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A NEW UPPER BOUND FOR THE COMPLEX GROTHENDIECK CONSTANT

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ABSTRACT

Let φ denote the real function

$$\varphi(k) = k\int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1-k^2\sin^2 t}}\,dt, \qquad -1 \leqq k \leqq 1$$

and let $K_G^{\mathbf{C}}$ be the complex Grothendieck constant. It is proved that $K_G^{\mathbf{C}} \leqq 8/\pi(k_0+1)$, where k_0 is the (unique) solution to the equation $\varphi(k) = \frac{1}{8}\pi(k+1)$ in the interval [0, 1]. One has $8/\pi(k_0+1) \approx 1.40491$. The previously known upper bound is $K_G^{\mathbf{C}} \leqq e^{1-\gamma} \approx 1.52621$ obtained by Pisier in 1976.

§1. Introduction

In [3], Grothendieck proved the following fundamental inequality: Let F be the real or the complex scalar field. There are universal constants $K^{\mathbf{R}}$ and $K^{\mathbf{C}}$ such that for every pair of compact spaces S, T and every bounded bilinear form V: $C(S,F)\times C(T,F)\to F$ there exist probability measures μ, ν on S and T respectively, such that

$$|V(f,g)| \leqq K^F \,\|V\|\, \mu(|f|^2)^{1/2}\nu(|g|^2)^{1/2}$$

for all $f\in C(S,F)$ and all $g\in C(T,F)$. The smallest possible values for $K^{\mathbf{R}}$ and $K^{\mathbf{C}}$ are usually denoted $K_G^{\mathbf{R}}$ and $K_G^{\mathbf{C}}$ respectively. Grothendieck's inequality has important applications in the theory of Banach lattices (cf. [6], [9]) and there exist natural generalizations of the inequality to C^*-algebras (cf. [11], [4]). The exact values of $K_G^{\mathbf{R}}$ and $K_G^{\mathbf{C}}$ are not known, although the hunt for these constants has been going on for several years. Grothendieck proved that

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$$\pi/2 \leqq K_G^{\mathbf{R}} \leqq \sinh(\pi/2) \approx 2.301.$$

In [12], Rietz pushed the upper bound down to 2.261. Finally Krivine proved by a very elegant method that

$$K_G^{\mathbf{R}} \leqq \frac{\pi}{2\log(1+\sqrt{2})} \approx 1.782$$

(cf. [7]). Moreover he showed that $K_G^{\mathbf{R}} > \pi/2$ (unpublished). A straightforward generalization of Grothendieck's proof of $K_G^{\mathbf{R}} \geqq \pi/2$ gives $K_G^{\mathbf{C}} \geqq 4/\pi$. Kaiser proved by use of Rietz' method that $K_G^{\mathbf{C}} \leqq 1.607$ (cf. [5]) and in 1976 Pisier proved that $K_G^{\mathbf{C}} \leqq e^{1-\gamma} \approx 1.526$ (γ is Euler's constant). Recently Davie [2] has proved that $K_G^{\mathbf{C}} > 1.338$. (In particular $K_G^{\mathbf{C}} > 4/\pi$.)

The basic idea in this paper is to generalize Krivine's method for the proof of $K_G^{\mathbf{R}} \leqq \frac{1}{2}\pi(\log(1+\sqrt{2}))^{-1}$ to the complex case, but in the course of doing this, one runs into several technical problems, which are not present in the real case:

The starting point of Krivine's proof is that if (X_1, X_2) are random variables that form a two-dimensional (real) joint normal distribution, such that $E(X_1) = E(X_2) = 0$, $E(X_1^2) = E(X_2^2) = 1$, then

$$E(\operatorname{sign} X_1 \cdot \operatorname{sign} X_2) = \frac{2}{\pi} \operatorname{Arcsin} E(X_1 X_2).$$

