PRIME ENDS AND QUASICONFORMAL MAPPINGS[†]

By

RAIMO NÄKKI

1. Introduction

Carathéodory [3] introduced the concept of a prime end, which enabled him to establish in a satisfactory way the correspondence of the boundaries under a conformal mapping between the unit disk and a bounded simply connected plane domain. Subsequently, 2-dimensional prime ends have been extensively studied from various points of view by several authors. In higher dimensions there are prime end theories by Kaufmann [15], Mazurkiewicz [21], and Freudenthal [8], but only Zorič [36]–[38] studied prime ends from the point of view of mapping theory, the original motivation for Carathéodory's work. Zorič established the Carathéodory theorem on the correspondence of boundaries for quasiconformal mappings of a ball.

In this paper we investigate prime ends in n-space, $n \geqq 2$, and, in Section 4, obtain the Carathéodory correspondence theorem for quasiconformal mappings of collared domains. In contrast with the definition of Zorič, we define prime ends in terms of quasiconformally invariant concepts. In Section 5 we give a metric characterization of prime ends, an n-dimensional analogue of the characterization used by Collingwood and Piranian [6] in the plane. A third approach to prime ends will be examined in Section 6. This is via the well-known Cantor–Meray–Hausdorff completion of a domain. The idea of using this approach for obtaining prime ends is due to Mazurkiewicz [20]. We generalize his methods to n dimensions. In Sections 7 and 8 we study the impression of a prime end, that is, the set of boundary points which is naturally associated with the prime end. We prove a quasiconformal analogue of Koebe's theorem concerning arcwise limits and give a simple normal family argument for Gehring's quasiconformal version of Lindelöf's theorem concerning angular limits [9]. We also give a form of Lindelöf's theorem expressed in terms of boundary cluster sets and show that the boundary cluster set and the cluster set coincide for quasiconformal mappings of collared domains. An extension

[†] Part of this research was done while the author was visiting at the University of Michigan in 1974–75. The research was supported by the Academy of Finland.

of Lindelöf's theorem, due to Hall [13] in 2-space, will be proved for quasiconformal mappings in n-space. An analogue of Tsuji's theorem on the correspondence of boundary sets of capacity zero will also be established. Carathéodory's question whether every prime end of a domain can be accessible and nondegenerate will be answered negatively in n-space. In 2-space this was first proved by Weniaminoff [34] and Urysohn [31]. In Section 9 this result will be combined with the fact that the boundary homeomorphisms of quasiconformal self-mappings of a collared domain act doubly transitively on each boundary component to show that a quasiconformal mapping of a collared domain has a continuous boundary extension if and only if all arcwise limits exist for every quasiconformal self-mapping of the image domain. As a consequence we observe that a quasiconformal mapping of a collared domain has a homeomorphic boundary extension if and only if every quasiconformal self-mapping of the image domain has a continuous boundary extension. In Section 10 we briefly discuss difficulties arising in attempts to develop prime end theories for domains that are not quasiconformally equivalent to collared domains.

2. Preliminaries

2.1. *Notation.* We consider sets in $\bar{R}^n$, $n \geqq 2$, the Möbius space obtained by adding the point ∞ to Euclidean n-space R^n. Stereographic projection from the n-sphere induces a natural metric q on $\bar{R}^n$, the spherical metric, and all topological considerations in this paper refer to $\bar{R}^n$ and the topology induced on it by q. We also use the Euclidean metric d in R^n and the relative spherical metric q_D in $\bar{R}^n$. The latter is needed only for domains D and it is defined by setting for $x_1, x_2 \in D$,

$$q_D(x_1, x_2) = \inf q(|\gamma|),$$

where the infimum is taken over the spherical diameters of the loci of all paths γ in D joining x_1 and x_2. By a path we mean a continuous nonconstant mapping of a closed line interval into D, and $\Delta(E, F\colon D)$ will denote the family of all paths in D joining the set E to the set F. The modulus of a family Δ of paths is designated by $M(\Delta)$. Given a number $r > 0$ and a point $b \in R^n$, we let $B^n(b, r)$ denote the open (Euclidean) ball of radius r with center at b and we let $S^{n-1}(b, r)$ denote the boundary of $B^n(b, r)$. We write $B^n(r)$ for $B^n(0, r)$, B^n for $B^n(1)$ and $S^{n-1}(r)$ for $S^{n-1}(0, r)$. Let f be a mapping of a domain D and let $b \in \partial D$. The cluster set of f at b is defined as

$$C(f, b) = \bigcap_U \overline{f(U \cap D)},$$

where U ranges over all neighborhoods of b. The cluster set $C(f, E)$ of f on a nonempty set $E \subset \partial D$ is defined as the union of the sets $C(f, b)$, $b \in E$. A homeomorphism f of a domain D is said to be K-*quasiconformal*, $1 \leqq K < \infty$, if

$$\frac{1}{K} M(\Delta) \leqq M(f\Delta) \leqq K M(\Delta)$$

for each family Δ of paths in D. A homeomorphism is *quasiconformal* if it is K-quasiconformal for some K.

2.2. *Collared domains.* Prime ends will be defined in Section 4 for domains that are quasiconformally equivalent to collared domains. A domain is said to be *quasiconformally collared*, or briefly, *collared* if each boundary point of the domain has an arbitrarily small neighborhood such that the part of the neighborhood inside the domain is quasiconformally equivalent to a ball. Collared domains have only finitely many boundary components, each of which is a compact $(n-1)$-dimensional manifold. Conversely, if a domain has only finitely many boundary components, each of which is an $(n-1)$-dimensional C^1-manifold, then the domain is collared. In particular, a ball is collared. A plane domain is collared if and only if its boundary consists of a finite number of disjoint Jordan curves. For proofs of these remarks and for further discussion of collaredness, the reader is referred to [24] and Väisälä [32, § 17].

We list three extremal length results needed in subsequent sections.

2.3. Lemma. *Let D be a collared domain and let F and F^* be nondegenerate connected subsets of D. Then*

(1) $M(\Delta(F, F^*: D)) = \infty$ *if and only if* $q(F, F^*) = 0$.

(2) *For each* $r > 0$ *there is a* $\delta > 0$ *such that* $M(\Delta(F, F^*: D)) \geqq \delta$ *whenever* $q(F) \geqq r$ *and* $q(F^*) \geqq r$.

Let D be quasiconformally equivalent to a collared domain and let F and F^* be nondegenerate connected subsets of D. Then

(3) $M(\Delta(F, F^*: D)) = \infty$ *if and only if* $q_D(F, F^*) = 0$.

Condition (1) was proved by Väisälä [32, § 17] (see also [24]), while conditions (2) and (3) were proved in [25] and [24], respectively.

3. Chains

3.1. *Cross-set.* A connected set E in a domain D is called a *cross-set* of D if E is closed in D, if $\bar{E}$ intersects ∂D, and if $D - E$ consists of two components, the boundary of each meeting ∂D.

3.2. *Chain.* A sequence $(E_k) = E_1, E_2, \cdots$ of cross-sets of a domain D is called a *chain* if E_{k+1} separates E_k and E_{k+2} in D for each k. Given a chain (E_k) in D, we denote by D_k the component of $D - E_k$ containing E_{k+1}. The set

$$\mathrm{I}(E_k) = \bigcap \bar{D}_k$$

is called the *impression* of the chain (E_k). As an intersection of a decreasing sequence of continua, $\mathrm{I}(E_k)$ is either a continuum or a point.

3.3. *Prime chain.* A chain (E_k) in a domain D is called a *prime chain* if

$$M(\Delta(E_k, E_{k+1}: D)) < \infty$$

for each k and

$$\lim M(\Delta(A, E_k: D)) = 0$$

for some (each, cf. [25]) continuum A in D.

3.4. Remark. Let f be a homeomorphism of a domain D onto a domain D' and let (E_k) be a chain in D. Then (fE_k) is a chain in D'. If f is quasiconformal and if (E_k) is a prime chain, then (fE_k) is a prime chain.

3.5. Lemma. *A chain (E_k) in a collared domain D is a prime chain if and only if $q(E_k, E_{k+1}) > 0$ for each k and $\mathrm{I}(E_k)$ reduces to a single boundary point. Moreover, for each boundary point b of D there is a prime chain (E_k) in D with $\{b\} = \mathrm{I}(E_k)$.*

Proof. The first part follows immediately from conditions (1) and (2) in 2.3. For the second part, we may, by [24, 2.3], choose a neighborhood U of b so that there is a homeomorphism g of $U \cap \bar{D}$ onto $\{x \in B^n : x_n \geqq 0\}$ which is quasiconformal in $U \cap D$ and maps b to the origin. Setting

$$E_k = g^{-1}(R^n_+ \cap S^{n-1}(1/2^k)),$$

$k = 1, 2, \cdots$, where R^n_+ is the upper half-space $\{x \in R^n : x_n > 0\}$, we obtain a prime chain (E_k) in D with $\{b\} = \mathrm{I}(E_k)$.

