QUADRATIC DIVISORS OF HARMONIC POLYNOMIALS IN $\mathbb{R}^n$

By

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Abstract. Necessary and sufficient conditions are given for a quadratic polynomial to be a divisor of a nonzero harmonic polynomial in $\mathbb{R}^n$.

1 Formulation of the problem and main results

Let Q be a polynomial in $\mathbb{R}^n$ with real coefficients. We are concerned with the question: When does Q divide a nonzero harmonic polynomial?

We call such a polynomial Q *a harmonic divisor*. For Q homogeneous, being a divisor of a nonzero harmonic polynomial is equivalent to being a divisor of a nonzero harmonic function in $\mathbb{R}^n$.

The problem of characterizing harmonic divisors arises in the study of the Cauchy problem in the category of formal power series (see the question in P. Ebenfelt, H. Shapiro [ES], 6.1, and question 3.24 in the problem list [BBH]), investigation of stationary sets for the wave and heat equations ([AQ], [A]), injectivity of the spherical Radon transform [AVZ] and other questions. One should also mention the paper [FNS], which contains many interesting observations about zeros of entire harmonic functions and, in particular, polynomials, in $\mathbb{R}^2$. For homogeneous Q, this problem can be viewed as describing nodal sets for the Laplace–Beltrami operator on the unit sphere in $\mathbb{R}^n$.

In the general setting, the problem of describing harmonic divisors seems very difficult, except for the case of homogeneous polynomials Q in $\mathbb{R}^2$. Armitage [Ar] obtained the characterization of quadratic forms of n real variables with only two distinct eigenvalues which divide a nonzero harmonic polynomial in $\mathbb{R}^n$ in terms of zeros of Gegenbauer polynomials. Note that the case $n = 3$ was studied long ago by F. Klein in [K]; see the remark in [Ho], p. 481.

In this article, we give a complete characterization of all quadratic harmonic divisors in $\mathbb{R}^n$, for arbitrary n. The characterization is given in terms of polynomial

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solvability of the initial value problem for a Fuchsian ordinary differential equation and, in equivalent terms, of the solvability of a system of algebraic equations (Niven equations; cf. [KM]).

Geometrically, we describe all quadratic cones in $\mathbb{R}^n$ on which a nonzero harmonic function in $\mathbb{R}^n$ can vanish. The obtained characterization includes the result of [Ar] as a particular case. Our approach exploits separation of variables for the Laplace operator in n-dimensional ellipsoidal coordinates.

Let us formulate the main result. Let Q be a polynomial in $\mathbb{R}^n$ of degree 2,

$$Q(x) = \sum_{i,j=1}^{n} a_{ij}x_ix_j + \sum_{j=1}^{n} b_jx_j + c, \quad a_{ij} = a_{j,i}. \tag{1.1}$$

Denote by $\lambda_1, \dots, \lambda_\ell$, $\ell \geq 2$, the eigenvalues of the symmetric matrix $A = (a_{ij})_{i,j=1}^n$ and by m_i the multiplicity of λ_i. We assume that all λ_i are different from 0, since otherwise the problem reduces to a smaller number of variables (Proposition 2.1), and set $a_i = 1/\lambda_i$, $i = 1, \dots, \ell$.

In the sequel we use notation $\deg f$ for the degree of polynomial f.

Theorem 1.1. *The quadratic polynomial Q in* (1.1) *is a harmonic divisor in $\mathbb{R}^n$ if and only if there exist nonnegative integers $k_i, i = 1, \dots, \ell$ and a polynomial $\psi(s)$, $\deg \psi \leq \ell - 2$, such that the Fuchsian ordinary differential equation*

$$E'' + \frac{1}{2}\left(\sum_{i=1}^{\ell} \frac{m_i}{s - a_i}\right) E' + \frac{1}{\prod_{i=1}^{\ell}(s - a_i)} \left[\sum_{i=1}^{\ell} \mu_i \frac{\prod_{j\neq i}(a_i - a_j)}{4(s - a_i)} + \psi(s)\right] E = 0 \tag{1.2}$$

has a nonzero solution of the form $E(s) = \rho^\varepsilon(s)w(s)$, $\rho^\varepsilon(s) = \prod_{j=1}^{\ell}(s - a_j)^{\varepsilon_j/2}$, where w is a polynomial satisfying $w(0) = 0$.

Here $\mu_i = -k_i(k_i + m_i - 2)$; and $\varepsilon_i = 0$ if k_i is even, $\varepsilon_i = 1$ if k_i is odd.

In this case, Q divides a nonzero harmonic polynomial of degree $2\deg w + |\varepsilon|$, $|\varepsilon| = \sum_{i=1}^{\ell} \varepsilon_i$.

We refer to [AAR] for general information about Fuchsian differential equations.

If all the eigenvalues λ_i are simple, i.e., $m_i = 1$, then (1.2) simplifies to the Heun equation (cf. [E], p. 62)

$$E'' + \frac{1}{2}\left(\sum_{i=1}^{n} \frac{1}{s - a_i}\right) E' + \frac{1}{\prod_{i=1}^{n}(s - a_i)}\psi E = 0, \quad E = \rho^\varepsilon w,$$

$\deg \psi \leq n - 2$, whose polynomial solutions w are known as Lamé polynomials of order n (for $n = 3$, they coincide with the classical Lamé polynomials; cf. [E]).