(The function $(2/\pi)$ arcsin also plays a key role in Grothendieck's proof of $K_G^{\mathbf{R}} \leqq \sinh(\frac{1}{2}\pi)$, cf. [3], [8].) We prove that for complex symmetric normal distributions, the corresponding formula is

$$E(\operatorname{sign} X_1 \cdot \overline{\operatorname{sign} X_2}) = k \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1-|k|^2 \sin^2 t}}\, dt$$

where $k = E(X_1 \bar{X}_2)$ (cf. Lemma 3.2). Now, put

$$\varphi(k) = k \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1-k^2 \sin^2 t}}, \qquad -1 \leqq k \leqq 1.$$

The function φ can be expressed in terms of the complete elliptic integrals $E(k)$ and $K(k)$ (see, e.g., [1]), namely

$$\varphi(k) = \frac{1}{k}(E(k) - (1-k^2)K(k)), \qquad -1 < k < 1, \quad k \neq 0.$$

It is easy to check that $\varphi(k)$ is a homeomorphism of $[-1, 1]$ onto $[-1, 1]$, and that it can be expressed by the Taylor series

$$\varphi(k)=\frac{\pi}{2}\left(k+\left(\frac{1}{2}\right)^2\frac{k^3}{2}+\left(\frac{1\cdot 3}{2\cdot 4}\right)^2\frac{k^5}{3}+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^2\frac{k^7}{4}+\cdots\right)$$

for all $k\in[-1, 1]$. The crucial part in the proof of our new upper bound for $K_G^{\mathbf{C}}$ is to prove that the Taylor series for the inverse function

$$\varphi^{-1}(u)=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty} b_n u^n$$

converges to $\varphi^{-1}(u)$ for all $u\in[-1, 1]$, and that $b_n\leqq 0$ for $n\geqq 3$. This is in marked contrast to the real case, where the function corresponding to $\varphi^{-1}(u)$ is $\sin(\frac{1}{2}\pi u)$, which has an alternating Taylor series. The first few b_n's are easily computed:

$$b_1=\frac{4}{\pi},\quad b_3=-\frac{1}{8}\left(\frac{4}{\pi}\right)^3,\quad b_5=0,\quad b_7=-\frac{1}{1024}\left(\frac{4}{\pi}\right)^7.$$

To prove that $b_n\leqq 0$ for $n\geqq 9$, we first observe that φ has an analytic continuation to the disk $|z|<1$ and that

$$b_n=\frac{1}{n}\operatorname{Res}\left(\frac{1}{\varphi^n}, 0\right)$$

($\operatorname{Res}(f, z_0)$ denotes the residue of f at $z_0\in\mathbf{C}$). Next it is proved that φ can be extended further to a continuous function φ^+ in the upper half plane $\operatorname{Im} z\geqq 0$, such that φ^+ is analytic in the interior. This yields

$$b_n=\frac{2}{\pi n}\operatorname{Im}\left(\int_{\Gamma_\alpha}\frac{ds}{\varphi^+(s)^n}\right)\qquad\text{for } n \text{ odd},$$

where Γ_α is the arc consisting of the line segment $[1, \alpha]$ $(\alpha>1)$ and the quarter circle $\{\alpha e^{i\theta}\mid 0\leqq\theta\leqq\pi/2\}$. We put $\alpha=5\sqrt{2}$, and prove that for $n\geqq 9$ the main part of the above counter integral stems from a small interval $[1, \alpha_n]$ to the right of 1, where $(\varphi^+(s))^{-n}$ has a negative imaginary part. Thus $b_n<0$ for $n\geqq 9$.

We can now argue almost as in Krivine's paper [7, pp. 23–25] to see that if $\beta_0\in[0, 1]$ is the number for which

$$\sum_{n \text{ odd}} |b_n| \beta_0^n = 1$$

then $K_G^{\mathbf{C}} \leqq 1/\beta_0$ (cf. Section 3). Since $b_1 = 4/\pi$ and $b_n \leqq 0$ for $n \geqq 3$, the identity can also be written

$$\frac{8}{\pi}\beta_0 - \varphi^{-1}(\beta_0) = 1.$$

Putting $k_0 = \varphi^{-1}(\beta_0)$, we get the following equation:

$$\varphi(k_0) = \frac{\pi}{8}(k_0 + 1),$$

which can be solved numerically by use of tables of elliptic integrals. One has $k_0 \approx 0.81256$, from which

$$K_G^{\mathbf{C}} \leqq \frac{1}{\beta_0} = \frac{8}{\pi(k_0 + 1)} \approx 1.40491.$$

