WHEN ARE THE SPATIAL LEVEL SURFACES OF SOLUTIONS OF DIFFUSION EQUATIONS INVARIANT WITH RESPECT TO THE TIME VARIABLE?

By

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Abstract. We consider solutions of the initial-Neumann problem for the heat equation on bounded Lipschitz domains in $\mathbb{R}^N$ and classify the solutions whose spatial level surfaces are invariant with respect to the time variable. (Of course, the values of each solution on its spatial level surfaces vary with time.) The prototype of such classification is a result of Alessandrini, which proved a conjecture of Klamkin. He considered the initial-Dirichlet problem for the heat equation on bounded domains and showed that if all the spatial level surfaces of the solution are invariant with respect to the time variable under the homogeneous Dirichlet boundary condition, then either the initial data is an eigenfunction or the domain is a ball and the solution is radially symmetric with respect to the space variable. His proof is restricted to the initial-Dirichlet problem for the heat equation. In the present paper, in order to deal with the initial-Neumann problem, we overcome this obstruction by using the invariance condition of spatial level surfaces more intensively with the help of the classification theorem of *isoparametric hypersurfaces in Euclidean space* of Levi-Civita and Segre. Furthermore, we can deal with nonlinear diffusion equations, such as the porous medium equation.

1 Introduction

Alessandrini proved a number of symmetry results [1, 2] which settled a conjecture of Klamkin [16] (see also [29]). We quote a theorem from [2] (see [2, Theorem 1.3, p. 254]).

Theorem A (Alessandrini). *Let* Ω *be a bounded domain in* $\mathbb{R}^N$ $(N \geqq 2)$ *all of whose boundary points are regular with respect to the Laplacian. Let* $\varphi \in L^2(\Omega)$ *satisfy* $\varphi \not\equiv 0$ *and let* $u = u(x,t)$ *be the unique solution of*

$$(1.1)\qquad \begin{cases} \partial_t u = \Delta u & \textit{in } \Omega \times (0,\infty), \\ u(x,0) = \varphi(x) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega \times (0,\infty). \end{cases}$$

If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\cdot, t)$ is constant on every level surface $\{x \in \Omega : u(x, \tau) = \text{ const.}\}$ of $u(\cdot, \tau)$ in Ω, then one of the following two cases occurs.

(i) *φ is an eigenfunction of $-\Delta$ under the homogeneous Dirichlet boundary condition.*

(ii) *Ω is a ball, $u(\cdot, t)$ is radially symmetric for each $t \geqq 0$, and u never vanishes in $\Omega \times [\tau, \infty)$.*

Klamkin's conjecture [16] was that if all the spatial level surfaces of the solution u of (1.1) are invariant with respect to the time variable t for positive constant initial data, then the domain must be a ball. Therefore Theorem A proved Klamkin's conjecture [16].

In the present paper we consider the analogous problem under the homogeneous Neumann boundary condition and the problems for nonlinear diffusion equations such as the porous medium equation. Our first result is

Theorem 1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$ $(N \geqq 2)$ with boundary $\partial\Omega$, and let $\varphi \in L^2(\Omega)$ satisfy $\varphi \not\equiv 0$ and $\int_\Omega \varphi \, dx = 0$. Let $u = u(x, t)$ be the unique solution of the following initial-Neumann problem:*

$$(1.2) \qquad \begin{cases} \partial_t u = \Delta u & \text{in } \ \Omega \times (0, \infty), \\ u(x, 0) = \varphi(x) & \text{in } \ \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \ \partial\Omega \times (0, \infty), \end{cases}$$

where ν denotes the exterior normal unit vector to $\partial\Omega$. If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\cdot, t)$ is constant on every level surface $\{x \in \Omega : \ u(x, \tau) = \text{ const.}\}$ of $u(\cdot, \tau)$ in Ω, then one of the following two cases occurs.

(i) *φ is an eigenfunction of $-\Delta$ under the homogeneous Neumann boundary condition.*

(ii) *By a rotation and a translation of coordinates we have one of the following:*

(a) *There exists a finite interval (a, b) such that u extends as a function of x_1 and t only, say $u = u(x_1, t)$ $((x_1, t) \in [a, b] \times (0, \infty))$, where $x = (x_1, ..., x_N)$, and there exist an integer $n \geqq 1$ and a finite sequence $\{s_j\}_{j=0}^n$ satisfying*

$$s_0 = a, \quad s_n = b, \quad \text{and} \quad s_{j+1} - s_j = \frac{b - a}{n} \quad \text{for } 0 \leqq j \leqq n - 1,$$

such that $\partial u / \partial x_1$ does not vanish on $\bigcup_{j=0}^{n-1} (s_j, s_{j+1}) \times (\tau, \infty)$ but vanishes identically on $\{s_j\}_{j=0}^n \times (0, \infty)$. When $n \geqq 2$, u is symmetric with respect to the hyperplane

$\{x \in \mathbb{R}^N : x_1 = s_j\}$ *for each* $1 \leqq j \leqq n-1$. *Furthermore, the boundary* $\partial\Omega$ *consists of at most the following:*

(a-1) *a part of the hyperplane* $\{x \in \mathbb{R}^N : x_1 = b\}$,

(a-2) *a part of the hyperplane* $\{x \in \mathbb{R}^N : x_1 = a\}$,

(a-3) *a part of the hyperplane* $\{x \in \mathbb{R}^N : x_1 = s_j\}$ *for each* $1 \leqq j \leqq n-1$ *when* $n \geqq 2$,

(a-4) *a collection of straight line segments* ℓ *given by*

$$\ell = \{x \in \mathbb{R}^N : x = (x_1, y) \ \text{ and } \ s_j \leqq x_1 \leqq s_{j+1}\},$$

where y *is a point in* $\mathbb{R}^{N-1}$ *and* $0 \leqq j \leqq n-1$.

Here (a-1), (a-2), *and* (a-4) *are nonempty and there is a case in which* (a-3) *is empty.*

(b) *There exist a finite interval* (a, b) *with* $a \geqq 0$ *and a natural number* k *with* $2 \leqq k \leqq N$ *such that* u *extends as a function of* $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ *and* t *only, say* $u = u(r, t)$ $((r, t) \in [a, b] \times (0, \infty))$, *whose derivative* $\frac{\partial u}{\partial r}(r, t)$ *does not vanish on* $(a, b) \times (\tau, \infty)$ *but vanishes identicially on* $\{a, b\} \times (0, \infty)$. *Furthermore, when* $2 \leqq k \leqq N-1$, *the boundary* $\partial\Omega$ *consists of the following:*

(b-1) *a part of the hypersurface* $\{x \in \mathbb{R}^N : r = b\}$,

(b-2) *a part of the hypersurface* $\{x \in \mathbb{R}^N : r = a\}$ *when* $a > 0$,

(b-3) *a collection of straight line segments* ℓ *given by*

$$\ell = \{x \in \mathbb{R}^N : (x_1, \ldots, x_k) = r\omega, a \leqq r \leqq b, \text{ and } (x_{k+1}, \ldots, x_N) = y\},$$

where y *is a point in* $\mathbb{R}^{N-k}$ *and* ω *is a point in the* $(k-1)$*-dimensional unit sphere* S^{k-1} *in* $\mathbb{R}^k$.

When $k = N$, *there exists a Lipschitz domain* S *in* S^{N-1} *(*S *can be the whole sphere* S^{N-1}*) such that* $\Omega = \{r\omega \in \mathbb{R}^N : r \in (a, b)$ *and* $\omega \in S\}$ *when* $a > 0$, *and either* $\Omega = \{r\omega \in \mathbb{R}^N : r \in (0, b)$ *and* $\omega \in S\}$ *with* $S \neq S^{N-1}$ *or* $\Omega = \{x \in \mathbb{R}^N : r < b\}$ *when* $a = 0$.

In particular, in case (ii), *if* $\partial\Omega$ *is* C^1, *then* Ω *must be either a ball or an annulus.*

We refer the reader to [10] for existence and uniqueness of solutions of the initial-Neumann problem in Lipschitz cylinders. Since any constant function is a trivial solution of the initial-Neumann problem (1.2) with constant initial data, and since adding any constant function to the solution u in Theorem 1 does not have any influence on the invariance condition of spatial level surfaces of u, we have assumed for simplicity that $\varphi \not\equiv 0$ and $\int_\Omega \varphi \, dx = 0$ for the initial data φ.

Alessandrini used an eigenfunction expansion and a special case of a well-known theorem of symmetry for elliptic equations of Serrin [27, Theorem 2, pp. 311–312] in order to prove Theorem A.

Theorem S (Serrin). *Let D be a bounded domain with C^2 boundary ∂D and let $v \in C^2(\overline{D})$ satisfy*

$$\begin{cases} \Delta v = f(v) \text{ and } v > 0 & \text{in } D, \\ v = 0 \text{ and } \partial v/\partial \nu = c & \text{on } \partial D, \end{cases}$$

where $f = f(s)$ is a C^1 function of s, c is a constant, and ν denotes the exterior normal unit vector to ∂D. Then D is a ball, and v is radially symmetric and decreasing in D.