An equivalent characterization of quadratic divisors can be given in algebraic terms. For the case of simple eigenvalues we have

Theorem 1.2. *Suppose that the matrix A in* (1.1) *has simple eigenvalues (different from zero) and $a_i = 1/\lambda_i$, $i = 1, \dots, n$. Then Q is a harmonic divisor if and only if for some $N \in \mathbb{N}$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_j = 0$ or $\varepsilon_j = 1$, the system of N equations (Niven equations, cf. [Sz],6.81; [KM], [V])*

$$\sum_{j=1}^{n} \frac{2\varepsilon_j + 1}{z_s - a_j} + \sum_{q=1, q\neq s}^{N} \frac{4}{z_s - z_q} = 0, \quad s = 1, \dots, N, \tag{1.3}$$

has a solution $(z_1, \dots, z_N) \in \mathbb{R}^N$ with $z_s = 0$ for some s.

In this case, Q divides a harmonic polynomial of degree $2N + |\varepsilon|$.

2 Preliminary facts

For two polynomials Q, h the notation $Q|h$ will mean Q divides h.

We require the following observation of H. Shapiro, which we give here in the (weaker) form we need.

Theorem ([Sh]). *Let Q be a quadratic polynomial in $\mathbb{R}^n$. Then $Q|h$ for some harmonic polynomial h, $\deg h = m$, if and only if $Q_2|h_m$, where Q_2 and h_m are the leading homogeneous terms of Q and h respectively.*

(Of course, the nontrivial part is the "if" statement.)

Therefore, it suffices to study only homogeneous Q, i.e., quadratic forms. Since the Laplace operator commutes with each linear transformation in the orthogonal group $\mathcal{O}(n)$, we can assume Q to be diagonalized:

$$Q(x) = \sum_{i=1}^{n} \lambda_i x_i^2. \tag{2.1}$$

The eigenvalues λ_i can be assumed nonzero, as follows from

Proposition 2.1. *If one of the λ_i in* (2.1) *vanishes, then Q is a harmonic divisor in $\mathbb{R}^n$ if and only if Q is a harmonic divisor in $\mathbb{R}^{n-1}$.*

Proof. Suppose $\lambda_n = 0$. Then Q can be regarded as a quadratic form in $\mathbb{R}^{n-1}$. If $Q|h$, $h = Qg$, where g and h do not depend on x_n and h is harmonic, then clearly h is harmonic in $x_1, \dots, x_n$ and therefore Q is a harmonic divisor in $\mathbb{R}^n$.

Conversely, suppose $Q|h$, $Qg = h$, $\Delta_{\mathbb{R}^n} h = 0$. Represent g and h as

$$g = \sum_{k=0}^{\ell} g_k x_n^k, \quad h = \sum_{k=0}^{m} h_k x_n^k,$$

where g_k, h_k are polynomials in $x_1, \dots, x_{n-1}$. Then we have for the leading terms $Qg_\ell = h_m$. The equation $\Delta_{\mathbb{R}^n} h = 0$ implies $\Delta_{\mathbb{R}^{n-1}} h_k = (k+2)(k+1)h_{k+2}$, and therefore h_m is a harmonic polynomial in $\mathbb{R}^{n-1}$. □

From now on, Q will be assumed to be of the form (2.1) with $\lambda_i \neq 0$ for all i.

We will call Q an *m-divisor* if Q divides some nonzero harmonic homogeneous polynomial of degree m. Clearly, Q is a harmonic divisor if and only if Q is a harmonic m-divisor for some m.

Proposition 2.2. *The set of all points $(\lambda_1, \dots, \lambda_n)$, corresponding to harmonic m-divisors* (2.1), *constitutes an algebraic conic hypersurface in $\mathbb{R}^n$ of degree* $\binom{n+m-3}{m-2}$.

Proof. Let P_m be the space of all homogeneous polynomials in $\mathbb{R}^n$ of degree m. The equation

$$\Delta(Qf) = 0, \quad f \in P_{m-2},$$

can be regarded as a linear $d_{m-2} \times d_{m-2}$ system for the coefficients of the polynomial f. Here

$$d_m = \dim P_m = \binom{n+m-1}{m}.$$

Denote by $B_m(\lambda)$, $\lambda = (\lambda_1, \dots, \lambda_n)$, the matrix of this linear system and set $b_m(\lambda) = \det B_m(\lambda)$. The entries of the matrix $B_m(\lambda)$ are linear forms in $\lambda_1, \dots, \lambda_n$; therefore, the determinant $b_m(\lambda)$ is a homogeneous (symmetric) polynomial of degree $\deg b_m = d_{m-2}$.

It remains to note that harmonic m-divisors are described by the equation $b_m(\lambda) = 0$. □

Remark. For small m and n, the determinants b_m can be written down explicitly. For instance,

$$b_2(\lambda_1, \dots, \lambda_n) = \lambda_1 + \dots + \lambda_n, \quad b_3(\lambda_1, \dots, \lambda_n) = \prod_{j=1}^{n} (2\lambda_j + \lambda_1 + \lambda_2 + \dots + \lambda_n).$$

In dimension $n = 3$, we have

$$b_4(\lambda_1, \lambda_2, \lambda_3) = (3\sigma_1 - 2\lambda_1)(3\sigma_1 - 2\lambda_2)(3\sigma_1 - 2\lambda_3)(3\sigma_1^3 + 12\sigma_1\sigma_2 + 56\sigma_3),$$

where σ_i are the basic symmetric polynomials,

$$\sigma_1 = \lambda_1 + \lambda_2 + \lambda_3, \quad \sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \quad \sigma_3 = \lambda_1\lambda_2\lambda_3.$$

Using the Mathematica package, we have computed, for $n = 3$, some further determinants b_m, which turn out to have rather cumbersome expressions.