We doubt that the above upper estimate for $K_G^{\mathbf{C}}$ is an equality. A perhaps more plausible candidate for $K_G^{\mathbf{C}}$ is the slightly smaller number

$$|\varphi(i)|^{-1} = \left(\int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 + \sin^2 t}}\, dt\right)^{-1} \approx 1.40458.$$

This can be considered as a formed analogue of Krivine's upper bound $\frac{1}{2}\pi(\log(1 + \sqrt{2}))^{-1}$ for $K_G^{\mathbf{R}}$, because

$$\left|\frac{2}{\pi}\operatorname{Arcsin}(i)\right|^{-1} = \frac{\pi}{2\operatorname{Arsinh}(1)} = \frac{\pi}{2}(\log(1 + \sqrt{2}))^{-1}.$$

§2. Power series expansions of φ and φ^{-1}

Let φ be the function

$$\varphi(k) = k\int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 - k^2 \sin^2 t}}\, dt, \qquad -1 \leqq k \leqq 1.$$

It is easily checked that φ is a continuous, strictly increasing function and that $\varphi(1) = 1$, $\varphi(-1) = -1$. Hence φ is a homeomorphism of $[-1, 1]$ onto itself. Using the expansion

$$(1-k^2\sin^2 t)^{-1/2}=\sum_{m=0}^{\infty}\frac{1\cdot 3\cdot\ \cdots\ \cdot(2m-1)}{2\cdot 4\cdot\ \cdots\ \cdot 2m}k^{2m}\sin^{2m}t$$

($|k|\leqq 1$, $0\leqq t<\pi/2$) and the formula

$$\int_0^{\pi/2}\cos^2 t\sin^{2m}t\,dt=\frac{\pi}{4(m+1)}\left(\frac{1\cdot 3\cdot\ \cdots\ \cdot(2m-1)}{2\cdot 4\cdot\ \cdots\ \cdot 2m}\right),$$

φ can be expressed by the power series

$$\varphi(k)=\sum_{n\text{ odd}}a_nk^n,\qquad -1\leqq k\leqq 1$$

where

$$a_{2m+1}=\frac{\pi}{4(m+1)}\left(\frac{1\cdot 3\cdot\ \cdots\ \cdot(2m-1)^2}{2\cdot 4\cdot\ \cdots\ \cdot 2m}\right)^2.$$

The first few terms of the series are

$$\varphi(k)=\frac{\pi}{4}\left(k+\frac{1}{8}k^3+\frac{3}{64}k^5+\frac{25}{1024}k^7+\cdots\right).$$

For $k\in]-1,1[$, $k\neq 0$, $\varphi(k)$ can also be expressed in terms of the complete elliptic integrals

$$E(k)=\int_0^{\pi/2}(1-k^2\sin^2 t)^{1/2}dt,\quad K(k)=\int_0^{\pi/2}(1-k^2\sin^2 t)^{-1/2}dt,$$

namely

$$\varphi(k)=\frac{1}{k}(E(k)-(1-k^2)K(k)).$$

THEOREM 2.1. (1) *The inverse function φ^{-1} of φ can be expressed by an absolutely convergent power series*

$$\varphi^{-1}(s)=\sum_{n\text{ odd}}b_ns^n,\qquad -1\leqq s\leqq 1.$$

(2) $b_1=4/\pi$ *and* $b_n\leqq 0$ *for all* $n\geqq 3$.

