i.e.,

$$\begin{aligned}
i\Phi_t + \Phi_{xx} + \Phi V_2 &= i\Phi_t + \Phi_{xx} + P_{\mathcal{B}}[\Phi V_2] \\
&= O\left(e^{-10 \frac{N^{\rho_1}}{M_1}} \left(\sum_\alpha \|\psi_0'(0)\|_2\right)\right) \\
&\overset{(8.14)}{<} O(N^{C(d)} e^{-10 \frac{N^{\rho_1}}{M_1}} \|f\|_2) \\
&\leq O(e^{-9 \frac{N^{\rho_1}}{M_1}} \|f\|_2) \\
&\leq O(e^{-T^{\delta \rho_1}} \|f\|_2). \qquad (8.31)
\end{aligned}$$

Next, we need an estimate on $\|\Phi(t)\|_2$. Write for $0 \le t, t' \le 4T$

$$\Big| \|\Phi(t)\|_2^2 - \|\Phi(t')\|_2^2 \Big| \le \int_0^{4T} \left| \frac{d}{d\tau} \|\Phi(\tau)\|_2^2 \right| d\tau. \tag{8.32}$$

Hence

$$\|\Phi(t)\|_2^2 \le \frac{1}{4T} \int_0^{4T} \|\Phi(\tau)\|_2^2 d\tau \tag{8.33}$$

$$+ \int_0^{4T} \left| \frac{d}{d\tau} \|\Phi(\tau)\|_2^2 \right| d\tau. \tag{8.34}$$

Let $\varphi$ be a smooth bump function such that

$$0 \le \varphi \le 1, \tag{8.35}$$

$$\varphi = 1 \quad \text{on } [0,1], \tag{8.36}$$

$$\varphi = 0 \quad \text{outside } [-2,2], \tag{8.37}$$

$$|\widehat{\varphi}(\lambda)| < C e^{-\sqrt{\lambda}}. \tag{8.38}$$

By (8.11)

(8.39)

$$\frac{1}{4T} \int \|S(t) f\|_2^2 \varphi\Big(\frac{t}{4T}\Big) dt = \frac{1}{4T} \sum_n \int \Big| \sum_k \sum_\alpha \widehat{\psi'_\alpha}(n,k) e^{i(\frac{\pi k}{2T} + \tilde{E}_\alpha) t} \Big|^2 \varphi\Big(\frac{t}{4T}\Big) dt.$$

Write in (8.39)

$$\sum_k = \sum_{k \in \mathcal{D}_2(n)} + \sum_{k \notin \mathcal{D}_2(n)},$$

$$\Big| \sum_k \Big|^2 = \Big| \sum_{k \in \mathcal{D}_2(n)} \Big|^2 + \Big| \sum_{k \notin \mathcal{D}_2(n)} \Big|^2 + 2 \operatorname{Re} \Big( \sum_{k \in \mathcal{D}_2(n)} \Big) \Big( \overline{\sum_{k \notin \mathcal{D}_2(n)}} \Big). \tag{8.40}$$

To estimate the contribution of the last term in (8.40), denote

$$\mathcal{D}_2(n) \supset \mathcal{D}'_2(n) = \{k : \Delta - \operatorname{dist}\big((n,k), \mathcal{D}_1\big) \le 10^{-2} N^{\rho_1}\} \tag{8.41}$$

and write

$$\Big[ \sum_{k \in \mathcal{D}_2(n)} \Big] \Big[ \overline{\sum_{k \notin \mathcal{D}_2(n)}} \Big] =$$

$$\Big[ \sum_{k \in \mathcal{D}'_2(n)} \Big] \Big[ \overline{\sum_{k \notin \mathcal{D}_2(n)}} \Big] \tag{8.42}$$

$$+ \Big[ \sum_{k \in \mathcal{D}_2(n) \setminus \mathcal{D}'_2(n)} \Big] \Big[ \overline{\sum_{k \notin \mathcal{D}_2(n)}} \Big]. \tag{8.43}$$

Observe that by (8.22), (8.41), if $k \in \mathcal{D}_2'(n)$, $k' \notin \mathcal{D}_2(n)$, then

$$\frac{1}{T}|k-k'| > \left(\frac{1}{20} - \frac{1}{100}\right) N^{\rho_1} > \frac{1}{30} N^{\rho_1}. \tag{8.44}$$

