HOLOMORPHIC SYNTHESIS OF MONOGENIC FUNCTIONS OF SEVERAL QUATERNIONIC VARIABLES

By

VICTOR P. PALAMODOV

Abstract. The system of differential equations for polymonogenic functions of several quaternionic variables is an analogue of the $\bar{\partial}$-equation in complex analysis. We give a representation of polymonogenic functions by means of integration of a family of σ-holomorphic functions as σ runs over the variety Σ of all complex structures $\mathbb{H} \cong \mathbb{C}^2$ which are consistent with the metric and an orientation in $\mathbb{H}$. The variety Σ is isomorphic to the manifold of all proper right ideals in the complexified quaternionic algebra and has a natural complex analytic structure. We construct a $\bar{\partial}$-complex on Σ that provides a resolvent for the sheaf of polymonogenic functions.

1 Introduction

Hamilton's algebra $\mathbb{H}$ of quaternions can be supplied with complex coordinates by means of a $\mathbb{R}$-linear bijection $\mathbb{H} \to \mathbb{C}^2$. This approach was systematically exploited for study of the Yang–Mills equations, which intrinsically relate to quaternions [3]. There is no distinguished complex structure in $\mathbb{H}$, but a family. Fix a Euclidean structure and an orientation in $\mathbb{H}$ and consider a linear isometry $\mathbb{H} \to \mathbb{C}^2$ which is consistent with the orientation. It defines a complex structure σ in $\mathbb{H}$. The variety Σ of all such complex structures in $\mathbb{H}$ is equivalent to the sphere S^2 ([3]). We show here that these structures are indispensable for the study of monogenic functions of a quaternionic variable, which play the role of holomorphic functions in quaternionic analysis. Indeed, there is a bijection of the variety Σ to the set of proper right ideals R of the complexified quaternion algebra $\mathbb{H}_{\mathbb{C}}$ such that for any structure $\sigma \in \Sigma$, any σ-holomorphic function h and any element r of the corresponding ideal R, the product hr is a monogenic function. The linear span $Q(U, R)$ of functions of this form is in fact an algebra (Section 2). We show that the space of all monogenic functions in a convex open set $U \subset \mathbb{H}$ is equal to the integral over Σ of the family of algebras $Q(U, R)$. For a polymonogenic function of n quaternionic variables we get a similar representation by means of a family of holomorphic functions of $2n$ complex variables. Unlike holomorphic functions of several variables, polymonogenic functions are not generated by tensor products of monogenic functions. The reason, of course, is the noncommutativity of the Hamilton algebra.

Recall that the algebra $\mathbb{H}$ of quaternions is an extension of the field $\mathbb{R}$ by the imaginary units $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. We henceforth denote the generators by $e_1 = \mathbf{i}$, $e_2 = \mathbf{j}$, $e_3 = \mathbf{k}$ and the unit element of $\mathbb{H}$ by e_0 for convenience. Recall the multiplication table in $\mathbb{H}$: $e_1e_2 = e_3$, $e_2e_3 = e_1$, $e_3e_1 = e_2$; $e_je_i = -e_ie_j$, $i \neq j$, $ij > 0$; $e_1^2 = e_2^2 = e_3^2 = -1$. The *complexified* quaternion algebra is the tensor product

$$\mathbb{H}_{\mathbb{C}} = \mathbb{H} \otimes_{\mathbb{R}} \mathbb{C},$$

whereas the algebras $\mathbb{C}$, $\mathbb{H}$ are considered as independent extensions of the field $\mathbb{R}$. Hamilton called an element $q \in \mathbb{H}_{\mathbb{C}}$ a *biquaternion* ([5]), since it can be written in the form $q = q' + \sqrt{-1}q''$, where $\sqrt{-1}$ denotes the imaginary unit in the centre $\mathbb{C}$ of the algebra $\mathbb{H}_{\mathbb{C}}$ and q', q'' are in $\mathbb{H}$. The equation

$$\frac{\partial u}{\partial \bar{q}} = f \tag{1.1}$$

for the quaternion-valued functions $u = u_0 + u_1e_1 + u_2e_2 + u_3e_3$, $f = \sum_0^4 f_ie_i$, is a formal analogue of the Cauchy–Riemann system, where

$$\frac{\partial}{\partial \bar{q}} = \frac{\partial}{\partial x_0} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}$$

is the quaternionic $\bar{\partial}$-operator. Here the partial derivatives commute with the imaginary units e_1, e_2, e_3; and the quaternionic coefficients of this operator act by left multiplication. Equation (1.1) was studied first by R. Fueter in several papers beginning with [4]. We extend it for $\mathbb{H}_{\mathbb{C}}$-valued functions. Equation (1.1) is equivalent to the following system for $\mathbb{C}$-valued functions:

$$\begin{aligned}
\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3} &= f_0,\\
\frac{\partial u_0}{\partial x_1} + \frac{\partial u_1}{\partial x_0} - \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} &= f_1,\\
\frac{\partial u_0}{\partial x_2} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_0} - \frac{\partial u_3}{\partial x_1} &= f_2,\\
\frac{\partial u_0}{\partial x_3} - \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_3}{\partial x_0} &= f_3
\end{aligned} \tag{1.2}$$

We call both systems (1.1), (1.2) the Cauchy–Fueter system. A solution of this system with $f = 0$ is called a *left monogenic* (or regular) function of the quaternionic variable q. Any monogenic function is an analytic function of four real variables since the Cauchy–Fueter system is of elliptic type. See [11] for a survey of the theory of regular functions.

Example. The function $u = \tilde{q} \equiv x_0 - x_1 e_1 - x_2 e_2 - x_3 e_3$ is not monogenic since $\partial \tilde{q} / \partial \tilde{q} = 4$, whereas the functions $e_j \tilde{q}$, $j = 1, 2, 3$ are monogenic. Thus the set of all monogenic functions is not a left $\mathbb{H}$-module and is not an algebra. On the other hand, it is always a right $\mathbb{H}_{\mathbb{C}}$-module.

For an arbitrary natural number n, consider the following system of equations with quaternionic variables $q_1, ..., q_n$:

$$\frac{\partial u}{\partial \tilde{q}_1} = \cdots = \frac{\partial u}{\partial \tilde{q}_n} = 0. \tag{1.3}$$

This is an analogue of the $\bar{\partial}$-system in complex analysis. Here u is a function in an open set $U \subset \mathbb{H}^n \cong \mathbb{R}^{4n}$ with values in $\mathbb{H}_{\mathbb{C}}$; thus (1.3) is a system of $4n$ first-order equations with 4 unknown $\mathbb{C}$-valued functions. This system was studied in [1], [10], [2]. We call a solution of (1.3) a *polymonogenic* function.

A right (or left) ideal R in the algebra $\mathbb{H}_{\mathbb{C}}$ is called *proper*, if $\{0\} \neq R \neq \mathbb{H}_{\mathbb{C}}$. Consider the variety of all proper right ideals R in $\mathbb{H}_{\mathbb{C}}$. We denote this variety by $\mathcal{R}(\mathbb{H})$ for short. This variety is a 2-sphere with a canonical complex algebraic structure (Proposition 3.6). Consider the subbundle $r : \mathcal{R} \to \mathcal{R}(\mathbb{H})$ of π such that the fibre of r over $R \in \mathcal{R}(\mathbb{H})$ is equal to the ideal $R \subset \mathbb{H}_{\mathbb{C}}$ (tautological bundle). Take the trivial bundle $\pi : \mathbb{H}_{\mathbb{C}} \times \mathcal{R}(\mathbb{H}) \to \mathcal{R}(\mathbb{H})$. For an open set $U \subset \mathbb{H}^n$, we denote by $Q(U)$ the space of all polymonogenic functions in U and by $\mathcal{Q}$ the sheaf of germs of generalized functions $u : \mathbb{H}^n \times \mathcal{R}(\mathbb{H}) \to \mathbb{H}_{\mathbb{C}}$ which are polymonogenic in fibres of the bundle π. Take an open set $U \subset \mathbb{H}^n$ and consider the restriction of this sheaf to $U \times \mathcal{R}(\mathbb{H})$. We denote $\mathcal{Q}(U) = \pi_*(\mathcal{Q}|U \times \mathcal{R}(\mathbb{H}))$; this is the sheaf of germs on $\mathcal{R}(\mathbb{H})$ of generalized function with values in $Q(U)$. Set

$$\mathcal{Q}(U)^{*,*} = \mathcal{Q}(U) \otimes_{\mathcal{E}} \mathcal{E}^{*,*},$$

where $\mathcal{E}^{p,q}$ denotes the sheaf of germs of smooth p, q-forms on $\mathcal{R}(\mathbb{H}) \cong \mathrm{P}_1(\mathbb{C})$, $p, q = 0, 1$, with values in $\mathbb{C}$, $\mathcal{E} = \mathcal{E}^{0,0}$. There is a trivial connection ∇ in the bundle π. The operator $\bar{\partial}$ in the complex $\mathcal{E}^{*,*}$ induces a sheaf morphism

$$\nabla(\bar{\partial}) : \mathcal{Q}(U)^{1,0} \to \mathcal{Q}(U)^{1,1}.$$

Let $R \in \mathcal{R}(\mathbb{H})$; we denote by $Q(U, R)$ the space of R-valued polymonogenic functions in U. This is an *algebra* with respect to pointwise multiplication (unlike the space $Q(U)$, which is not an algebra). The family of algebras $Q(U, R)$, $R \in$

$\mathcal{R}(\mathbb{H})$, is a fibre bundle on $\mathcal{R}(\mathbb{H})$. Denote by $\mathcal{Q}(U, \mathcal{R})$ the subsheaf of $\mathcal{Q}(U)$ of germs of generalized sections of this bundle. Consider the sheaf of $\mathcal{Q}(U, \mathcal{R})$-valued differential forms $\mathcal{Q}(U, \mathcal{R})^{*,*} = \mathcal{Q}(U, \mathcal{R}) \otimes_{\mathcal{E}} \mathcal{E}^{*,*}$. This subsheaf is invariant under the operator $\nabla(\bar\partial)$, since $\mathcal{R}$ is an analytic fibre bundle. The section space $\Gamma(\mathcal{R}(\mathbb{H}), \mathcal{Q}(U, \mathcal{R})^{*,*})$ is the space of forms on $\mathcal{R}(\mathbb{H})$ with values in the bundle with fibres $Q(U, R)$. Hence the following sequence is well-defined:

$$(1.4) \qquad 0 \to \Gamma(\mathcal{R}(\mathbb{H}), \mathcal{Q}(U, \mathcal{R})^{1,0}) \overset{\nabla(\bar\partial)}{\to} \Gamma(\mathcal{R}(\mathbb{H}), \mathcal{Q}(U, \mathcal{R})^{1,1}) \overset{\int}{\to} \mathcal{Q}(U) \to 0;$$

here $\int$ denotes the integration operator

$$\omega = \int_{\mathcal{R}(\mathbb{H})} \omega^{1,1}.$$

Our main result is

Theorem 1.1. *The sequence (1.4) is exact for any convex open $U \subset \mathbb{H}^n$.*

We see that $\int \nabla(\bar\partial) = 0$ since $\mathcal{R}(\mathbb{H})$ is compact. To see that the first mapping is an injection, let $\omega \in \mathcal{Q}(U, \mathcal{R})^{1,0}$ be a solution of the equation $\nabla(\bar\partial)\omega = 0$. This means the equation $\bar\partial\omega(\cdot; q_1, ..., q_n) = 0$ for any $q_1, ..., q_n \in \mathbb{H}^n$, from whence it follows that $\omega(\cdot; q_1, ..., q_n)$ is a holomorphic form on $\mathrm{P}_1(\mathbb{C})$. This form vanishes, which shows that $\omega = 0$. In Sections 7–10 we shall prove exactness of (1.4) in the third and second terms.