Under the hypothesis that case (i) of Theorem A does not hold, Alessandrini showed that there exists a level set $D = \{x \in \Omega : \psi(x) > s\}$ with $s > 0$ of an eigenfunction $\psi = \psi(x)$ of $-\Delta$ under the homogeneous Dirichlet boundary condition such that the function $v = \psi - s$ satisfies the overdetermined boundary conditions as in Theorem S. Applying Theorem S to v then implies that D is a ball and that v is radially symmetric and decreasing in D. A little more reasoning yields the case (ii) of Theorem A. In this proof, essential use is made of the fact that the boundary of D does not touch the boundary $\partial\Omega$. This fact arises from the homogeneous Dirichlet boundary condition of the eigenfunction ψ. Therefore, in our problem (1.2) we cannot use Theorem S because of the homogeneous Neumann boundary condition. We overcome this obstruction by using the invariance condition of spatial level surfaces more intensively with the help of the classification theorem of *isoparametric hypersurfaces in Euclidean space* of Levi-Civita and Segre (see [18, 26]). Besides, we can give another proof of Theorem A which does not depend on Theorem S.

In fact, the introduction of isoparametric surfaces was motivated by Somigliana [28] and Segre [25] in terms of similar questions of the geometry of solutions of partial differential equations.

Next we want to consider nonlinear diffusion equations. For the porous medium equation under the homogeneous Neumann boundary condition we have

Theorem 2. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$ with smooth boundary $\partial\Omega$, and let $u = u(x,t) \in C^\infty(\overline{\Omega} \times (0,\infty))$ satisfy*

$$
(1.3) \qquad \begin{cases} \partial_t \beta(u) = \Delta u & \text{in } \Omega \times (0,\infty), \\ u > 0 & \text{in } \overline{\Omega} \times (0,\infty), \\ \partial u / \partial \nu = 0 & \text{on } \partial\Omega \times (0,\infty), \end{cases}
$$

where $\beta(s) = s^{1/m}$ $(m > 0, m \neq 1)$ and ν denotes the exterior normal unit vector to $\partial\Omega$. If there exists $\tau > 0$ such that, for every $t > \tau$, $u(\cdot,t)$ is constant on every level surface $\{x \in \Omega : u(x,\tau) = \text{ const.}\}$ of $u(\cdot,\tau)$ in Ω, then one of the following two cases occurs.

(i) *u is a positive constant.*

(ii) *Ω is either a ball or an annulus; for each $t \geqq \tau$, $u(\cdot,t)$ is radially symmetric with respect to the center; and, for $t > \tau$, the derivative with respect to the radial direction, say $\partial u/\partial r$, vanishes in Ω only at the center of the ball.*

For the generalized porous medium equation under the homogeneous Dirichlet boundary condition we have

Theorem 3. *Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geqq 2)$ with smooth boundary $\partial\Omega$, and let $u = u(x,t) \in C(\overline{\Omega} \times (0,T)) \cap C^\infty(\Omega \times (0,T))$ satisfy*

$$
(1.4) \qquad \begin{cases} \partial_t \beta(u) = \Delta u \ \text{ and } \ u > 0 & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \end{cases}
$$

where β is a continuous function on $[0,\infty)$ such that

(1) *β is real analytic on $(0,\infty)$,*

(2) *$\beta(0) = 0$ and $\beta'(s) > 0$ for any $s > 0$.*

If there exists $\tau \in (0,T)$ such that, for every $t > \tau$, $u(\cdot,t)$ is constant on every level surface $\{x \in \Omega : u(x,\tau) = \text{ const.}\}$ of $u(\cdot,\tau)$ in Ω, then one of the following two cases occurs.

(i) *There exists a positive C^∞ function $\lambda = \lambda(t)$ on $[\tau,T)$ such that $u(x,t) = \lambda(t)u(x,\tau)$ for any $(x,t) \in \overline{\Omega} \times [\tau,T)$.*

(ii) *Ω is a ball; for each $t \in [\tau,T), u(\cdot,t)$ is radially symmetric with respect to the center; and, for each $t \in (\tau,T)$, the derivative with respect to the radial direction, say $\partial u/\partial r$, is negative in Ω except at the center of Ω.*

See [9, 24, 3] for the existence and uniqueness of weak solutions of the initial-boundary value problems for $\partial_t\beta(u) = \Delta u$, and [23] for the continuity of bounded weak solutions. When $\beta(s) = s^{1/m}$ with $0 < m < 1$, if the initial data $u(x, 0) \in L^\infty(\Omega)$ for the initial-Dirichlet problem, there exists a finite extinction time T^* such that $u \equiv 0$ for $t \geqq T^*$ (see, for example, [7, p. 176]). Therefore, in Theorem 3 we consider the finite time interval $(0, T)$. Concerning case (i), see [6, 8] for separable solutions of (1.4) when $\beta(s) = s^{1/m}$ with $m > 0$.

In Section 2, we prove Theorems 1, 2, and 3 simultaneously. Section 3 is devoted to some remarks concerning these theorems.

2 Proofs of theorems

First of all, let us quote the classification theorem of isoparametric hypersurfaces in Euclidean space $\mathbb{R}^N$, which was proved by Levi-Civita [18] for $N = 3$, and by Segre [26] for arbitrary N. See [20, 21] for a survey of isoparametric surfaces.

Theorem LcS (Levi-Civita and Segre). *Let D be a bounded domain in $\mathbb{R}^N$ ($N \geqq 2$) and let f be a real-valued smooth function on D satisfying $\nabla f \neq 0$ on D. Suppose that there exist two real-valued functions $g = g(\cdot)$ and $h = h(\cdot)$ of a real variable such that*

$$|\nabla f|^2 = g(f) \quad \text{and} \quad \Delta f = h(f) \quad \text{on } D. \tag{2.1}$$

Then the family of level surfaces $\{x \in D : f(x) = s\}$ ($s \in f(D)$) of f must be either parallel hyperplanes, concentric spheres, or concentric spherical cylinders. In particular, by a rotation and a translation of coordinates one of the following holds:

(a) *There exists a finite interval (a_1, b_1) such that f extends as a function of x_1 only, say $f = f(x_1)$ ($x_1 \in (a_1, b_1)$), and $D \subset (a_1, b_1) \times \mathbb{R}^{N-1}$ with $\partial D \cap (\{a_1\} \times \mathbb{R}^{N-1}) \neq \emptyset$ and $\partial D \cap (\{b_1\} \times \mathbb{R}^{N-1}) \neq \emptyset$.*

(b) *There exist a finite interval (a_1, b_1) with $a_1 \geqq 0$ and a natural number k with $2 \leqq k \leqq N$ such that f extends as a function of $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ only, say $f = f(r)$ ($r \in (a_1, b_1)$), and furthermore, when $a_1 > 0$, $D \subset \{(x_1, \dots, x_k) \in \mathbb{R}^k : a_1 < r < b_1\} \times \mathbb{R}^{N-k}$ with $\partial D \cap (\{(x_1, \dots, x_k) \in \mathbb{R}^k : r = a_1\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial D \cap (\{(x_1, \dots, x_k) \in \mathbb{R}^k : r = b_1\} \times \mathbb{R}^{N-k}) \neq \emptyset$, and when $a_1 = 0$, $D \subset \{(x_1, \dots, x_k) \in \mathbb{R}^k : 0 \leqq r < b_1\} \times \mathbb{R}^{N-k}$ with $\overline{D} \cap (\{0\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial D \cap (\{(x_1, \dots, x_k) \in \mathbb{R}^k : r = b_1\} \times \mathbb{R}^{N-k}) \neq \emptyset$. Here, when $k = N$, $\mathbb{R}^{N-k}$ is disregarded.*

In this theorem, the function f is called an *isoparametric function* and the level surfaces of f are called *isoparametric surfaces*. For our application, we have assumed that the domain D is bounded.

Let us put $u(x,\tau) = \psi(x)$ for $x \in \overline{\Omega}$. By the common assumption of Theorems 1, 2, and 3 (the invariance condition of spatial level surfaces) as in [1, (2.2), p. 231] we have

$$(2.2) \qquad u(x,t) = \mu(\psi(x),t) \quad \text{for any } (x,t) \in \overline{\Omega} \times [\tau,\infty) \ ([\tau,T) \text{ in Theorem 3})$$

for some function $\mu = \mu(s,t) : \mathbb{R} \times [\tau,\infty) \to \mathbb{R}$ satisfying

$$(2.3) \qquad \mu(s,\tau) = s \quad \text{for any } s \in \mathbb{R}.$$

Although the time interval is $[\tau,T)$ in Theorem 3, for simplicity let us use the time interval $[\tau,\infty)$. In Theorems 1 and 3, ψ is not constant; and in Theorem 2, if ψ is constant, then u is constant for $t \geq \tau$ and moreover by the uniqueness theorem [12, Chapter 6, Theorem 7, p. 178] for linear backward parabolic equations we have case (i). Therefore, we may assume that ψ is not constant. Hence there exist a point $x_0 \in \Omega$ and an open ball B in $\mathbb{R}^N$ centered at x_0 such that

$$(2.4) \qquad \nabla\psi \neq 0 \quad \text{on } \overline{B}(\subset \Omega).$$

Then, by a standard difference quotient argument (see [1, Lemma 1, p.232] and [2, Lemma 2.1, p. 255]), we have

Lemma 2.1. *There exists $\delta > 0$ such that for $I = [\psi(x_0) - \delta, \psi(x_0) + \delta]$ we have $I \subset \psi(B)$ and $\mu \in C^\infty(I \times [\tau,\infty))$.*