Thus, from (8.38), the following bound on the contribution of (8.42) is obtained:

(8.45)
$$\frac{1}{4T}\int \Big[\sum_{k\in\mathcal{D}_2'(n)}\sum_{\alpha}\widehat{\psi_\alpha'}(n,k)e^{i(\frac{\pi k}{2T}+\tilde{E}_\alpha)t}\Big]\Big[\overline{\sum_{k'\notin\mathcal{D}_2(n)}\sum_{\alpha'}\widehat{\psi_{\alpha'}'}(n,k')e^{i(\frac{\pi k'}{2T}+\tilde{E}_{\alpha'})t}}\Big]\varphi\Big(\frac{t}{4T}\Big)$$

bounded by

$$\sum_{\substack{\alpha,\alpha' \\ |k-k'|>\frac{1}{30}TN^{\rho_1}}} |\widehat{\psi_\alpha'}(n,k)|\,|\widehat{\psi_{\alpha'}'}(n,k')| e^{-[|k-k'|+\frac{2T}{\pi}(|\tilde{E}_\alpha|+|\tilde{E}_{\alpha'}|)]^{1/2}} \tag{8.46}$$

$$\le Ce^{-\frac{1}{6}(TN^{\rho_1})^{1/2}}\sum_{\alpha,\alpha'}\Big(\sum_k|\widehat{\psi_\alpha'}(n,k)|^2\Big)^{1/2}\Big(\sum_k|\widehat{\psi_{\alpha'}'}(n,k)|^2\Big)^{1/2}$$

$$\le N^C e^{-\frac{1}{6}(TN^{\rho_1})^{1/2}}\Big[\sum_\alpha\sum_k|\widehat{\psi_\alpha'}(n,k)|^2\Big]. \tag{8.47}$$

Summing (8.47) over $n$ gives by (8.14) the following bound on the (8.42)-contribution:

$$N^C e^{-\frac{1}{6}(TN^{\rho_1})^{1/2}}\Big(\sum_\alpha\|\psi_\alpha'(0)\|_2^2\Big) < e^{-\frac{1}{7}(TN^{\rho_1})^{1/2}}\|f\|_2^2. \tag{8.48}$$

For (8.43), estimate by (8.37) and Cauchy-Schwarz

$$\frac{1}{4T}\sum_{\alpha,\alpha'}\int\Big|\sum_{k\in\mathcal{D}_2(n)\backslash\mathcal{D}_2'(n)}\widehat{\psi_\alpha'}(n,k)e^{i\frac{\pi k}{2T}t}\Big|\Big|\sum_{k\notin\mathcal{D}_2(n)}\widehat{\psi_{\alpha'}'}(n,k)e^{i\frac{\pi k}{2T}t}\Big|\varphi\Big(\frac{t}{4T}\Big) \tag{8.49}$$

$$\le C\sum_{\alpha,\alpha'}\Big(\sum_{k\in\mathcal{D}_2(n)\backslash\mathcal{D}_2'(n)}|\widehat{\psi_\alpha'}(n,k)|^2\Big)^{1/2}\Big(\sum_k|\widehat{\psi_{\alpha'}'}(n,k)|^2\Big)^{1/2}. \tag{8.50}$$

Summing (8.50) over $n$ and applying estimate (8.27) to the first factor (taking definitions (8.22), (8.41) into account) gives the following bound on the (8.43)-contribution:

$$C\sum_{\alpha,\alpha'}e^{-10\frac{N^{\rho_1}}{M_1}}\|\psi_\alpha'(0)\|_2\|\psi_{\alpha'}'(0)\|_2 < e^{-N^{\rho_1/2}}\|f\|_2^2. \tag{8.51}$$

From (8.48), (8.51), it follows that the contribution of (8.40) in (8.39) is at most

$$[e^{-\frac{1}{7}(TN^{\rho_1})^{1/2}} + e^{-N^{\rho_1/2}}]\,\|f\|_2^2 \le \|f\|_2^2. \tag{8.52}$$