Example. The function

$$E(q) = \frac{1}{2\pi^2 q^2 \tilde q} \equiv \frac{\tilde q}{2\pi^2 (q\tilde q)^2}$$

is a fundamental solution of the Cauchy–Fueter operator, since it satisfies the equation

$$\frac{\partial E}{\partial \tilde q} = \delta_0,$$

where δ_0 is the delta-function in $\mathbb{H}$. This is a monogenic function in an arbitrary open halfspace $H \subset \mathbb{H}$. According to Theorem 1.1, there exists a form $\omega_H \in \Gamma(\mathcal{R}(\mathbb{H}), \mathcal{Q}(H, \mathcal{R}))$ of type $(1,1)$ such that $\int \omega_H = E$ in H. Suppose that this representation is unique; then we have $\omega_H = \omega_G$ for arbitrary halfspaces H and G. It follows that the family of forms defines a form ω in the domain $\mathbb{H}_0 \doteq \mathbb{H} \setminus \{0\}$. By Proposition 2.1, the space $Q(\mathbb{H}_0, R)$ is for any $R \in \mathcal{R}$ isomorphic to the space of σ-holomorphic functions $h : \mathbb{H}_0 \to \mathbb{C}^2$ for the corresponding complex structure σ. These are holomorphic functions of two variables and hence have holomorphic extensions to the whole space $\mathbb{H}$. Therefore ω would have an extension to $\mathbb{H}$ as well. But this is impossible since the function $E = \int \omega$ has a singularity at the origin. This shows how sizeable is the kernel of the mapping $\int$ in (1.4).

Note that the substitution $x_0 \mapsto \sqrt{-1}ct$ transforms the Cauchy–Fueter system (1.1) into the hyperbolic system

$$\left(-\frac{\sqrt{-1}}{c}\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x_1} + e_2\frac{\partial}{\partial x_2} + e_3\frac{\partial}{\partial x_3}\right) u = f \tag{1.5}$$

in the space-time $\mathbb{R}^4$, which has several physical interpretations. One of these is the basic system of the relativistic mechanics in the absence of electromagnetic forces; another is Maxwell's system. It can be treated as well as the Dirac–Weyl operator acting on spinors in space-time. The algebraic structure of the hyperbolic system (1.5) is equivalent to that of the system (1.1); but its analysis is, of course, different. Nevertheless, the system (1.5) can be studied along lines parallel to the forthcoming analysis of (1.1). We do not touch on the hyperbolic counterpart of the theory, which is beyond the framework of the present paper.

I thank Prof. L.Avramov and Prof. D.Struppa for stimulating discussions.

2 Complex structures in the quaternionic line

Endow the algebra $\mathbb{H}$ with the Euclidean structure $|q|^2 \doteqdot x_0^2 + x_1^2 + x_2^2 + x_3^2$, where $q = \sum x_k e_k$, and with the orientation by means of the basis e_1, e_2, e_3, e_0. We now construct a mapping from the variety Σ of all complex structures defined by positively oriented isometries $\mathbb{H} \to \mathbb{C}^2$ to the variety $\mathcal{R}(\mathbb{H})$ of all proper right ideals in $\mathbb{H}_\mathbb{C}$. Let S^2 be the unit sphere in the Euclidean subspace $\operatorname{Im}\mathbb{H} = \{q \in \mathbb{H},\ x_0 = 0\}$. Given a point $s = (s_1, s_2, s_3) \in S^2$, consider the quaternion

$$q(s) = \sqrt{-1} + s_1e_1 + s_2e_2 + s_3e_3 \in \mathbb{H}_\mathbb{C}. \tag{2.1}$$

Take the right ideal $R(q(s)) = q(s) \cdot \mathbb{H}_\mathbb{C}$ generated by this quaternion. It is proper, since the quaternion $q(s)$ is not invertible. On the other hand, there is a complex structure σ with complex coordinates

$$z_\sigma = x_0 + \sqrt{-1}\langle s, x\rangle, \quad w_\sigma = \langle t, x\rangle - \sqrt{-1}\langle v, x\rangle \tag{2.2}$$

in $\mathbb{H}$, where t, v are vectors in $\operatorname{Im}\mathbb{H}$ such that the triple s, t, v is a positively oriented orthonormal frame. These coordinates define a complex structure σ which is consistent with the orientations and with the metrics, since $|q|^2 = |z_\sigma(q)|^2 + |w_\sigma(q)|^2$. The complex coordinates do not depend on the choice of t, v up to a rotation in the w_σ-plane; hence the complex structure σ depends only on s. Thus we have the mapping $\Sigma \cong S^2 \to \mathcal{R}(\mathbb{H})$. We show in the next section that this is a bijection.

For arbitrary n, we endow $\mathbb{H}^n$ with the analytic structure $\mathbb{H}^n \cong \mathbb{C}^{2n}$ by means of coordinates $z_{\sigma,j}, w_{\sigma,j}$; $j = 1, ..., n$, which are related for each j to the real coordinates x_{jk}, $k = 0, 1, 2, 3$ as in (2.2).

Proposition 2.1. *Given a complex structure $\sigma \in \Sigma$, two arbitrary $\mathbb{C}$-independent elements r_1, r_2 of the corresponding right ideal $R = R(q(s))$ and an arbitrary nonempty open set $U \subset \mathbb{H}^n$, the functions $h_1, h_2 : U \to \mathbb{C}$ are σ-holomorphic if and only if the sum $u = h_1 r_1 + h_2 r_2$ is polymonogenic.*

Proof. Choose an orthogonal matrix $A = \{a_{ij}\} \in \mathrm{SO}(3, \mathbb{R})$, whose first row coincides with s, and apply it to the imaginary units in $\mathbb{H}$ and simultaneously to the real coordinates in $\mathbb{H}^n$. We get new units and coordinates:

$$e'_i = \sum a_{ij} e_j, \quad x'_{k0} = x_{k0}, \; x'_{ki} = \sum_j a_{ij} x_{kj}, \; k = 1, ..., n.$$

The fields $\partial/\partial x_i$ transform by the conjugated representation of the orthogonal group. Therefore,

$$\frac{\partial}{\partial \bar{q}_i} = \frac{\partial}{\partial x_{i0}} + e'_1 \frac{\partial}{\partial x'_{i1}} + e'_2 \frac{\partial}{\partial x'_{i2}} + e'_3 \frac{\partial}{\partial x'_{i3}} = \frac{\partial}{\partial \bar{q}'_i},$$

i.e., the system (1.3) keeps its form. Now we have $q(s) = \sqrt{-1} + e'_1$. By Proposition 3.2, we can set $r_1 = q(s), r_2 = q(s)e'_2 = \sqrt{-1}e'_2 + e'_3$ since this is a basis in R. Then we have $u = \sum u_k e_k$, where $u_0 = \sqrt{-1}h_1$, $u_1 = h_1$, $u_2 = \sqrt{-1}h_2, u_3 = h_2$. Substituting the function u in (1.2) with $x_j = x_{kj}$, $k = 1, ..., n$ yields

$$\left(\frac{\partial}{\partial x_{k0}} + \sqrt{-1}\frac{\partial}{\partial x'_{k1}}\right) h = \left(\frac{\partial}{\partial x'_{k2}} - \sqrt{-1}\frac{\partial}{\partial x'_{k3}}\right) h = 0, \quad k = 1, ..., n \tag{2.3}$$

for $h = h_1, h_2$. Conversely, (1.3) follows from (2.3). A function h is σ-holomorphic if and only if it satisfies (2.3). □

For an open set $U \subset \mathbb{H}^n$ we denote by $H_\sigma(U)$ the space of σ-holomorphic functions in U, i.e., of solutions to (2.3).

Corollary 2.2. *For any $\sigma \in \Sigma$, any $\mathbb{C}$-basis r_1, r_2 of the corresponding ideal R and any open $U \subset \mathbb{H}^n$, the mapping*

$$H_\sigma(U)^2 \to Q(U, R), \quad (h_1, h_2) \mapsto h_1 r_1 + h_2 r_2$$

is a bijection.

In fact, we can write any function $u \in Q(U, R)$ in a form $h_1 r_1 + h_2 r_2$ with some $\mathbb{C}$-valued coefficients $h_{1,2}$. They are σ-holomorphic by Proposition 2.1.

Corollary 2.3. *$Q(U, R)$ is an algebra for any proper right ideal R.*

Proof. Indeed, the product of holomorphic functions is again a holomorphic function and $r_i r_j \in R$. □

Example. For $s = (1, 0, 0)$, the ideal R has the basis $r_1 = \sqrt{-1} + e_1$, $r_2 = \sqrt{-1}e_2 + e_3$; hence the function $zr_1 + wr_2$ is monogenic for the σ-holomorphic coordinates $z = x_0 + \sqrt{-1}x_1$, $w = x_2 - \sqrt{-1}x_3$. For the opposite point $-s$, the complex conjugated quaternions $\bar{r}_1$, $\bar{r}_2$ form a basis in the ideal $R(q(-s))$ and the conjugated functions $\bar{z}$, $\bar{w}$ are σ-holomorphic coordinates. Hence $2e_3 q = zr_2 + wr_1 + \overline{zr_2 + wr_1}$, which gives a representation of the monogenic function $e_3 q$ by means of holomorphic functions. This is in fact a particular case of the representation in Theorem 1.1 with a distribution $u^{1,1}$ supported by two points $(\pm 1, 0, 0)$ in the sphere.

3 Ideals in $\mathbb{H}_\mathbb{C}$

We can write an arbitrary element ζ of the complexified quaternion algebra $\mathbb{H}_\mathbb{C}$ in the form $\zeta = \sum \zeta_i e_i$, $\zeta_i \in \mathbb{C}$. The notations $\operatorname{Re}\zeta = \zeta_0, \operatorname{Im}\zeta = \zeta_1 e_1 + \zeta_2 e_2 + \zeta_3 e_3$ are related to the quaternionic structure in $\mathbb{H}_\mathbb{C}$. The numbers $\operatorname{Re}\zeta, \operatorname{Im}\zeta$ are called the scalar and vector parts of ζ and $\tilde{\zeta} = \operatorname{Re}\zeta - \operatorname{Im}\zeta$ is called the conjugated quaternion. On the other hand, the operations $\Re$, $\Im$ and $\bar{\ }$ relate to the $\mathbb{C}$-structure. Consider the following quadratic cone in $\mathbb{C}^4 \cong \mathbb{H}_\mathbb{C}$:

$$V = \{\zeta : \zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2 \equiv \zeta\tilde{\zeta} \equiv \tilde{\zeta}\zeta = 0\}.$$

Proposition 3.1. *The cone V is the set of all quaternions $\zeta \in \mathbb{H}_\mathbb{C}$ that have no inverse.*

Proof. For any $\zeta \in \mathbb{H}_\mathbb{C}$, the product $\zeta\tilde{\zeta}$ belongs to the subfield $\mathbb{C}$, which is the centre of the algebra $\mathbb{H}_\mathbb{C}$. If $\zeta \in \mathbb{H}_\mathbb{C} \setminus V$, the quaternion $\eta = (\zeta\tilde{\zeta})^{-1}\zeta$ is its two-sided inverse. On the other hand, if $\zeta \in V$, then for any $\eta \in \mathbb{H}_\mathbb{C}$, the products $\eta\zeta$ and $\zeta\eta$ are in V as well, since $(\eta\zeta)\widetilde{\eta\zeta} = \eta\zeta\tilde{\zeta}\tilde{\eta} = 0$. Therefore, $\eta\zeta \neq 1, \zeta\eta \neq 1$, which means that ζ has no inverse. □

The operation $\zeta \mapsto \tilde{\zeta}$ is an involution in $\mathbb{H}_\mathbb{C}$, i.e., $\widetilde{\zeta\eta} = \tilde{\eta}\tilde{\zeta}$. It transforms an arbitrary left ideal L to the right ideal $R = \tilde{L}$ and vice versa. Therefore, one can change right ideals to left ones in the following statements.

Proposition 3.2. *Any proper right (or left) ideal R is contained in V and* $\dim_{\mathbb{C}} R = 2$.

Proof. Any element $\zeta \in R$ is noninvertible; hence $\zeta \in V$. This implies the inclusion $R \subset V$. Since $\dim V = 3$, the dimension of the linear subspace R is ≤ 2. Suppose that $\dim R(\zeta) = 1$. Then, for an arbitrary $\zeta \in R$, the vectors $\zeta, \zeta e_1, \zeta e_2, \zeta e_3$ are collinear. This is not possible unless $\zeta = 0$. □

For an arbitrary $\zeta \in V$, we denote by $L(\zeta) = \mathbb{H}_{\mathbb{C}} \cdot \zeta$ the left ideal and by $R(\zeta) = \zeta \cdot \mathbb{H}_{\mathbb{C}}$ the right ideal in $\mathbb{H}_{\mathbb{C}}$.