REMARK. Since φ is a real analytic function, $\varphi(0)=0$ and $\varphi'(0)\neq 0$, it is clear that φ^{-1} can be expanded in a power series

$$\varphi^{-1}(s) = \sum_{n=1}^{\infty} b_n s^n$$

in some neighbourhood of 0. Moreover $b_n = 0$ for n even, because φ^{-1} is an odd function of s. By solving the equation

$$s = \frac{\pi}{4}\left(k + \frac{1}{8}k^3 + \frac{3}{64}k^5 + \frac{25}{1024}k^7 + \cdots\right)$$

up to 7th power in s, one finds

$$k = \varphi^{-1}(s) = \frac{4s}{\pi} - \frac{1}{8}\left(\frac{4s}{\pi}\right)^3 - \frac{1}{1024}\left(\frac{4s}{\pi}\right)^7 + O(s^9).$$

Hence

$$b_1 = \frac{4}{\pi}, \quad b_3 = -\frac{1}{8}\left(\frac{4}{\pi}\right)^3, \quad b_5 = 0, \quad b_7 = -\frac{1}{1024}\left(\frac{4}{\pi}\right)^7.$$

The rest of this section is used to prove that $\Sigma_{n\,\mathrm{odd}} | b_n | < \infty$, and that $b_n < 0$ for $n \geqq 9$, n odd.

Following standard notation of elliptic integrals, we put $E'(k) = E(\sqrt{1-k^2})$ and $K'(k) = K(\sqrt{1-k^2})$, $0 \leqq k \leqq 1$.

LEMMA 2.2. (1) *The function* $\varphi(k)$, $-1 \leqq k \leqq 1$ *can be extended to a continuous function* $\varphi^+(k)$ *in the closed upper half plane* $\operatorname{Im} k \geqq 0$ *such that* φ^+ *is analytic in the open half plane* $\operatorname{Im} k > 0$.

(2) *For* $k \in \mathbf{R}$, $k \geqq 1$,

$$\operatorname{Re} \varphi^+(k) = E\left(\frac{1}{k}\right), \quad \operatorname{Im} \varphi^+(k) = K'\left(\frac{1}{k}\right) - E'\left(\frac{1}{k}\right).$$

PROOF. For $k \in]-1, 1[$,

$$\frac{d}{dt}\operatorname{Arcsin}(k \sin t) = \frac{k \cos t}{\sqrt{1 - k^2 \sin^2 t}}.$$

Thus, by partial integration,

$$\varphi(k) = \int_0^{\pi/2} \sin t \operatorname{Arcsin}(k \sin t)dt, \qquad -1 \leqq k \leqq 1.$$

The analytic function $\sin z$ is a bijection of $[-\pi/2, \pi/2] \times [0, \infty[$ onto the upper closed half plane. Let Arcsin^+ be the inverse of this map. Then Arcsin^+

is analytic in the open half plane $\operatorname{Im} z > 0$, continuous in the closed half plane, and for $z \in \mathbf{R}$,

$$\operatorname{Arcsin}^+ z = \operatorname{Arcsin} z, \qquad -1 \leqq z \leqq 1,$$

$$\operatorname{Arcsin}^+ z = \tfrac{1}{2}\pi + i \operatorname{Arcosh}|z|, \quad |z| > 1.$$

Now, define

$$\varphi^+(k) = \int_0^{\pi/2} \sin t \operatorname{Arcsin}^+(k \sin t) dt, \qquad \operatorname{Im} k \geqq 0.$$

Then it is easily checked that φ^+ is a continuous extension of φ, and that φ^+ is analytic in the open half plane $\operatorname{Im} k > 0$.

For $k > 1$ we have by partial integration

$$\begin{aligned} \operatorname{Re} \varphi^+(k) &= \int_0^{\sin t = 1/k} \sin t \operatorname{Arcsin}(k \sin t) dt + \int_{\sin t = 1/k}^{\pi/2} \sin t \, \tfrac{1}{2}\pi \, dt \\ &= k \int_0^{\sin t = 1/k} \frac{\cos^2 t}{\sqrt{1 - k^2 \sin^2 t}} dt \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im} \varphi^+(k) &= \int_{\sin t = 1/k}^{\pi/2} \sin t \operatorname{Arcosh}(k \sin t) dt \\ &= k \cdot \int_{\sin t = 1/k}^{\pi/2} \frac{\cos^2 t}{\sqrt{k^2 \sin^2 t - 1}} dt. \end{aligned}$$