From (8.36), (8.52) and definitions (8.28), (8.30), we may conclude that

$$(8.33) \le \frac{1}{4T}\int \varphi\Big(\frac{t}{4T}\Big)\sum_n \Big|\sum_{k\in\mathcal{D}_2(n)}\sum_\alpha \widehat{\psi'_\alpha}(n,k)e^{i(\frac{\pi k}{2T}+\tilde{E}_\alpha)t}\Big|^2$$
$$\le \frac{1}{4T}\int \varphi\Big(\frac{t}{4T}\Big)\|S(t)f\|_2^2 + \|f\|_2^2$$
$$\overset{(8.37)}{\le} 10\|f\|_2^2. \tag{8.53}$$

Next, consider (8.34). Writing

$$\frac{d}{dt}\|\Phi(t)\|_2^2 = 2\,\mathrm{Re}\,\langle\Phi(t),\dot{\Phi}(t)\rangle$$

and substituting (8.31) gives an estimate

$$C\|\Phi(t)\|_2\, e^{-T^{\delta\rho_1}}\,\|f\|_2. \tag{8.54}$$

Thus

$$(8.34) \le CTe^{-T^{\delta\rho_1}}\,\|f\|_2\Big(\sup_{0\le t\le 4T}\|\Phi(t)\|_2\Big)$$
$$\le \|f\|_2\Big(\sup_{0\le t\le 4T}\|\Phi(t)\|_2\Big). \tag{8.55}$$

Finally, from (8.33), (8.34), (8.53), (8.55),

$$\sup_{0\le t\le 4T}\|\Phi(t)\|_2^2 \le 10\|f\|_2^2 + \|f\|_2\Big(\sup_{0\le t\le 4T}\|\Phi(t)\|_2\Big); \tag{8.56}$$

hence

$$\sup_{0\le t\le 4T}\|\Phi(t)\|_2 \le 5\|f\|_2. \tag{8.57}$$

From the integral equation and $L^2$-conservation,

$$\|S(t)f - e^{it\Delta}f\|_2 = \Big\|\int_0^t e^{i(t-\tau)\Delta}[S(\tau)f.V_2(\tau)]d\tau\|_2$$
$$< \frac{1}{100}\|f\|_2 \tag{8.58}$$

for $|t| < c$, $c$ some constant.

Take $\varphi$ satisfying (4.19)–(4.21). Thus

$$\Big|\int\langle S(t)f,\, e^{it\Delta}f\rangle\varphi(-t)dt\Big| \ge \frac{1}{2}\|f\|_2^2. \tag{8.59}$$

Since

$$e^{it\Delta}f = \sum_{n\in\mathcal{D}} \widehat{f}(n)\, e^{i(nx-|n|^2t)}, \tag{8.60}$$

we have

$$\begin{aligned}
&\int \langle S(t)f,\, e^{it\Delta}f\rangle\varphi(-t)dt \overset{(8.11)}{=} \\
&\sum_\alpha \int e^{i\tilde{E}_\alpha t}\langle\psi'_\alpha(t), e^{it\Delta}f\rangle\varphi(-t)dt \overset{(8.16),(8.60)}{=} \\
&\sum_\alpha \sum_{\substack{n\in\mathcal{B}\\ k\in\mathbb{Z}}} \widehat{\psi'_\alpha}(n,k)\overline{\widehat{f}(n)}\widehat{\varphi}\Big(\frac{k}{4T}+\frac{|n|^2}{2\pi}+\frac{\tilde{E}_\alpha}{2\pi}\Big) \overset{(8.28),(8.18),(4.21)}{=} \\
&\sum_\alpha \sum_{n,k} \widehat{\Psi}_\alpha(n,k)\overline{\widehat{f}(n)}\widehat{\varphi}\Big(\frac{k}{4T}+\frac{|n|^2}{2\pi}+\frac{\tilde{E}_\alpha}{2\pi}\Big) + \\
&O\Big(\sum_\alpha \|\psi_\alpha(0)\|_2\,\|f\|_2\Big[\sum_{\substack{|n|\sim N\\ |n^2+\frac{\pi k}{2T}|>\frac{1}{200}N^{\rho_1}}} e^{-|\frac{k}{4T}+\frac{|n|^2}{2\pi}+\frac{\tilde{E}_\alpha}{2\pi}|^{1/2}}\Big]\Big) \\
&\overset{(8.30)}{=} \int\langle\Phi(t), e^{it\Delta}f\rangle\varphi(-t)dt + 0(Te^{-N^{\rho_1/3}})\|f\|_2^2.
\end{aligned} \tag{8.61}$$