Proposition 3.3. *For distinct proper right ideals R, R' in $\mathbb{H}_{\mathbb{C}}$ we have $R \cap R' = \{0\}$. Each proper right ideal R is principal, i.e., $R = R(\zeta)$ for some $\zeta \in V \setminus \{0\}$.*

Proof. Take an arbitrary element $\zeta \in R \setminus \{0\}$ and consider the principal right ideal $R(\zeta)$. It is proper and is contained in R. On the other hand, we have $\dim R = \dim R(\zeta)$, hence $R = R(\zeta)$. If $\zeta \in R'$, then the three right ideals coincide. □

Proposition 3.4. *For an arbitrary $\zeta \in V \setminus \{0\}$, the ideal $R(\zeta)$ coincides with the set of quaternions α such that $\tilde{\zeta}\alpha = 0$.*

Proof. The equation $\tilde{\zeta}\alpha = 0$ obviously holds for any $\alpha \in R(\zeta)$. For the inverse statement, we use the equation $\zeta\tilde{\eta} + \eta\tilde{\zeta} = 1$, where $\eta = \bar{\zeta}/2|\zeta|^2$, which follows from the identity $\tilde{\zeta}\bar{\zeta} + \zeta\tilde{\bar{\zeta}} = 2|\zeta|^2$. Multiply it by α to the right to get $\alpha = \zeta\tilde{\eta}\alpha \in R$. □

Proposition 3.5. *There is a holomorphic fibre bundle $\mathcal{R} : V \setminus \{0\} \to C$, whose fibres are planes $R(\zeta) \setminus \{0\}$ and whose base C is a nonsingular conic in $\mathbf{P}_2(\mathbb{C})$.*

Proof. Let $\mathbf{P}_4(\mathbb{C})$ be a projective closure of $\mathbb{H}_{\mathbb{C}} = \mathbb{C}^4$ and P be an arbitrary projective 2-subspace that does not contain the vertex of V and is not tangent to V. The intersection of P with the projective closure of V is a non-singular conic C. Any proper right ideal R is a subspace of $\mathbb{C}^4$ of dimension 2 and has at least one common point with P. The intersection $P \cap R$ cannot contain more than a single point since otherwise it would be a line in C. This is not possible since C is non-singular. It follows that the curve C parameterizes the variety $\mathcal{R}(\mathbb{H})$ of all proper right ideals. □

Proposition 3.6. *The mapping*

$$s = (s_1, s_2, s_3) \mapsto R(q(s)), \quad q(s) = \sqrt{-1} + s_1 e_1 + s_2 e_2 + s_3 e_3 \tag{3.1}$$

defines a diffeomorphism $S^2 \cong \mathcal{R}(\mathbb{H})$.

Proof. The ideal $R(q(s))$ is proper since $\tilde{q}(s)q(s) = 0$. This ideal does not contain another quaternion q with $\operatorname{Re} q = \sqrt{-1}$, $\Im \operatorname{Im} q = 0$. Indeed, if q were such a quaternion, then the equation $\tilde{q}(s)q = 0$ would imply $\operatorname{Im} q(s) \operatorname{Im} q = 1$. This would mean that the vector $\operatorname{Im} q$ belongs to the tangent plane to S^2 at the point $\operatorname{Im} q(s)$. This implies $q = q(s)$, since $|\operatorname{Im}| = 1$. Thus the mapping $S^2 \to \mathcal{C}$ is an injection. We show that it is a surjection, as well. Take an ideal $R \in \mathcal{R}(\mathbb{H})$ and choose an element $q \in R$, $\operatorname{Re} q = \sqrt{-1}$. Write $\operatorname{Im} q = u + \sqrt{-1}v$, $u, v \in \operatorname{Im} \mathbb{H}$. From the equation $0 = q\tilde{q} = -1 + |u|^2 - |v|^2 + 2\sqrt{-1}\langle u, v\rangle$ it follows that u is orthogonal to v and $|u|^2 - |v|^2 = 1$. Setting $r = (1 + |v|^2)^{-1}(1 - v)$, $a = qr$, we have $a \in R$ and $\operatorname{Re} a = \sqrt{-1}$, $\Im \operatorname{Im} a = 0$, because $v \times v = 0$. □

Remark 3.7. By means of the isomorphism $S^2 \cong \mathcal{C} \cong \mathrm{P}_1(\mathbb{C})$ we get a structure of the Riemann sphere in S^2. Compare the coordinates in the sphere with complex algebraic coordinates in the curve $\mathcal{C}$. For a point $s \in S^2$, we consider a right ideal $R = R(q(s))$, where $q(s)$ is given by (3.1). The intersection of $R(q(s))$ with the plane $\zeta = 1 + \zeta_1 e_1 + \zeta_2 e_2$ is a point with the coordinates $\zeta_1 = (s_2 s_3 - \sqrt{-1}s_1)(s_1^2 + s_2^2)^{-1}$, $\zeta_2 = (-s_1 s_3 - \sqrt{-1}s_2)(s_1^2 + s_2^2)^{-1}$ and the curve $\mathcal{C}$ is given by the equation $\zeta_1^2 + \zeta_2^2 + 1 = 0$. The function $\lambda = \zeta_1^{-1}(\zeta_2 + \sqrt{-1})$ is a projective coordinate in the curve. It relates to coordinates in the sphere $S^2 \subset \operatorname{Im} \mathbb{H}$ by the formula $\lambda = (s_1 + \sqrt{-1}s_3)/(1 + s_2)$, which is a standard stereographic projection from the sphere to a complex plane.

The $\mathbb{C}$-bundle $\mathcal{R}$ is of rank 2, consequently, it is a direct sum of two line bundles.

Proposition 3.8. *Any holomorphic isomorphism* $\varepsilon : \mathcal{R}(\mathbb{H}) \to \mathrm{P}_1(\mathbb{C})$ *induces a sheaf isomorphism* $\mathcal{R} \cong \varepsilon^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$.

Proof. Consider the family of quaternions

$$q_1(\lambda) = \sqrt{-1} + e_1 + \lambda e_2 + \sqrt{-1}\lambda e_3, \quad \lambda \in \mathrm{P}_1(\mathbb{C}).$$

In a neighbourhood of the point $\lambda = \infty$, we normalize this family as follows: $\hat{q}_1(\lambda) = \lambda^{-1}q_1(\lambda) = \sqrt{-1}\lambda^{-1} + \lambda^{-1}e_1 + e_2 + \sqrt{-1}e_3$, where $\hat{q}_1(\infty) = e_2 + \sqrt{-1}e_3$. This family belongs to V and meets each proper right ideal R. Therefore it generates over $\mathbb{H}_{\mathbb{C}}$ the bundle $\mathcal{R}$. Consider the holomorphic line subbundle $\mathcal{R}_1$ generated by the same family over the field $\mathbb{C}$. It is isomorphic to $\mathcal{O}(-1)$. Similarly, the subbundle $\mathcal{R}_2$ generated by the family

$$q_2(\lambda) = \sqrt{-1}\lambda - \lambda e_1 + e_2 - \sqrt{-1}e_3, \quad \hat{q}_2(\lambda) = \lambda^{-1}q_2(\lambda)$$

generates over $\mathbb{C}$ a line subbundle $\mathcal{R}_2$ of $\mathcal{R}$, which is isomorphic to $\mathcal{O}(-1)$ as well. For any λ, we have $\tilde{q}_2(\lambda)q_1(\lambda) = 0$; hence the quaternions $q_1(\lambda)$ and $q_2(\lambda)$ generate the same right ideal according to Proposition 3.4. They are independent over $\mathbb{C}$. Therefore, $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2 \cong \sigma^*(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$. □

4 Characteristic variety for the Cauchy–Fueter system

Put $q_i = \sum x_{ij} e_j$, $i = 1, ..., n$ and write (1.3) as

$$p(\partial/\partial x_{10}, ..., \partial/\partial x_{n3})u = 0,$$

where p is a $4n \times 4$-matrix of first order differential operators with constant coefficients in the space $\mathbb{R}^{4n}$ with coordinates $x_{ij}, j = 0, 1, 2, 3;\ i = 1, ..., n$. Consider the dual complexified space $\mathbb{C}^{4n}$ with complex coordinates ζ_{ij} such that the pairing $\mathbb{R}^{4n} \times \mathbb{C}^{4n} \to \mathbb{C}$ is given by a bilinear form $(x, \zeta) \mapsto x\xi \doteqdot \sum x_{ij}\zeta_{ij}$. We have $(\partial/\partial x_{ij}) \exp(x\zeta) = \zeta_{ij} \exp(x\zeta)$, whence the symbol matrix $p(\zeta)$ of the system (1.3) is of size $4n \times 4$ and consists of n blocks

$$p_k(\zeta) = \begin{vmatrix} \zeta_{k0} & -\zeta_{k1} & -\zeta_{k2} & -\zeta_{k3} \\ \zeta_{k1} & \zeta_{k0} & -\zeta_{k3} & \zeta_{k2} \\ \zeta_{k2} & \zeta_{k3} & \zeta_{k0} & -\zeta_{k1} \\ \zeta_{k3} & -\zeta_{k2} & \zeta_{k1} & \zeta_{k0} \end{vmatrix}, \quad k = 1, ..., n. \tag{4.1}$$

The characteristic variety of (1.3) is by definition the algebraic variety $N \subset \mathbb{C}^{4n}$ given by the condition $\operatorname{rank} p(\zeta) < 4$. For the case $n = 1$, we have $\det p(\zeta) = (\zeta_0^2 + \zeta_1^2 + \zeta_2^2 + \zeta_3^2)^2$; hence the characteristic variety is equal to V. For the general case, we identify the space $\mathbb{C}^{4n}$ with $\mathbb{H}_{\mathbb{C}}^n$ by means of the coordinates $\zeta_1, ..., \zeta_n$.

Denote by $\mathcal{L}(\mathbb{H})$ the variety of all proper left ideals in $\mathbb{H}_{\mathbb{C}}$. The following statement was essentially proved in [10], [2].

Proposition 4.1. *The cone $N \in \mathbb{H}_{\mathbb{C}}^n$ coincides with the set of points $(\zeta_1, ..., \zeta_n)$ such that the quaternions $\zeta_1, ..., \zeta_n$ belong to the same proper left ideal in $\mathbb{H}_{\mathbb{C}}$. It follows that there exists a fibre bundle*

$$\nu : N \setminus \{0\} \to \mathcal{L}(\mathbb{H}) \cong C, \quad \nu(\zeta_1, ..., \zeta_n) = \mathbb{H}_{\mathbb{C}} \cdot (\zeta_1, ..., \zeta_n) \tag{4.2}$$

whose fibres are $\mathbb{C}$-linear subspaces of $\mathbb{C}^{4n}$ of dimension $2n$. We have $\dim_{\mathbb{C}} N = 2n + 1$.

Proof. Multiply the columns of the matrix $p(\zeta)$ by e_0, e_1, e_2, e_3 respectively and take the sum. We get the column of quaternions

$$\tilde{\zeta}_1, \tilde{\zeta}_1 e_1, \tilde{\zeta}_1 e_2, \tilde{\zeta}_1 e_3; \tilde{\zeta}_2, \tilde{\zeta}_2 e_1, ..., \tilde{\zeta}_n e_2, \tilde{\zeta}_n e_3. \tag{4.3}$$

Suppose that $(\zeta_1, ..., \zeta_n) \in N$. This means that the quaternions $\zeta_1, ..., \zeta_n$ belong to the same proper left ideal L. This ideal has dimension 2. Therefore, any three quaternions in (4.3) are $\mathbb{C}$-dependent. It follows that $\operatorname{rank} p(\xi) < 3$; hence

$\zeta = (\zeta_1, ..., \zeta_n) \in N$. Now take an arbitrary point $\zeta = (\zeta_1, ..., \zeta_n) \in N \setminus \{0\}$ and show that the quaternions $\zeta_1, ..., \zeta_n$ belong to the same left ideal. We may suppose that $\zeta_1 \neq 0$. The condition $\operatorname{rank} p(\zeta) < 4$ implies a $\mathbb{C}$-linear relation between any four quaternions in (4.3). This implies that the quaternion ζ_1 is not invertible and there are two independent quaternions out of the four $\tilde{\zeta}_1 e_k$, $k = 0, 1, 2, 3$ (Proposition 3.2). Take these quaternions and any two of the quaternions $\tilde{\zeta}_j e_k$, $k = 0, 1, 2, 3$ for arbitrary $j > 1$ such that $\zeta_j \neq 0$. These four quaternions are $\mathbb{C}$-dependent, which implies a linear equation $\tilde{\zeta}_1 \alpha = \tilde{\zeta}_j \beta \neq 0$ for some $\alpha, \beta \in \mathbb{H}_{\mathbb{C}}$. The equation $\tilde{\alpha}\zeta_1 = \tilde{\beta}\zeta_j \neq 0$ shows that the ideals $L(\zeta_1)$ and $L(\zeta_2)$ have a common nonzero element. By Proposition 3.3, they must coincide. □