Proof. For the reader's convenience, we give a proof. The partial differentiability of μ with respect to t is a straightforward consequence of (2.2). It follows from (2.4) that there exists an interval $I = [\psi(x_0) - \delta, \psi(x_0) + \delta]$ with some $\delta > 0$ such that $I \subset \psi(B)$. Let $s \in I$. Then there exists a point $y \in B$ such that $\psi(y) = s$ and $\nabla\psi(y) \neq 0$. For $h \in \mathbb{R}$ with $|h|$ sufficiently small, put $x(h) = y + h\nabla\psi(y) \in B$. Hence $\psi(x(h)) = s + h|\nabla\psi(y)|^2 + O(h^2)$ as $h \to 0$. Thus for every $k \in \mathbb{R}$ with $|k|$ sufficiently small there exists a unique $h \in \mathbb{R}$ such that $\psi(x(h)) = s + k$, and $h = k|\nabla\psi(y)|^{-2} + O(k^2)$ as $k \to 0$. Consequently, we have for each $t \in [\tau,\infty)$

$$(2.5) \qquad \begin{aligned} \mu(s+k,t) - \mu(s,t) &= u(x(h),t) - u(y,t) \\ &= k\frac{\nabla u(y,t) \cdot \nabla\psi(y)}{|\nabla\psi(y)|^2} + O(k^2) \quad \text{as } k \to 0. \end{aligned}$$

This means that there exists a partial derivative μ_s $(= \partial\mu/\partial s)$ given by

$$\mu_s(s,t) = \mu_s(\psi(y),t) = \frac{\nabla u(y,t)\cdot\nabla\psi(y)}{|\nabla\psi(y)|^2}. \tag{2.6}$$

On the other hand, we have from (2.2)

$$\mu_t(s,t) = \mu_t(\psi(y),t) = \partial_t u(y,t). \tag{2.7}$$

In view of (2.4), since the right-hand sides of both (2.6) and (2.7) are bounded on $\overline{B}\times[\tau,\tilde{t}]$ for each $\tilde{t} > \tau$, by using the mean value theorem we get $\mu \in C^0(I\times[\tau,\infty))$. Because of (2.4), the right-hand side of (2.6) is smooth in $\overline{B}\times[\tau,\infty)$. Therefore, we can repeat the same process from (2.6) to prove the existence of the partial derivatives μ_{ss} and μ_{st}; and we get $\mu_s \in C^0(I\times[\tau,\infty))$ with the help of the mean value theorem. Similarly, we can start the same process from (2.7) and get $\mu_t \in C^0(I\times[\tau,\infty))$. Consequently, by repeating the process as many times as we want, we obtain $\mu \in C^\infty(I\times[\tau,\infty))$. □

In view of Lemma 2.1, we can substitute (2.2) into the differential equation and get

$$\beta'(\mu)\mu_t = \operatorname{div}(\mu_s\nabla\psi) = \mu_s\Delta\psi + \mu_{ss}|\nabla\psi|^2 \quad \text{on } \psi^{-1}(I)\times[\tau,\infty), \tag{2.8}$$

where $\psi^{-1}(I) = \{x\in\Omega : \psi(x)\in I\}$ and in Theorem 1 we recognize that $\beta(s)\equiv s$. Differentiating (2.8) with respect to t yields

$$\beta''(\mu)(\mu_t)^2 + \beta'(\mu)\mu_{tt} = \mu_{st}\Delta\psi + \mu_{sst}|\nabla\psi|^2 \quad \text{on } \psi^{-1}(I)\times[\tau,\infty). \tag{2.9}$$

Let us introduce the function $\mathfrak{D}$ by

$$\mathfrak{D} = \det\begin{pmatrix}\mu_s & \mu_{ss}\\ \mu_{st} & \mu_{sst}\end{pmatrix} \equiv \mu_s\mu_{sst} - \mu_{ss}\mu_{st}. \tag{2.10}$$

We distinguish two cases:

(1) $\mathfrak{D}\equiv 0$ on $I\times[\tau,\infty)$,

(2) $\mathfrak{D}\not\equiv 0$ on $I\times[\tau,\infty)$.

Note that these cases are slightly different from the cases in the paper [1], where the time is fixed, that is, $t=\tau$. This modification is useful in dealing with nonlinear diffusion equations (Theorems 2 and 3).

Case (1). In this case, let us show that the solution u must be a separable solution, which implies case (i) of Theorems 1 and 3. It follows from (2.3) that $\mu_s(s,\tau)\equiv 1$. Therefore, there exists a time $T_1 > \tau$ such that

$$\mu_s > 0 \quad \text{on } I\times[\tau,T_1]. \tag{2.11}$$

Hence we have

$$(2.12)\qquad (\log \mu_s)_{st} = \mathfrak{D}/(\mu_s)^2 = 0 \quad \text{on } I \times [\tau, T_1].$$

Solving this equation with $\mu_s(s,\tau) \equiv 1$ yields

$$(2.13)\qquad \mu(s,t) = \lambda(t)s + \eta(t) \quad \text{for any } (s,t) \in I \times [\tau, T_1]$$

for some C^∞ functions $\lambda = \lambda(t) \equiv \mu_s > 0$ and $\eta = \eta(t)$ on $[\tau, T_1]$ satisfying

$$(2.14)\qquad \lambda(\tau) = 1 \quad \text{and} \quad \eta(\tau) = 0.$$

On the other hand, we know that for each time $t > 0$, $u(\cdot, t)$ is analytic in x (see Friedman [13]). Therefore, by (2.13) and (2.2), we see that

$$(2.15)\qquad u(x,t) = \lambda(t)\psi(x) + \eta(t) \quad \text{for any } (x,t) \in \overline{\Omega} \times [\tau, T_1].$$

Now we distinguish Theorems 1, 2, and 3. Let us consider Theorem 3 first. The homogeneous Dirichlet boundary condition implies that $\eta \equiv 0$ on $[\tau, T_1]$. Namely, we have

$$(2.16)\qquad u(x,t) = \lambda(t)\psi(x) \quad \text{for any } (x,t) \in \overline{\Omega} \times [\tau, T_1].$$

Let $T^* = \sup\{T_1 \in (\tau, T) : \mu_s > 0 \text{ on } I \times [\tau, T_1]\}$. Suppose that $T^* < T$. Since $u > 0$ in $\Omega \times (0, T)$, in view of (2.13) and (2.16) we have by continuity

$$(2.17)\qquad \mu_s(s, T^*) = \lim_{t \uparrow T^*} \lambda(t) = u(x_0, T^*)/\psi(x_0) > 0 \quad \text{for any } s \in I.$$

This contradicts the definition of T^* and the continuity of μ_s. Therefore, we get $T^* = T$ and have case (i) of Theorem 3.

Next we consider Theorem 1. Since $\int_\Omega \varphi\, dx = 0$, we have $\int_\Omega u(x,t)\, dx = 0$ for any $t > 0$. Therefore, by integrating (2.15), we see that $\eta \equiv 0$ on $[\tau, T_1]$. Hence we get (2.16). By substituting (2.16) into the heat equation and letting $t = \tau$, we get from (2.14)

$$(2.18)\qquad \Delta\psi = \lambda'(\tau)\psi \quad \text{in } \Omega.$$

Since ψ is not constant and satisfies the homogeneous Neumann boundary condition, by separating variables we have

$$u(x,t) = e^{-\lambda'(\tau)(t-\tau)}\psi(x) \quad \text{for any } (x,t) \in \Omega \times [0, \infty).$$

This implies case (i) of Theorem 1.

Finally, let us consider Theorem 2. Substituting (2.15) into the diffusion equation yields

$$\frac{1}{m}(\lambda(t)\psi(x)+\eta(t))^{\frac{1}{m}-1}(\lambda'(t)\psi(x)+\eta'(t))=\lambda(t)\Delta\psi(x). \tag{2.19}$$

Dividing this by $\lambda(t)$ and differentiating the resulting equation with respect to t give

$$\begin{aligned}&\Big(\frac{1}{m}-1\Big)(\lambda'(t)\psi(x)+\eta'(t))^2+(\lambda(t)\psi(x)+\eta(t))(\lambda''(t)\psi(x)+\eta''(t))\\&\quad-\frac{\lambda'(t)}{\lambda(t)}(\lambda(t)\psi(x)+\eta(t))(\lambda'(t)\psi(x)+\eta'(t))=0.\end{aligned} \tag{2.20}$$

A further calculation gives

$$I(t)\psi^2(x)+II(t)\psi(x)+III(t)=0, \tag{2.21}$$

where

$$\begin{cases}I(t)=\Big(\frac{1}{m}-2\Big)(\lambda'(t))^2+\lambda(t)\lambda''(t),\\ II(t)=\Big(\frac{2}{m}-3\Big)\lambda'(t)\eta'(t)+\lambda(t)\eta''(t)+\lambda''(t)\eta(t)-\frac{(\lambda'(t))^2}{\lambda(t)}\eta(t),\\ III(t)=\Big(\frac{1}{m}-1\Big)(\eta'(t))^2+\eta(t)\eta''(t)-\frac{\lambda'(t)}{\lambda(t)}\eta(t)\eta'(t).\end{cases} \tag{2.22}$$

Therefore, by (2.21), we have

$$I(t)\equiv II(t)\equiv III(t)\equiv 0. \tag{2.23}$$

Solving $I(t)\equiv 0$ with $\lambda(\tau)=1$ gives

$$\lambda'(t)=\lambda^{2-\frac{1}{m}}(t)\lambda'(\tau). \tag{2.24}$$

By solving $II(t)\equiv 0$ with respect to $\eta''(t)$, we get

$$\eta''(t)=-\Big(\frac{2}{m}-3\Big)\frac{\lambda'(t)}{\lambda(t)}\eta'(t)-\Big(\frac{\lambda'(t)}{\lambda(t)}\Big)'\eta(t). \tag{2.25}$$

Substituting this into $III(t)\equiv 0$ gives

$$\Big(\frac{1}{m}-1\Big)(\eta'(t))^2-2\Big(\frac{1}{m}-1\Big)\frac{\lambda'(t)}{\lambda(t)}\eta(t)\eta'(t)-\Big(\frac{\lambda'(t)}{\lambda(t)}\Big)'\eta^2(t)=0. \tag{2.26}$$