Now we introduce the class of ε-odd polynomials. Given a vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_j = 0$ or $\varepsilon_j = 1$, denote by $P_{m,\varepsilon}$ the space of all polynomials in P_m of the form

$$x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n} P(x_1^2, \dots, x_n^2),$$

where P is a homogeneous polynomial.

Invariance of the Laplace operator and the quadratic form Q with respect to reflections $x_i \to -x_i$, $i = 1, \dots, n$, imply

Proposition 2.3. *If Q is a harmonic m-divisor, then Q divides a nonzero harmonic polynomial in $P_{m,\varepsilon}$ for some ε.*

3 Generic case (simple eigenvalues)

3.1 We start by characterizing quadratic divisors (2.1) with simple eigenvalues λ_i, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$. According to Proposition 1.1, all λ_i can be assumed nonzero, so we write $a_i = 1/\lambda_i$.

The numbers a_i cannot all be of the same sign, since by the Brelot–Choquet theorem [BC] no nonnegative or nonpositive polynomial in $\mathbb{R}^n$ can divide a nonzero harmonic polynomial. Therefore, after renumbering, we can assume

$$a_1 < a_2 < \cdots < a_{n_0} < 0 < a_{n_0+1} < \cdots < a_n.$$

Now introduce ellipsoidal coordinates, associated with the quadratic form Q, by

$$x_i^2 = \frac{\prod_{j=1}^{n} (t_j - a_i)}{\prod_{j \neq i} (a_i - a_j)}, \quad i = 1, \dots, n, \tag{3.1}$$

$$t_i \in (a_i, a_{i+1}), \quad a_{n+1} = \infty$$

(see, e.g., [E], [Ho], [Sz], [KM], [SW]).

Formula (3.1) establishes a diffeomorphism between

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0\}$$

and the domain $\prod_{i=1}^n (a_i, a_{i+1})$.

The ellipsoidal coordinates $t_1, \dots, t_n$ of the point $(x_1, \dots, x_n) \in \mathbb{R}^n$ are the n (real) roots $\theta = t_i$ of the equation

$$\sum_{i=1}^{n} \frac{x_i^2}{\theta - a_i} = 1.$$

The coordinate section $t_{n_0} = 0$ describes the hyperboloid $Q + 1 = 0$.

The Riemannian metric has in t-coordinates the form

$$ds^2 = -\frac{1}{4} \sum_{i=1}^{n} \frac{r'(t_i)}{q(t_i)} dt_i^2,$$

where

$$q(s) = \prod_{j=1}^{n} (s - a_j), \quad r(s) = \prod_{j=1}^{n} (s - t_j);$$

and the Laplace operator is

$$\Delta = \sum_{i=1}^{n} \frac{\sqrt{q(t_i)}}{r'(t_i)} \frac{\partial}{\partial t_i} \left(\sqrt{q(t_i)} \frac{\partial}{\partial t_i} \right). \tag{3.2}$$

3.2 Let u be an ε-odd polynomial, $u \in P_{m,\varepsilon}$.

Then u can be represented in the ellipsoidal coordinates $t = (t_1, \dots, t_n)$ in the form

$$u(t) = \prod_{i=1}^{n} \rho^{\varepsilon}(t_i) w(t), \tag{3.3}$$

where $\rho^{\varepsilon}(s) = \prod_{j=1}^{n} (s - a_j)^{\varepsilon_j/2}$ and w is a polynomial, $m = 2 \deg w + |\varepsilon|$.

Suppose that w in (3.3) can be written as

$$w(t) = \prod_{i=1}^{n} w_i(t_i). \tag{3.4}$$

Then $u(t) = \prod_{i=1}^{n} E_i(t_i)$, where $E_i(s) = \rho^{\varepsilon}(s) w_i(s)$; and, by (3.2), the equation $\Delta u = 0$ takes the form

$$\Delta u = \sum_{i=1}^{n} \frac{D_i E_i(t_i)}{r'(t_i)} \prod_{j \neq i} E_j(t_j) = 0,$$

with $D_i = \sqrt{q(t_i)}\, \partial_{t_i} \sqrt{q(t_i)}\, \partial_{t_i}$, or equivalently,

$$\sum_{i=1}^{n} \frac{1}{r'(t_i)} \frac{D_i E_i(t_i)}{E_i(t_i)} = 0. \tag{3.5}$$

Taking into account the explicit expression $r'(t_i) = \prod_{j\neq i}(t_i - t_j)$, we obtain $D_iE_i(t_i) = \psi(t_i)E_i(t_i)$, $i = 1,\dots,n$, where ψ is a fixed polynomial of degree $\leq n-2$.

Conversely, if $D_iE_i = \psi E_i$, $\deg\psi \leq n-2$, then the residue theorem yields

$$\sum_{i=1}^{n}\frac{\psi(t_i)}{r'(t_i)} = \sum_{i=1}^{n}\mathrm{res}_{t_i}\frac{\psi}{r} = -\mathrm{res}_{\infty}\frac{\psi}{r} = 0,$$

and (3.5) holds.