Proposition 4.2. *The cone N coincides with the set of solutions of equations*

$$\zeta_i \tilde{\zeta}_j = 0, \quad i, j = 1, ..., n. \tag{4.4}$$

Proof. Let $\zeta = (\zeta_1, ..., \zeta_n) \in N$ and $\zeta_i \neq 0$ for some i. Any coordinate ζ_j belongs to the ideal $L(\zeta_i)$, i.e., $\zeta_j = \alpha\zeta_i$. Therefore, $\zeta_i \tilde{\zeta}_j = \zeta_i \tilde{\zeta}_i \tilde{\alpha} = 0$. Conversely, suppose that (4.4) is valid and $\zeta_i \neq 0$ for some i. These equations imply $(\alpha\zeta_i + \beta\zeta_k)(\widetilde{\alpha\zeta_i} + \widetilde{\beta\zeta_k}) = 0$ for any k and $\alpha, \beta \in \mathbb{H}_{\mathbb{C}}$. Hence the left ideal $L = \{\alpha\zeta_i + \beta\zeta_k,\ \alpha, \beta \in \mathbb{H}_{\mathbb{C}}\}$ is contained in V and contains both ideals $L(\zeta_i)$ and $L(\zeta_k)$. The ideal L is proper and consequently coincides with $L(\zeta_i)$. Therefore $L(\zeta_k) \subset L(\zeta_i)$ for any k. □

Proposition 4.3. *The cone $N_* \doteq N \setminus \{0\}$ is nonsingular and irreducible. Moreover, for an arbitrary point $\lambda \in N_*$, there are $2n + 1$ independent forms among*

$$\operatorname{Re}(e_k d(\zeta_i \tilde{\zeta}_j)), \quad i, j = 1, ..., n, \quad k = 0, 1, 2, 3.$$

Proof. Let $\lambda = (\lambda_1, ..., \lambda_n)$ and $\lambda_1 \neq 0$. Consider $\mathbb{H}_{\mathbb{C}}$-valued forms

$$d(\zeta_1 \tilde{\zeta}_i) = d(\zeta_1)\tilde{\zeta}_i + \zeta_1 d(\tilde{\zeta}_i), \quad i = 2, ..., n. \tag{4.5}$$

By Proposition 3.2, there are two $\mathbb{C}$-independent vectors out of $e_j\lambda_1$, $j = 0, 1, 2, 3$, say, λ_1 and $e_1\lambda_1$. Multiplying (4.5) by e_1 and taking scalar parts, we get $2n - 2$ forms at the point λ:

$$\operatorname{Re}(d(\zeta_1\tilde{\zeta}_i)) \equiv \operatorname{Re}(\lambda_1 d\tilde{\zeta}_i), \ \operatorname{Re}(e_1 d(\zeta_1\tilde{\zeta}_i)) \equiv \operatorname{Re}(e_1\lambda_1 d\tilde{\zeta}_i) \pmod{Z_1},$$

for $i = 2, ..., n$, where Z_1 is the $\mathbb{C}$-linear span of the forms $d\zeta_{10}, d\zeta_{11}, d\zeta_{12}, d\zeta_{13}$. These are linearly independent modulo Z_1. The form $d(\zeta_1\tilde{\zeta}_1) = d\zeta_1\tilde{\lambda}_1 + \lambda_1 d\tilde{\zeta}_1$ is yet another independent form. The total number is thus $2n - 1$ which proves our assertion. This shows that the cone N_* is nonsingular. The variety N_* is connected, since the base $\mathcal{C}$ in (4.2) and the fibre $\cong \mathbb{C}^{2n} \setminus \{0\}$ are connected. Therefore, the variety N is irreducible. □

5 Nöther operator for the characteristic module

Let $A = \mathbb{C}[\zeta]$ be the algebra of complex-valued polynomials in $\mathbb{C}^{4n}$. The transpose ${}^t p$ of the matrix p defines a morphism of A-modules

$$ {}^t p : A^{4n} \to A^4, \quad a \mapsto {}^t p a, $$

where we take elements of A^{4n} and of A^4 as columns. Set $A(N) = A/I(N)$, where $I(N)$ is the ideal in A of polynomials that vanish on N. Consider the morphism $p^\flat : A^4 \to A(N)^{4n}$ that is equal to the composition of multiplication by the matrix p and of the natural projection $\pi : A^{4n} \to A(N)^{4n}$.

Theorem 5.1. *The sequence*

$$ A^{4n} \xrightarrow{{}^t p} A^4 \xrightarrow{p^\flat} A(N)^{4n} \tag{5.1} $$

is exact.

Proof. To simplify calculations, we multiply the rows of the matrix ${}^t p$ and the columns of the matrix p by e_j, $j = 0, 1, 2, 3$ respectively and sum them. We get from ${}^t p$ the row of quaternions $\tilde{\zeta}_i, \tilde{\zeta}_i e_1, \tilde{\zeta}_i e_2, \tilde{\zeta}_i e_3$, $i = 1, ..., n$; and the matrix p gives the column of the same quantities. An entry of the product $p\,{}^t p$ is an inner product of a column of ${}^t p$ and of a row of p, i.e.,

$$ p\,{}^t p = \{\mathrm{Re}(\widetilde{\tilde{\zeta}_i e_k}\, \tilde{\zeta}_j e_l),\ i, j = 1, ..., n;\ k, l = 0, 1, 2, 3\}. $$

We find that $\widetilde{\tilde{\zeta}_i e_k}\, \tilde{\zeta}_j e_l = \tilde{e}_k \zeta_i \tilde{\zeta}_j e_l = 0$ in N by virtue of (4.2). This implies the equation $p^\flat\, {}^t p = 0$.

Now we show that the equation $p^\flat a = 0$ for a polynomial $a \in A^4$ implies the inclusion $a \in {}^t p A^{4n}$. The above equation is equivalent to the system $\mathrm{Re}(e_k \zeta_i \alpha) = 0$ in N for $k = 0, 1, 2, 3,\ i = 1, ..., n$ for the $\mathbb{H}_\mathbb{C}$-valued polynomial $\alpha = \sum a_k e_k$. The latter is equivalent to the quaternionic equation $\zeta_i \alpha = 0$ in N for $i = 1, ..., n$. We prove that this equation implies that $\alpha = \sum_j \tilde{\zeta}_j \beta_j$ for some $\mathbb{H}_\mathbb{C}$-valued polynomials $\beta_1, ..., \beta_n$. We localize this problem, considering the affine scheme $(\mathrm{Spec}\, A, \hat{A})$ of the algebra A. Denote by $\mathfrak{m}$ the maximal ideal of the point $\zeta = 0$ in A and set $\hat{A}(N) = \hat{A}/I(N)\hat{A}$.

Lemma 5.2. *The sequence of sheaf morphisms*

$$\hat{A}^{4n} \xrightarrow{{}^t p} \hat{A}^4 \xrightarrow{p^\flat} \hat{A}(N)^{4n} \tag{5.2}$$

is exact in $\operatorname{Spec} A \setminus \{\mathfrak{m}\}$.

Proof. We check first the inclusion

$$I(N)\hat{A} \subset {}^t p\hat{A}^{4n} \quad \text{in } \operatorname{Spec} A \setminus \{\mathfrak{m}\}. \tag{5.3}$$

According to Proposition 4.3, the sheaf is generated over $\operatorname{Spec} A \setminus \{\mathfrak{m}\}$ by the polynomials $\operatorname{Re}(e_k \zeta_i \tilde{\zeta}_j)$, $k = 0, 1, 2, 3$, $i = 1, \ldots, n$. We show that each of them can be written in the form $\sum \tilde{\zeta}_j b_j$. This follows from the identities

$$2\operatorname{Re}(\zeta_i \tilde{\zeta}_j) = \tilde{\zeta}_i \zeta_j + \tilde{\zeta}_j \zeta_i, \quad 2\operatorname{Re}(e_k \zeta_i \tilde{\zeta}_j) = \tilde{\zeta}_i e_k \zeta_j - \tilde{\zeta}_j e_k \zeta_i.$$

By (5.3), the homology of the complex (5.2) coincides with the homology of the complex

$$\hat{A}(N)^{4n} \xrightarrow{{}^t p} \hat{A}(N)^4 \xrightarrow{p^\flat} \hat{A}(N)^{4n} \tag{5.4}$$

of free $\hat{A}(N)$-sheaves. The sequence of sheaves (5.4) is generated by free $\mathbb{C}$-vector bundles on the algebraic manifold N_*. The middle term is the bundle with the fibre $\mathbb{H}_\mathbb{C}$. The kernel of the mapping $p^\flat$ of the corresponding vector bundles is a subbundle whose fibre at an arbitrary point ζ is the proper right ideal $R = \widetilde{\nu(\zeta)}$. Indeed, if, for example, $\zeta_1 \neq 0$, then the equation $p^\flat \alpha = 0$ is equivalent to the equation $\zeta_1 \alpha = 0$ for a quaternion $\alpha \in \mathbb{H}_\mathbb{C}$. The set of solutions coincides with the ideal $R(\tilde{\zeta}_1)$. This follows from Proposition 3.4. The ideal $R(\tilde{\zeta}_1)$ is equal to the image of the linear mapping ${}^t p$ at ζ since $\tilde{\zeta}_j \in R(\tilde{\zeta}_1)$ for all j. Therefore, the sequence (5.4) is exact as a complex of vector bundles. Consequently, it is exact as a complex of algebraic sheaves. □

Lemma 5.2 implies that

(i) the cone N is associated with the A-module $M = \operatorname{Cok} {}^t p$,

(ii) the morphism $p^\flat$ is a $I(N)$-Nöther operator for M (see the next section),

(iii) no other simple ideal $\mathfrak{p} \neq \mathfrak{m}$ is associated with M. In fact, the ideal $\mathfrak{m}$ is not associated with M either, since the depth of the module M is positive. □

Theorem 5.3. *The sequence* (5.1) *can be extended to an exact sequence*

$$A^4 \xrightarrow{p^\flat} A(N)^{4n} \xrightarrow{q^\flat} A(N)^m. \tag{5.5}$$

Proof. Arguing as in the proof of Lemma 5.2, we see that the image of the mapping $p^\flat$ restricted to N_* is the sheaf of sections of the locally free vector bundle $\nu^*(\mathcal{R}(\mathbb{H}))$. Therefore, the associated set of the module $Q \doteq \operatorname{Cok} p^\flat$ consists of the ideal $I(N)$ and possibly of the maximal ideal $\mathfrak{m}$ of the origin. We show that the latter is, in fact, not associated with Q. Indeed, we have an exact sequence

$$0 \to M \to A(N)^{4n} \to Q \to 0,$$

which induces another exact sequence

$$0 = \operatorname{Hom}(A/\mathfrak{m}, A(N))^{4n} \to \operatorname{Hom}(A/\mathfrak{m}, Q) \to \operatorname{Ext}^1(A/\mathfrak{m}, M).$$

The last term vanishes because $\operatorname{Prof}(M) = 2n+1 > 1$ ([2, Theorem 2.4]). Therefore, the middle term is equal to zero, which implies our assertion.