Here, by using (2.24), we have

$$\begin{cases}\frac{\lambda'(t)}{\lambda(t)}=\lambda^{1-\frac{1}{m}}(t)\lambda'(\tau),\\ \Big(\frac{\lambda'(t)}{\lambda(t)}\Big)'=\Big(1-\frac{1}{m}\Big)\lambda^{2(1-\frac{1}{m})}(t)(\lambda'(\tau))^2.\end{cases} \tag{2.27}$$

By substituting these into (2.26) we get

$$\left(\eta'(t) - \lambda^{1-\frac{1}{m}}(t)\lambda'(\tau)\eta(t)\right)^2 = 0. \tag{2.28}$$

Therefore, by using the first equation of (2.27) once more, we conclude that

$$\left(\frac{\eta(t)}{\lambda(t)}\right)' = 0. \tag{2.29}$$

Since $\eta(\tau) = 0$ (see (2.14)), this implies

$$\eta(t) \equiv 0 \quad \text{on } [\tau, T_1]. \tag{2.30}$$

Namely, we get (2.16). Since $\int_\Omega u^{\frac{1}{m}}(x,t)\,dx = \int_\Omega \psi^{\frac{1}{m}}(x)\,dx > 0$ for any $t > 0$, we have from (2.16)

$$\lambda(t) \equiv 1 \quad \text{for any } t \in [\tau, T_1].$$

Then the diffusion equation implies that $\Delta\psi = 0$ in Ω. In view of the homogeneous Neumann boundary condition, we see that ψ is a positive constant. This contradicts (2.4), that is, we cannot have case (1) in the situation of Theorem 2.

Case (2). In this case, by supposing that each case (i) of Theorems 1, 2, and 3 does not hold, we show that each case (ii) of the theorems holds. It follows from the continuity of $\mathfrak{D}$ that there exist a nonempty open subinterval $J \subset I$ and a time $t_0 \geqq \tau$ such that $\mathfrak{D} \neq 0$ on $\overline{J} \times \{t_0\}$. Hence we can solve equations (2.8) and (2.9) with respect to $|\nabla\psi|^2$ and $\Delta\psi$ for $(x, t_0) \in \psi^{-1}(\overline{J}) \times \{t_0\}$. Specifically, there exists a nonempty bounded domain $D \subset \psi^{-1}(\overline{J})(\subset \Omega)$ in $\mathbb{R}^N$ such that

$$|\nabla\psi|^2 = g(\psi) \quad \text{and} \quad \Delta\psi = h(\psi) \quad \text{on } D \tag{2.31}$$

for some functions g and h as in (2.1). Then it follows from Theorem LcS that, after a rotation and translation of coordinates, there exists a finite interval (a_1, b_1) such that either (a) or (b) of Theorem LcS holds for $f = \psi$ and (a_1, b_1). Consequently, since ψ is analytic in Ω, by (2.2), we have one of the following two possibilities.

(a) There exists a finite interval $(a, b) \supset (a_1, b_1)$ such that u extends as a function of x_1 and t only, say $u = u(x_1, t)$ $((x_1, t) \in [a, b] \times [\tau, \infty))$. Furthermore, $\Omega \subset (a, b) \times \mathbb{R}^{N-1}$ with $\partial\Omega \cap (\{a\} \times \mathbb{R}^{N-1}) \neq \emptyset$ and $\partial\Omega \cap (\{b\} \times \mathbb{R}^{N-1}) \neq \emptyset$.

(b) There exist a finite interval $(a, b) \supset (a_1, b_1)$ with $a \geqq 0$ and a natural number k with $2 \leqq k \leqq N$ such that u extends as a function of $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$ and t only, say $u = u(r, t)$ $((r, t) \in [a, b] \times [\tau, \infty))$. Furthermore, when $a > 0$, $\Omega \subset \{(x_1, \ldots, x_k) \in \mathbb{R}^k : a < r < b\} \times \mathbb{R}^{N-k}$ with $\partial\Omega \cap (\{(x_1, \ldots, x_k) \in \mathbb{R}^k : r =$

$a\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial\Omega \cap (\{(x_1, ..., x_k) \in \mathbb{R}^k : r = b\} \times \mathbb{R}^{N-k}) \neq \emptyset$, and when $a = 0$, $\Omega \subset \{(x_1, ..., x_k) \in \mathbb{R}^k : 0 \leqq r < b\} \times \mathbb{R}^{N-k}$ with $\overline{\Omega} \cap (\{0\} \times \mathbb{R}^{N-k}) \neq \emptyset$ and $\partial\Omega \cap (\{(x_1, ..., x_k) \in \mathbb{R}^k : r = b\} \times \mathbb{R}^{N-k}) \neq \emptyset$. Here, when $k = N$, $\mathbb{R}^{N-k}$ is disregarded.

Lemma 2.2. *In case* (b), $u(r, \tau)(= \psi(r))$ *is monotone on* $[a, b]$ *(provided each case* (i) *of Theorems 1, 2, and 3 does not hold).*

Proof. Suppose that ψ is not monotone. Then ψ has either a local maximum point or a local minimum point. So suppose that ψ has a local maximum point. Since ψ is analytic and not constant, there exist three numbers in (a, b), say $r_1 < r_2 < r_3$, such that

$$\psi(r_1) = \psi(r_3) \text{ and } \psi'(r) \begin{cases} > 0 & \text{if } r_1 \leqq r < r_2, \\ < 0 & \text{if } r_2 < r \leqq r_3. \end{cases} \tag{2.32}$$

Hence, using Lemma 2.1 once more and putting $\tilde{I} = [\psi(r_1), \frac{1}{2}(\psi(r_1) + \psi(r_2))]$, we have $\tilde{I} \subset \psi((a, b))$ and $\mu \in C^\infty(\tilde{I} \times [\tau, \infty))$. Therefore, we get (2.8) and (2.9), where I is replaced by $\tilde{I}$. If $\mathfrak{D} \equiv 0$ on $\tilde{I} \times [\tau, \infty)$, we have already proved that cases (i) of both Theorem 1 and Theorem 3 hold as in Case (1); and in Theorem 2 this leads to a contradiction. Therefore, we see that $\mathfrak{D} \not\equiv 0$ on $\tilde{I} \times [\tau, \infty)$. By proceeding as in the beginning of Case (2), we see that there exist a nonempty open subinterval $J \subset \tilde{I}$ and a time $t_0 \geqq \tau$ such that $\mathfrak{D} \neq 0$ on $\overline{J} \times \{t_0\}$. By solving equations (2.8) and (2.9) with respect to $|\nabla\psi|^2$ and $\Delta\psi$ for $(x, t_0) \in \psi^{-1}(\overline{J}) \times \{t_0\}$, we have in particular that

$$(\psi'(r))^2 = g(\psi(r)) \quad \text{on } \psi^{-1}(\overline{J}) \cap [r_1, r_3] \tag{2.33}$$

for some function $g = g(\cdot)$ of a real variable as in (2.31). In view of (2.32), we see that

$$\psi^{-1}(\overline{J}) \cap [r_1, r_3] = [r_4, r_5] \cup [r_6, r_7], \tag{2.34}$$

where $r_1 \leqq r_4 < r_5 < r_2 < r_6 < r_7 \leqq r_3$. Since $\psi(r_4) = \psi(r_7)$ and $\psi(r_5) = \psi(r_6)$, by using (2.33) we see that $r_5 - r_4 = r_7 - r_6 \left(= \int_{\psi(r_4)}^{\psi(r_5)} (g(s))^{-\frac{1}{2}}\, ds\right)$ and

$$\psi(r) = \psi(2r_* - r) \quad \text{for any } r \in [r_4, r_5] \cup [r_6, r_7], \tag{2.35}$$

where $r_* = \frac{1}{2}(r_4 + r_7)$. Furthermore, by (2.2),

$$u(r, t) = u(2r_* - r, t) \quad \text{for any } (r, t) \in ([r_4, r_5] \cup [r_6, r_7]) \times [\tau, \infty). \tag{2.36}$$

On the other hand, since u satisfies the diffusion equation, we have

$$\partial_t \beta(u) = \partial_r^2 u + \frac{k-1}{r}\partial_r u \quad \text{in } (a,b) \times [\tau, \infty). \tag{2.37}$$

Since $k \geqq 2$, it follows from (2.36) and (2.37) that

$$\partial_r u \equiv 0 \quad \text{in } ([r_4, r_5] \cup [r_6, r_7]) \times [\tau, \infty). \tag{2.38}$$

In particular, this implies that $\psi' \equiv 0$ on $[r_4, r_5] \cup [r_6, r_7]$, which contradicts (2.32).

Similarly, if we suppose that ψ has a local minimum point, we also get a contradiction. Consequently, we have proved that $u(r,\tau)(=\psi(r))$ is monotone on $[a,b]$. □

We now distinguish Theorems 1, 2, and 3. Since $\partial\Omega$ is only Lipschitz continuous in Theorem 1, careful consideration is required to prove Theorem 1.

Completion of the proof of Theorem 1. Especially in Theorem 1, since problem (1.2) is solved by an eigenfunction expansion, we see that $u = u(x_1, t)((x_1,t) \in [a,b] \times [0,\infty))$ in case (a) and $u = u(r,t)((r,t) \in [a,b] \times [0,\infty))$ in case (b).