Thus, the equation $\Delta u = 0$ for u as in (3.3), (3.4) is equivalent to the n ordinary differential equations

$$D_i\rho^\varepsilon w_i = \psi\rho^\varepsilon w_i, \quad i = 1,\dots,n, \quad \deg\psi \leq n-2.$$

In fact, we deal with the same differential equation

$$\sqrt{q}\,\frac{\partial}{\partial s}\left(\sqrt{q}\,\frac{\partial E}{\partial s}\right) = \psi E, \quad E = \rho^\varepsilon w_i, \tag{3.6}$$

which can be reduced, by introducing the hyperelliptic variable

$$\xi = \int\frac{ds}{\sqrt{\prod_{j=1}^{n}(s-a_j)}}, \quad s = F(\xi),$$

to

$$\frac{\partial^2 E}{\partial\xi^2} = \psi(F(\xi))E.$$

Explicitly, equation (3.6) is the Fuchsian equation

$$E'' + \frac{1}{2}\left(\sum_{j=1}^{n}\frac{1}{s-a_j}\right)E' = \frac{\psi(s)}{\prod_{j=1}^{n}(s-a_j)}E, \quad E = \rho^\varepsilon w. \tag{3.7}$$

Note that after substituting $E = \rho^\varepsilon w$ in (3.7) and differentiation, all the radicals cancel and we obtain for w a differential equation with polynomial coefficients.

3.3 There is a finite set of spectral polynomials ψ for which (3.7) admits a nontrivial polynomial solution w, called *Lamé polynomials*. The admissibility conditions for w are obtained by solving recurrence relations for the coefficients w_k of w. For instance, for $\varepsilon = 0$ we have by substituting into (3.7)

$$(N(N-1+\frac{n}{2}) - \psi_{n-2})w_N = 0, \quad N = \deg w,$$

$$[(N-1)(N-2+\frac{n}{2}) - \psi_{n-2}]w_{N-1}$$
$$- [(a_1+\cdots+a_n)N(N-\frac{n+1}{2}) + \psi_{n-3}]w_N = 0,$$

and so on.

The equations for the coefficients $\psi_0, \dots, \psi_{n-2}$ of the polynomial ψ result from expressing all w_k's through the leading coefficient w_N (which can be assumed to equal 1) and then setting $w_{-1} = \dots = w_{-1-N} = 0$. Due to the recurrence, we then have $w_j = 0$ for all $j < 0$, so w is a polynomial of degree N. In particular, the leading coefficient is $\psi_{n-2} = N(N-1+n/2)$ (for $\varepsilon = 0$) and, more generally, $\psi_{n-2} = (N+|\varepsilon|)(N+|\varepsilon|-1+n/2)$.

The $n-1$ admissible parameters $\psi_0, \dots, \psi_{n-2}$ satisfy $n-1$ algebraic equations and constitute a finite set. These parameters become algebraic functions of the singular values $a_1, \dots, a_n$ (and therefore of the eigenvalues $\lambda_1, \dots, \lambda_n$).

Now we are ready to give necessary and sufficient conditions for the quadratic form Q to be a harmonic divisor. The following statement is a particular case of Theorem 1 for the case of simple eigenvalues ($m_i = 1$ in (1.2)).

Theorem 3.1. *The quadratic form* $Q(x) = \sum_{i=1}^n \lambda_i x_i^2$, $\lambda_i \neq 0$, $\lambda_i \neq \lambda_j$ *for* $i \neq j$, *is a harmonic divisor if and only if, for some polynomial* ψ, $\deg \psi \leq n-2$, *the Heun equation* (3.7) *with* $a_i = 1/\lambda_i$ *has a nonzero solution of the form*

$$E(s) = \prod_{j=1}^{n} (s - q_j)^{\varepsilon_j/2} w(s),$$

where w *is a polynomial with* $w(0) = 0$, *and* $\varepsilon_j = 0$ *or* $\varepsilon_j = 1$.

In this case, Q is a harmonic m-divisor with $m = 2\deg w + |\varepsilon|$.

Proof. By Proposition 2.3, it suffices to look for ε-odd harmonic polynomials $u \in P_{m,\varepsilon}$ divisible by Q. Every such polynomial can be decomposed in the ellipsoidal coordinates $t_1, \dots, t_n$ into a finite sum of separable solutions of the Laplace equation (cf. [KM], [V]):

$$u(t) = \sum_j E_1^{\psi^j}(t_1) \cdots E_n^{\psi^j}(t_n), \tag{3.8}$$

where $E_i^{\psi^j} = \rho^{\varepsilon} w_i^j$ and w_i^j is a polynomial corresponding to an admissible spectral polynomial ψ^j in (3.7).

Using the result of H. Shapiro [Sh] mentioned in Section 1, we can replace Q by $Q+1$. Divisibility of the real harmonic polynomial by $Q+1$ is equivalent to vanishing on the surface $Q+1=0$ which is given in t-coordinates by the equation $t_{n_0} = 0$.