From (8.57), (8.58), (8.59), (8.61),

$$\begin{aligned}
\Big|\int\langle S(t)^{-1}\Phi(t), f\rangle\varphi(-t)dt\Big| &> \Big(\frac{1}{2}-\frac{1}{20}+0(Te^{-\overline{N}^{\rho_1/3}})\Big)\|f\|_2^2 \\
&> \frac{1}{4}\|f\|_2^2.
\end{aligned} \tag{8.62}$$

Thus, for some $|t_0| < c$,

$$|\langle S(t_0)^{-1}\Phi(t_0), f\rangle| \geq \frac{1}{4}\|f\|_2^2. \tag{8.63}$$

From (8.31) and the integral equation, it follows that, for $0 \leq t \leq T$,

$$\begin{aligned}
\|\Phi(t) - S(t)\Phi(0)\|_2 &< O(Te^{-T^{\delta\rho_1}}\,\|f\|_2) \\
&< e^{-\frac{1}{2}T^{\delta\rho_1}}\,\|f\|_2.
\end{aligned} \tag{8.64}$$

In particular, letting $t = t_0$, we have

$$\|S(t_0)^{-1}\Phi(t_0) - \Phi(0)\|_2 < e^{-\frac{1}{2}T^{\delta\rho_1}}\,\|f\|_2; \tag{8.65}$$

and (8.63), (8.65) imply

$$|\langle\Phi(0), f\rangle| \geq \tfrac{1}{5}\|f\|_2^2. \tag{8.66}$$

Moreover, by (8.57),

$$\|\Phi(0)\|_2 \leq 5\|f\|_2; \tag{8.67}$$

and by (8.23), (8.28), (8.30),

$$\widehat{\Phi}(n) = 0 \quad \text{if } n \notin \mathcal{B}. \tag{8.68}$$

Observe that by (8.4), (8.7),

$$n \in \mathcal{B} \Rightarrow N - 2N^{1/2} < |n| < 2N + 2N^{1/2}. \tag{8.69}$$

Also, from (8.66), (8.67) and an appropriate choice of $\gamma$, $|\gamma| = \frac{1}{100}$, we get

$$\begin{aligned} \|f - \gamma\Phi(0)\|_2^2 &\leq \|f\|_2^2 + 25\gamma^2\|f\|_2^2 - 2\operatorname{Re}\overline{\gamma}\langle f, \Phi(0)\rangle \\ &< \left(1 - \frac{1}{1000}\right)\|f\|_2^2. \end{aligned} \tag{8.70}$$

A straightforward approximation procedure allows us to produce $F = F(x,t)$ satisfying (8.64),

$$\|f - F(0)\|_2 < e^{-N^{1/10}}\|f\|_2 \tag{8.71}$$

and

$$\widehat{F}(n) = 0 \quad \text{unless } \tfrac{1}{2}N < |n| < 4N. \tag{8.72}$$

Thus, by (8.64), (8.71),

$$\begin{aligned} \|F(t) - S(t)f\|_2 &\leq (e^{-\frac{1}{2}T^{\delta\rho_1}} + e^{-N^{1/10}})\|f\|_2 \\ &< (e^{-\frac{1}{3}T^{\delta\rho_1}})\|f\|_2. \end{aligned} \tag{8.73}$$

If we replace $V_2$ by $V_1$ and redefine $S(t)$ as the flowmap for the equation

$$iu_t + u_{xx} + Vu = 0,$$

it follows from (1.9) and the integral equation that also

$$\|F(t) - S(t)f\|_2 \leq (e^{-\frac{1}{3}T^{\delta\rho_1}} + T\|V_1 - V_2\|_\infty)\|f\|_2 < C_\alpha T^{-\alpha} \tag{8.74}$$

for $0 \leq t \leq T$ and all $\alpha > 0$.

Assume as in (5.1) that

$$\widehat{f}(n) = 0 \quad \text{unless } T^{2\delta} \leq |n| \leq T^A. \tag{8.75}$$

Take $N_0 = T^2$. Write

$$f_\ell = \sum_{|n|\sim 2^\ell} \widehat{f}(n) e^{inx} \; (T^{2\delta} < 2^\ell < T^A) \tag{8.76}$$

and denote $F_\ell = F_\ell(x,t)$ the approximate solution obtained above in (8.74), replacing $f$ by $f_\ell$ and $N$ by $2^\ell$.