Let us consider case (b) first. Lemma 2.2 implies that either $\psi' \geqq 0$ or $\psi' \leqq 0$. Consider the case where $\psi' \geqq 0$. Since ψ is analytic and not constant, there exists a sequence of positive numbers $\{\varepsilon_j\}_{j=1}^{\infty}$ with $\varepsilon_j \downarrow 0$ as $j \uparrow \infty$ such that

$$\psi'(a+\varepsilon_j) > 0 \quad \text{and} \quad \psi'(b-\varepsilon_j) > 0 \quad \text{for any } j \geqq 1.$$

By continuity, we see that for each $j \geqq 1$ there exists $\tau_j > \tau$ satisfying

$$\partial_r u > 0 \quad \text{on } \{a+\varepsilon_j, b-\varepsilon_j\} \times [\tau, \tau_j].$$

Hence, since $\psi' \geqq 0$, it follows from the strong maximum principle (see [12, Chapter 2] or [22, Chapter 4.4, pp. 121–124] for the maximum principle) that for each $j \geqq 1$, $\partial_r u > 0$ in $[a+\varepsilon_j, b-\varepsilon_j] \times (\tau, \tau_j]$. By dealing with the case where $\psi' \leqq 0$ similarly, we conclude that there exist two sequences $\{\varepsilon_j\}_{j=1}^{\infty}$ and $\{\tau_j\}_{j=1}^{\infty}$ with $\varepsilon_j \downarrow 0$ as $j \uparrow \infty$ and $\tau_j > \tau$ such that for each $j \geqq 1$

$$\partial_r u \neq 0 \quad \text{in } [a+\varepsilon_j, b-\varepsilon_j] \times (\tau, \tau_j]. \tag{2.39}$$

Since u satisfies the homogeneous Neumann boundary condition, this will determine the boundary $\partial\Omega$ as in case (ii) (b) of Theorem 1. If

$$\partial\Omega \cap \{x \in \mathbb{R}^N : r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}} \in (a,b)\} = \emptyset,$$

then $k = N$ and Ω must be either a ball or an annulus in $\mathbb{R}^N$. So let us consider the case in which $\partial\Omega \cap \{x \in \mathbb{R}^N : r \in (a, b)\} \neq \emptyset$. Take sufficiently large $j \geqq 1$ and an arbitrary point $x^* \in \partial\Omega$ with

$$r = \left((x_1^*)^2 + \cdots + (x_k^*)^2\right)^{\frac{1}{2}} \in (a + \varepsilon_j, b - \varepsilon_j).$$

Since Ω is a bounded Lipschitz domain, we can find an orthogonal matrix $\mathfrak{R} = (\mathfrak{r}_{ij})$ and a neighborhood V of x^* in $\mathbb{R}^N$ with

$$V \subset \{x \in \mathbb{R}^N : r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}} \in (a + \varepsilon_j, b - \varepsilon_j)\}$$

such that by introducing the rotation of coordinates $z = \mathfrak{R}x$, we have in z-coordinates

$$\begin{cases} \partial\Omega \cap V = \{z = (\tilde{z}, \phi(\tilde{z})) \in \mathbb{R}^N : \tilde{z} = (z_1, ..., z_{N-1}) \in \tilde{B}\}, \\ \Omega \cap V = \{z = (\tilde{z}, z_N) \in \mathbb{R}^N : c < z_N < \phi(\tilde{z}) \text{ and } \tilde{z} \in \tilde{B}\}, \\ V = \tilde{B} \times (c, d), \end{cases} \tag{2.40}$$

where $\tilde{B}$ is an open ball in $\mathbb{R}^{N-1}$, (c, d) is a bounded open interval, and ϕ is a Lipschitz continuous function on $\overline{\tilde{B}}$. Then, by Rademacher's theorem on the almost everywhere total differentiability of Lipschitz functions (see [30, Theorem 2.21, p. 50] for example), we see that the exterior unit normal vector ν to $\partial\Omega$ is given by

$$\nu(\tilde{z}, \phi(\tilde{z})) = (1 + |\nabla_{\tilde{z}}\phi(\tilde{z})|^2)^{-\frac{1}{2}}(-\nabla_{\tilde{z}}\phi(\tilde{z}), 1) \quad \text{for almost every } \tilde{z} \in \tilde{B}, \tag{2.41}$$

where $\nabla_{\tilde{z}}\phi = (\partial_{z_1}\phi, ..., \partial_{z_{N-1}}\phi)$. Therefore, it follows from the homogeneous Neumann boundary condition that for any $t \in (\tau, \tau_j]$

$$(-\nabla_{\tilde{z}}\phi(\tilde{z}), 1) \cdot \nabla_z (u(\mathfrak{R}^* z, t)) = 0 \quad \text{for almost every } \tilde{z} \in \tilde{B}, \tag{2.42}$$

where $z = (\tilde{z}, \phi(\tilde{z}))$ and $\mathfrak{R}^*$ denotes the transposed matrix of $\mathfrak{R}$. Since $u = u(r, t)$ with $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$,

$$\nabla_x u = \frac{\partial_r u(r, t)}{r}(x_1, ..., x_k, 0, ..., 0). \tag{2.43}$$

Then, using $\nabla_z = \mathfrak{R}\nabla_x$ we get

$$\nabla_z (u(\mathfrak{R}^* z, t)) = \frac{\partial_r u(r, t)}{r}\Big(\sum_{\alpha=1}^{k}\sum_{j=1}^{N} \mathfrak{r}_{1\alpha}\mathfrak{r}_{j\alpha}z_j, ..., \sum_{\alpha=1}^{k}\sum_{j=1}^{N} \mathfrak{r}_{N\alpha}\mathfrak{r}_{j\alpha}z_j\Big). \tag{2.44}$$

Since $\partial_r u(r, t)/r \neq 0$ for any $t \in (\tau, \tau_j]$ and $r \in (a + \varepsilon_j, b - \varepsilon_j)$, we have from (2.42)

$$(-\nabla_{\tilde{z}}\phi(\tilde{z}), 1) \cdot \vec{a}(\tilde{z}, \phi(\tilde{z})) = 0 \quad \text{for almost every } \tilde{z} \in \tilde{B}, \tag{2.45}$$

where we put

$$\begin{aligned}\vec{a}(\tilde{z}, \phi(\tilde{z})) &= (a_1(z), ..., a_N(z)) \\ &= \left(\sum_{\alpha=1}^{k} \sum_{j=1}^{N} \mathfrak{r}_{1\alpha} \mathfrak{r}_{j\alpha} z_j, ..., \sum_{\alpha=1}^{k} \sum_{j=1}^{N} \mathfrak{r}_{N\alpha} \mathfrak{r}_{j\alpha} z_j \right) \quad \text{with } z_N = \phi(\tilde{z}).\end{aligned}$$

Equality (2.45) is regarded as a first-order quasilinear partial differential equation for the function ϕ. We can solve this by the method of characteristics (see [11, pp. 343–344], for example). For each $x \in \mathbb{R}^N$ with $r = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}} \in (a, b)$, let $z = z(s)$ $(s \in \mathbb{R})$ be the curve satisfying

$$\text{(2.46)} \qquad \begin{cases} \dfrac{d}{ds} z(s) = \vec{a}(z(s)) & \text{for } s \in \mathbb{R}, \\ z(0) = \mathfrak{R} x. \end{cases}$$

This curve is called a *characteristic curve*. By putting $z(s) = \mathfrak{R} x(s)$, we get

$$\frac{d}{ds} x(s) = (x_1(s), ..., x_k(s), 0, ..., 0).$$

Solving this yields

$$\text{(2.47)} \qquad \mathfrak{R}^* z(s) = x(s) = (x_1(0)e^s, ..., x_k(0)e^s, x_{k+1}(0), ..., x_N(0)).$$

Namely, $z = z(s)$ $(s \in \mathbb{R})$ is a straight half-line through a point $z(0) \in \mathbb{R}^N$ with direction $\frac{1}{r_0}\mathfrak{R}(x_1(0), ..., x_k(0), 0, ..., 0)$, where $r_0 = (x_1^2(0) + \cdots + x_k^2(0))^{\frac{1}{2}} \in (a, b)$. We call such a line a *characteristic line*. Let $\mathcal{L}$ be the set of all characteristic lines intersecting V. Since V is convex, for each $\ell \in \mathcal{L}$ the intersection $\ell \cap V$ is a line segment. By introducing polar coordinates for the first k coordinates $(x_1, ..., x_k)$ in x-coordinates and using Fubini's theorem, we see that for almost every $\left(\frac{1}{r_0}(x_1(0), ..., x_k(0)), (x_{k+1}(0), ..., x_N(0))\right) \in S^{k-1} \times \mathbb{R}^{N-k}$, ϕ has a total differential at almost every point $z(s)$ on the intersection of V and the line $z = z(s)$ with $z(0) = \mathfrak{R} x(0)$, provided that the intersection is nonempty. Let $\mathcal{G}$ be the set of such lines intersecting V and let $\mathcal{B} = \mathcal{L} \setminus \mathcal{G}$. We call an element of $\mathcal{G}$ a *good line* and that of $\mathcal{B}$ a *bad line*, respectively. Almost all elements of $\mathcal{L}$ are good lines.