Since the separable solutions in (3.8) are linearly independent, $u(t) = 0$ on $t_{n_0} = 0$ is equivalent to $E_{n_0}^{\psi^j}(0) = 0$ for all j. Take $E = E_{n_0}^{\psi^j} \neq 0$. Then E has the form $E = \rho^\varepsilon w$, where $w = w^j_{n_0}$ is a nonzero polynomial solution to (3.7) satisfying the initial condition $w(0) = 0$. □

Remark. The admissible parameters in (3.7) for polynomial solutions w, $\deg w = N$, are algebraic functions of the eigenvalues λ_i. Let $\psi^i(\lambda_1, \dots, \lambda_n, \varepsilon_1, \dots, \varepsilon_n, s)$ be a continuous branch of this algebraic function and $w^i(\lambda_1, \dots \lambda_n, \varepsilon_1, \dots, \varepsilon_n, s)$ the (unique) polynomial solution of (3.7), corresponding to the polynomial ψ^i and having leading coefficient 1.

By Theorem 3.1, the variety V_m, $m = 2N + |\varepsilon|$, of all harmonic m-divisors is determined by the condition

$$F_m(\lambda_1, \dots, \lambda_n) = \prod_i \prod_\varepsilon w^i(\lambda_1, \dots, \lambda_n, \varepsilon_1, \dots, \varepsilon_n, 0) = 0,$$

where the product is over all branches ψ^i and all 2^n vectors $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_j = 0$ or $\varepsilon_j = 1$.

The function F_m is constructed from transcendental functions, while V_m is a real algebraic variety by Proposition 2.2. It would be interesting to understand relations between the functions $F_m(\lambda)$ and $b_m(\lambda)$ defined in Section 2 having the same zero set $V_m \subset \mathbb{R}^n$.

4 Characteristic (Niven) equations

4.1 According to Section 3, the quadratic form $Q+1$ is a harmonic divisor if and only if Q divides a nonzero harmonic polynomial u separating in the ellipsoidal coordinates, $u(t) = \rho^\varepsilon u_1(t_1) \cdots u_n(t_n)$.

Each polynomial u_i has only real roots, as otherwise u_i has a divisor of the form $(t_i - z)(t_i - \overline{z})$, $\operatorname{Im} z \neq 0$, which would contradict the Brelot–Choquet theorem [BC] about the nonexistence of nonnegative harmonic divisors. Since the equation $t_i = z$ is equivalent to

$$\sum_{i=1}^n \frac{x_i^2}{z - a_i} - 1 = 0,$$

it follows that the polynomial u has in x-coordinates the form

$$u(x) = x^\varepsilon \prod_{s=1}^N \left(\sum_{i=1}^n \frac{x_i^2}{z_s - a_i} - 1 \right), \quad x^\varepsilon = x_1^{\varepsilon_1} \cdots x_n^{\varepsilon_n},$$

where z_s are roots of the polynomials $u_1, \dots, u_n$.

We have that $u = 0$ on the hyperboloid $Q + 1 = 0$ if and only if $z_s = 0$ for some s, for instance, $z_1 = 0$. Then the leading term of u divides Q.

Thus, we have

Corollary 4.1. *If Q is a harmonic divisor, then Q divides a nonzero harmonic polynomial u representable as a product of quadratic forms*

$$u(x) = x^{\varepsilon} \prod_{s=1}^{N} Q_s, \quad Q_s(x) = \sum_{i=1}^{n} \frac{x_i^2}{a_i - z_s}, \tag{4.1}$$

where $Q_1 = Q$.

Note that the quadratic factors Q_s are confocal with the original quadratic form Q. (Two quadratic forms $\sum_{i=1}^{n} \lambda_i x_i^2$ and $\sum_{i=1}^{n} \mu_i x_i^2$ are confocal if $1/\lambda_i - 1/\mu_i = 1/\lambda_j - 1/\mu_j$ for all $i, j = 1, \dots, n$.)

4.2 Proof of Theorem 2. The condition of harmonicity of the product of quadratic forms (4.1) can be rewritten as the (Niven) system of algebraic equations (cf. [Sz], 6.81, [KM], [V]):

$$\sum_{j=1}^{n} \frac{2\varepsilon_j + 1}{z_s - a_j} + \sum_{i \neq s} \frac{4}{z_s - z_i} = 0, \quad s = 1, \dots, N. \tag{4.2}$$

Divisibility of (4.1) by Q is equivalent to the condition $z_s = 0$ for some s. □

Theorem 2 can be reworded as follows: Q is a harmonic divisor if and only if the polynomial

$$\psi(z_1, \dots, z_p) = \prod_{i=1}^{n} \prod_{s=0}^{p} (z_s - a_i)^{2\varepsilon_i + 1} \prod_{i \neq j} (z_i - z_j)^4$$

has at least one critical point on a coordinate plane $z_s = 0$. Indeed, the system (4.1) is equivalent to $\nabla\psi = 0$ (cf. [Sz], 6.82).

5 General case (multiple eigenvalues)

In this section, we consider the general case of multiple eigenvalues and prove Theorem 1.