4. Prime ends. Quasiconformally invariant definition

Let D be a domain which can be mapped quasiconformally onto some collared domain. Two chains (E_k) and (E'_k) in D, with (D_k) and (D'_k) the corresponding sequences of subdomains of D, are *equivalent* if each domain D_k contains all but a finite number of the cross-sets E'_k and each domain D'_k contains all but a finite number of the cross-sets E_k. An equivalence class P in the collection of all prime chains in D is called a *prime end* of D. (By Remark 3.4 and Lemma 3.5, there are prime chains in D.) The *impression* $I(P)$ of P is defined as the common impression of all chains belonging to P. A sequence of points b_j (or sets F_j) in D is said to *converge* to a prime end P of D if, given a chain (E_k) belonging to P with (D_k) the corresponding sequence of subdomains of D, each D_k contains all but a finite number of the points b_j (or the sets F_j). (Note that (b_j) or (F_j) need not converge to any point in the ordinary sense.)

Since we have defined the prime ends in a quasiconformally invariant fashion, an n-dimensional analogue of Carathéodory's prime end theorem is easily verified:

4.1. Theorem. *Under a quasiconformal mapping f of a collared domain D_0 onto a domain D, there exists a one-to-one correspondence between the boundary points of D_0 and the prime ends of D. Moreover, the cluster set $C(f, b)$, $b \in \partial D_0$, coincides with the impression I(P) of the corresponding prime end P of D.*

Proof. We need only show that the boundary points and the prime ends of D_0 are in one-to-one correspondence. Let $\mathscr{C}$ be the collection of all prime ends of D_0. In view of Lemma 3.5, we can define a surjective mapping g of $\mathscr{C}$ onto ∂D_0 by setting

$$g(P) = I(P)$$

for each $P \in \mathscr{C}$. We claim that g is bijective. If not, there is a point b in ∂D_0 and a prime chain (E_k) in D_0, with (D_k) the corresponding sequence of subdomains of D_0, such that $\bigcap \bar{D}_k = b = \lim b_j$, where (b_j) is a sequence of points belonging to $D_0 - D_k$ for some fixed k. But since D_0 is collared, this implies that $b \in \bar{E}_i$ for $i \geqq k$. Thus $q(E_k, E_i) = 0$, contrary to Lemma 3.5.

In Theorem 4.1, we may consider $D^* = D \cup \mathscr{C}$, the set obtained by adding to D all the prime ends of D, as a compact Hausdorff space by extending the ordinary topology in D to D^* as follows: Let P be a prime end of D, let (E_k) be a chain determining P, and let (D_k) be the corresponding sequence of subdomains of D. Each D_k shall be a neighborhood of P and of each prime end of D that contains a chain whose elements all lie in D_k. This defines a Hausdorff topology on D^* and, if D is collared, the inclusion mapping of $\bar{D}$ onto D^* is a continuous bijection. Theorem 4.1 may accordingly be stated in the following form:

4.2. Theorem. *A quasiconformal mapping between a collared domain D_0 and a domain D can be extended to a homeomorphism between $\bar{D}_0$ and the prime end compactification D^*.*

The set $\mathscr{E}$ of prime ends of D can be made into a metric space by choosing a quasiconformal mapping f of a collared domain D_0 onto D and by adopting as the metric on $\mathscr{E}$ the spherical distance between the pair of points on ∂D_0 corresponding to a given pair of prime ends of $\mathscr{E}$. Under this metric we have:

4.3. Theorem. *$\mathscr{E}$ is a complete metric space and thus of the second category.*

Proof. By Theorem 4.2, $\mathscr{E}$ is a complete metric space. By Baire's theorem, $\mathscr{E}$ is of the second category.

The metric employed above depends on the mapping f. In Section 10 we will define an intrinsic metric on $\mathscr{E}$ without the introduction of an auxiliary mapping. For a simply connected domain D another such metric will be presented in Section 6.

As consequences of Theorem 4.1 we have:

4.4. Corollary. *Every sequence (F_j) of connected sets in D tending to ∂D with $q(F_j)\to 0$ has a subsequence converging to a prime end.*

Proof. Let f be a quasiconformal mapping of a collared domain D_0 onto D and let A be a continuum in D. Then

$$\lim M(\Delta(A, F_j\colon D)) = 0.$$

By the quasiconformality of f,

$$\lim M(\Delta(f^{-1}A, f^{-1}F_j\colon D_0)) = 0.$$

The modulus condition (2) in 2.3 implies

$$\lim q(f^{-1}F_j) = 0.$$

Hence a subsequence of $(f^{-1}F_j)$ converges to a boundary point of D_0. Theorem 4.1 concludes the proof.

4.5. Corollary. *Every sequence of points in D tending to ∂D contains a subsequence converging to a prime end.*

4.6. Remark. In 2-space, a somewhat different (quasi-) conformally invariant definition of prime ends, also given in terms of extremal length, was exhibited by Schlesinger [29] (cf. Ahlfors [1, 4–6]).

5. Metric characterization of prime ends

We next give a metric characterization of prime ends. This is an n-dimensional analogue of the definition used by Collingwood and Piranian [6] for prime ends in the plane. It shows, in particular, that our theory, restricted to the case of a ball, is equivalent to the theory of Zorič. (See [38, theorem 1].) Let D again be a domain which can be mapped quasiconformally onto a collared domain.

5.1. Theorem. *Each prime end of D contains a chain (E_k) such that $q(E_k)\to 0$ and $q_D(E_k, E_{k+1})>0$ for all k. Conversely, a chain (E_k) in D with these properties is a prime chain and thus determines a prime end of D.*

Proof. The second part of the theorem is obvious. For the first part, let f be a quasiconformal mapping of D onto a collared domain D_0, let P be a prime end of D, and let b be the point of ∂D_0 corresponding to P under f. As in the proof of Lemma 3.5, we may choose a neighborhood U of b and a homeomorphism g of $U\cap\bar{D}_0$ onto $\{x\in B^n\colon x_n\geqq 0\}$ so that g is quasiconformal in $U\cap D_0$ and maps b to the origin. A well-known lemma of Gehring [9, p. 18] implies that there is a decreasing sequence of numbers $r_k\in(0,1)$ such that $r_k\to 0$ and

$$q(E_k)\to 0,$$

where $E_k=(g\circ f)^{-1}S_k$ with $S_k=R^n_+\cap S^{n-1}(r_k)$. For each k, $q(g^{-1}S_k, g^{-1}S_{k+1})>0$ and therefore

$$M(\Delta(fE_k, fE_{k+1}\colon D_0))<\infty.$$

By the quasiconformality of f,

$$M(\Delta(E_k, E_{k+1}\colon D))<\infty$$

and by condition (3) in 2.3,

$$q_D(E_k, E_{k+1})>0.$$

Since (E_k) obviously belongs to P, the proof is complete.

6. The Mazurkiewicz definition for prime ends

In this section we generalize Mazurkiewicz' paper [20] to n dimensions. We obtain prime ends by completion of a metric space, where the definition of the metric is based on the intrinsic properties of the domain.

Let D be a simply connected domain which can be mapped quasiconformally onto a collared domain. Fix a closed ball B in D and for $x_1, x_2 \in D - B$ let

$$m(x_1, x_2) = \inf\{q(F): F \in \mathcal{F}(x_1, x_2)\},$$

where $\mathcal{F}(x_1, x_2)$ is the collection of all connected, relatively closed sets F in D separating B from $\{x_1, x_2\}$ in D so that x_1 and x_2 lie in one component of $D - F$.

6.1. Lemma. *m defines a metric in $D - B$.*

Proof. Obviously m assumes nonnegative finite values, $m(x_1, x_2) = m(x_2, x_1)$, and $m(x_1, x_2) = 0$ if and only if $x_1 = x_2$. Thus it remains to prove the triangle inequality. For this, let $x_1, x_2, x_3 \in D - B$ and let $\eta > 0$. Choose $F_1 \in \mathcal{F}(x_1, x_2)$ and $F_2 \in \mathcal{F}(x_2, x_3)$ so that

$$q(F_1) \leqq m(x_1, x_2) + \eta/2, \qquad q(F_2) \leqq m(x_2, x_3) + \eta/2.$$

Let G_1 be the component of $D - F_1$ containing x_1 and x_2, and let G_2 be the component of $D - F_2$ containing x_2 and x_3. Then $G = G_1 \cup G_2$ is a domain containing x_1, x_2, and x_3. Let H be the component of $D - \bar{G}$ containing the ball B and let $F_3 = D \cap \partial H$. Since D is simply connected, F_3 is connected and closed in D by the Phragmén–Brouwer theorem. Therefore $F_3 \in \mathcal{F}(x_1, x_3)$. Since $F_3 \subset F_1 \cup F_2$, we see that

$$m(x_1, x_3) \leqq q(F_3) \leqq q(F_1) + q(F_2) \leqq m(x_1, x_2) + m(x_2, x_3) + \eta.$$

Thus m defines a metric in $D - B$.

6.2. *Equivalence of the metrics q and m.* Since each point $x \in D - B$ has a neighborhood U such that $q(x_1, x_2) = m(x_1, x_2)$ for $x_1, x_2 \in U$, the metrics q and m are equivalent in $D - B$, i.e. $q(x_0, x_k) \to 0$ if and only if $m(x_0, x_k) \to 0$, $x_0, x_k \in D - B$.