First, let us show that if ℓ is a good line intersecting $\partial\Omega \cap V$, then $\ell \cap V$ is contained in $\partial\Omega$. Let ℓ be a good line, given by $z = z(s)$ $(s \in \mathbb{R})$, intersecting $\partial\Omega \cap V$. Set

$$\text{(2.48)} \qquad w(s) = z_N(s) - \phi(\tilde{z}(s)).$$

Then $w(s_0) = 0$ for some $s_0 \in \mathbb{R}$, and for almost every $s \in \mathbb{R}$ with $z(s) \in V$ we have

$$\begin{aligned}\frac{d}{ds}w(s) &= \frac{d}{ds}z_N(s) - \nabla_{\tilde{z}}\phi(\tilde{z}(s)) \cdot \frac{d}{ds}\tilde{z}(s)\\ &= a_N(z(s)) - \nabla_{\tilde{z}}\phi(\tilde{z}(s)) \cdot (a_1(z(s)), ..., a_{N-1}(z(s)))\\ &= (-\nabla_{\tilde{z}}\phi(\tilde{z}(s)), 1) \cdot \vec{a}\,(\tilde{z}(s), \phi(\tilde{z}(s)) + w(s))\,.\end{aligned}$$

Observe that

$$\vec{a}\,(\tilde{z}(s), \phi(\tilde{z}(s)) + w(s)) = \vec{a}(\tilde{z}(s), \phi(\tilde{z}(s))) + \left(\sum_{\alpha=1}^{k} \mathfrak{r}_{1\alpha}\mathfrak{r}_{N\alpha}, ..., \sum_{\alpha=1}^{k} \mathfrak{r}_{N\alpha}\mathfrak{r}_{N\alpha}\right) w(s).$$

Therefore, it follows from (2.45) that for almost every $s \in \mathbb{R}$ with $z(s) \in V$

$$\frac{d}{ds}w(s) = (-\nabla_{\tilde{z}}\phi(\tilde{z}(s)), 1) \cdot \left(\sum_{\alpha=1}^{k} \mathfrak{r}_{1\alpha}\mathfrak{r}_{N\alpha}, ..., \sum_{\alpha=1}^{k} \mathfrak{r}_{N\alpha}\mathfrak{r}_{N\alpha}\right) w(s). \tag{2.49}$$

Hence, since the Lipschitz continuity of ϕ implies that the absolute value of the right-hand side of this equality is bounded from above by $K|w(s)|$ for some constant $K > 0$, by integrating this equality from s_0 to s, we get from $w(s_0) = 0$

$$|w(s)| \leqq K \left| \int_{s_0}^{s} |w(s')|\, ds' \right|. \tag{2.50}$$

This implies that

$$w(s) = 0 \qquad \text{for all } s \in \mathbb{R} \text{ with } z(s) \in V.$$

In view of the definition of w (see (2.48)), we see that $\ell \cap V$ is contained in $\partial\Omega$.

Next, let us show that $\partial\Omega \cap V$ consists of characteristic lines $z = z(s)$ in V. Suppose that there exists a line $\ell \in \mathcal{L}$ intersecting $\partial\Omega \cap V$ such that $\ell \cap V$ is not contained in $\partial\Omega$. Then ℓ is a bad line. Let ℓ be given by $z = z(s)$ $(s \in \mathbb{R})$. If necessary, by choosing another characteristic line sufficiently close to ℓ, we may assume that there exist two numbers s_1 and s_2 satisfying

$$\begin{cases} z(s_i) \in V \ \text{ for } i = 1, 2, \\ z_N(s_1) < \phi(\tilde{z}(s_1)), \ \text{ and } \ z_N(s_2) > \phi(\tilde{z}(s_2)). \end{cases} \tag{2.51}$$

Since almost all elements of $\mathcal{L}$ are good lines, from the continuity of ϕ we can find a good line sufficiently close to ℓ which still satisfies (2.51). Therefore, by continuity, this good line must intersect $\partial\Omega \cap V$. This is a contradiction. Consequently, we see that $\partial\Omega \cap V$ consists of characteristic lines $z = z(s)$ in V.

On the other hand, we know that in the original x-coordinates these characteristic lines are given by (2.47) globally. Therefore, since $\varepsilon_j \downarrow 0$ as $j \uparrow \infty$ and

x^* was an arbitrary point in $\partial\Omega$ with $r = \left((x_1^*)^2 + \cdots + (x_k^*)^2\right)^{\frac{1}{2}} \in (a + \varepsilon_j, b - \varepsilon_j)$, we see that $\partial\Omega \cap \{x \in \mathbb{R}^N : a < r < b\}$ consists of characteristic lines $z = z(s)$ in $\{x \in \mathbb{R}^N : a < r < b\}$. Consequently, it follows that the boundary $\partial\Omega$ consists of at most (b-1), (b-2), and (b-3) in (ii) (b) of Theorem 1. When $k = N$, there is a case in which (b-3) is empty. To be precise, Ω is either a ball or an annulus in $\mathbb{R}^N$ if and only if (b-3) is empty. (Note that $\int_\Omega \varphi \, dx = C(\Omega) \int_a^b \varphi(r) r^{k-1} \, dr$ for some positive constant $C(\Omega)$ depending only on Ω. Then $\int_\Omega \varphi \, dx = 0$ if and only if $\int_a^b \varphi(r) r^{k-1} \, dr = 0$.) Since the characteristic lines in (b-3) are parallel to the normal direction of both hypersurfaces $\{x \in \mathbb{R}^N : r = b\}$ and $\{x \in \mathbb{R}^N : r = a\}$ when $a > 0$, then (b-1) has positive area and (b-2) does when $a > 0$. Hence, from the homogeneous Neumann boundary condition we get

(2.52) $$\partial_r u(b, t) = 0 \quad \text{for any } t \in (0, \infty),$$

(2.53) $$\partial_r u(a, t) = 0 \quad \text{for any } t \in (0, \infty) \text{ when } a > 0.$$

When $a = 0$, let $\tilde{u} = \tilde{u}(r, t)$ be the unique solution of the problem

$$\begin{cases} \partial_t \tilde{u} = \Delta \tilde{u} & \text{in } B' \times (0, \infty), \\ \tilde{u}(x, 0) = \varphi(x) & \text{in } B', \\ \partial \tilde{u} / \partial \nu = 0 & \text{on } \partial B' \times (0, \infty), \end{cases}$$

where $B' = \{(x_1, \ldots, x_k) \in \mathbb{R}^k : r < b\}$ and ν denotes the exterior normal unit vector to $\partial B'$. Then $\tilde{u}$ satisfies (1.2), and by uniqueness $u \equiv \tilde{u}$. Since of course $\partial_r \tilde{u}(0, t) = 0$ for any $t > 0$, we get

(2.54) $$\partial_r u(0, t) = 0 \quad \text{for any } t \in (0, \infty).$$

Therefore, in view of Lemma 2.2, (2.52), (2.53), and (2.54), we have from the strong maximum principle

(2.55) $$\partial_r u \neq 0 \quad \text{in } (a, b) \times (\tau, \infty).$$

Consequently, we obtain conclusion (ii) (b) of Theorem 1.

Next let us consider case (a). For $u = u(x_1, t)$ consider the set $\mathfrak{Z}$ given by

(2.56) $$\mathfrak{Z} = \{x_1 \in (a, b) : \partial_{x_1} u(x_1, t) = 0 \text{ for all } t \geqq \tau\}.$$

Since $\mathfrak{Z}$ is contained in $\{s \in (a, b) : \psi'(s) = 0\}$ and ψ is an analytic non-constant function, so $\mathfrak{Z} \cap (a + \varepsilon, b - \varepsilon)$ is at most finite for each $\varepsilon > 0$. Suppose that there exists a point $s \in \mathfrak{Z} \cap (\frac{a+b}{2}, b)$. Then, by setting, for any $t \geqq \tau$,

$$v(x_1, t) = \begin{cases} u(x_1, t) & \text{if } x_1 \in (a, s), \\ u(2s - x_1, t) & \text{if } x_1 \in [s, 2s - a), \end{cases}$$

we see that v also satisfies the one-dimensional heat equation on $(a, 2s-a)\times(\tau,\infty)$. Since $2s-a>b$ and $u\equiv v$ in $(a,s)\times[\tau,\infty)$, by analyticity we get $u\equiv v$ in $(a,b)\times[\tau,\infty)$. In particular, u is symmetric with respect to $x_1=s$. Also, by supposing that there exists a point $s\in\mathfrak{Z}\cap(a,\frac{a+b}{2})$, we get the same conclusion. These observations imply that $\mathfrak{Z}$ is itself at most finite, and its elements are located at regular intervals if $\mathfrak{Z}\neq\emptyset$. Let

$$s_{\max}=\begin{cases}\max\mathfrak{Z} & \text{if } \mathfrak{Z}\neq\emptyset,\\ a & \text{if } \mathfrak{Z}=\emptyset.\end{cases}$$

Then $s_{\max}<b$. Take an arbitrary point $x^*\in\partial\Omega$ with $x_1^*\in(s_{\max},b)$. Then by the continuity of $\partial_{x_1}u$ and the definition of $s_{\max}$, there exist a time $t^*>\tau$ and $\delta>0$ such that

$$\text{(2.57)}\qquad \partial_{x_1}u(x_1,t^*)\neq 0\quad\text{for any } x_1\in(x_1^*-\delta,x_1^*+\delta).$$

Since u satisfies the homogeneous Neumann boundary condition, this will determine $\partial\Omega\cap\{x\in\mathbb{R}^N : s_{\max}<x_1<b\}$. As in the proof of case (b), by the method of characteristics we can determine a part of the boundary $\partial\Omega\cap\{x\in\mathbb{R}^N : x_1^*-\delta<x_1<x_1^*+\delta\}$. Observe that the characteristic curves are lines parallel to the x_1-axis. Then, since $x^*\in\partial\Omega$ is an arbitrary point with $x_1^*\in(s_{\max},b)$, we see that $\partial\Omega\cap\{x\in\mathbb{R}^N : s_{\max}<x_1<b\}$ consists of lines parallel to the x_1-axis such that x_1 varies from $s_{\max}$ to b. Since these lines are parallel to the normal direction of hyperplane $\{x\in\mathbb{R}^N : x_1=b\}$, $\partial\Omega\cap\{x\in\mathbb{R}^N : x_1=b\}$ has positive area. Therefore, it follows from the homogeneous Neumann boundary condition that

$$\text{(2.58)}\qquad \partial_{x_1}u(b,t)=0\quad\text{for any } t\in(0,\infty).$$