5.1 Suppose Q has multiple eigenvalues $\lambda_1, \dots, \lambda_\ell \neq 0$ of multiplicities $m_1, \dots, m_\ell$, $m_1 + \dots + m_\ell = n$, respectively:

$$Q(x_1, \dots, x_n) = \sum_{i=1}^{\ell} \lambda_i \left(\sum_{s=m_{i-1}+1}^{m_{i-1}+m_i} x_s^2 \right), \quad m_0 = 0.$$

Denote by $S^{m_i - 1}$ the unit sphere in the space $\mathbb{R}^{m_i}$ of the variables $x_{m_{i-1}+1}, \dots, x_{m_{i-1}+m_i}$. Decomposition into irreducible representations of the group

$\mathcal{O}(m_1) \times \cdots \times \mathcal{O}(m_\ell) \subset \mathcal{O}(n)$ yields that any polynomial h in $\mathbb{R}^n$ has a finite decomposition

$$h(x) = \sum_k F_k(r_1^2, \ldots, r_\ell^2) r_1^{\varepsilon_1} \cdots r_n^{\varepsilon_n} \psi_1^{k_1}(s_1) \cdots \psi_\ell^{k_\ell}(s_\ell), \tag{5.1}$$

where F_k are polynomials, $k = (k_1, \ldots, k_n)$, $\varepsilon_i = 0$ for k_i even and $\varepsilon_i = 1$ for k_i odd,

$$r_i^2 = \sum_{s=m_{i-1}+1}^{m_{i-1}+m_i} x_s^2, \quad s_i \in S^{m_i-1},$$

and $\psi_i^{k_i}$ is a spherical harmonic of degree k_i on the sphere S^{m_i-1}.

The functions Ψ_i are eigenfunctions of the Laplace–Beltrami operator $\Delta_{S^{m_i-1}}$ on the unit sphere S^{m_i-1} :

$$\Delta_{S^{m_i-1}} \Psi_i^{k_i} = \mu_i \Psi_i^{k_i}$$

with the eigenvalues $\mu_i = -k_i\ (k_i + m_i - 2)$. In the degenerate case $m_i = 1$, we have $\mu_i = 0$, $k_i = 0$ or $k_i = 1$, and $\Psi_i^{k_i} \equiv 1$.

Decompose the Laplace operator Δ in $\mathbb{R}^n$ into the sum of partial Laplacians Δ_i in $\mathbb{R}^{m_i}$, written in the spherical coordinates in $\mathbb{R}^{m_i}$:

$$\Delta = \sum_{i=1}^{\ell} \left(\Delta_{r_i} + \frac{1}{r_i^2} \Delta_{S^{m_i-1}} \right),$$

where Δ_{r_i} is the radial part of Δ_i,

$$\Delta_{r_i} = \frac{1}{r_i^{m_i-1}} \frac{\partial}{\partial r_i} r_i^{m_i-1} \frac{\partial}{\partial r_i}.$$

If h is harmonic, $\Delta h = 0$, then applying the Laplace operator to (5.1) we obtain the system of equations

$$\sum_{i=1}^{\ell} \left(\Delta_{r_i} + \frac{\mu_i}{r_i^2} \right) u_k = 0, \quad \text{for all } k = (k_1, \ldots, k_n), \tag{5.2}$$

where we have written

$$u_k(r_1, \ldots, r_\ell) = F_k(r_1^2, \ldots, r_\ell^2) r_1^{\varepsilon_1} \cdots r_n^{\varepsilon_n}. \tag{5.3}$$

5.2 Proof of Theorem 1. Since $Q = \sum_{i=1}^{\ell} \lambda_i r_i^2$, then $h|Q$ if and only if $F_k|Q$ for all k, where F_k are the polynomials in (5.1). The harmonicity condition for h is given by the system (5.2), so we conclude that Q is a harmonic divisor, $Q|h$, if and only if the differential equation

$$\sum_{i=1}^{\ell}\left(\frac{\partial^2}{\partial r_i^2}+\frac{m_i-1}{r_i}\frac{\partial}{\partial r_i}+\frac{\mu_i}{r_i^2}\right)u=0 \tag{5.2'}$$

has a nonzero solution $u = u_k = r_1^{\varepsilon_1}\cdots r_\ell^{\varepsilon_\ell}F(r_1,\ldots,r_\ell^2)$, where $F = F_k$ is a polynomial vanishing on the quadric $\sum_{i=1}^{\ell}\lambda_i r_i^2 = 0$, and $\varepsilon_j = 0$ or $\varepsilon_j = 1$.

The relation between the degrees is $\deg h = 2\deg F + |\varepsilon|$. Thus, we arrive at the problem of divisibility by quadratic form in the space of ℓ variables $r_1,\ldots,r_\ell$, with simple eigenvalues, and with the differential operator (5.2′) instead of the Laplacian.

We repeat the arguments from Section 3. First, we replace Q by $Q+1$ in the condition of divisibility. Then we introduce the ellipsoidal coordinates

$$r_i^2=\frac{\prod_{j=1}^{\ell}(t_j-a_i)}{\prod_{j\neq i}(a_j-a_i)},\quad a_i=\frac{1}{\lambda_i},\quad i=1,\ldots,\ell. \tag{5.4}$$

The surface $Q+1=0$ is written now as $t_{n_0}=0$, where $0\in(a_{n_0},a_{n_0+1})$.

A solution u of (5.2′) can be represented as a finite sum of (nonzero) separated solutions of the form $E_1\otimes\cdots\otimes E_\ell$, and each E_j is represented, due to the form of u, as

$$E_j=\rho^{\varepsilon}w,\quad \rho^{\varepsilon}(s)=\prod_{j=1}^{\ell}(s-a_j)^{\varepsilon_j/2},$$

where w is a polynomial. In addition, each E_j satisfies the separation equation (cf. [SW], Prop. 1)

(5.5)

$$E''+\frac{1}{2}\left(\sum_{i=1}^{\ell}\frac{m_i}{s-a_i}\right)E'+\frac{1}{4\prod_{i=1}^{\ell}(s-a_i)}\left[\sum_{i=1}^{\ell}\mu_i\frac{\prod_{q\neq i}(a_i-a_q)}{s-a_i}+\psi(s)\right]E=0,$$

where ψ is a polynomial, $\deg\psi\le\ell-2$.