6.3. *Extension of m to D.* Given any neighborhood V of B, $\bar{V} \subset D$, we can extend, by a method of Hausdorff [14], the restriction $m \mid (D - V)$ to all of D so that it remains equivalent to q. (Since we are interested in the behavior of m only near ∂D, the extension of m to all of D has nothing but technical relevance.) The domain D with metric m is denoted by D_m.

6.4. *Completion of D_m.* The metric space D_m is not complete. We complete it by using the well-known Cantor–Meray–Hausdorff method. For this, we divide the fundamental sequences of D_m into equivalence classes by regarding two such sequences (x_k) and (y_k) as equivalent if

$$\lim m(x_k, y_k) = 0.$$

It is easy to see that all sequences of the same equivalence class either converge to a certain point of D, or do not converge at all. In the second case the class of sequences is said to determine a *boundary element* of D. The sequences determining a boundary element are said to converge to this boundary element.

Let D^* be the set obtained by adding to D all the boundary elements of D and for $x_1^*, x_2^* \in D^*$ let

$$m^*(x_1^*, x_2^*) = \lim m(x_{1k}, x_{2k}),$$

where (x_{1k}) and (x_{2k}) are sequences in D converging to x_1^* and x_2^*, respectively. Then m^* defines a metric in D^*, D is contained isometrically in D^*, i.e. $m(x_1, x_2) = m^*(x_1, x_2)$ for $x_1, x_2 \in D$, D is dense in D^*, and D^* is complete. The set D^* with metric m^* will be called the *Mazurkiewicz completion* of D and denoted by D_m^*.

6.5. Remark. The above completion method applied to the domain D with metric q would lead to the ordinary boundary ∂D, while the metric q_D would give the accessible boundary points of D. The following theorem shows that $D_m^* - D$, the Mazurkiewicz boundary of the simply connected domain D, can be identified with the set of prime ends of D.

6.6. Theorem. *The prime end completion D^* of D and the Mazurkiewicz completion D_m^* of D are equivalent, i.e. there exists a homeomorphism of D^* onto D_m^* which reduces to the identity in D.*

Proof. Let D_0 be a collared domain equivalent to D and let f be a quasiconformal mapping of D_0 onto D. By Theorem 4.2, f can be extended to a homeomorphism f^* of $\bar{D}_0$ onto D^*. Thus we need only show that f can be extended to a homeomorphism f_m^* of $\bar{D}_0$ onto D_m^*. For this, let $x_0 \in \partial D_0$, let (x_j) and (y_j) be two sequences of points in D_0 converging to x_0, and let P be the prime end of D for which $f^*(x_0) = P$. By Theorem 5.1, P contains a chain (E_k), with (D_k) the corresponding sequence of subdomains of D, such that $q(E_k) \to 0$. Since $f(x_j), f(y_j) \in D_k$ for all j greater than some $j(k)$, it follows from the definition of the metric m that $(f(x_j))$ and $(f(y_j))$ are equivalent fundamental sequences in D_m^*. Thus they converge in D_m^* to an element $x_0^* \in D_m^* - D$ and, consequently, f can be extended to a continuous mapping f_m^* of $\bar{D}_0$ onto D_m^*.

If f_m^* is not a homeomorphism, there exist sequences (x_k) and (y_k) in D converging in D_m^* to an element $x^* \in D_m^* - D$ so that

$$\lim f^{-1}(x_k) \neq \lim f^{-1}(y_k).$$

Fix a continuum A in D. Since

$$\lim m(x_k, y_k) = 0,$$

there exists, by the definition of the metric m, a sequence (D_k) of subdomains of D such that $x_k, y_k \in D_k$ and

$$M(\Delta(A, D_k : D)) \to 0.$$

But since $f^{-1}A$ and each $f^{-1}D_k$ are connected sets in $f^{-1}D = D_0$ with $q(f^{-1}A) \geqq r$ and $q(f^{-1}D_k) \geqq r$ for some $r > 0$,

$$M(\Delta(f^{-1}A, f^{-1}D_k : D_0)) \not\to 0,$$

by condition (2) in 2.3. This contradiction to the quasiconformality of f shows that f_m^* is a homeomorphism. The proof is complete.

7. Theorems of Koebe, Lindelöf, and Tsuji

7.1. *Accessible prime ends. Koebe's Theorem.* Let D again be a domain quasiconformally equivalent to a collared domain. A point $b \in \partial D$ is *accessible* from D if there is a closed Jordan arc lying in D except for one endpoint, b. Such an arc is called an *end-cut* of D from b. The point b is *accessible relative to a prime end* P of D if b belongs to the impression $I(P)$ of P and there is an end-cut $\gamma : [0,1] \to D \cup \{b\}$ with $\gamma(1) = b$ such that γ converges to P, i.e. $\gamma_k \to P$ where $\gamma_k = \gamma[1 - 1/k, 1)$. A prime end is called *accessible* if its impression contains an accessible point relative to the prime end. By Corollary 4.4, every end-cut converges to an accessible prime end. Let f be a mapping of D into $\bar{R}^n$. The cluster set of f at b along an end-cut γ from b is denoted by $C_\gamma(f, b)$. If $C_\gamma(f, b) = \{b'\}$, then b' is called an *asymptotic value* or *arcwise limit* of f at b.

A well-known theorem of Koebe [16] states that a conformal mapping of a simply connected plane domain D onto the unit disk has arcwise limits along all end-cuts of D. An analogous result holds for quasiconformal mappings in n-space.

7.2. Theorem. *A quasiconformal mapping f of a domain D onto a collared domain has arcwise limits along all end-cuts of D.*

Proof. Let γ be an end-cut of D from a point $b \in \partial D$. Choose a continuum A in D and a sequence of neighborhoods U_k of b so that $\bigcap U_k = \{b\}$ and $\gamma_k = U_k \cap D \cap \gamma$ is connected. Then

$$\lim M(\Delta(A, \gamma_k : D)) = 0.$$

By the quasiconformality of f,

$$\lim M(\Delta(fA, f\gamma_k : fD)) = 0.$$

The modulus condition (2) in 2.3 implies

$$\lim q(f\gamma_k) = 0,$$

i.e. f has a limit along γ.

The proof shows, in fact, that a quasiconformal mapping of D onto any domain D' satisfying the modulus condition (2) in 2.3 has arcwise limits along all end-cuts of D. Such domains D' were studied in [25] and Palka [26].

7.3. *Principal and subsidiary points. Lindelöf's Theorem.* A point $b \in \mathrm{I}(P)$ is called a *principal point* (relative to the prime end P) if every neighborhood of b contains a cross-set of a chain belonging to P. Thus b is the limit of a convergent chain in P. The set of principal points of $\mathrm{I}(P)$ is denoted by $\Pi(P)$. Other points of $\mathrm{I}(P)$ are called *subsidiary points.* By Theorem 5.1, $\mathrm{I}(P)$ always contains at least one principal point. Obviously

$$\Pi(P) \subset C(\gamma, 1)$$

for every half-open path $\gamma: [0,1) \to D$ converging to P. If $\Pi(P) = C(\gamma, 1)$, then γ is called a *principal path.* The following analogue of Lindelöf's theorem [19] concerning angular cluster values guarantees the existence of principal paths. An end-cut of B^n from $b \in \partial B^n$ is called *angular* if it is contained in a cone $\{x \in R^n : (b \mid b - x) > |b - x| \cos \varphi\}$ for some $\varphi \in (0, \pi/2)$, where $(\cdot \mid \cdot)$ denotes the usual inner product.

7.4. Theorem. *Under a quasiconformal mapping f of the ball B^n, the cluster set $C_\gamma(f, b)$ on any angular end-cut γ of B^n from $b \in \partial B^n$ is the set $\Pi(P)$ of principal points of the prime end P of fB^n corresponding to b under f.*

Proof. This result follows from the proof of a similar theorem of Gehring [9, p. 19]. However, we give here a slightly different proof using normal families.

By performing a preliminary Möbius transformation, we may replace the ball B^n by the upper half-space R^n_+ and assume that b is the origin. Let K_α denote the cone $\{x \in R^n_+ : |x| < \alpha d(x, \partial R^n_+)\}$, where $\alpha > 0$ is chosen so that γ lies entirely in K_α. Since $\Pi(P) \subset C_\gamma(f, 0)$, it suffices to show that given a sequence (b_k) on γ with $b_k \to 0$ and $f(b_k) \to b'$, the point b' is a principal point of $\mathrm{I}(P)$. For this, let A_k denote the closed spherical annulus $|b_k|/2 \leqq |x| \leqq |b_k|$ and let $H_k = A_k \cap R^n_+$. For each k let $S_k \subset A_k$ be a sphere centered at the origin chosen so that

$$q(fS^+_k) \to 0,$$

where $S^+_k = S_k \cap R^n_+$ (Gehring [9, p. 18]). For $x \in R^n_+$ let

$$g_k(x) = \frac{|b_k|}{|b_1|} x, \qquad f_k = f \circ g_k.$$

Then f_k assumes the same values in H_1 that f assumes in H_k. Since each f_k omits two fixed values, (f_k) is a normal family (Väisälä, [32, § 20]). Hence there is a subsequence (f_{k_j}) of (f_k) converging to a constant or to a homeomorphism uniformly on $H_1 \cap \bar{K}_\alpha$ (Väisälä [32, § 21]). Since $C(f, 0) \subset \partial f R^n_+$, the limit mapping must be constant, and since $f(b_k) \to b'$, we have $f_{k_j}(H_1 \cap \bar{K}_\alpha) \to b'$. This, together with the facts that $q(fS^+_{k_j}) \to 0$ and $fS^+_{k_j}$ meets $f_{k_j}(H_1 \cap \bar{K}_\alpha)$, implies

$$fS^+_{k_j} \to b'.$$

Hence b' is a principal point of $\mathrm{I}(P)$, as desired.