Thus, for $0 \le t \le T$,

$$\begin{aligned}
&\Big(\sum_{|m|<N_0} |m|^{2s}\, |\widehat{S(t)f}(m)|^2\Big)^{1/2} \le \\
&\sum_\ell \Big(\sum_{|m|<N_0} |m|^{2s}\, |\widehat{S(t)f_\ell}(m)|^2\Big)^{1/2} \le \\
&\sum_\ell \Big(\sum_{|m|<N_0} |m|^{2s} |\widehat{F_\ell}(m)(t)|^2\Big)^{1/2} + N_0^s \sum_\ell \|S(t)f_\ell - F_\ell(t)\|_2 \\
&\overset{(8.72),(8.74)}{\le} \\
&C\sum_\ell 2^{\ell s}\|F_\ell(t)\|_2 + C_\alpha A(\log T) N_0^s T^{-\alpha}\|f\|_2 \le \\
&C\sum_\ell 2^{\ell s}\|S(t)f_\ell\|_2 + C_\alpha A T^{2s}(\log T) T^{-\alpha}\|f\|_2 \overset{(8.76)}{\le} \\
&CA(\log T)\|f\|_{H^s} + C_\alpha T^{-\alpha}\|f\|_2 \le
\end{aligned}$$

$$CA(\log T)\|f\|_{H^s}. \tag{8.77}$$

This gives inequality (5.21).

Since the remaining ingredients in Sections 5, 6 and 7 do not depend on the dimension, we may again conclude inequality (7.16).

## II. Construction of examples with random potential

In this section, we consider the linear Schrödinger equation (1D with periodic bc)

$$iu_t + u_{xx} + Vu = 0, \tag{0.1}$$

with $V$ a random potential of the form

$$V(x,t) = \sum_j [g_j(\omega)e^{ix} + \overline{g_j(\omega)}e^{-ix}]\gamma_j(t), \tag{0.2}$$

where the $\{g_j\}$ are independent normalized complex Gaussian random variables and the $\{\gamma_j\}$ are appropriately chosen disjointly supported bump functions satisfying ($s \geq 0$),

$$\sup_j \|\gamma_j\|_{H^s} < 1. \tag{0.3}$$

(As will be clear from the considerations below, there are many variants on this construction.) We show that for any initial data $\phi \in H^1$, $\phi \neq 0$,

$$\overline{\lim_{t\to\infty}} \frac{\|S(t)\phi\|_{H^1}}{t^{1/2(1+s)}} > 0, \quad \text{almost surely.} \tag{0.4}$$

The main point of the argument is a simple local analysis in time of the increment of $\|S(t)\phi\|^2_{H^1}$ and, more specifically, of the quadratic term in $V$ when considering the multilinear expansion of $\|S(t+T)\phi\|^2_{H^1} - \|S(t)\phi\|^2_{H^1}$ $(T = \Delta t < 1)$ as a power series in $V$.

§**1.** Denote by $S(t)$ the flowmap corresponding to (0.1) and write $u(x,t) = \big(S(t)\phi\big)(x)$. Then

$$\begin{aligned}\frac{d}{dt}\Big[\int |u_x|^2 dx\Big] &= 2\operatorname{Re}\int u_{xt}\,\overline{u}_x = -2\operatorname{Re}\int u_t\,\overline{u}_{xx} = -2\operatorname{Im}\int(-u_{xx} - Vu)\,\overline{u}_{xx}\\ &= 2\operatorname{Im}\int Vu\,\overline{u}_{xx} = -2\operatorname{Im}\int V_x u\,\overline{u}_x = 2\operatorname{Im}\int V_x\,\overline{u}u_x. \end{aligned}\tag{1}$$

§**2.** From the integral equation,

$$u = e^{it\Delta}\phi + i\int_0^t e^{i(t-\tau)\Delta}\big(Vu(\tau)\big)d\tau. \tag{2}$$