Computing degrees gives $\deg h = 2\deg w+|\varepsilon|$. It remains to note that, according to (5.1) and (5.3), $(Q+1)|h$ is equivalent to $(Q+1)|u$, which in turn means that the factor E_{n_0} vanishes at $s=0$, $E_{n_0}(0)=0$, in each separated solution in the decomposition of u. Since u is nonzero, E_{n_0} also can be assumed nonzero. Then $E=E_{n_0}$ is the needed solution of the Fuchsian equation (1.2). □

5.3 Particular cases. Consider the case of two eigenvalues $\lambda_1 \neq \lambda_2$, of multiplicities m_1, m_2 respectively. Then

$$Q(x_1, \dots, x_n) = \lambda_1(x_1^2 + \dots + x_{m_1}^2) + \lambda_2(x_{m_1+1}^2 + \dots + x_n^2). \tag{5.6}$$

Then (5.4) is the Fuchsian differential equation with three singularities a_1, a_2 and ∞, $a_i = 1/\lambda_i$:

$$\begin{aligned} &E'' + \frac{1}{2}\left(\frac{m_1}{s-a_1} + \frac{m_2}{s-a_2}\right)E' \\ &+ \frac{1}{4(s-a_1)(s-a_2)}\left[\frac{\mu_1(a_1-a_2)}{s-a_1} + \frac{\mu_2(a_2-a_1)}{s-a_2} + \psi\right]E = 0, \end{aligned} \tag{5.7}$$

where $\mu_i = -k_i(k_i + m_i - 2)$, $\psi = \text{const}$.

We are looking for a nonzero solution of the form

$$E(s) = (s-a_1)^{\varepsilon_1/2}(s-a_2)^{\varepsilon_2/2}w(s),$$

where w is a polynomial and ε_i as above. The condition of admissibility is

$$\psi = 4\left(N + \frac{|\varepsilon|}{2}\right)\left(N + \frac{|\varepsilon| + n - 2}{2}\right),$$

where $N = \deg w$.

The change of variables $z = (s-a_1)/(a_2-a_1)$ in (5.7) and consequent substitution $E = z^{k_1/2}(1-z)^{k_2/2}v$ lead to the hypergeometric equation (cf. [AAR], 2.3)

$$v'' + [(a+b+1)z - c]v' + abv = 0$$

with parameters $a = C_1 + A_1 + B_1$, $b = C_2 + A_1 + B_1$, $c = 1 - A_1 - A_2$, where $A_1 = k_1/2$, $B_1 = k_2/2$, $A_2 = 1 - (k_1 + m_1)/2$, $B_2 = 1 - (k_2 + m_2)/2$, $C_1 = -N - |\varepsilon|/2$, $C_2 = N + (|\varepsilon| + n - 2)/2$.

The solution v is the hypergeometric function $v(z) = {}_2F_1(a, b, c; z)$; since $a = -N + (k_1 + k_2 - |\varepsilon|)/2$ is an integer, it can be expressed via Jacobi polynomials (cf. [AAR], p. 99): $v(z) = \text{const}\, P_M^{(\alpha,\beta)}(1-2z)$, where $M = N - (k_1 + k_2 - |\varepsilon|)/2$, $\alpha = c - 1 = k_1 + (m_1 - 2)/2$, $\beta = b - M - \alpha - 1 = k_2 + (m_2 - 2)/2$.

The condition of divisibility by Q is $E(0) = 0$, which can be translated as

$$v\left(\frac{a_1}{a_1 - a_2}\right) = \text{const}\, P_M^{(\alpha,\beta)}\left(\frac{a_1 + a_2}{a_2 - a_1}\right) = 0.$$

Thus, we have

Corollary 5.1. *The quadratic form Q in* (5.6) *with two multiple eigenvalues λ_1 and λ_2 is a harmonic divisor if and only if there exist nonnegative integers k_1, k_2, N, such that*

$$P^{(k_1+(m_1-2)/2, k_2+(m_2-2)/2)}_{N-(k_1+k_2-|\varepsilon|)/2}\left(\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right) = 0, \quad |\varepsilon| = \varepsilon_1+\varepsilon_2+1. \tag{5.8}$$

In this case, Q is a harmonic $(2N+|\varepsilon|)$-divisor.

Take, in particular, $m_1 = n-1$, $m_2 = 1$ and write $\lambda_1 = \gamma^2$, $\lambda_2 = \gamma^2 - 1$, so that Q has the form

$$Q(x_1, \dots, x_n) = \gamma^2(x_1^2 + \cdots + x_n^2) - x_n^2. \tag{5.9}$$

Then $k_2 = 0$ or $k_2 = 1$, and (5.8) transforms to

$$P^{(k_1+(n-3)/2, k_2-1/2)}_{N-(k_1-\varepsilon_1)/2}(2\gamma^2 - 1) = 0.$$

The Jacobi polynomials $P^{(\alpha,\pm 1/2)}_M$ are related to the Gegenbauer polynomials ([E], 10.9, formulas (2.1), (2.2)), so the last condition can be rewritten as

$$C^{k_1+(n-2)/2}_{2N-k_1+\varepsilon_1}(\gamma) = 0, \quad \text{if } k_2 = 0,$$
$$C^{k_1+(n-2)/2}_{2N-k_1+\varepsilon_1+1}(\gamma) = 0, \quad \text{if } k_2 = 1.$$

If $m = 2N + |\varepsilon|$, both cases can be unified by the condition

$$C^{k_1+(n-2)/2}_{m-k_1}(\gamma) = 0; \tag{5.10}$$

and we obtain the result of [Ar]: the quadratic form (5.9) is a harmonic m-divisor if and only if γ is a root of the Gegenbauer polynomial, $C^{k+(n-2)/2}_{m-k}(\gamma) = 0$, for some integer k, $0 \le k \le m$ (in fact, $k \le m-2$).