Consider the $\hat{A}(N)$-sheaf $\operatorname{Hom}(Q, \hat{A}(N))$. It is an algebraic sheaf on the affine variety N and is generated by its sections. Choose a finite set of sections, say $q_1, ..., q_m$, that generate this sheaf at each point. These sections can be lifted to mappings $q_i : A(N)^{4n} \to A(N)$, $i = 1, ..., m$. The direct sum is a mapping $q^\flat = \oplus q_i : A(N)^{4n} \to A(N)^m$ which vanishes on the image of the mapping $p^\flat$ and hence defines a morphism $q^\sharp : Q \to A(N)^m$. By construction, this mapping is locally injective at each point of N_*. Therefore, $q^\sharp$ is injective, since the ideal $\mathfrak{m}$ is not associated with the module Q. □

6 Exponential representation of solutions

Recall the representation theorem proved in [7], [8]. Let

$$p(\partial/\partial x_1, ..., \partial/\partial x_n)u = 0 \tag{6.1}$$

be an arbitrary system of r linear differential equations with constant coefficients in $\mathbb{R}^n$ with s unknown functions $u = (u_1, ..., u_s)$. Consider the dual complex space $\mathbb{C}^n$ with the dual coordinates $\xi_1, ..., \xi_n$ and the polynomial algebra $A = \mathbb{C}[\xi_1, ..., \xi_n]$. The symbol $p(\zeta)$ of the differential operator (6.1) defines an A-morphism ${}^tp : A^r \to A^s$, $a \mapsto {}^tpa$, where ${}^tp$ means the transposed matrix. Consider the A-module $M = \operatorname{Cok} {}^tp$ and the associated set $\operatorname{Ass}(M) \subset \operatorname{Spec} A$. Recall that a simple ideal $\mathfrak{p}$ in the algebra A is associated with M if there exists an element $m \in M$ such that $am = 0$, $a \in A$, if and only if $a \in \mathfrak{p}$. According to [7], [8], there exists for each $\mathfrak{p} \in \operatorname{Ass}(M)$ a differential operator $\delta_\mathfrak{p} : A^s \to (A/\mathfrak{p})^{r(\mathfrak{p})}$ for some natural $r(\mathfrak{p})$ such that $\operatorname{Ker} \delta_\mathfrak{p}$ is a submodule of A^s and $\bigcap_\mathfrak{p} \operatorname{Ker} \delta_\mathfrak{p} = {}^tpA^r$. The mapping $\delta_\mathfrak{p}$ is called a $\mathfrak{p}$-*Nöther operator* for M or the $Z(\mathfrak{p})$-Nöther operator, where $Z(\mathfrak{p})$ denotes the corresponding irreducible variety. It can be represented as the composition of the mappings

$$d_{\mathfrak{p}}(\zeta, \partial/\partial\zeta) : A^s \to A^{r(\mathfrak{p})}, \quad \pi : A^{r(\mathfrak{p})} \to (A/\mathfrak{p})^{r(\mathfrak{p})},$$

where π is the natural projection and the entries of the matrix $d_{\mathfrak{p}}$ are polynomials in ζ and $\partial/\partial\zeta$. Denote by ${}^t d_{\mathfrak{p}}$ the transpose matrix.

For an arbitrary compact set $K \subset \mathbb{R}^n$, we define the pseudonorm $\lceil \xi \rceil_K = \max\{\Re(\xi x), x \in K\}$ in the dual space $\mathbb{C}^n$. This is a real convex positively homogeneous function which need not to be positive.

Theorem 6.1. *Let U be an arbitrary convex open set in $\mathbb{R}^n$. Any distribution-solution of* (6.1) *can be written in the form*

$$u(x) = \sum_{\mathfrak{p} \in \mathrm{Ass}(M)} \int_{Z(\mathfrak{p})} \exp(\zeta x)\, {}^t d_{\mathfrak{p}}(\zeta, x) \mu_{\mathfrak{p}}(\zeta), \tag{6.2}$$

where $\mu_{\mathfrak{p}}$ is a $\mathbb{C}^{r(\mathfrak{p})}$-valued density in $Z(\mathfrak{p})$ such that

$$\sum_{\mathfrak{p}} \int_{Z(\mathfrak{p})} \exp(\lceil \xi \rceil_K) |\mu_{\mathfrak{p}}| < \infty \tag{6.3}$$

for any compact set $K \subset U$; here $|\cdot|$ denotes a norm in the space $\mathbb{C}^{r(\mathfrak{p})}$.

Moreover, given for each $\mathfrak{p} \in \mathrm{Ass}(M)$ an arbitrary proper algebraic subvariety W of $Z(\mathfrak{p})$ that contains the singular part of $Z(\mathfrak{p})$, there exist a positive polynomial ρ in $\mathbb{R}^{2n}$ such that the density as above can be chosen to satisfy the following additional conditions:

$$\mathrm{supp}\, \mu_{\mathfrak{p}} \subset Z(\mathfrak{p}) \setminus W_\rho, \quad W_\rho = \{\xi \in \mathbb{C}^n,\, \rho(\xi)\, \mathrm{dist}(\xi, W) \leq 1\}, \tag{6.4}$$

and

$$\int_{Z(\mathfrak{p})} \exp(\lceil \xi \rceil_K) |L(v)^k \mu_{\mathfrak{p}}| < \infty, \quad k = 0, 1, 2... \tag{6.5}$$

for an arbitrary regular algebraic tangent field v in the variety $Z(\mathfrak{p}) \setminus W$.

Proof. A representation satisfying (6.2), (6.3) and (6.4) was constructed in [7], [8]. To fulfill the condition (6.5) we choose a linear projector $\pi : \mathbb{C}^n \to \Pi$ to a subspace Π in $\mathbb{C}^n$ such that the restriction $\pi_Z : Z(\mathfrak{p}) \to \Pi$ is surjective and finite. We can suppose that the subvariety W in (6.4) contains the critical set of the mapping π_Z. Choose a smooth even density g in Π such that $\mathrm{supp}\, g$ belongs to the unit ball B and $\int g(w)h(w) = h(0)$ for an arbitrary harmonic function h in B. For an arbitrary $\xi \in \Pi$ and positive r, we have

$$\int g_r(\eta) h(\xi - \eta) = h(\xi) \tag{6.6}$$

for any harmonic function h in the ball $\xi + rB$, where $g_r(\eta) = g(r^{-1}\eta)$. Therefore, for an arbitrary compactly supported density μ in Π and for an arbitrary variable parameter r, the convolution $g_r * \mu$ is smooth and defines the same functional on holomorphic functions as does μ. We can apply this convolution to the density $\mu_{\mathfrak{p}}$ supported by $Z(\mathfrak{p}) \setminus W_\rho$ by means of the pull-down operation with respect to the projection π_Z. The property (6.6) is preserved since the mapping π_Z is locally conformal. We need only ensure that the support of the convolution does not touch the set $W_{\rho'}$ for a positive polynomial ρ' in $\mathbb{R}^{2n}$. This is the case if we choose the parameter r to be $\sigma \doteq 2\rho(p(\xi))$. Thus we get a smooth density $g_r * \mu_{\mathfrak{p}}$, which can be substituted for $\mu_{\mathfrak{p}}$ in (6.2), since the kernel satisfies (6.6). To prove inequalities (6.5), we note that any ξ, η-derivative of the kernel $g_\sigma(\eta)$ is of polynomial growth in $Z_{\mathfrak{p}} \setminus W_\rho$. Hence, for an arbitrary tangent field v as above and arbitrary k, we have

$$\int \exp(\lceil \xi \rceil_K)|L(v)^k(g_\sigma * \mu_{\mathfrak{p}})| \leq \int \exp(\lceil \xi \rceil_K) P_k(\xi)|\mu_{\mathfrak{p}}|, \tag{6.7}$$

where P_k is a real polynomial in $\mathbb{R}^{2n}$. Now apply the estimate

$$\exp(\lceil \xi \rceil_K)|\xi|^m = O(\exp(\lceil \xi \rceil_L)) \quad \text{in } N, \tag{6.8}$$

which follows from ellipticity of the cone N, where m is an arbitrary integer. It implies that the right hand side of (6.7) is bounded by the integral $C_k \int \exp(\lceil \xi \rceil_L)|\mu_{\mathfrak{p}}|$, which is finite because of (6.3); and (6.5) follows. □

7 Surjectivity

Now we prove surjectivity in (1.4) in a stronger form.

Theorem 7.1. *Let U be an arbitrary open convex set in $\mathbb{H}^n$. For any polymonogenic function u in U, there exists a smooth $(1,1)$-form $u^{1,1}$ in $\mathcal{R}(\mathbb{H}) \cong \mathrm{P}_1(\mathbb{C})$ with values in the sheaf $\mathcal{Q}(U, \mathcal{R})$ such that*

$$u = \int_{\mathcal{R}(\mathbb{H})} u^{1,1}. \tag{7.1}$$

This statement can be rephrased as follows: there exists a smooth family of polymonogenic functions $u_R : U \to R$, $R \in \mathcal{R}(\mathbb{H})$ such that $u = \int_{\mathcal{R}(\mathbb{H})} u_R \,\Omega$, where Ω is the Kähler $(1,1)$-form in $\mathrm{P}_1(\mathbb{C})$.

Lemma 7.2. *We have*

$$u(x) = \int_N \exp(\zeta x) \sum \tilde{\zeta}_j \omega_j, \tag{7.2}$$

where ω_j, $j = 1, ...n$ *are* $\mathbb{H}_{\mathbb{C}}$*-valued densities in* N *satisfying the condition*

$$\bigcup_i \operatorname{supp}\omega_i \subset N_* \doteq N \setminus \{0\} \tag{7.3}$$

and the inequalities (6.5).

Proof of Lemma 7.2. According to Theorem 5.1 the set Ass(M) for the module $M = A^4/{}^t pA^{4n}$ consists only of the ideal $\mathfrak{p} = I(N)$, and the mapping $\delta = p^\flat$ is a Nöther operator. Hence we can take $d(\zeta, \partial/\partial\zeta) = p(\zeta)$ and ${}^t d = {}^t p$ (see 4.1). Fix an affine complex coordinate λ in $\mathcal{R}(\mathbb{H}) \cong \mathrm{P}_1(\mathbb{C})$ and set

$$W = \{0\} \cup \nu^{-1}(\{\lambda = 0, \infty\}).$$

This is an algebraic subvariety in N. Now apply Theorem 6.1 to u and W. To represent the function (6.2) in the quaternionic form $u = \sum u_k e_k$, we multiply the columns of (4.1) by e_k, $k = 0, 1, 2, 3$, respectively. Then the matrix ${}^t d$ turns into the row of $\mathbb{H}_{\mathbb{C}}$-valued polynomials ${}^t d(\zeta) = \{\tilde{\zeta}_i e_k,\ i = 1, ..., n,\ k = 0, 1, 2, 3\}$. The product ${}^t d\mu$ with a $\mathbb{C}^{4n}$-valued density $\mu_{I(N)}$ can be written in the form

$${}^t d(\zeta)\mu = \sum_i \tilde{\zeta}_i \sum_j e_k \mu_{ik} = \sum \tilde{\zeta}_i \omega_i,$$

where μ_{ik} are components of $\mu_{\mathfrak{p}}$ and $\omega_i = \sum_k \mu_{ik} e_k$ are $\mathbb{H}_{\mathbb{C}}$-valued densities, i.e., $(2n+1, 2n+1)$-forms in N_*. They satisfy (6.5) for any compact $K \subset U$. Now (6.2) implies the representation (7.2). □

Proof of Theorem 7.1. The density $e \doteq \exp(\zeta x) \sum \tilde{\zeta}_j \omega_j$ is a $(2n+1, 2n+1)$-form in N and its direct image under the projection (4.2) is equal to u. The image is a $(1,1)$-form $u^{1,1}$ in the base $\mathbb{R}(\mathbb{H})$ since the general fibre is a complex manifold of dimension n. It satisfies (7.1) in view of Fubini's Theorem and can be explicitly calculated as follows. Divide the densities $\omega_1, ..., \omega_n$ by the Kähler form Ω and integrate along fibres of the projection (4.2):

$$w_R(x) \doteq \int_{N_{\tilde{R}}} \exp(\zeta x) \sum \tilde{\zeta}_j \frac{\omega_j}{\Omega}, \quad N_R = \nu^{-1}(\tilde{R}). \tag{7.4}$$

The function $w_R(\cdot)$ is polymonogenic as

$$\frac{\partial}{\partial \bar{q}_i} \int_{N_R} \exp(\zeta x) \sum \tilde{\zeta}_j \omega_j = \int_{N_R} \sum_j \exp(\zeta x) \zeta_i \tilde{\zeta}_j \omega_j = 0$$

according to (4.4). The function w_R takes values in R. Indeed, the quaternions $\zeta_1, ..., \zeta_n$ specified in (7.4) belong to the left ideal $L = \tilde{R}$; hence the quaternions

$\tilde{\zeta}_1, ..., \tilde{\zeta}_n$ reside in the right ideal $R = \tilde{L}$. By the construction we have $u^{1,1} = w\Omega$, where $w(R, \cdot) = w_R(\cdot)$.