Set

$$u_0 = e^{it\Delta}\phi, \tag{3}$$

$$\delta u_0 = i\int_0^t e^{i(t-\tau)\Delta}\big(Vu_0(\tau)\big). \tag{4}$$

Thus

$$(1) = 2\operatorname{Im}\int V_x\,\overline{u}_0(u_0)_x + \tag{5}$$

$$2\operatorname{Im}\int V_x\,\overline{u}_0(\delta u_0)_x + \tag{6}$$

$$2\operatorname{Im}\int V_x\,\overline{\delta u_0}(u_0)_x + \tag{7}$$

$$O(\|V\|^3). \tag{8}$$

Write

$$(6) = -2\,\mathrm{Im}\int (V_{xx}\,\overline{u}_0 + V_x(\overline{u}_0)_x)\delta u_0,$$

$$\begin{aligned}(6)+(7) &= 2\,\mathrm{Im}\int V_x[\overline{\delta u}_0(u_0)_x - \delta u_0(\overline{u_0})_x] - 2\,\mathrm{Im}\int V_{xx}\,\overline{u}_0(\delta u_0)\\ &= 4\,\mathrm{Im}\int V_x(u_0)_x\,\overline{\delta u_0} - 2\,\mathrm{Im}\int V_{xx}\,\overline{u}_0(\delta u_0)\\ &= -4\,\mathrm{Re}\,\langle V_x e^{it\Delta}\phi_x, \int_0^t e^{i(t-\tau)\Delta}\,(Ve^{i\tau\Delta}\phi)d\tau\rangle\\ &\quad - 2\,\mathrm{Re}\,\langle V_{xx}\int_0^t e^{i(t-\tau)\Delta}(Ve^{i\tau\Delta}\phi)d\tau, e^{it\Delta}\phi\rangle\end{aligned}$$

$$= -4\,\mathrm{Re}\int_0^t \langle e^{-it\Delta}V_x e^{it\Delta}\phi_x, e^{-i\tau\Delta}\,Ve^{i\tau\Delta}\phi\rangle d\tau \tag{9}$$

$$- 2\,\mathrm{Re}\int_0^t \langle e^{-i\tau\Delta}\,Ve^{i\tau\Delta}\phi, e^{-it\Delta}\,V_{xx}e^{it\Delta}\phi\rangle d\tau. \tag{10}$$

§**3.** Take

$$\phi(x) = \sum_k \widehat{\phi}(k)e^{ikx}, \quad \phi \in H^1 \quad \text{and} \quad \|\phi\|_2 = 1,$$

$$V(x,t) = \Big[\sum_{\substack{n\neq 0\\ n \text{ odd}}} \widehat{V}(n)\,e^{inx}\Big]\gamma(t) \text{ where } \widehat{V}(n) = \overline{\widehat{V}(-n)} \text{ are independent Gaussians.}$$

Average over the Gaussians. We get

$$(9) = 4\,\mathrm{Re}\int_0^t \Big[\sum_n n\,\mathbb{E}[|\widehat{V}(n)|^2]\Big(\sum_k k|\widehat{\phi}(k)|^2\,e^{i((k+n)^2-k^2)(t-\tau)}\Big)\Big]\gamma(\tau)\gamma(t)d\tau$$

$$= 4\sum_{n,k} nk\,\mathbb{E}[|\widehat{V}(n)|^2]|\widehat{\phi}(k)|^2\int_0^t \cos(2kn+n^2)(t-\tau)\gamma(\tau)\gamma(t)d\tau, \tag{11}$$

$$(10) = 2\,\mathrm{Re}\int_0^t \Big[\sum_n n^2\,\mathbb{E}[|\widehat{V}(n)|^2]\Big(\sum_k |\widehat{\phi}(k)|^2\,e^{i((k+n)^2-k^2)(\tau-t)}\Big)\Big]\gamma(\tau)\gamma(t)d\tau$$

$$= 2\sum_{n,k} n^2\,\mathbb{E}[|\widehat{V}(n)|^2]\,|\widehat{\phi}(k)|^2\int_0^t \cos(2kn+n^2)(t-\tau)\gamma(\tau)\gamma(t)d\tau. \tag{12}$$

Thus

$$(9)+(10) = (11)+(12)$$

$$= 2\sum_{n,k}(2nk+n^2)\,\mathbb{E}[|\widehat{V}(n)|^2]|\widehat{\phi}(k)|^2\int_0^t \cos(2kn+n^2)(t-\tau)\gamma(\tau)\gamma(t)d\tau. \tag{13}$$