Substituting $\sin u = k \sin t$ in the integral for $\operatorname{Re} \varphi^+(k)$ and $\sin v = (1 - k^{-2})^{-1/2} \cos t$ in the integral for $\operatorname{Im} \varphi^+(k)$, one finds

$$\operatorname{Re} \varphi^+(k) = \int_0^{\pi/2} \sqrt{1 - k^{-2} \sin^2 u} \, du = E(1/k),$$

$$\begin{aligned} \operatorname{Im} \varphi^+(k) &= (1 - k^{-2}) \int_0^{\pi/2} \frac{\sin^2 v}{\sqrt{1 - (1 - k^{-2}) \sin^2 v}} dv \\ &= K(\sqrt{1 - k^{-2}}) - E(\sqrt{1 - k^{-2}}). \end{aligned}$$

Lemma 2.3. (1) $\operatorname{Im} \varphi^+(k) \geqq \operatorname{Im} \varphi^+(|k|)$ *for* $|k| \geqq 1$, $\operatorname{Im} k \geqq 0$.
(2) φ^+ *has no zeros in the closed half plane* $\operatorname{Im} k \geqq 0$ *except* $k = 0$.

Proof. (1) The analytic function sin maps the line segment

$$\{t + ia \mid -\pi/2 \leqq t \leqq \pi/2\}$$

onto the half ellipsoid

$$\{z \in \mathbf{C} \mid |z-1| + |z+1| = 2\cosh a, \operatorname{Im} z \geqq 0\}.$$

Therefore

$$\operatorname{Im} \operatorname{Arcsin}^+(z) = \operatorname{Arcosh}(\tfrac{1}{2}(|z-1| + |z+1|))$$

for $\operatorname{Im} z \geqq 0$. Since Arcosh is an increasing function on $[1, \infty[$, we get

$$\operatorname{Im} \operatorname{Arcsin}^+(z) \geqq \begin{cases} \operatorname{Arcosh}|z|, & |z| \geqq 1, \\ 0, & |z| < 1. \end{cases}$$

Hence, for $\operatorname{Im} k \geqq 0$, $|k| \geqq 1$,

$$\begin{aligned} \operatorname{Im} \varphi^+(k) &= \int_0^{\pi/2} \sin t \operatorname{Im} \operatorname{Arcsin}^+(k \sin t) dt \\ &\geqq \int_{\sin t = 1/|k|}^{\pi/2} \sin t \operatorname{Arcosh}(|k| \sin t) dt \\ &= \operatorname{Im} \varphi^+(|k|). \end{aligned}$$

(2) Since $\operatorname{Im} \operatorname{Arcsin}^+(z) > 0$ for $\operatorname{Im} z > 0$, $\varphi^+(k)$ has strictly positive imaginary part when $\operatorname{Im} k > 0$.

For $k \in [-1, 1]$, $\varphi^+(k) = \varphi(k)$ is zero only at $k = 0$, and for $k > 1$ or $k < -1$,

$$\operatorname{Im} \varphi^+(k) = \int_{\sin t = 1/|k|}^{\pi/2} \sin t \operatorname{Arcosh}(|k| \sin t) dt > 0.$$

This proves (2).

LEMMA 2.4. *Let $\alpha > 1$. For $n \in \mathbf{N}$, n odd*

$$b_n = \frac{2}{\pi n} \int_1^\alpha \operatorname{Im}(\varphi^+(k)^{-n}) dk + r_n(\alpha)$$

where

$$|r_n(\alpha)| \leqq \frac{\alpha}{n} (\operatorname{Im} \varphi^+(\alpha))^{-n}.$$

PROOF. The Taylor series

$$\varphi(k) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} a_n k^n$$

for φ defines an analytic function in the complex disk $|k| < 1$, which coincides with $\varphi^+(k)$ for $|k| < 1$, $\operatorname{Im} k \geqq 0$. Since $\varphi(0) = 0$ and $\varphi'(0) = \pi/4 \neq 0$, there exists $\delta_0 \in]0, 1]$, such that $\varphi(k)$ has an analytic inverse in the disk $|k| < \delta_0$. Let C_δ be the circle with radius δ with usual (counter-clockwise) orientation. For $0 < \delta < \delta_0$, $\varphi(C_\delta)$ is a simple closed curve with winding number $+1$. Hence by Cauchy's integral formula

$$b_n = \frac{1}{2\pi i} \int_{\varphi(C_\delta)} \frac{\varphi^{-1}(s)}{s^{n+1}} ds.$$