To prove that the coefficient w is smooth in $\mathrm{P}_1(\mathbb{C})$, take the tangent field $\tau_0 = \partial/\partial\lambda$ in $\mathcal{C}$. The bundle $N \setminus \nu^{-1}(\infty) \to \mathrm{P}_1(\mathbb{C}) \setminus \infty$ is algebraically trivial; consequently, the field τ_0 can be lifted to an algebraic regular field t in $N \setminus W$. Take the Lie derivative $L(\tau_0)$ of the form $u^{1,1}$ and evaluate it on an arbitrary function $\psi \in \mathcal{D}(\mathrm{P}_1(\mathbb{C}))$:

$$\begin{aligned} \int_{\mathrm{P}_1(\mathbb{C})} L(\tau_0)u^{1,1}\,\psi &= -\int u^{1,1}\,\tau_0\psi = -\int_N \exp(\zeta x)\tilde{\zeta}\omega\,\nu^*(\tau_0\psi) \\ &= -\int_N \exp(\zeta x)\tilde{\zeta}\omega\, t(\nu^*(\psi)) = \int_N L(t)(\exp(\zeta x)\tilde{\zeta}\omega)\,\nu^*(\psi) \\ &= \int_N t(\exp(\zeta x)\tilde{\zeta})\omega\,\nu^*(\psi) + \int \exp(\zeta x)\tilde{\zeta}L(t)(\omega)\,\nu^*(\psi), \end{aligned} \tag{7.5}$$

where we write $\tilde{\zeta}\omega$ for $\sum \tilde{\zeta}_j\omega_j$. Since the field t is regular algebraic we have $t(\exp(\zeta x)\zeta) = T(x,\zeta)\exp(\zeta x)$, where $T(x,\zeta)$ is a linear function of x whose coefficients are rational functions which are regular in $N \setminus W$. They are of polynomial growth in $N \setminus W_\rho$, where ρ is a real polynomial as in Theorem 6.1. When x runs over a compact set K, the right hand side is bounded by $C\exp(\lceil \xi \rceil)_L$ for an arbitrary compact L such that $K \Subset L$. This follows from (6.8). Consequently, the first integral in the right hand side of (7.5) converges and is bounded by $C\|\psi\|_{L_1}$. The second term in (7.5) is estimated similarly by means of (6.5). As a result, we obtain the inequality

$$\left| \int_{\mathrm{P}_1(\mathbb{C})} L(\tau_0)u^{1,1}\psi \right| \le C\|\psi\|_{L_1},$$

which implies that the form $\int_{\mathrm{P}_1(\mathbb{C})} L(\tau_0)u^{1,1}$ is essentially bounded in $K \times \mathrm{P}_1(\mathbb{C})$. Repeating these arguments, we show that $L(\tau_0)^k u^{1,1}$ is bounded for arbitrary field τ and $k = 2, 3, \ldots$. This implies that the coefficient u is infinitely differentiable at least in the affine plane $\lambda \neq \infty$. To cover the point ∞, we consider the tangent field $\tau_\infty = \lambda^2\partial/\partial\lambda$ extended to infinity. It is nonsingular in the plane $\lambda \neq 0$. Arguing as above, we show that w is infinitely differentiable in this plane as well. □

8 The middle term

Now we prove that the sequence (1.4) is exact at the middle term.

Theorem 8.1. *Any current* $u^{1,1} \in \Gamma(\mathcal{R}(\mathbb{H}), Q(U,\mathcal{R})^{1,1})$ *such that*

$$\int_{\mathcal{R}(\mathbb{H})} u^{1,1} = 0$$

can be represented in the form

$$u^{1,1} = \nabla(\bar{\partial})u^{1,0}, \quad u^{1,0} \in \Gamma(\mathcal{R}(\mathbb{H}), \mathcal{Q}(U,\mathcal{R})^{1,0}). \tag{8.1}$$

Proof. For an arbitrary compact set $K \subset \mathbb{H}^n$, consider the space E_K of smooth forms φ in $\mathbb{C}^{4n}$ such that

$$\left(\frac{\partial}{\partial\bar{\zeta}}\right)^k \varphi_\alpha(\zeta) = O(\exp(\lceil \zeta \rceil_K)) \quad \text{for any } k = (k_1, ..., k_{4n}).$$

Here φ_α denotes an arbitrary coefficient of the form φ with respect to the basis generated by the forms $d\zeta_{ij}$, $d\bar{\zeta}_{ij}$, where the bar denotes complex conjugation. This space has a natural Fréchet topology. Let H_K be the subspace of holomorphic functions with the norm $\|f\|_K = \sup|f(\zeta)|\exp(-\lceil \zeta \rceil_K)$.

Lemma 8.2. *The current as above admits the representation*

$$u^{1,1} = \int_\nu \exp(\zeta x) \sum \tilde{\zeta}_j \omega_j, \tag{8.2}$$

where ω_j, $j = 1, ..., n$ are $(2n+1, 2n+1)$-currents in N with support in $N \setminus \{0\}$, which are continuous functionals on E_K for any $K \Subset U$.

The symbol $\int_\nu \omega$ means the direct image of the current ω with respect to the mapping ν. We prove this Lemma in the last section.

By the condition of Theorem 8.1, we have

$$0 = \int_{\mathcal{R}(\mathbb{H})} u^{1,1} = \int_N \exp(\zeta x)\tilde{\zeta}_j \omega_j. \tag{8.3}$$

Lemma 8.3. *For any compact set $K \subset U$, we have*

$$\zeta_j \omega_j = \sum_j \zeta_j \bar{\partial}\rho_j + {}^t q v \tag{8.4}$$

for some $(2n+1, 2n)$-currents ρ_i and a $\mathbb{C}^m$-valued $(2n+1, 2n+1)$-current v in N which are continuous functionals on E_K. Here q is the polynomial mapping from Lemma 5.3.

For the proof we require additional technique. For an arbitrary holomorphic subvariety $W \subset \mathbb{C}^{4n}$, denote by $H(W)$ the space of holomorphic functions $W \to \mathbb{C}$ which are traces of holomorphic functions in a neighbourhood of W. Let $U_z(\varepsilon)$ be the closed ball in $\mathbb{C}^{4n}$ with the centre z and radius ε. Denote by $U(\varepsilon)$ the covering of N by the balls $U_z(\varepsilon)$, $z \in N$. Let $H^r(N,\mathcal{U})$, $r = 0, 1, ...$ be the space of r-cochains with values in the spaces $H(N_{z_0,...,z_r})$, where $N_{z_0,...,z_r} \doteq N \cap U_{z_0}(\varepsilon) \cap \cdots \cap U_{z_r}(\varepsilon)$.

For any compact set $K \subset \mathbb{H}^n$, we denote by $H^r_K(N,\mathcal{U})$ the normed subspace of $H^r(N,\mathcal{U})$ of cochains with finite norm given by

$$(8.5) \qquad \|F\|_{K,\mathcal{U}} = \sup_{z_0,\dots,z_r} \sup_{\zeta \in N_{z_0,\dots,z_r}} \frac{|F_{z_0,\dots,z_r}(\zeta)|}{\exp(\lceil \zeta \rceil_K)}.$$

Consider the Čech complex

$$0 \to H_K(N) \xrightarrow{\delta} H^0_K(N,\mathcal{U}) \xrightarrow{\delta_0} H^1_K(N,\mathcal{U}) \to \cdots ,$$

where $\delta, \delta_0, \dots$ are coboundary mappings. Choose a sequence of convex compact sets $K_1 \Subset K_2 \Subset \cdots \subset U$ such that $\bigcup K_j = U$. Replace each term of (8.5) by the direct spectrum of the corresponding normed space with K running over the sequence K_j as above and $\mathcal{U}$ running over the sequence $\mathcal{U}(2^{-j})$, $j = 1, 2, \dots$ with the obvious continuous injections $H^r_{K_j}(N,\mathcal{U}(2^{-j}) \to H^r_{K_{j+1}}(N,\mathcal{U}(2^{-j-1}))$. We get the sequence of direct spectra

$$(8.6) \qquad 0 \to H_{\{K\}}(N) \xrightarrow{\delta} H^0_{\{K\}}(N,\mathcal{U}) \xrightarrow{\delta_0} H^1_{\{K\}}(N,\mathcal{U}) \to \cdots .$$

According to [7], we say that a sequence of direct spectra is *strictly exact* if it is exact and each mapping is a topological homomorphism in the category of direct spectra (cf. [9]).

Lemma 8.4. *The sequence of spectra* (8.6) *is strictly exact.*

Proof. This assertion is contained in [7] (proof of Theorem 2 of Ch. IV, Section 5). □

Any polynomial $a \in A$ defines a mapping of holomorphic cochains $F \mapsto aF$. For any $\mathcal{U}$ and compact sets $K \Subset L$ this mapping generates a continuous operator $a : H_K(N,\mathcal{U}) \to H_L(N,\mathcal{U})$. This follows from (6.8). Therefore, the algebraic mappings (5.5) generate the following commutative diagram of spectra:

$$\begin{array}{ccccc}
 & & (H^1_{\{K\}}(N,\mathcal{U}))^{4n} & & \\
 & & \uparrow \delta_0 & & \\
 & & (H^0_{\{K\}}(N,\mathcal{U}))^{4n} & \xrightarrow{q} & (H^0_{\{K\}}(N,\mathcal{U}))^m \\
 & & \uparrow \delta & & \uparrow \delta \\
(H_{\{K\}})^4 & \xrightarrow{p} & (H_{\{K\}}(N))^{4n} & \xrightarrow{q} & (H_{\{K\}}(N))^m \\
 & & & & \uparrow \\
 & & & & 0
\end{array}$$

This gives rise to the sequence

$$(8.7)\qquad (H_{\{K\}})^4 \xrightarrow{\Delta} (H^0_{\{K\}}(N,\mathcal{U}))^{4n} \xrightarrow{\Delta_0} (H^1_{\{K\}}(N))^{4n} \oplus (H^0_{\{K\}}(N,\mathcal{U}))^m,$$

where $\Delta = \delta p$, $\Delta_0 = \delta_0 \oplus q$.

Lemma 8.5. *Let L be an arbitrary convex compact set in $\mathbb{H}^n$ and $K \Subset L$. Then for sufficiently small ε the quotient norm in the space*

$$\frac{(H^0_L(N,\mathcal{U}(\varepsilon)))^{4n}}{\Delta(H_L)^4}$$

is majorized by the norm

$$(8.8)\qquad f \mapsto \|\delta_0 f\|_{K,\mathcal{U}} + \|qf\|_{K,\mathcal{U}},$$

where we denote $\mathcal{U} = \mathcal{U}(1)$.

Proof. In terms of direct spectra, the assertion means that the sequence (8.7) is strictly exact. To prove this, we use the diagram chase method ([7] Ch. I) and the strict exactness of the bottom row and columns. For the columns, this follows from [7] (Ch. IV, §5, Th. 2). For the bottom row, this follows from [7] applied to the sequence (5.5). □

Proof of Lemma 8.3. For an arbitrary open set $W \Subset N$, let $E(W)$ be the space of smooth functions F in $W \setminus \{0\}$ with bounded derivatives $\bar{t}_1 \cdots \bar{t}_k(F)$, where $t_1, \ldots, t_k$ are arbitrary algebraic tangent fields in N. It has a natural Fréchet topology. Any current v in N with support $\operatorname{supp} v \Subset W$ defines a continuous functional on $E(W)$. For a covering $\mathcal{U} = \mathcal{U}(\varepsilon)$, we define the spaces of smooth cochains $E^r_K(N,\mathcal{U})$, $r = 0, 1, \ldots$ by means of the sequence of seminorms $F \mapsto \|\bar{t}_1 \cdots \bar{t}_k(F)\|_{K,\mathcal{U}}$, where the fields $t_1, \ldots, t_k$ are as above and the norm $\|\cdot\|_{K,\mathcal{U}}$ is defined in (8.5).

Choose a bounded family of functions $\{h_z,\ z \in N\} \subset \mathcal{D}(U_0(1))$ such that the family of sets $\operatorname{supp} \eta_z$ is locally finite and $\sum_z \eta_z = 1$ in N, where $\eta_z(\zeta) = h_z(\zeta - z)$ for $z \in N$. Take the currents ω_j found in Lemma 8.2 and consider $\omega = (\omega_1, \ldots, \omega_n)$ as a $\mathbb{H}^n_{\mathbb{C}}$-valued current. For any $z \in N$, the current $\eta_z \omega$ is supported by N_z and defines a continuous functional $\tilde{\omega}_z$ on the space $E(N_z) \otimes_{\mathbb{C}} \mathbb{H}^n_{\mathbb{C}}$. Together they define a functional on the space of smooth 0-cochains:

$$\tilde{\omega}^0(F) = \sum_z \int_{N_z} \operatorname{Re}(\tilde{F}_z \eta_z \omega).$$

The sum is a continuous functional on the space $(H^0_L(N,\mathcal{U}) \otimes \mathbb{H}_{\mathbb{C}})^n$ for any $L \Subset U$ because ω_j are continuous on E_L.

Lemma 8.6. *The functional* $\check{\omega}^0$ *vanishes on the image of* Δ (see(8.7)).