By the same argument, we get

$$\text{(2.59)}\qquad \partial_{x_1}u(a,t)=0\quad\text{for any } t\in(0,\infty).$$

In view of (2.58) and (2.59), by using the above reflection argument for u which was used to show that $\mathfrak{Z}$ is at most finite, we see that elements of $\mathfrak{Z}\cup\{a,b\}$ are located at regular intervals. Hence, there exist an integer $n\geqq 1$ and a finite sequence $\{s_j\}_{j=0}^n$ satisfying

$$\text{(2.60)}\qquad s_0=a,\quad s_n=b,\quad\text{and}\quad s_{j+1}-s_j=\frac{b-a}{n}\quad\text{for } 0\leqq j\leqq n-1,$$

where $\mathfrak{Z}\cup\{a,b\}=\{s_j\}_{j=0}^n$. Note that $\mathfrak{Z}=\emptyset$ when $n=1$. Then, by using an eigenfunction expansion for $u=u(x_1,t)$, we have furthermore for any $j=1,\dots,n-1$

$$\text{(2.61)}\qquad \partial_{x_1}u(s_j,t)=0\quad\text{for any } t\in(0,\infty).$$

Hence, it follows from the above reflection argument for u that u is symmetric with respect to the hyperplane $\{x \in \mathbb{R}^N : x_1 = s_j\}$ for each $1 \leqq j \leqq n-1$, when $n \geqq 2$. Instead of Lemma 2.2 we have

Lemma 2.3. *$u(x_1, \tau)(= \psi(x_1))$ is monotone on each interval $[s_j, s_{j+1}]$ for any $j = 0, ..., n-1$ (provided case (i) of Theorem 1 does not hold).*

Proof. By the symmetry of u, it suffices to show that ψ is monotone on $[s_0, s_1]$. Suppose that ψ is not monotone on $[s_0, s_1]$. Then it follows from the same argument as in the proof of Lemma 2.2 that there exist four points $(s_0 <) r_4 < r_5 < r_6 < r_7 (< s_1)$ with $r_5 - r_4 = r_7 - r_6$ such that

$$u(x_1, t) = u(2r_* - x_1, t) \quad \text{for any } (x_1, t) \in ([r_4, r_5] \cup [r_6, r_7]) \times [\tau, \infty), \tag{2.62}$$

where $r_* = \frac{1}{2}(r_4 + r_7) \in (s_0, s_1)$ (see (2.36)). Therefore, by analyticity,

$$u(x_1, t) = u(2r_* - x_1, t) \quad \text{for any } (x_1, t) \in [r_4, r_7] \times [\tau, \infty).$$

This implies that $\partial_{x_1} u(r_*, t) = 0$ for any $t \in [\tau, \infty)$. Namely, $r_* \in \mathfrak{Z}$ (see (2.56)), which contradicts the fact that $\mathfrak{Z} \cup \{a, b\} = \{s_j\}_{j=0}^n$. This completes the proof. $\square$

In view of Lemma 2.3, (2.58), (2.59), and (2.61), by using the strong maximum principle we see that $\partial u / \partial x_1$ never vanishes in $\bigcup_{j=0}^{n-1} (s_j, s_{j+1}) \times (\tau, \infty)$. Finally, by the method of characteristics, we see that for each $j = 0, ..., n-1$, $\partial\Omega \cap \{x \in \mathbb{R}^N : s_j < x_1 < s_{j+1}\}$ consists of lines parallel to the x_1-axis such that x_1 varies from s_j to s_{j+1}. This implies that the boundary $\partial\Omega$ consists of at most (a-1), (a-2), (a-3), and (a-4) in the conclusion (ii) (a) of Theorem 1. (Then we note that $\int_\Omega \varphi \, dx = 0$ if and only if $\int_{s_0}^{s_1} \varphi(x_1) \, dx_1 = 0$.) The proof of Theorem 1 is now completed.

Completion of the proof of Theorem 2. Next we consider Theorem 2. Since $u \in C^\infty(\overline{\Omega} \times (0, \infty))$ and $\partial\Omega$ is smooth, by using the boundary condition of (1.3), Lemma 2.2, and the strong maximum principle, we immediately have (2.55), as in case (b) of Theorem 1. Furthermore, since $\partial\Omega$ is smooth, in view of (a) and (b) we see that (ii) of Theorem 2 holds, which completes the proof of Theorem 2.

Completion of the proof of Theorem 3. Finally, let us consider Theorem 3. In view of (a) and (b), it follows from Lemma 2.2 combined with the boundary condition of (1.4) that the domain Ω must be a ball. Let $\Omega = \{x \in \mathbb{R}^N : r < b\}$, where $r = |x| = (x_1^2 + \cdots + x_N^2)^{\frac{1}{2}}$ for $x = (x_1, ..., x_N)$. It remains to show that $\partial_r u$ is negative in $(0, b) \times (\tau, T)$. Note that the equation $\partial_t \beta(u) = \Delta u$ may be degenerate or singular parabolic depending on the behaviour of $\beta'(s)$ as $s \downarrow 0$, and therefore it is not clear whether the classical derivative $\partial_r u$ exists on $\partial\Omega \times (0, T)$ because

$u = 0$ there. We can overcome this obstruction by using a standard approximation technique. More precisely, for each $\varepsilon \in (0, \frac{1}{3}b)$, take a function $\chi_\varepsilon \in C^\infty([0,\infty))$ satisfying

$$\chi_\varepsilon' \leqq 0 \text{ on } [0,\infty), \text{ and } \chi_\varepsilon(r) = \begin{cases} 1 & \text{if } 0 \leqq r \leqq b - 2\varepsilon, \\ 0 & \text{if } r \geqq b - \varepsilon. \end{cases}$$

Consider the problem

$$\text{(2.63)} \qquad \begin{cases} \partial_t \beta(v) = \Delta v & \text{in } \Omega \times (\tau, \infty), \\ v(x,\tau) = u(x,\tau)\chi_\varepsilon(|x|) + \varepsilon & \text{in } \Omega, \\ v = \varepsilon & \text{on } \partial\Omega \times (\tau,\infty). \end{cases}$$

(This problem is useful for showing the existence of solutions of the initial-Dirichlet problems for the degenerate or singular parabolic equation $\partial_t \beta(u) = \Delta u$.) Then by the theory of quasilinear uniformly parabolic equations (see [17]), there exists a unique bounded classical solution $v = v_\varepsilon \in C^\infty(\overline{\Omega} \times [\tau, \infty))$ of (2.63) satisfying

$$\varepsilon \leqq v_\varepsilon \leqq \max_{x \in \Omega} u(x,\tau) + \varepsilon \quad \text{in } \overline{\Omega} \times [\tau, \infty).$$

It follows from this inequality combined with the regularity result of [23] that the family $\{v_\varepsilon\}_{0<\varepsilon<\frac{1}{3}b}$ is equicontinuous on each compact subset of $\Omega \times (\tau,\infty)$. By a diagonalization argument, the Arzela-Ascoli theorem, and the uniqueness of the solution u, we see that

$$v_\varepsilon \to u \text{ as } \varepsilon \to 0 \text{ uniformly on each compact subset of } \Omega \times (\tau, T).$$

Furthermore, since $u \in C(\overline{\Omega} \times [\tau, T))$ and $u > 0$ in $\Omega \times (\tau, T)$, by the theory of uniformly parabolic equations ([17]) this convergence implies in particular that

$$\text{(2.64)} \quad \partial_r v_\varepsilon \to \partial_r u \text{ as } \varepsilon \to 0 \text{ uniformly on each compact subset of } \Omega \times (\tau, T).$$

Observe that for $v_\varepsilon = v_\varepsilon(r,t)$

$$\partial_r v_\varepsilon(0,t) = 0 \quad \text{and} \quad \partial_r v_\varepsilon(b,t) \leqq 0 \quad \text{for any } t \geqq \tau.$$

It follows from Lemma 2.2 and the maximum principle that

$$\partial_r v_\varepsilon \leqq 0 \quad \text{in } [0,b] \times [\tau,\infty).$$

Therefore, we get from (2.64)

$$\text{(2.65)} \qquad \partial_r u \leqq 0 \quad \text{in } (0,b) \times (\tau, T).$$

Since $u > 0$ in $\Omega \times (\tau, T)$, we can apply the strong maximum principle to $\partial_r u$; and we see that $\partial_r u < 0$ in $(0,b) \times (\tau,T)$. This completes the proof of Theorem 3.

3 Concluding remarks

We offer three remarks in this final section.