Substituting $s = \varphi(k)$ we get

$$b_n = \frac{1}{2\pi i} \int_{C_\delta} \frac{k}{\varphi(k)^{n+1}} \varphi'(k) dk.$$

Using

$$-n \int_{C_\delta} \frac{k\varphi'(k)}{\varphi(k)^{n+1}} dk + \int_{C_\delta} \frac{1}{\varphi(k)^n} dk = \int_{C_\delta} \frac{d}{dk}\left(\frac{k}{\varphi(k)^n}\right) dk = 0$$

we get

$$b_n = \frac{1}{2\pi i n} \int_{C_\delta} \varphi(k)^{-n} dk,$$

i.e., nb_n is the residue of φ^{-n} at 0. Since $b_n \in \mathbf{R}$,

$$b_n = \frac{1}{2\pi n} \operatorname{Im}\left(\int_{C_\delta} \varphi(k)^{-n} dk\right)$$

$$= \frac{1}{2\pi n} \operatorname{Im}\left(\int_0^{2\pi} \varphi(\delta e^{i\theta})^{-n} \delta i e^{i\theta} d\theta\right).$$

Using that $\varphi(k)^{-n}$ is an odd function for n odd, and using that $\varphi(\bar{k}) = \overline{\varphi(k)}$, one gets that the imaginary parts of the integrals over the four intervals $[0, \pi/2]$, $[\pi/2, \pi]$, $[\pi, 3\pi/2]$, $[3\pi/2, 2\pi]$ are equal. Thus, if C'_δ denotes the quarter circle

$$k = \delta e^{i\theta}, \qquad 0 \leqq \theta \leqq \pi/2,$$

then

$$b_n = \frac{2}{\pi n} \operatorname{Im}\left(\int_{C'_\delta} \varphi(k)^{-n} dk \right).$$

Since $\varphi(k)$ coincides with $\varphi^+(k)$ on C'_δ and since $\varphi^+(k)$ has no zeros in the set

$$\{ z \in \mathbf{C} \mid \delta \leqq |z| \leqq \alpha, 0 \leqq \arg z \leqq \pi/2 \}$$

(cf. Lemma 2.3(2)), we get by Cauchy's integral formula that

$$b_n = \frac{2}{\pi n} \operatorname{Im}\left(\int_\delta^\alpha \varphi^+(k)^{-n} dk + \int_{C'_\alpha} \varphi^+(k)^{-n} dk + \int_{i\alpha}^{i\delta} \varphi^+(k)^{-n} dk \right)$$

where the last integral is taken along the imaginary axis. Moreover, since $\varphi^+(k)$ is real on $[\delta, 1]$ and purey imaginary on the imaginary axis,

$$\operatorname{Im}\left(\int_\delta^1 \varphi^+(k)^{-n} dk \right) = 0$$

and

$$\operatorname{Im}\left(\int_{i\alpha}^{i\delta} \varphi^+(k)^{-n} dk \right) = 0.$$

Hence

$$b_n = \frac{2}{\pi n} \int_1^\alpha \operatorname{Im} \varphi^+(k)^{-n} dk + \frac{2}{\pi n} \operatorname{Im}\left(\int_{C'_\alpha} \varphi^+(k)^{-n} dk \right).$$

By Lemma 2.3, $|\varphi^+(k)| \geqq \operatorname{Im} \varphi^+(k) \geqq \operatorname{Im} \varphi^+(|k|)$. Thus

$$\left| \int_{C'_\alpha} \varphi^+(k)^{-n} dk \right| \leqq \frac{\pi \alpha}{2} (\operatorname{Im} \varphi^+(\alpha))^{-n}.$$

This completes the proof of Lemma 2.4.