Proof of Lemma 8.6. Write an arbitrary element $f = (f_0, f_1, f_2, f_3) \in (H_K)^4$ in the quaternionic form $f = f + f_1 e_1 + f_2 e_2 + f_3 e_3$. We have $\Delta f = \delta(\zeta_1 f, ..., \zeta_n f)$ and

$$\omega^0(\Delta f) = \sum_z \int_{N_z} \operatorname{Re}(\tilde{f}\tilde{\zeta}_j \eta_z \omega_j) = \int_N \operatorname{Re}(\tilde{f}\tilde{\zeta}_j \omega) = \sum_k \operatorname{Re}(\tilde{e}_k \int_N f_k \zeta_j \omega_j), \tag{8.9}$$

where the summation is over $j = 1, ..., n$. An arbitrary function $g \in H_K^4$ can be written in the form $g = g_N + \hat{\phi}$, where

$$\hat{\phi}(\zeta) = \int \exp(\zeta x)\phi(x)dx \tag{8.10}$$

is the Fourier–Laplace transformation of a function $\phi \in \mathcal{D}(U)^4$ and the function $g_N \in (H_L)^4$ vanishes in N. This follows from [7] (Ch. IV, §5). We apply this decomposition to the functions $g = f_k$ in (8.9) The term with g_N obviously vanishes. Substitute the absolutely convergent integral (8.10) for f_k in (8.9). The integral is equal to zero because of (8.3). □

We resume the proof of Theorem 8.1. Fix an arbitrary compact set $K \subset U$. By Lemma 8.2, the functional $\check{\omega}^0$ is bounded on the space $H_L(N, \mathcal{U}) \otimes \mathbb{H}_{\mathbb{C}}^n$ for any $L \subset U$. Therefore, according to Lemmas 8.6 and 8.5, it is bounded with respect to the norm $\|\delta_0 f\|_{K,\mathcal{U}} + \|qf\|_{K,\mathcal{U}}$ as well. There exists a bounded functional ω' on the image of the mapping Δ_0 such that $\check{\omega}^0 = \Delta_0^*(\omega')$ (where Δ_0^* means the dual operator). By means of the Hahn–Banach theorem, ω' can be extended to a bounded functional $\rho^1 \oplus \sigma^0 \in (H_K^1(N, \mathcal{U})^*)^{4n} \oplus (H_K^0(N, \mathcal{U}))^*)^m$. As a result, we have

$$\check{\omega}^0(F) = \rho^1(\delta_0 F) + \sigma^0(qF) \tag{8.11}$$

for an arbitrary $F \in H_K^0(N, \mathcal{U})^{4n}$. Write

$$\rho^1(F) = \sum_{z,w} \rho_{z,w}(F_{z,w}),$$

where $\rho_{z,w}$ is for each z, w a continuous functional in the space $H(N_{z,w})$ and $\rho_{z,w} = 0$ except for a countable number of pairs z, w. Again by means of the Hahn–Banach theorem, we can extend this functional to a $(2n+1, 2n+1)$-density $\check{\rho}_{z,w} \in E(N_{z,w})$ which allows the integral representation

$$\rho_{z,w}(f) = \int_{N_{z,w}} f \check{\rho}_{z,w}$$

such that $\check{\rho}_{w,z} = -\check{\rho}_{z,w}$ and

$$\sum_{z,w} \int \exp(\uparrow \zeta \, 1_K)|\rho_{z,w}| < \infty. \tag{8.12}$$

Proceeding with σ^0 in the same way, we get a sequence of $(2n+1, 2n+1)$-densities $\check{\sigma}_z$ in N_z, $z \in N$ which satisfy similar conditions. For any z we set

$$v_z \doteq \eta_z \omega - \sum_w \check{\rho}_{z,w} - q\check{\sigma}_z. \tag{8.13}$$

This is again a current in N_z; it vanishes on the space $H(N_z) \otimes \mathbb{H}^n_{\mathbb{C}}$ by virtue of (8.11), and the series $\sum \check{v}_z$ converges strongly in the dual space to E_K.

Lemma 8.7. *The above construction can be specified so that*

$$0 \notin \bigcup_z \operatorname{supp} v_z. \tag{8.14}$$

Proof. The support of $\eta_z \omega$ does not contain the origin because of (7.3). The selection of the currents $\check{\sigma}_z$ can be subjected to a condition like (8.14), if we pass from $\mathcal{U}$ to the covering $\mathcal{U}' = \mathcal{U}(3)$. Indeed, take the inverse image of the density $\check{\sigma}_z$ with respect to the restriction mapping $E^0_K(N,\mathcal{U}') \to E^0_K(N,\mathcal{U})$. Define modified densities $\check{\sigma}'_z$ as follows: if $|z| > 2$, we set $\check{\sigma}'_z = \check{\sigma}_z$. If $|z| \leq 2$, we take the density $\check{\sigma}'_z$ with support in $\partial N'_z$, $N'_z = N \cap U_z(3)$ such that $\int f\check{\sigma}'_z = \int f\check{\sigma}_z$ for any function $f \in H(N \cap U_z(3))$ and $\|\check{\sigma}'_z\| = \|\check{\sigma}'_z\|$ on the space of continuous functions in N'_z. The modified density can be found by means of the Hahn–Banach theorem and of the following inequality, which holds for functions $f \in H(N'_z)$:

$$\sup_{N'_z} |f| = \sup_{\partial N'_z} |f|. \tag{8.15}$$

The set N is a union of $2n$-subspaces in $\mathbb{C}^{4n}$. Therefore, (8.15) is a corollary of the maximum principle for harmonic functions. We modify the family of densities $\check{\rho}_{z,w}$ in a similar way. By means of (8.13), we obtain a cochain v which is defined on the covering $\mathcal{U}'$ and satisfies (8.14). □

Lemma 8.8. *For any z, the equation $\check{v}_z = \bar{\partial}\psi_z$ has a solution ψ_z which is a $(2n+1, 2n)$-current with compact support in $N_z \setminus \{0\}$ such that the series $\sum \psi_z$ converges in E^0_K.*

Proof of Lemma 8.8. If $|z| > 4$, the support of v_z is contained in $H \setminus U_0(1)$; and a solution ψ with the same property can be constructed by standard methods. The convergence of the series $\sum \psi_z$ is easy to control, since the variety N is a cone.

Now we pass to the covering $\mathcal{U}(7)$ and take the sum v_0' of currents v_z for $|z| \leq 4$. The series converges on the space $E(N \cap U_0(7))$, and $0 \notin \operatorname{supp} v_0'$. To solve the equation $v_0' = \bar{\partial}\psi_0$ in the singular space $N \cap U_0(7))$, we apply Theorem 9.1. □

Set $\check{\sigma}^0 = \{\check{\sigma}_z\}$ and similarly for $\check{\rho}^1$ and ψ^0. We have by Lemma 8.8

$$\omega = \delta^*\check{\omega}^0 = \delta^* q^* \check{\sigma}^0 + \delta^*\bar{\partial}\psi^0.$$

Setting $\psi = \delta^*\psi^0 = (\psi_1, ..., \psi_n)$, we have

$$\tilde{\zeta}\omega = \tilde{\zeta}\bar{\partial}\psi, \tag{8.16}$$

since $\tilde{\zeta}^t q = p^* q^* = (qp)^* = 0$. (Here we mean $\tilde{\zeta}\omega = \sum \tilde{\zeta}_j \omega_j$ and so on.) We put

$$u_K^{1,0} = \int_\nu \exp(\zeta x)\tilde{\zeta}\psi,$$

where the integral again means the pull-down of a current by the mapping ν. This is, in fact, a $(1,0)$-current in $\mathrm{P}_1(\mathbb{C})$ with values in the sheaf $\mathcal{Q}(\operatorname{int}K, \mathbb{R})$, which means that its local coefficient satisfies (1.3) in $\operatorname{int}K$. Indeed, we have

$$\frac{\partial}{\partial \bar{q}_j} u_K^{1,0} = \int_\nu \exp(\zeta x)\zeta_j \sum_k \tilde{\zeta}_k \psi_k = 0$$

for $j = 1, ..., n$ by Proposition 4.2. Now we check the equation

$$\nabla(\bar{\partial}) u_K^{1,0} = u^{1,1}. \tag{8.17}$$

Evaluating the left hand side on a function $\phi \in \mathcal{D}(\mathrm{P}_1(\mathbb{C}))$, we have

$$\begin{aligned}\int_{\mathrm{P}_1(\mathbb{C})} \phi\bar{\partial}u_K^{1,0} &= -\int \bar{\partial}(\phi) \wedge u_K^{1,0} = -\int \bar{\partial}(\phi) \wedge \int_\nu \exp(\zeta x)\tilde{\zeta}\psi \\ &= -\int_N \nu^*(\bar{\partial}\phi) \wedge \exp(\zeta x)\tilde{\zeta}\psi = -\int_N \bar{\partial}(\nu^*(\phi) \wedge \exp(\zeta x)\tilde{\zeta}\psi \\ &= -\int \bar{\partial}(\nu^*(\phi) \wedge \exp(\zeta x)\tilde{\zeta}\psi) + \int \nu^*(\phi) \wedge \exp(\zeta x)\tilde{\zeta}\bar{\partial}\psi.\end{aligned}$$

The commutation relation $\nu^*(\bar{\partial}\phi) = \bar{\partial}(\nu^*\phi)$, which was used above, is a corollary of the fact that ν is holomorphic. The first integral in the right hand side vanishes since the integrand equals $d\chi$ for a current χ. The second one equals

$$\int_{\mathrm{P}_1(\mathbb{C})} \phi \int_\nu \exp(\zeta x)\tilde{\zeta}\omega = \int \phi u^{1,1}$$

because of (8.16), and (8.17) follows. For a larger compact $L \subset U$, we have $\nabla(\bar{\partial})(u_K^{1,0} - u_L^{1,0}) = 0$ in $\operatorname{int}K$; hence the difference $u_L^{1,0} - u_K^{1,0}$ is a holomorphic form on the projective line, evaluated in the space of continuous functions $\operatorname{int} K \to \mathbb{H}_\mathbb{C}$. It equals zero; consequently, the form $u_L^{1,0}$ is an extension of $u_K^{0,1}$ to a larger compact. Taking a sequence of successive extensions, we get a form $u^{0,1}$ which gives a solution to (8.1) in U. □

9 $\bar{\partial}$-equation in an analytic set

Theorem 9.1. *Let U be a Stein manifold and A a closed complex analytic subset of U of pure dimension $m > 1$ which has only a finite set S of singular points. For an arbitrary (m, m)-current α in $A \setminus S$ with compact support such that*

$$\int f\alpha = 0 \tag{9.1}$$

for any holomorphic function f in U, the equation $\alpha = \bar{\partial}_A \beta$ has a solution that is a $(m, m-1)$-current in A with compact support in $A \setminus S$.