Theorem 7.4, combined with 7.3, gives

7.5. Corollary. *Let f be a quasiconformal mapping of B^n and let b' be an arcwise limit of f at a point $b \in \partial B^n$. Then b' is the angular limit of f at b.*

In the case of a collared domain, we obtain from Theorem 7.4

7.6. Corollary. *Let f be a quasiconformal mapping of a collared domain D and let P be the prime end of fD corresponding to a given point b of ∂D under f. Then*

$$\Pi(P) = \bigcap_\gamma C_\gamma(f, b),$$

where γ ranges over all end-cuts of D from b. Moreover, there are end-cuts γ of D from b such that $\Pi(P) = C_\gamma(f, b)$.

7.7. Corollary. *The set of principal points of a prime end is either a point or a continuum, while the set of subsidiary points is either empty or has the cardinality of the continuum.*

Corollary 7.6, together with the fact that an end-cut converging to a prime end meets all but a finite number of the cross-sets of any chain in the prime end, gives

7.8. Corollary. *A prime end is accessible if and only if its impression contains only one principal point.*

7.9. Corollary. *The impression of a prime end contains at most one accessible point relative to the prime end, which, if it exists, is the sole principal point of the prime end.*

7.10. *Correspondence of sets of capacity zero. Tsuji's Theorem.* Let F be a compact proper subset of $\bar{R}^n$. Then $M(\Delta(F, \partial U\colon \bar{R}^n)) > 0$ either for each or for no neighborhood U of F, $\bar{U} \neq \bar{R}^n$. (See e.g. [25, 3.2].) The set F is said to be of *positive* (conformal) *capacity* in the first case and of (conformal) *capacity zero* in the second case. An arbitrary set A is of capacity zero, denoted $\operatorname{cap} A = 0$, if each of its compact subsets is of capacity zero.

Rešetnjak [28] uses capacities of condensers for classifying compact sets of positive capacity and compact sets of capacity zero. Both methods lead to the same classification, because $M(\Delta(F, \partial U\colon \bar{R}^n)) = \operatorname{cap}(F, U)$ by Ziemer [35]. By a result of Rešetnjak [28], the α-dimensional Hausdorff measure of a compact set of capacity zero is zero for every $\alpha > 0$.

A well-known theorem of Tsuji [30] states that, under a conformal mapping of the unit disk B^2 onto a domain D', $\operatorname{cap} A = 0$ whenever $A \subset \partial B^2$ corresponds to a compact set A' of accessible boundary points of D' with $\operatorname{cap} A' = 0$. An analogous result holds for quasiconformal mappings in n-space. To prove this, we introduce

7.11. *Asymptotic extension.* For a quasiconformal mapping f of a collared domain D and for the set A_f of points in ∂D where f has an asymptotic value, we let $\dot{f}$ denote the *asymptotic extension* of f to $D \cup A_f$. That is, $\dot{f}(x) = f(x)$ for $x \in D$, while $\dot{f}(x)$ equals the asymptotic value of f for $x \in A_f$. By Corollary 7.9, $\dot{f}$ is well-defined.

7.12. Theorem. *Let f be a quasiconformal mapping of B^n and let A' be a compact set of asymptotic values of f with* $\operatorname{cap} A' = 0$. *Then* $\operatorname{cap} \dot{f}^{-1}A' = 0$.

Proof. Choose a closed arc F' in $D' = fB^n$ and let Δ' be the family of all open paths in D' that have endpoints in A' and F'. Then $M(\Delta') = 0$, because $\operatorname{cap} A' = 0$.

Let $\Delta = f^{-1}\Delta'$. By the quasiconformality of f,

$$M(\Delta) = 0.$$

By Theorem 7.2, each path in Δ has two endpoints, one in $A = \dot{f}^{-1}A'$ and the other in $F = f^{-1}F'$. Let Δ_0 be the family of all open paths in B^n, not belonging to Δ but having two endpoints, one in A and the other in F. Then each path in $f\Delta_0$ is nonrectifiable, because $\dot{f}A = A'$. Hence $M(f\Delta_0) = 0$ by Väisälä [32, 6.11]. The quasiconformality of f implies

$$M(\Delta_0) = 0.$$

By Väisälä [32, 7.10], $M(\Delta_0 \cup \Delta)$ is equal to the modulus of the corresponding family $\Delta(A, F\colon B^n)$ of closed paths. Hence

$$M(\Delta(A, F\colon \bar{R}^n))/2 \leqq M(\Delta(A, F\colon B^n)) \leqq M(\Delta_0) + M(\Delta) = 0$$

by the symmetry principle of the modulus. (See Gehring [10].) Therefore $\operatorname{cap} A = 0$.

7.13. Corollary. *Let f be a continuous mapping of $\bar{B}^n$ such that f maps B^n quasiconformally onto a domain D and let A be a set in ∂D with* $\operatorname{cap} A = 0$. *Then* $\operatorname{cap} f^{-1}A = 0$.

A quasiconformal analogue, due to Zorič [39], of Beurling's theorem [2] concerning angular limits of a conformal mapping follows easily:

7.14. Corollary. *Let f be a quasiconformal mapping of B^n onto a domain D and let A be the set of all points in ∂B^n where f has an angular limit. Then* $\operatorname{cap}(\partial B^n - A) = 0$ *and the set of angular limits coincides with the set of accessible boundary points of D.*

Proof. By Corollary 7.5, the set of asymptotic values of f in $\partial B^n - A$ is empty. If Δ_0 is the family of all end-cuts of B^n with one endpoint in $\partial B^n - A$, then each path in $f\Delta_0$ is nonrectifiable and, therefore, $M(\Delta_0) = M(f\Delta_0) = 0$. This implies that $\operatorname{cap}(\partial B^n - A) = 0$. The second assertion follows from Theorem 7.2.

7.15. *Extensions of Lindelöf's Theorem, Hall's Theorem.* Given a set F and a point $b \neq \infty$, let $A = \{r > 0\colon F \cap S^{n-1}(b, r) \neq \varnothing\}$. If A is measurable with respect to Lebesgue 1-measure m_1, we define the lower *radial density* of F at b as

$$\operatorname{rad\,dens}(F, b) = \liminf_{r \to 0} \frac{m_1(A \cap (0, r))}{r}.$$

In terms of radial density, theorem 6 of Gehring [9] and Theorem 7.4 in this paper can be strengthened as follows:

7.16. Theorem. *Let f be a quasiconformal mapping of B^n, let P be the prime end of fB^n corresponding to a given point b of ∂B^n under f, and let $\dot{f}$ be the asymptotic extension of f to $B^n \cup A_f$. Then*

$$\Pi(P) = \bigcap_F C_F(\dot{f}, b),$$

where the intersection is taken over the cluster sets of $\dot{f}$ at b along all sets F in $B^n \cup A_f$ for which $\operatorname{rad\,dens}(F, b) > 0$.

Proof. It suffices to show that $\Pi(P)$ is a subset of $\bigcap C_F(\dot{f}, b)$ (Theorem 7.4). By performing a preliminary Möbius transformation, we may replace the ball B^n by the upper half-space R^n_+ and assume that b is the origin and that $f(B^n \cap R^n_+)$ is bounded. Let F be a set in $R^n_+ \cup A_f$ for which $\operatorname{rad\,dens}(F, 0) > 0$. Choose $c \in (0, 1)$ and $r_0 > 0$ so that

$$m_1(A \cap (0, r)) \geqq cr$$

for $r \in (0, r_0)$, where $A = \{r > 0 \colon F \cap S^{n-1}(r) \neq \emptyset\}$. Fix a point b' in $\Pi(P)$. Since b' is a principal point of the prime end P, there is a sequence (b_k) of points in $R^n_+ \cap B^n(r_0)$ converging to the origin angularly so that $f(b_k) \to b'$. Let A_k denote the closed spherical annulus $c|b_k| \leqq |x| \leqq |b_k|$, let $H_k = A_k \cap R^n_+$, let L_k denote the closed line interval $[c|b_k|/2, |b_k|]$, and let I_0 be the set of all $r > 0$ such that f fails to have an asymptotic value for at least one point of $\partial R^n_+ \cap S^{n-1}(r)$. From Corollary 7.14 it follows, by Fubini's theorem, that $m_1(I_0) = 0$.