§4. The contribution to $\|u_x(T)\|_2^2 - \|u_x(0)_2^2$ is obtained by integration of (13) in $t$. If $\gamma(t) = 0$ for $t \notin [0, T]$, this clearly yields

$$\sum_{n,k} (2kn + n^2)\, \mathbb{E}[|\widehat{V}(n)|^2]\, |\widehat{\phi}(k)|^2 |\widehat{\gamma}(2kn + n^2)|^2. \tag{14}$$

Take

$$\begin{cases} \mathbb{E}|\widehat{V}(1)| = 1 = \mathbb{E}|\widehat{V}(-1)|, \\ \widehat{V}(n) = 0 \quad \text{ for } |n| \neq 1. \end{cases} \tag{15}$$

Fix $M$ and take $\gamma$ satisfying

$$\text{supp } \gamma \subset \left[0, \frac{1}{M}\right], \tag{16}$$

$$|\widehat{\gamma}(m)| = |\widehat{\gamma}(-m)| = M^{-s-\frac{1}{2}} \quad \text{ for } |m| < 2M + 1, \tag{17}$$

$$= 0 \text{ for } |m| > 3M; \tag{18}$$

$$\big|\partial_m(|\widehat{\gamma}(m)|)\big| \lesssim M^{-s-\frac{3}{2}}. \tag{19}$$

From the definition of $V$ and $\gamma$, it follows that $V(x, t)$ is analytic in $x$ and $H^s$ in $t \in [0, T]$. From (14) and (15)–(19), we get, for $T = 1/M$,

$$\begin{aligned} (14) &\sim \sum_k |\widehat{\phi}(k)|^2[(2k+1)|\widehat{\gamma}(2k+1)|^2 - (2k-1)|\widehat{\gamma}(2k-1)|^2] \\ &= 2\Big(\sum_{|k|<M} |\widehat{\phi}(k)|^2\Big) M^{-2s-1} + O\Big(\sum_{|k|\geq M} |\widehat{\phi}(k)|^2 . M^{-2s-1}\Big) \\ &\sim M^{-2s-1} + O(M^{-2s-3}\|\phi\|_{H^1}^2). \end{aligned} \tag{20}$$

Accordingly, take $M \gtrsim \|\phi\|_{H^1}$. This gives

$$(14) \sim M^{-2s-1}. \tag{21}$$

The contribution of the higher orders in $V$ (hence order $\geq 4$) coming from (8) may easily be estimated by

$$\|\phi\|_2\, \|\phi\|_{H^1} \Big(\int \gamma\Big)^4 \lesssim \|\phi\|_{H^1} \Big(M^{-s-\frac{1}{2}}\Big)^4 < M^{-4s-1}. \tag{22}$$

Assume $s > 0$. From (14), (21), (22), it follows that one may ensure

$$\|u_x(T)\|_2^2 - \|u_x(0)\|_2^2 > cM^{-2s-1} \sim T^{2s+1}. \tag{23}$$

Now (23) corresponds to the inequality

$$\dot{I} > cI^{-s}, \tag{24}$$

implying, for $t \to \infty$,

$$I(t) \gtrsim t^{1/(1+s)}, \ \|u(t)\|_{H^1} \gtrsim t^{1/2(1+s)}. \tag{25}$$

Thus, coming back to (0.2), define

$$t_{j+1} = t_j + \Delta t_j, \tag{26}$$

$$\Delta t_j \equiv t_j^{-1/2(1+s)}, \tag{27}$$

and let $\gamma_j = \gamma$ be as above (taking $T = 1/M \sim \Delta t_j$), supp $\gamma_j \subset [t_j, t_{j+1}]$.

The preceding then allows us easily to conclude (0.4).

## REFERENCES

[B1] J. Bourgain, *Growth of Sobolev norms in linear Schrödinger equations with quasi-periodic potential*, Comm. Math. Phys., to appear.

[S] T. Spencer, private communications.

*J. Bourgain*
SCHOOL OF MATHEMATICS
INSTITUTE FOR ADVANCED STUDY
PRINCETON, NJ 08540, USA
email: bourgain@math.ias.edu

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