Proof. Let $\mathcal{O}$ be the structure sheaf of U and $\mathcal{O}(A) = \mathcal{O}/\mathcal{I}(A)$, where $\mathcal{I}(A)$ is the sheaf of holomorphic functions in U that vanish in A. Take a Stein submanifold $U' \Subset U$ containing $\operatorname{supp}\alpha$ and such that the holomorphic functions in U are dense in $\Gamma(U', \mathcal{O})$. The sheaf $\mathcal{O}(A)$ has a resolvent $(\mathcal{L}^*, d)$ in U', where all $\mathcal{L}^q$, $q = 0, 1, ...$ are free $\mathcal{O}$-modules of finite rank and $\mathcal{L}^0 = \mathcal{O}$. Let $\mathcal{E}^*$ be the $\bar{\partial}$-complex in U of sheaves of smooth forms of type $(0, *)$. This is a flat $\mathcal{O}$-module according to [6]. Take the tensor product $\mathcal{B} \doteqdot \mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{E}^*$. This is a bicomplex, with the first degree and differential coming from $(\mathcal{L}^*, d)$ and the second degree and differential inherited from $(\mathcal{E}^*, \bar{\partial})$. We have for the first differential

$$H^q(\mathcal{B}, d) = 0,\ q > 0; \qquad H^0(\mathcal{B}, d) \cong \mathcal{O}(A) \otimes \mathcal{E}^*.$$

For the second differential, we have $H^k(\mathcal{B}, \bar{\partial}) = 0,\ k > 0$ and $H^0(\mathcal{B}, \bar{\partial}) = \mathcal{L}^*$. Comparing two spectral sequences, we find that the complex $\mathcal{O}(A) \otimes \mathcal{E}^*$ is acyclic in positive degree. Taking into account that the functor $\Gamma(U', \cdot)$ is exact with respect to the bicomplex $\mathcal{B}$, we conclude that the following sequence of Fréchet spaces is exact:

$$0 \to \Gamma(U', \mathcal{O}(A)) \to \Gamma(U', \mathcal{O}(A) \otimes \mathcal{E}^0) \xrightarrow{\bar{\partial}} \Gamma(U', \mathcal{O}(A) \otimes \mathcal{E}^1) \xrightarrow{\bar{\partial}} \cdots .$$

Each mapping in this sequence is continuous and open. Hence the sequence of dual spaces is exact as well:

$$0 \leftarrow \Gamma(U', \mathcal{O}(A))^* \leftarrow \Gamma(U', \mathcal{O}(A) \otimes \mathcal{E}^0)^* \xleftarrow{\bar{\partial}} \Gamma(U', \mathcal{O}(A) \otimes \mathcal{E}^1)^* \xleftarrow{\bar{\partial}} \cdots . \tag{9.2}$$

The current α defines a continuous functional $\alpha^\sharp$ on the space $\Gamma(U', \mathcal{O}(A) \otimes \mathcal{E}^0)$. The latter can be written by means of local generators $g_1, ..., g_l$, $l = n - m$, $n = \dim U$ of the sheaf $\mathcal{I}(A)$ as

$$\alpha^\sharp = \alpha\, \delta(g_1)...\delta(g_l) dg_1 \wedge \cdots \wedge dg_l \wedge d\bar{g}_1 \wedge \cdots \wedge d\bar{g}_l.$$

By assumption, the functional $\alpha^\sharp$ vanishes on the space of holomorphic functions in U, which is dense in $\Gamma(U', \mathcal{O}(A))$. Therefore, $\alpha^\sharp$ vanishes on the space $\Gamma(U', \mathcal{O})$, which by means of (9.2) implies the existence of a solution $\beta^\sharp \in \Gamma(U', O(A) \otimes \mathcal{E}^1)^*$ to the equation $\alpha^\sharp = \bar\partial\beta^\sharp$. Take the pull-back $\beta^\sharp$ of this solution with respect to the surjection $\Gamma(U', O \otimes \mathcal{E}^1) \to \Gamma(U', O(A) \otimes \mathcal{E}^1)$. The functional $\beta^\sharp$ is a current in U' of type $(n, n-1)$ with compact support which vanishes on any form $g\varepsilon$, where $g \in \Gamma(U', \mathcal{I}(A))$, $\varepsilon \in \Gamma(U', \mathcal{E}^1)$. Consequently, $\operatorname{supp}\beta^\sharp \subset A$; and the condition of the theorem implies that $\bar\partial\beta^\sharp = 0$ in a small Stein neighbourhood $W \subset U'$ of S. Now we seek a solution to the equation $\bar\partial\gamma^\sharp = \beta^\sharp$ in W with a current $\gamma^\sharp \in \Gamma_c(W, \mathcal{O}(A) \otimes \mathcal{E}^2))^*$ of the type $(n, n-2)$. To solve this equation, we argue as above with the functor $\Gamma(U', \cdot)$ replaced by $\Gamma_c(W, \cdot)$ using exactness of the sequence

$$\Gamma_c(W, \mathcal{O}(A) \otimes \mathcal{E}^1) \xrightarrow{\bar\partial} \Gamma_c(W, \mathcal{O}(A) \otimes \mathcal{E}^2) \xrightarrow{\bar\partial} \Gamma_c(W, \mathcal{O}(A) \otimes \mathcal{E}^3).$$

Set $\beta' = \beta^\sharp - \bar\partial(h\gamma^\sharp)$ for a function $h \in \mathcal{D}(W)$ which is equal to 1 in a neighbourhood of S. We now have $\bar\partial\beta' = \alpha^\sharp$ for a current β' with compact support in $U' \setminus S$, which vanishes on forms $g\varepsilon$ as above. We transform the solution β' to a current in the manifold A. First we take the representation $\beta' = \sum_k \beta'$, where $\{h_k\}$ is a finite partition of unity in U' with sufficiently small supports. Now we write each term with the help of a local coordinate system that includes the generators $g_1, ..., g_l$ of $\mathcal{I}(V)$:

$$h_k\beta' = \sum_j (\bar\partial_{\bar g})^j \beta_{kj} \delta(g_1)...\delta(g_l) dg_1 \wedge \cdots \wedge dg_l \wedge d\bar g_1 \wedge \cdots \wedge d\bar g_l,$$

where the sum is finite and β_{kj} are compactly supported $(m, m-1)$-currents in $A \setminus S$ and $(\bar\partial_{\bar g})^j = (\bar\partial_{\bar g_1})^{j_1} \cdots (\bar\partial_{g_l})^{j_l}$, $j = (j_1, ..., j_l)$. It is easy to see that the current $\beta = \sum \beta_{k0}$ in A is a solution of the equation $\bar\partial_A\beta = \alpha$. □

10 Exponential representation for a family of systems

We prove here Lemma 8.2. Let $\delta_1(r)$ be the delta-function in $\operatorname{Im}\mathbb{H} = \mathbb{R}^3$ supported by the unit sphere S^2. Consider the tensor product $u = u^{1,1} \otimes \delta_1(r)dr$. This is a distribution in $U \times \operatorname{Im}\mathbb{H}$ with values in the bundle $\mathcal{R}$ supported by $U \times S^2$. For any point $s \in S^2$, the fibre R of the bundle is equal to $R(q(s))$, where $q(s)$ is given by (2.1). We have $\bar q(s)R = 0$; hence $\bar q(s)u(s) = 0$, i.e.,

$$(\sqrt{-1} - y_1e_1 - y_2e_2 - y_3e_3)u(x, y) = 0, \tag{10.1}$$

where y_j, $j = 1, 2, 3$, are coordinates in $\mathrm{Im}\,\mathbb{H}$. Apply the Fourier transformation with respect to these coordinates:

$$\hat{u}(x; \xi_1, \xi_2, \xi_3) = \int_{\mathrm{Im}\,\mathbb{H}} \exp(-\imath \xi y) u(x; y).$$

The $\mathbb{H}_{\mathbb{C}}$-valued function $\hat{u}$ is defined in $U \times (\mathrm{Im}\,\mathbb{H})^*$. It is bounded in $K \times (\mathrm{Im}\,\mathbb{H})^*$ for any compact set $K \subset U$ and satisfies the equation

$$\left(1 - e_1 \frac{\partial}{\partial \xi_1} - e_2 \frac{\partial}{\partial \xi_2} - e_3 \frac{\partial}{\partial \xi_3}\right)\hat{u} = 0, \tag{10.2}$$

which follows from (10.1). The equation

$$\Delta \hat{u} + \hat{u} = 0 \tag{10.3}$$

is a corollary of $(y_1^2 + y_2^2 + y_3^2 - 1)\delta_1(r) = 0$. The function $\hat{u}$ is polymonogenic with respect to the coordinates $q_j = \sum x_{jk} e_k$, $j = 1, \ldots, n$. Thus $\hat{u}$ is a solution of the large system (1.3), (10.2), (10.3) of differential equations with constant coefficients. It belongs to the space of distributions in $U \times (\mathrm{Im}\,\mathbb{H})^*$ of moderate growth with respect to the coordinates ξ. Now we write an exponential representation in this space like that of Theorem 6.1. We get

$$\hat{u}(x, \xi) = \int_{\check{N}} \exp(\zeta x - \imath \xi y) \sum \tilde{\zeta}_j \lambda_j(\zeta, y), \tag{10.4}$$

where $\check{N}$ is the characteristic variety of the large system and λ_j, $j = 1, \ldots, n$ are $\mathbb{H}_{\mathbb{C}}$-valued currents in $\check{N}$ that are defined on the space of continuous functions $\varphi : \mathbb{C}^{4n} \times \mathrm{Im}\,\mathbb{H} \to \mathbb{H}_{\mathbb{C}}$ such that

$$\left(\frac{\partial}{\partial y}\right)^i \varphi = O(\exp(\lceil \zeta \rceil_K)), \quad \text{for each } i = (i_1, i_2, i_3) \quad \text{and some } K \Subset U.$$

The set $\check{N}$ is a subbundle of $S^2 \times \mathbb{C}^{4n}$, and the fibre $\check{N}_s$ over a point $s \in S^2$ is equal to the fibre N_L of the cone N over the corresponding left ideal $L = L(\tilde{q}(s))$, i.e., $\check{N}$ is the blow-up of the cone N with centre at the origin. The representation (10.4) can be proved by methods of [7]. Now by (10.4), we have for an arbitrary function $\psi \in \mathcal{D}(\mathrm{Im}\,\mathbb{H})$

$$u(x, \cdot)(\psi) = \hat{u}(x, \cdot)(\check{\psi}) = \int_{\check{N}} \exp(\zeta x) \sum \tilde{\zeta}_j \lambda_j(\zeta, y) \int \exp(-\imath \xi x) \check{\psi}(\xi) d\xi,$$

where $\check{\psi}$ denotes the inverse Fourier transform of ψ. The interior integral equals $\psi(y)$, whence

$$u(x, \cdot)(\psi) = \int_{\check{N}} \exp(\zeta x) \sum \tilde{\zeta}_j \lambda_j(\zeta, y) \psi(y). \tag{10.5}$$

Both sides vanish on functions of the form $(y_1^2+y_2^2+y_3^2-1)\psi$; consequently, (10.5) can be extended to the space $\mathcal{D}(S^2)$:

$$\int_{S^2} u^{1,1}(x,s)\phi(s) = \int_{S^2}\left(\int_{\tilde{N}_s} \exp(\zeta x)\sum \tilde{\zeta}_j\lambda_j(\zeta,y)\right)\phi(s).$$

This equation is equivalent to (8.2), and we need only specify the choice of the currents λ_j to satisfy the condition $\operatorname{supp}\lambda_j \subset N\setminus\{0\}$. We choose a function $e\in\mathcal{D}(U_0)$ such that $e=1$ in a neighbourhood of the origin (U_0 is the unit ball) and set $\lambda_j' = (1-e)\lambda_j + \mu_j$, where μ_j is a current supported by $\partial N\cap U_0$ which coincides with $e\lambda_j$ as a functional on the space $H(N\cap U_0)$. We produce μ_j by "sweeping away" $h\lambda_j$ from a neighbourhood of the origin, i.e., by applying (8.15) and succeeding arguments. □

REFERENCES

[1] W. W. Adams, C. A. Berenstein, P. Loustaunau, I. Sabadini and D. C. Struppa, *Regular functions of several quaternionic variables and the Cauchy–Fueter complex*, J. Geom. Anal., to appear.

[2] W. W. Adams, P. Loustaunau, V. P. Palamodov and D. C. Struppa, *Hartogs' phenomenon for polyregular functions and projective dimension of related modules over a polynomial ring*, Ann. Inst. Fourier (Grenoble) **47** (1997), 623–640.

[3] M. F. Atiyah, *Geometry of Yang–Mills Fields*, Accademia Nazionale dei Lincei, Scuola Normale Superiore, Lezioni Fermiane, Pisa, 1979.

[4] R. Fueter R, *Die Funktionentheorie der Differentialgleichungen $\Delta u = 0$ und $\Delta\Delta u = 0$ mit vier reellen Variablen*, Comment. Math. Helv. **7** (1935), 307–330.

[5] W. R. Hamilton, *The Mathematical Papers of Sir William Rowan Hamilton*, Vol. III, Cambridge, 1967.

[6] B. Malgrange, *Ideals of Differentiable Functions*, Oxford University Press, Bombay, 1968.

[7] V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Nauka, Moscow, 1967 (Russian); Springer, Berlin, 1970 (English).

[8] V. P. Palamodov, *A remark on exponential representation of solutions of differential equations with constant coefficients*, Math. USSR Sb. **5** (1968), 401–416.

[9] V. P. Palamodov, *Homological methods in the theory of locally convex spaces*, Russian Math. Surveys **26** (1971), 1–63.

[10] V. P. Palamodov, *On "holomorphic" functions of several quaternionic variables*, Linear Topological Spaces and Complex Analysis **2** (1995), 67–77 (METU-Tübitak, Ankara).

[11] A. Sudbery, *Quaternionic analysis*, Math. Proc. Camb. Phil. Soc. **85** (1979), 199–225.

Victor P. Palamodov
SCHOOL OF MATHEMATICS
TEL AVIV UNIVERSITY
RAMAT AVIV, TEL AVIV 69978, ISRAEL
email: palamodo@math.tau.ac.il

(Received August 19, 1998 and in revised form November 8, 1998)