Next let

$$E_k = A \cap L_k - I_0$$

and for $r \in (0, 1)$ let

$$\operatorname{osc}(f, r) = \sup_{x, y \in S(r)} |f(x) - f(y)|,$$

where $S(r) = R^n_+ \cap S^{n-1}(r)$. An n-dimensional analogue of a well-known lemma due to Gehring [9, p. 18] implies

$$\int_{E_k} [\operatorname{osc}(f,r)]^n \frac{dr}{r} \leqq \int_{L_k} [\operatorname{osc}(f,r)]^n \frac{dr}{r} \leqq C m_n(H_k'),$$

where $H_k' = fH_k$ and C is a positive constant depending only on n and the maximal dilatation of f. Since $m_1(A \cap (0, |b_k|)) \geqq c|b_k|$, it follows that

$$m_1(E_k) \geqq c|b_k|/2.$$

Therefore

$$\operatorname{osc}(f, r_k) \leqq (2Cm_n(H_k')/c)^{1/n}$$

for some $r_k \in E_k$. Since $m_n(H_k') \to 0$ as $k \to \infty$, we conclude that

$$q(fS(r_k)) \to 0.$$

The normal family argument of Theorem 7.4 shows that

$$fS(r_{k_j}) \to b'$$

for a subsequence (r_{k_j}) of (r_k). Since $\bar{R}^n_+ \cap S^{n-1}(r_{k_j})$ meets F and since $\dot{f}$ is defined in $\bar{R}^n_+ \cap S^{n-1}(r_{k_j})$, we conclude that $b' \in C_F(\dot{f}, b)$, as desired.

As a consequence we have a quasiconformal analogue in n-space of a theorem due to Hall [13]. (For additional results in this direction, see Vuorinen [33].)

7.17. Corollary. *Let f be a quasiconformal mapping of B^n and let the asymptotic extension of f to $B^n \cup A_f$ have a limit b' at a point $b \in \partial B^n$ along a set F in $B^n \cup A_f$ with* $\operatorname{rad\,dens}(F, b) > 0$. *Then b' is the angular limit of f at b.*

A well-known theorem of Tsuji states that under a conformal mapping f of the unit disk B^2, there exists for each point $b \in \partial B^2$ a sequence of circular arcs $\gamma_k = B^2 \cap S^1(b, r_k)$ such that $r_k \to 0$, $q(f\gamma_k) \to 0$, and f has angular limits at endpoints of γ_k. The proof of Theorem 7.16 shows that an analogous result holds for quasiconformal mappings in n-space, thus providing a slightly strengthened version of a result of Gehring [9, p. 18]:

7.18. Corollary. *Let f be a quasiconformal mapping of B^n, let $b \in \partial B^n$, and let P be the prime end of fB^n corresponding to b under f. Then for each point $b' \in \Pi(P)$ there exists a sequence of numbers $r_k \to 0$ such that $f(B^n \cap S^{n-1}(b, r_k)) \to b'$ and f has angular limits at all points of $\partial B^n \cap S^{n-1}(b, r_k)$.*

7.19. *Boundary cluster sets.* Let f be a mapping of a collared domain D and let $b \in \partial D$. The *boundary cluster set* of f at b is defined as

$$C_{\partial D}(f, b) = \bigcap_U \overline{C(f, U \cap \partial D - \{b\})},$$

where U ranges over all neighborhoods of b. The boundary cluster set of f at b along a set $F \subset \partial D$ is defined as

$$C_{\partial D, F}(f, b) = \bigcap_U \overline{C(f, U \cap F - \{b\})}.$$

Obviously,

$$C_{\partial D, F}(f, b) \subset C_{\partial D}(f, b) \subset C(f, b).$$

Corollary 7.18 implies the following form of Lindelöf's theorem in terms of boundary cluster sets:

7.20. Corollary. *Let f be a quasiconformal mapping of a collared domain D and let P be the prime end of fD corresponding to a given point b of ∂D under f. Then*

$$\Pi(P) \subset C_{\partial D, \gamma}(f, b)$$

for each arc γ in ∂D terminating at b.

It is known that $C(f, b)$ may differ from $C_{\partial D}(f, b)$ even if f is analytic and D is the unit disk. (See, for example, Collingwood–Piranian [6].) The following theorem shows that this cannot happen if f is quasiconformal.

7.21. Theorem. *Let f be a quasiconformal mapping of a collared domain D, let $b \in \partial D$, and let $\dot{f}$ be the asymptotic extension of f to $D \cup A_f$. Then*

$$C(f, b) = C_{\partial D}(f, b) = C(\dot{f} \mid A_f, b).$$

Proof. Obviously $C(f, b) \supset C_{\partial D}(f, b) \supset C(\dot{f} \mid A_f, b)$. Let $b' \in C(f, b)$, let P be the prime end of $D' = fD$ corresponding to b under f, and let $I(P) = \bigcap \bar{D}'_k$, where (D'_k) is a nested sequence of subdomains of D' determined by a chain (E'_k) in P with $q(E'_k) \to 0$ (Theorem 5.1). By collaredness of D and by Corollary 7.18, every principal point of $I(P) = C(f, b)$ belongs to $C(\dot{f} \mid A_f, b)$. Assume, therefore, that b' is a subsidiary point of $I(P)$ and that $b' \neq \infty$. Choose $x'_k \in D'_k$ so that $x'_k \to b'$, join x'_k

to b' by a line segment L'_k, and denote by b'_k the first point at which L'_k meets $\partial D'_k$. Obviously $b'_k \to b'$. Moreover $b'_k \in \partial D'$ for all k sufficiently large, because otherwise b'_k would lie in $\partial D'_k - \partial D'$, hence in E'_k, and b' would be a principal point of I(P). It is clear, furthermore, that $b'_k \neq b'$ for each k, because otherwise b' would again be a principal point of I(P). We conclude that, for large k, b'_k is accessible from D' and f^{-1} has an arcwise limit, b_k, at b'_k along L'_k (Theorem 7.2). Since $f^{-1}D'_k \to b$, it follows that $b_k \to b$, and therefore $b' \in C(\dot{f} \mid A_f, b)$.

7.22. Corollary. *A quasiconformal mapping f of a collared domain D has a limit at a boundary point b if and only if the asymptotic extension of f to $D \cup A_f$ has a limit at b along A_f.*

7.23. Corollary. *A quasiconformal mapping f of a collared domain D can be extended to a continuous mapping of $\bar{D}$ if and only if the asymptotic extension of f to $D \cup A_f$ has a limit along A_f at every point of ∂D.*

8. The classification of prime ends

8.1. *A problem of Carathéodory.* Let D again be a domain which can be mapped quasiconformally onto a collared domain. Following Carathéodory [3] we say that a prime end of D is of the *first, second, third*, or *fourth kind* according as its impression consists of

(1) only one point (necessarily a principal point),

(2) one principal point and some subsidiary points,

(3) more than one principal point and no subsidiary points,

(4) more than one principal point and some subsidiary points.

We have already agreed to denote the set of prime ends of D by $\mathscr{E}$. The subset of $\mathscr{E}$ of prime ends of i-th kind will be denoted by $\mathscr{E}_i$, $i = 1,2,3,4$.

By Corollary 7.8, a prime end is of the first or second kind if and only if it is accessible. We also recall that the principal points of a prime end of the third or fourth kind form a continuum, and the subsidiary points, whenever they exist, form an infinite set (Corollary 7.7).

By Theorem 4.1, $\mathscr{E} = \mathscr{E}_1$ if and only if a quasiconformal mapping of a collared domain onto D extends continuously to the boundary. Continuous boundary extension of quasiconformal mappings was studied in some detail in [24]. From results there it follows that $\mathscr{E} = \mathscr{E}_1$ if and only if D is *finitely connected* on the boundary, that is, each point in ∂D has an arbitrarily small neighborhood U such that $U \cap D$ contains only a finite number of components. Alternatively, $\mathscr{E} = \mathscr{E}_1$ if and only if D satisfies the *uniform modulus condition* (2) in 2.3.

Carathéodory raised the question of whether it is possible that $\mathscr{E} = \mathscr{E}_2$ for a simply connected plane domain. This question has been answered in the negative by Weniaminoff [34] and Urysohn [31]. The situation remains unchanged in higher dimensions. This follows from an n-dimensional version of a theorem due to Collingwood [4]:

8.2. Theorem. *The set $\mathscr{E}_1 \cup \mathscr{E}_3$ of prime ends having only principal points is a residual subset of the set $\mathscr{E}$.*

Proof. Let f be a quasiconformal mapping of a collared domain D_0 onto D. By [24, 2.3], each boundary point of D_0 has a neighborhood U such that there is a quasiconformal mapping g of $U \cap D_0$ onto B^n which extends to a homeomorphism g^* of $U \cap \bar{D}_0$. Theorem 7.4 implies that, for each point $b \in g^*(U \cap \partial D_0)$, the radial cluster set of $f \circ g^{-1}$ at b equals the set $\Pi(P)$ of principal points of the corresponding prime end P of D. On the other hand, a well-known maximality theorem of Collingwood [4, theorem 7] implies that the radial cluster set of $f \circ g^{-1}$ equals the complete cluster set $I(P)$ of $f \circ g^{-1}$ at each point of $g^*(U \cap \partial D_0)$, except possibly for a set of the first category on ∂B^n. It follows that $\Pi(P) = I(P)$ for all prime ends P of D corresponding to the points of $U \cap \partial D_0$, except possibly for a set of the first category. Since ∂D_0 can be covered by a finite number of such neighborhoods U, we conclude that $\Pi(P) = I(P)$ for all prime ends of D, except possibly for a set of the first category. This proves the theorem.