6 Harmonic divisors of higher degrees

6.1 In this section, we discuss the problem for higher degree divisors and give some partial results.

The following statement is presented in [A].

Theorem 6.1 *([A]). Let Q be a polynomial in $\mathbb{R}^n$ that splits into a product of linear forms:*

$$Q(x) = \prod_{i=1}^{p} (\nu_i, x), \quad \nu_i \neq 0.$$

Then Q is a harmonic divisor if and only if the group W generated by the reflections

$$\sigma_i : x \to x - 2\frac{(\nu_i, x)}{(\nu_i, \nu_i)}$$

around the hyperplanes $L_i = \{(\nu_i, x) = 0\}$ is finite.

Since the proof is quite simple and short, we repeat it here.

Proof. Let $Q|h$, where h is harmonic. By the reflection principle, h is σ_i-odd, i.e., $h \circ \sigma_i = -h$, $i = 1, \dots, p$. Therefore, $h = 0$ on each hyperplane $w(L_i)$, $w \in W$. If W is infinite, then $h = 0$.

Conversely, suppose that the group W is finite. Let $L_1, \dots, L_m$, $m \geq p$ be all the mirrors of the Coxeter group W. We can assume that the first p mirrors are just $L_1, \dots, L_p$.

Define

$$h(x) = \prod_{i=1}^{m} (\nu_i, x),$$

where $L_j = \{(\nu_j, x) = 0\}$, $j = 1, \dots, m$. Then $Q|h$, and h is harmonic. Indeed, h is σ_j-odd for any reflection σ_j around L_j, $h \circ \sigma_j = -h$, $j = 1, \dots, m$, since σ_j interchanges the mirrors L_j and changes the orientation in $\mathbb{R}^n$. The Laplace operator commutes with σ_j; hence $(\Delta h) \circ \sigma_j = -\Delta h$, and therefore $\Delta h|_{L_j} = 0$, $j = 1, \dots, m$. This implies that $h|\Delta h$, which is possible only if $\Delta h = 0$. □

6.2 Another class of harmonic divisors which it is possible to describe consists of the products of confocal quadratic forms:

$$Q(x) = \prod_{s=0}^{m} Q_s(x), \tag{6.1}$$

where $Q_s(x) = \sum_{j=1}^{n} \lambda_j^s x_j^s$, $\lambda_j^s \neq 0$.

The condition of confocality means that the differences

$$\frac{1}{\lambda_j^s} - \frac{1}{\lambda_j^q} = \theta^{sq}$$

do not depend on $j = 1, \dots, n$.

Let $a_j^s = 1/\lambda_j^2$. If we fix some factors, say Q_0 in (6.1), and denote $a_j = a_j^0$, $\theta^q = \theta^{0q}$, then for any $q = 0, \dots, m$ we have $a_j^q = a_j - \theta^q$.

Denote by $m_1, \dots, m_\ell$ the (common) multiplicities of the eigenvalues of the forms Q_s.

Theorem 6.2. *The product* (6.1) *of confocal quadratic forms is a harmonic divisor if and only if the Fuchsian differential equation* (1.2) *has a nonzero polynomial solution* w *with* $w(0) = w(\theta^1) = \cdots = w(\theta^m) = 0$.

The proof follows from separation of variables for the Laplace operator in sphero-ellipsoidal coordinates in a fashion similar to that of Sections 3 and 4. The only difference is that now the separated solutions must have $m + 1$ prescribed zeros instead of a single zero in the case of a single quadratic factor.

6.3 Open questions. We have described all polynomials Q in $\mathbb{R}^n$ of degree 2 for which the equation

$$\Delta(Qf) = 0 \tag{6.2}$$

has a nontrivial solution in $\mathbb{R}^n$. If Q is a quadratic form, then f can be taken to be a homogeneous polynomial.

In applications to the wave equation (see [A]), it is important to know whether the dimension d of the polynomial solutions of (6.2) is finite or infinite. For instance, if Q is a completely reducible polynomial (product of linear forms) of any degree and $d > 0$ (Q is a harmonic divisor), then $d = \infty$.

1. We conjecture that for $n > 2$ there exists a polynomial Q (even a quadratic one) with $0 < d < \infty$. In view of Theorem 1 and the statements in 5.3, this question may be highly nontrivial, as it is related to the question of common zeros of orthogonal (Jacobi, Gegenbauer) polynomials.

2. Let Q_1, Q_2 be two quadratic forms which are harmonic divisors. When is the product $Q_1 Q_2$ also a harmonic divisor?

When is the product LQ of linear and quadratic forms a harmonic divisor?

There are other natural questions concerning divisors of harmonic polynomials and the related problem of studying zeros of harmonic functions in $\mathbb{R}^n$.

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