Remark 1. In case (ii) (a) of Theorem 1, $u = u(x_1, t)$ need not be monotone with respect to $x_1 \in [a, b]$, nor is the domain Ω necessarily of the form $(a, b) \times \tilde{\Omega}$ for some bounded Lipschitz domain $\tilde{\Omega}$ in $\mathbb{R}^{N-1}$. Namely, we give an example where $n = 2$ and (a-3) is nonempty. For simplicity, let $N = 2$. Let $u = u(x_1, t)$ be the unique solution of

$$\text{(3.1)} \qquad \begin{cases} \partial_t u = \partial_{x_1}^2 u & \text{in } (0,1) \times (0,\infty), \\ \partial_{x_1} u = 0 & \text{on } \{0,1\} \times (0,\infty), \\ u(x_1, 0) = u_0(x_1) & \text{in } (0,1), \end{cases}$$

where $u_0 = u_0(x_1)$ is an arbitrary C^1 function on $[0, 1]$ satisfying

$$\text{(3.2)} \qquad \partial_{x_1} u_0(0) = \partial_{x_1} u_0(1) = 0, \ \int_0^1 u_0(x_1)\, dx_1 = 0, \ \text{and } \partial_{x_1} u_0 < 0 \text{ in } (0,1).$$

Then it follows from the maximum principle that $\partial_{x_1} u(x_1, t) < 0$ in $(0, 1) \times (0, \infty)$. By putting

$$\text{(3.3)} \qquad u(x_1, t) = u(2 - x_1, t) \ \text{ and } \ u_0(x_1) = u_0(2 - x_1) \ \text{ for any } x_1 \in [1, 2],$$

we see that

$$\text{(3.4)} \qquad \begin{cases} \partial_t u = \partial_{x_1}^2 u & \text{in } (0,2) \times (0,\infty), \\ \partial_{x_1} u = 0 & \text{on } \{0,1,2\} \times (0,\infty), \\ u(x_1, 0) = u_0(x_1) & \text{in } (0,2), \\ \partial_{x_1} u \begin{cases} < 0 & \text{in } (0,1) \times (0,\infty), \\ > 0 & \text{in } (1,2) \times (0,\infty). \end{cases} \end{cases}$$

Let Ω be the bounded Lipschitz domain in $\mathbb{R}^2$ defined by

$$\text{(3.5)} \qquad \Omega = ((0,1] \times (0,1)) \cup ((1,2) \times (0,2)).$$

Then $(a, b) = (0, 2)$. Put $u(x, t) = u(x_1, t)$ and $\varphi(x) = u_0(x_1)$ for any $(x, t) = (x_1, x_2, t) \in \Omega \times [0, \infty)$. Then u solves (1.2), $\varphi \not\equiv 0$, and $\int_\Omega \varphi \, dx = 0$. Here we see that u is not monotone in x_1 on $(0, 2)$; Ω is not a rectangle; all the spatial level curves of u are invariant with respect to $t \in (0, \infty)$; and, of course, u is not necessarily a separable solution.

Remark 2. In case (ii) (b) of Theorem 1, when $2 \leqq k \leqq N-1$, Ω is not necessarily of the form $\{x \in \mathbb{R}^N : x = (r\omega, y),\ \omega \in S,\ a < r < b,\ y \in \tilde{\Omega}\}$, where S is a domain in S^{k-1} and $\tilde{\Omega}$ is a bounded domain in $\mathbb{R}^{N-k}$. Furthermore, Ω is not necessarily a finite union of such sets. We give an example where $N = 3$, $k = 2$, and $0 < a < b$. Let

$$\Omega = \{x \in \mathbb{R}^3 : x = (r\cos\theta, r\sin\theta, y),\ a < r < b,\ 0 < y < \pi/3,\ y < \theta < \pi/2\}.$$

Then $\partial\Omega$ consists of the parts (b-1), (b-2), (b-3). In particular, $\partial\Omega$ contains a part of a *helicoid*

$$x = x(\theta, r) = (r\cos\theta, r\sin\theta, \theta) \quad (a \leqq r \leqq b,\ 0 \leqq \theta \leqq \pi/3).$$

Therefore, this domain Ω is not of the above form.

Remark 3. In Theorems 1, 2, and 3, it is natural to have $\tau > 0$. Namely, in case (ii) of the theorems we may have a case where the solution u becomes a monotone function with respect to r after a finite time $\tau > 0$ for some non-monotone initial data. In such a case, the invariance condition of spatial level surfaces of u holds after a finite time. Ni and Sacks [19] deal with such problems. Denote by $B_R(0)$ an open ball in $\mathbb{R}^N$ centered at the origin with radius $R > 0$. In particular in Theorem 3, if the domain Ω equals $B_R(0)$ for some $R > 0$ and the initial data $\varphi(x)(= u(x, 0))$ is radially symmetric and nonnegative, and if $\beta(s) \equiv s$, then there exists a time $\tau > 0$ such that $\partial_r u < 0$ in $(0, R] \times [\tau, \infty)$.

Here let $\Omega = B_R(0)$ for some $R > 0$ and let us consider Theorem 1. Suppose that φ is radially symmetric, $\varphi \in C^0([0, R])$, $\int_\Omega \varphi\, dx = 0$, and

$$\varphi(r) \begin{cases} > 0 & \text{if } 0 \leqq r < r_0 \\ < 0 & \text{if } r_0 < r \leqq R \end{cases} \tag{3.6}$$

for some $r_0 \in (0, R)$. Let $u = u(r, t)$ be the unique solution of (1.2). Then we have

Proposition 3.1. *There exists a time* $\tau > 0$ *such that*

$$\partial_r u < 0 \quad \textit{in } (0, R) \times [\tau, \infty).$$

Proof. Let $z = z(t)$ be the number of zeros of the function $r \mapsto u(r, t)$ in the interval $[0, R]$ for each $t \geqq 0$. Then, in view of the homogeneous Neumann boundary condition, since $z(0) = 1$, and using the results of Angenent concerning the zero sets of solutions of one-dimensional parabolic equations (see [4] and [5, Sections 3 and 4, pp. 342–346], and also [14, Proposition 3.2(A), p. 580]), we see that

$$(3.7) \qquad z(t) = 1, \quad u(r(t), t) = 0, \quad \text{and} \quad \partial_r u(r(t), t) < 0 \text{ for any } t > 0$$

for some smooth function $r = r(t)$ with $0 < r(t) < R$. Take an arbitrary small number $\tau_0 > 0$. Then from (3.7) we can define

$$(3.8) \qquad \delta = \min\{|u(r, \tau_0)| : \partial_r u(r, \tau_0) = 0 \text{ and } 0 \leqq r \leqq R\} > 0.$$

Since $\int_\Omega \varphi \, dx = 0$ and problem (1.2) is solved by an eigenfunction expansion, $u \to 0$ as $t \to \infty$ uniformly in Ω. Therefore, we can choose $\tau > \tau_0$ sufficiently large to get

$$(3.9) \qquad |u(r, \tau)| < \delta \quad \text{for any } r \in [0, R].$$

Let us suppose that there exists a point $(r_*, t_*) \in \bigcup_{t \geqq \tau} [0, r(t)) \times \{t\}$ such that

$$(3.10) \qquad \partial_r u(r_*, t_*) = 0 \quad \text{and} \quad \partial_r^2 u(r_*, t_*) \geqq 0.$$

Since we have precise information on the zero set of $\partial_r u$ by using the results of Angenent once more, we can trace a path along the bottom of the valley of the graph of the function u back to the past. (Precise information on the zero set of $\partial_r u$ near $r = 0$ is in [5, Sections 3 and 4, pp. 342–346], and the information for $0 < r \leqq R$ is in [4].) Therefore, as in [15, Lemma 3.2, p. 822] (see also [19, Proposition 3, p. 462]), in view of the homogeneous Neumann boundary condition we see that there exists a continuous function $\eta : [\tau_0, t_*] \to [0, R]$ satisfying

(1) $\eta(t_*) = r_*$, and $0 \leqq \eta(t) < r(t)$ for any $t \in [\tau_0, t_*]$,

(2) $\partial_r u(\eta(t), t) = 0$ and $\partial_r^2 u(\eta(t), t) \geqq 0$ for any $t \in [\tau_0, t_*]$,

(3) if $\eta(t_0) = 0$ for some $t_0 \in [\tau_0, t_*]$, then $\eta(t) = 0$ for any $t \in [t_0, t_*]$,

(4) η is smooth except at most at finitely many points.

Then, as long as $\eta(t) > 0$, except at most at finitely many points

$$(3.11) \qquad \begin{aligned} \frac{d}{dt}(u(\eta(t), t)) &= \partial_t u(\eta(t), t) + \partial_r u(\eta(t), t)\eta'(t) \\ &= \partial_r^2 u(\eta(t), t) \geqq 0. \end{aligned}$$

On the other hand, if $\eta(t_0) = 0$ for some $t_0 \in [\tau_0, t_*)$, then by (3), $\eta(t) = 0$ for any $t \in [t_0, t_*)$. Hence we have for $t \in [t_0, t_*)$

$$(3.12) \qquad \frac{d}{dt}(u(\eta(t), t)) = \Delta u(0, t) \geqq 0.$$

Therefore, in view of (3.8) and (3.9), we get from (3.11) and (3.12)

$$\delta > u(\eta(\tau), \tau) \geqq u(\eta(\tau_0), \tau_0) \geqq \delta.$$

This is a contradiction. It follows that if $\partial_r u(r,t) = 0$ for some point $(r,t) \in \bigcup_{t \geqq \tau}[0, r(t)) \times \{t\}$, then $\partial_r^2 u(r,t) < 0$. Therefore, since $\partial_r u(0,t) = 0$ for any $t > 0$, $\partial_r u < 0$ if $t \geqq \tau$ and $0 < r < r(t)$. Similarly, if we suppose that there exists a point $(r_*, t_*) \in \bigcup_{t \geqq \tau}(r(t), R] \times \{t\}$ such that

$$\partial_r u(r_*, t_*) = 0 \quad \text{and} \quad \partial_r^2 u(r_*, t_*) \leqq 0,$$

then we can get a contradiction and see that $\partial_r u < 0$ if $t \geqq \tau$ and $r(t) < r < R$. This completes the proof of Proposition 3.1. □

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