8.3. Corollary. $\mathscr{E} \neq \mathscr{E}_2$.

8.4. Remark. Corollary 8.3 says that it is not possible for all prime ends of D to be accesible and nondegenerate. In fact, it is not possible for all prime ends of D to have subsidiary points, i.e. $\mathscr{E} \neq \mathscr{E}_2 \cup \mathscr{E}_4$, for $\mathscr{E}_2 \cup \mathscr{E}_4$ is of the first category (Theorem 8.2), while $\mathscr{E}$ is of the second category (Theorem 4.3). We also note that $\mathscr{E} \neq \mathscr{E}_3 \cup \mathscr{E}_4$, because D contains accessible boundary points, hence accessible prime ends.

The *diameter* of a prime end P is defined as the diameter of the impression $I(P)$. A subset of $\mathscr{E}$ is called an *arc* if it corresponds to an arc on the boundary of a collared domain under a quasiconformal mapping. We conclude this section with the following observations, the 2-dimensional versions of which are due to Piranian [27].

8.5. Theorem. *$\mathscr{E}_1$ is a G_δ-set, while $\mathscr{E}_1 \cup \mathscr{E}_2$ is an $F_{\sigma\delta}$-set having the cardinality of the continuum on each arc of $\mathscr{E}$.*

Proof. Obviously $\mathscr{E}_1 = \cap M_j$, where M_j denotes the set of prime ends of D of diameter less than $1/j$. Let P be a prime end of D, let (E_k) be a chain in P with (D_k) the corresponding sequence of subdomains of D, and suppose that each neighborhood of P in $D \cup \mathscr{E}$ (in the sense of Theorem 4.2) contains a prime end of diameter at least $1/j$. Then every set $\bar{D}_k$ is of diameter at least $1/j$ and therefore the diameter of P is at least $1/j$. Consequently, each $\mathscr{E} - M_j$ is closed, hence M_j is open in $\mathscr{E}$, and therefore $\mathscr{E}_1$ is a G_δ-set.

Let f be a quasiconformal mapping of a collared domain D_0 onto D. By [24, 2.3], each boundary point of D_0 has a neighborhood U such that there is a quasiconformal mapping of $U \cap D_0$ onto B^n which can be extended to a homeomorphism of $\overline{U \cap D_0}$ onto $\bar{B}^n$. The set of points on ∂B^n where a continuous mapping of B^n fails to have radial limits is a $G_{\delta\sigma}$-set (see, for example, [5, p. 23]). Therefore, since ∂D_0 can be covered by a finite number of the neighborhoods U, the set of points in ∂D_0 where f fails to have asymptotic values is a $G_{\delta\sigma}$-set, so its complement in ∂D_0 is an $F_{\sigma\delta}$-set. This set corresponds to $\mathscr{E}_1 \cup \mathscr{E}_2$ under f. Since D is quasiconformally equivalent to a collared domain, every arc on $\mathscr{E}$ contains a subarc corresponding to an arc on ∂B^n under a quasiconformal mapping of B^n. Therefore, since the complement of any set of capacity zero with respect to any arc has the cardinality of the continuum, the last assertion follows from Corollary 7.14.

The first part of the proof gives

8.6. Corollary. *For every positive number h, the set of prime ends of diameter at least h is closed.*

9. Transitive action and boundary extension

9.1. *Quasiconformally homogeneous boundaries.* Let $\mathscr{F}$ be a family of homeomorphisms of a set F onto itself. We say that *$\mathscr{F}$ acts m-transitively* on a set $E \subset F$ if, given any m points $a_1, \cdots, a_m$ of E and any m points $b_1, \cdots, b_m$ of E, there is a mapping f in $\mathscr{F}$ such that $f(a_i) = b_i$, $i = 1, \cdots, m$. The family $\mathscr{F}$ is said to *act transitively* if it acts 1-transitively. We will show that the induced boundary homeomorphisms of the group of quasiconformal self-mappings of a collared domain $D \subset \bar{R}^n$ act m-transitively, $m = 1, 2, \cdots$, on each boundary component of D, provided $n \geqq 3$. For $n = 2$ the boundary homeomorphisms act 2-transitively. This result will be combined with Corollary 8.3 to yield two theorems on the boundary extension of quasiconformal mappings. We begin with the following local result:

9.2. Lemma. *Let b be a boundary point of a collared domain D and let U be a neighborhood of b. Then there is a neighborhood $N \subset U$ of b and a $K \geqq 1$ with the following property: given a point b' in $N \cap \partial D$, there is a homeomorphism f of $\bar{D}$ onto itself such that f is K-quasiconformal in D, $f(x) = x$ for $x \in \bar{D} - U$, and $f(b) = b'$.*

Proof. Since D is collared, there is a neighborhood $V \subset U$ of b and a quasiconformal mapping g of $V \cap D$ onto B^n. By [24, 2.3], we may assume that g is a homeomorphism of $V \cap \bar{D}$ into $\bar{B}^n$. For $0 < \alpha < 1$ let C_α denote the open cone

$$C_\alpha = \{tx : t > 0, x \in B^n(g(b), \alpha) \cap \partial B^n\}$$

with vertex at the origin. Fix $\alpha \in (0, 1)$ such that $C_{2\alpha} \cap \partial B^n$ lies in $g(V \cap \partial D)$. Let $N' = C_\alpha \cap \bar{B}^n$ and $N = g^{-1}N' \cup (V - \bar{D})$. We show that the lemma holds for N.

Fix $b' \in N \cap \partial D$. Since $g(b') \in C_\alpha \cap \partial B^n$, there is a bi-Lipschitzian homeomorphism h of $\bar{B}^n$ onto itself (a "modified rotation") such that $h(g(b)) = g(b')$,

$$h(x) = x$$

for each x in $\bar{B}^n - C_{2\alpha}$,

$$|h(x)| = |x|$$

for each x in $\bar{B}^n$, and

$$L^{-1}|x - y| \leqq |h(x) - h(y)| \leqq L|x - y|$$

for all x, y in $\bar{B}^n$. Here L is a constant depending only on α and n.

Define

$$f(x) = \begin{cases} g^{-1} \circ h \circ g & \text{if } x \in \bar{D} \cap V, \\ x & \text{if } x \in \bar{D} - V. \end{cases}$$

Since h is L^{2n}-quasiconformal in B^n, it follows that f is K-quasiconformal in D with $K = L^{2n}K(g)^2$, where $K(g)$ denotes the maximal dilatation of g in $V \cap D$. Moreover, $f(x) = x$ for $x \in \bar{D} - U$ and $f(b) = b'$. Hence f is a mapping with the desired properties.

9.3. Theorem. *Given a collared domain D, there is a $K \geqq 1$ such that the boundary homeomorphisms of the K-quasiconformal self-mappings of D act transitively on each boundary component of D.*

Proof. For $U = \bar{R}^n$, $b \in \partial D$ choose $N = N_b$ and $K = K_b$ as in Lemma 9.2. Since ∂D is compact, a finite number of such N_b's, say $N_1, \cdots, N_p$, covers ∂D. We show that the theorem holds for $K = (K_1K_2\cdots K_p)^2$, where K_i, $i = 1, \cdots, p$, is the constant corresponding to N_i.

Let a and b be two points in a boundary component, B, of ∂D. We may assume that $a \in N_1$. Choose a subset, $\{N_1, \cdots, N_m\}$, of $\{N_1, \cdots, N_p\}$ so that $b \in N_m$ and $B \cap N_j \cap N_{j+1} \neq \emptyset$. Set $a_0 = a$, $a_m = b$, and for $j = 1, \cdots, m-1$ choose $a_j \in B \cap N_j \cap N_{j+1}$. By Lemma 9.2, for each $j = 1, \cdots, m$ there is a homeomorphism f_j of $\bar{D}$ onto itself such that f_j is K_j^2-quasiconformal in D and $f_j(a_{j-1}) = a_j$. The mapping $f = f_m \circ \cdots \circ f_1$ is a homeomorphism of $\bar{D}$ onto itself, K-quasiconformal in D, and $f(a) = b$.

9.4. Theorem. *The boundary homeomorphisms of the group of quasiconformal self-mappings of a collared domain $D \subset \bar{R}^n$, $n > 2$, act m-transitively, $m = 1, 2, \cdots$, on each boundary component of D. If $n = 2$, the boundary homeomorphisms act 2-transitively on each boundary component.*

Proof. Let B be a boundary component of D and let $a_1, \cdots, a_m \in B$, $b_1, \cdots, b_m \in B$. By Theorem 9.3, there is a homeomorphism f_1 of $\bar{D}$ onto itself which is quasiconformal in D and takes a_1 to b_1. Since B is a connected $(n-1)$-manifold — for $n = 2$, B is a closed Jordan curve — we can join $f_1(a_2)$ to b_2 by a closed path L_2 in B so that L_2 does not meet b_1. Let U_2 be a neighborhood of L_2 with $b_1 \notin \bar{U}_2$. Applying Lemma 9.2 repeatedly, we first find a homeomorphism f_2 of $\bar{D}$ onto itself such that $f_2(f_1(a_2)) = b_2$, f_2 is quasiconformal in D, and $f_2(x) = x$ for $x \in \bar{D} - \bar{U}_2$. The composed mapping $f_2' = f_2 \circ f_1$ is quasiconformal in D and $f_2'(a_i) = b_i$, $i = 1, 2$. Thus the boundary homeomorphisms act 2-transitively on B. If $n \geqq 3$ and $m \geqq 3$, we can join $f_2'(a_3)$ to b_3 by a closed path L_3 in B so that L_3 does not meet $\{b_1, b_2\}$. The above procedure gives a homeomorphism f_3' of $\bar{D}$ onto itself which is quasiconformal in D and takes a_i to b_i, $i = 1, 2, 3$. The desired mapping is obtained after $m - 1$ such steps.

9.5. Theorem. *A quasiconformal mapping f of a collared domain D can be extended to a continuous mapping of $\bar{D}$ if and only if all arcwise limits exist for every quasiconformal self-mapping of fD.*

Proof. If f can be extended to a continuous mapping of $\bar{D}$, then all the prime ends of $D' = fD$ are of the first kind. Since every end-cut of D' converges to a prime end, so does its image under a quasiconformal self-mapping of D', and the necessity of the condition follows.

For the sufficiency, observe first that every boundary component of D' contains accessible points, hence accessible prime ends. By Theorems 4.2 and 9.3, any two prime ends of the same boundary component correspond to each other under some quasiconformal self-mapping of D'. Therefore, the extension condition implies that every prime end of D' is accessible, hence either of the first or of the second kind. Let P_i, $i = 1,2$, be a prime end of i-th kind on a fixed boundary component of D', let

b_i be the point of ∂D corresponding to P_i under f, and let (x_k) be a sequence of points in D converging to b_2 so that the points $f(x_k)$ do not converge. Since D is collared, we can construct an end-cut γ of D from b_2 containing all the points x_k. By Theorem 9.3, there is a quasiconformal mapping g of D onto itself whose homeomorphic extension to $\bar{D}$ takes b_1 to b_2. Then $\gamma' = f \circ g^{-1}(\gamma)$ is an end-cut of D' from the point $I(P_1)$ and $f \circ g \circ f^{-1}$ is a quasiconformal mapping of D' onto itself which does not have a limit along γ'. Consequently, either all the prime ends of a given boundary component of D' are of the first kind or all are of the second kind. By Corollary 8.3, all the prime ends of D' must be of the first kind. Thus f has an extension to a continuous mapping of $\bar{D}$.

As a consequence we have the following result, proved by Erkama [7] for $D = B^n$ using a somewhat different argument.

9.6. Theorem. *A quasiconformal mapping f of a collared domain D can be extended to a homeomorphism of $\bar{D}$ if and only if every quasiconformal self-mapping of fD can be extended to a continuous mapping of $\overline{fD}$.*

Proof. If f can be extended to a homeomorphism of $\bar{D}$, then $D' = fD$ is collared and, therefore, the necessity of the condition follows [24], [32]. For the sufficiency, observe first that f has an extension to a continuous mapping f^* of $\bar{D}$ (Theorem 9.5). If f^* is not injective, there are three distinct points b_1, b_2, and b_3 on a boundary component of D with $f^*(b_1) = f^*(b_2) \neq f^*(b_3)$. By Theorem 9.4, there is a quasiconformal mapping h of D onto itself whose homeomorphic extension to $\bar{D}$ takes b_1 to b_2 and b_2 to b_3. Then $f \circ h \circ f^{-1}$ is a quasiconformal mapping of D' onto itself which does not extend continuously to the point $f^*(b_1)$, because the cluster set of $f \circ h \circ f^{-1}$ at $f^*(b_1)$ contains the points $f^*(b_2)$ and $f^*(b_3)$. Consequently, f^* is injective and the proof is complete.

10. Prime ends in noncollared domains

Prime ends have been defined above for domains quasiconformally equivalent to collared domains. In terms of extremal length one can define a "prime end metric" for an arbitrary domain D as follows. (The question has also been discussed recently by Goldstein and Vodopjanov [12].)

10.1. *A prime end metric.* Fix a continuum A in D and choose $r > 0$ so that the closure of

$$B = \{x : q(x, A) < r\}$$

lies in D. For $x_1, x_2 \in D - B$ let

$$e(x_1, x_2) = \inf M(\Delta(A, F\colon D)),$$

where the infimum is taken over all connected sets F in D containing x_1 and x_2, cf. [17].

10.2. Lemma. *e defines a metric in $D - B$ equivalent to the spherical metric q.*

Proof. Obviously e assumes nonnegative finite values, $e(x_1, x_2) = e(x_2, x_1)$, and $e(x_1, x_2) = 0$ if $x_1 = x_2$. By results in [25], $e(x_1, x_2) = 0$ implies $x_1 = x_2$. For the triangle inequality, let $x_1, x_2, x_3 \in D - B$ and let $\eta > 0$. Choose connected sets F_1 and F_2 in D such that $x_1, x_2 \in F_1$ and $x_2, x_3 \in F_3$ and such that

$$M(\Delta(A, F_1\colon D)) \leqq e(x_1, x_2) + \eta/2,$$

$$M(\Delta(A, F_2\colon D)) \leqq e(x_2, x_3) + \eta/2.$$

Then $F = F_1 \cup F_2$ is a connected set in D containing x_1 and x_3. The subadditivity of the modulus implies

$$e(x_1, x_3) \leqq M(\Delta(A, F\colon D)) \leqq M(\Delta(A, F_1\colon D)) + M(\Delta(A, F_2\colon D))$$

$$\leqq e(x_1, x_2) + e(x_2, x_3) + \eta.$$

Thus e defines a metric in $D - B$.

Since A lies at a positive distance from $D - B$, $\lim q(x_0, x_k) = 0$ implies $\lim e(x_0, x_k) = 0$ for any sequence (x_k) of points in $D - B$. By results in [25], $\lim e(x_0, x_k) = 0$ implies $\lim q(x_0, x_k) = 0$. Therefore, the metrics e and q are equivalent in $D - B$.

As in 6.3, we can extend e to all of D so that it remains equivalent to q. The domain D with metric e is denoted by D_e. We complete the metric space D_e as in 6.4 and denote the completion by D_e^*. Theorem 4.1 in [25] implies that the boundary elements in D_e^* are independent of the choice of the continuum A and the number r in 10.1.

10.3. Theorem. *If D is quasiconformally equivalent to a collared domain, then the prime end completion D^* of D is equivalent to the completion D_e^*.*

Proof. The proof of Theorem 6.6 applies verbatim.

Theorem 10.3 shows that the boundary $D_e^* - D$ can be identified with the set of prime ends of D whenever D is quasiconformally equivalent to a collared domain.

Let $D \neq \bar{R}^n$ once again be an arbitrary domain and let D' be quasiconformally equivalent to D. Fix a continuum A' in D', choose $r'>0$ so that the closure of $B'=\{x: q(x,A')<r'\}$ lies in D', and define a metric e' in $D'-B'$ as in 10.1. Extend e' to all of D' as in 6.3 and let $D^{*}_{e'}$ denote the completion of the metric space $D'_{e'}$.

10.4. Theorem. *A quasiconformal mapping f of D onto D' can be extended to a homeomorphism of D^{*}_{e} onto $D^{*}_{e'}$.*

Proof. As noted before, the boundary elements of $D^{*}_{e'}$ are independent of the choice of the continuum A' and the number r' used in defining the metric e'. Therefore, we may assume that the continuum used in defining e' is fA, where A is as in 10.1. There is a neighborhood U of ∂D such that e and e' are defined in $U\cap D$ and $f(U\cap D)$, respectively, in terms of extremal length as indicated in 10.1. Since $M(\Gamma)/K\leqq M(f\Gamma)\leqq KM(\Gamma)$ for each path family Γ in D, where K is the maximal dilatation of f, we have

$$K^{-1}e(x,y)\leqq e'(f(x),f(y))\leqq Ke(x,y)$$

for all x,y in $U\cap D$. Hence f is bi-Lipschitzian in $U\cap D$ with respect to the metrics e and e', and the theorem follows.

10.5. Remark. Lelong-Ferrand [18] defined metrics δ_D and $\delta_{D'}$ for proper subdomains D and D' of R^n in such a way that every quasiconformal mapping of D onto D' is bi-Lipschitzian with respect to these metrics. We note, however, that her metrics are complete.

The question immediately arises whether new results on pointwise boundary extension can be deduced as corollaries of Theorem 10.4. We will give an example of such a result. To this end, let $g:[0,\infty)\to R^1$ be a function satisfying the following conditions for some $0<a<\infty$:

(i) g is continuous, $g(0)=0$, $g(u)>0$ for $u>0$, and $g(u)=g(a)$ for $u\geqq a$.

(ii) g' is continuous and increasing in $(0,a)$.

(iii) $\lim_{u\to 0}g'(u)=0$.

Let

$$D=R^3_+-\{x=(r,\theta,x_3)\in R^3: 0\leqq r\leqq g(a-x_3), 0\leqq x_3\leqq a\},$$

where (r,θ,x_3) are cylindrical coordinates in R^3. The domain D is called an inward directed *spire*. The point ae_3 is called the *vertex* of the spire. The above terminology is taken from Gehring and Väisälä [11]. It was shown in [11] that D cannot be mapped quasiconformally onto a ball. Hence D is a noncollared Jordan domain in $\bar{R}^3$ whose boundary is a flat 2-sphere.

10.6. Corollary. *A quasiconformal mapping of a spire D onto a spire D' can be extended to a homeomorphism of $\bar{D}$ onto $\bar{D}'$.*

Proof. Since a spire is collared at each boundary point except for the vertex, it follows that two fundamental sequences clustering at the boundary of the spire are equivalent, in the metric defined in 10.1, if and only if they converge, in the spherical metric, to one and the same boundary point. The assertion follows, therefore, from Theorem 10.4. (Observing that the vertices must correspond to one another under the mapping, the existence of the extension would also follow from Väisälä [32, 17.17].)

10.7. *Concluding remarks.* To illustrate the difficulties, from the point of view of mapping theory, arising in attempts to develop satisfactory prime end theories for domains which are not quasiconformally equivalent to collared domains, consider domains of the form

$$D = \{x = (x_1, x_2, x_3) \in R^3 : |x_2| < g(x_1), x_1 > 0\},$$

where the function g satisfies the conditions (i)–(iii) in 10.5. Such a domain D is called an *outward directed wedge* of angle zero. The set

$$E = \{x : x_1 = x_2 = 0\} \cup \{\infty\}$$

is called the *edge* of D. The terminology is again taken from Gehring and Väisälä [11]. It was shown in [11] that D cannot be mapped quasiconformally onto a ball. Hence D is a noncollared Jordan domain in $\bar{R}^3$ whose boundary is a flat 2-sphere. The edge E is a continuum on ∂D. It follows from [22, 5.4] that

$$M(\Delta(A, E : D)) = 0$$

for every continuum A in D if (and only if)

$$\int_0^d \frac{du}{g(u)^{\frac{1}{2}}} = \infty$$

for all $d > 0$. This is the case, in particular, if D is defined by the function $g(u) = u^p$, $p \geqq 2$. Therefore, the completion of the metric e defined in 10.1 yields the whole edge E as one "boundary element" for $p \geqq 2$. We conclude that, for $p \geqq 2$, Theorem 10.4 does not reveal whether or not a quasiconformal mapping of a wedge D onto a wedge D' admits a pointwise extension to $\bar{D}$. By other methods

[23], however, one can show that such an extension does indeed exist. It remains an open question whether a quasiconformal mapping between two arbitrary Jordan domains in $\bar{R}^n$, $n > 2$, must admit an extension to a homeomorphism between the closures. It is this fact, demonstrated above, that, even in the case of a Jordan domain D in $\bar{R}^n$, $n > 2$, $M(\Delta(A, E\colon D))$ may be zero for continua A in D and E on ∂D, which prevents one from carrying over to the general problem the proof used to establish the homeomorphic extension in the case where one of the domains is a collared Jordan domain.

References

1. L. V. Ahlfors, *Conformal Invariants; Topics in Geometric Function Theory*, McGraw–Hill, New York, 1973.

2. A. Beurling, *Ensembles exceptionelles*, Acta Math. **72** (1940), 1–13.

3. C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Ann. Math. **73** (1913), 323–370.

4. E. F. Collingwood, *Cluster set theorems for arbitrary functions with applications to function theory*, Ann. Acad. Sci. Fenn. A I **336/8** (1963), 1–15.

5. E. F. Collingwood and A. J. Lohwater, *The Theory of Cluster Sets*, Cambridge Univ. Press, 1966.

6. E. F. Collingwood and G. Piranian, *The mapping theorems of Carathéodory and Lindelöf*, J. Math. Pures Appl. (9) **43** (1964), 187–199.

7. T. Erkama, *Group actions and extension problems for maps of balls*, Ann. Acad. Sci. Fenn. A I **556** (1973), 1–31.

8. H. Freudenthal, *Enden and Primenden*, Fund. Math. **39** (1952), 189–210.

9. F. W. Gehring, *The Carathéodory convergence theorem for quasiconformal mappings in space*, Ann. Acad. Sci. Fenn. A I **336/11** (1963), 1–21.

10. F. W. Gehring, *A remark on domains quasiconformally equivalent to a ball*, Ann. Acad. Sci. Fenn. A I **2** (1976), 147–155.

11. F. W. Gehring and J. Väisälä, *The coefficients of quasiconformality of domains in space*, Acta Math. **114** (1965), 1–70.

12. V. M. Goldstein and S. K. Vodopjanov, *Metric completion of a domain by using a conformal capacity invariant under quasi-conformal mappings*, Dokl. Akad. Nauk SSSR **238** (1978), 1042–1044 (Russian).

13. T. Hall, *Sur la mesure harmonique de certains ensembles*, Ark. Mat. Astr. Fys. 25A, **28** (1937), 1–8.

14. F. Hausdorff, *Erweiterung einer stetigen Abbildung*, Fund. Math. **30** (1938), 40–47.

15. B. Kaufmann, *Über die Berandung ebener und räumlicher Gebiete*, *Math. Ann.* **103** (1930), 70–144.

16. P. Koebe, *Abhandlung zur Theorie der konformen Abbildung*, J. Reine Angew. Math. **145** (1915), 177–225.

17. T. Kuusalo, *Generalized conformal capacity and quasiconformal metrics*, Proc. of the Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Brasov, 1971, pp. 193–202.

18. J. Lelong–Ferrand, *Construction de métriques pour lesquelles les transformations quasi-conformes sont Lipschitziennes*, Symp. Math. **18** (1976), 407–420.

19. E. Lindelöf, *Sur un principe général de l'analyse et ses applications à la théorie de la représentation conforme*, Acta Soc. Sci. Fenn. **46**, no. 4 (1915), 1–35.

20. S. Mazurkiewicz, *Über die Definition der Primenden*, Fund. Math. **26** (1936), 272–279.

21. S. Mazurkiewicz, *Recherches sur la théorie des bouts premiers*, Fund. Math. **33** (1945), 177–228.

22. R. Näkki, *Boundary behavior of quasiconformal mappings in n-space*, Ann. Acad. Sci. Fenn. A I **484** (1970), 1–50.

23. R. Näkki, *Extension of quasiconformal mappings between wedges*, Proc. of the

Romanian–Finnish Seminar on Teichmüller Spaces and Quasiconformal Mappings, Brasov, 1971, pp. 229–233.

24. R. Näkki, *Continuous boundary extension of quasiconformal mappings*, Ann. Acad. Sci. Fenn. A I **511** (1972), 1–10.

25. R. Näkki, *Extension of Loewner's capacity theorem*, Trans. Amer. Math. Soc. **180** (1973), 229–236.

26. B. Palka, *On the uniform convergence of quasiconformal mappings*, Trans. Amer. Math. Soc. **184** (1973), 137–152.

27. G. Piranian, *The boundary of a simply connected domain*, Bull. Amer. Math. Soc. **64** (1958), 45–55.

28. Ju. G. Rešetnjak, *Spatial mappings with bounded distortion*, Sibirsk Mat. Ž. **8** (1967), 629–658 (Russian).

29. E. C. Schlesinger, *Conformal invariants and prime ends*, Amer. J. Math. **80** (1958), 83–102.

30. M. Tsuji, *Beurling's theorem on exceptional sets*, Tohoku Math. J. **2** (1950), 113–125.

31. P. Urysohn, *Über ein Problem von Herrn C. Carathéodory*, Fund. Math. **6** (1924), 229–235.

32. J. Väisälä, *Lectures on n-dimensional quasiconformal mappings*, Lecture Notes in Math. **229**, Springer-Verlag, Berlin and New York, 1971.

33. M. Vuorinen, *On the existence of angular limits of n-dimensional quasiconformal mappings*, to appear.

34. V. Weniaminoff, *Sur un problème de la représentation conforme de M. Carathéodory*, – Rec. Math. Soc. Moscou **31** (1922), 91–93.

35. W. P. Ziemer, *Extremal length and p-capacity*, Michigan Math. J. **16** (1969), 43–51.

36. V. A. Zorič, *Correspondence of the boundaries in Q-quasiconformal mappings of a sphere*, Dokl. Akad. Nauk SSSR **145** (1962), 1209–1212 (Russian).

37. V. A. Zorič, *Boundary properties of a class of mappings in space*, Dokl. Akad. Nauk SSSR **153** (1963), 23–26 (Russian).

38. V. A. Zorič, *Determination of boundary elements by means of sections*, Dokl. Akad. Nauk SSSR **164** (1965), 736–739 (Russian).

39. V. A. Zorič, *Angular boundary values of quasiconformal mappings of a ball*, Dokl. Akad. Nauk SSSR **179** (1967), 1492–1494 (Russian).

UNIVERSITY OF HELSINKI
HELSINKI, FINLAND
AND
UNIVERSITY OF MICHIGAN
ANN ARBOR, U.S.A.

(Received June 15, 1978 and in revised form March 8, 1979)