LEMMA 2.5. *Let* $\varphi_1(k) = \operatorname{Re} \varphi^+(k)$, $\varphi_2(k) = \operatorname{Im} \varphi^+(k)$, *and* $\varphi'_1 = d\varphi_1/dk$, $\varphi'_2 = d\varphi_2/dk$ *for* $k > 1$. *Then*:

(1) $\varphi_1(k)\varphi'_2(k) - \varphi'_1(k)\varphi_2(k) = \pi/2k$, $k > 1$.

(2) *Let* $q = (4/\pi)E(1/\sqrt{2})(K(1/\sqrt{2}) - E(1/\sqrt{2})) \approx 0.86575$, *then*

$$\varphi_1(k)\varphi'_1(k) + \varphi_2(k)\varphi'_2(k) \geqq \frac{\pi}{2k} q, \qquad k > 1$$

and equality holds for $k = \sqrt{2}$.

PROOF. (1) By Lemma 2.2,

$$\varphi_1(k) = E\left(\frac{1}{k}\right) \quad \text{and} \quad \varphi_2(k) = K'\left(\frac{1}{k}\right) - E'\left(\frac{1}{k}\right).$$

Using

$$\frac{dE}{dk} = -\frac{1}{k}(K - E) \quad \text{and} \quad \frac{d(K-E)}{dk} = \frac{k}{1-k^2}E$$

(cf. [1, Formula 710.02 and 710.05]) one gets

$$\varphi_1'(k) = \frac{1}{k}\left(K\left(\frac{1}{k}\right) - E\left(\frac{1}{k}\right)\right),$$

$$\varphi_2'(k) = \frac{1}{k}E'\left(\frac{1}{k}\right).$$

Thus (1) follows from Legendre's relation $EK' + E'K - KK' = \pi/2$ (cf. [1, Formula 110.10]).

(2) We have

$$k(\varphi_1(k)\varphi_1'(k) + \varphi_2(k)\varphi_2'(k)) = (E(K-E) + E'(K'-E'))(1/k).$$

In particular

$$\sqrt{2}(\varphi_1\varphi_1' + \varphi_2\varphi_2')(\sqrt{2}) = 2E\left(\frac{1}{\sqrt{2}}\right)\left(K\left(\frac{1}{\sqrt{2}}\right) - E\left(\frac{1}{\sqrt{2}}\right)\right) = \frac{\pi}{2}q.$$

It remains to be proved that

$$(E(K-E) + E'(K'-E'))(k) \geqq (E(K-E) + E'(K'-E'))(1/\sqrt{2})$$

for $0 < k < 1$. Since the function

$$f(m) = (E(K-E) - E'(K'-E'))(\sqrt{m}), \qquad 0 < m < 1$$

is symmetric around $m = \frac{1}{2}$, it is sufficient to prove that f is convex. Using the above-mentioned formulas for dE/dk and $d(K-E)/dk$ one gets

$$\frac{d^2}{dm^2}E(\sqrt{m})(K-E)(\sqrt{m}) = \frac{1}{2}\left(\frac{E(\sqrt{m})}{1-m} - \frac{(K-E)(\sqrt{m})}{m}\right)^2$$
$$\geqq 0.$$

Since $E'(\sqrt{m})(K'-E')(\sqrt{m}) = E(\sqrt{1-m})(K-E)(\sqrt{1-m})$ also

$$\frac{d^2}{dm^2} E'(\sqrt{m})(K' - E')(\sqrt{m}) \geqq 0.$$

This proves that f is convex.

LEMMA 2.6. *$\Sigma_{n \text{ odd}} |b_n| < \infty$ and $\varphi^{-1}(s) = \Sigma_{n \text{ odd}} b_n s^n$ for $s \in [-1, 1]$.*

PROOF. By Lemma 2.5

$$\frac{d}{dk} |\varphi(k)|^2 = 2(\varphi_1 \varphi_1' + \varphi_2 \varphi_2')(k) \geqq \frac{\pi q}{k}.$$

Fix $\alpha > 1$, then for $1 \leqq k \leqq \alpha$,

$$|\varphi(k)|^2 \geqq |\varphi(1)|^2 + \int_1^k \frac{\pi q}{\alpha} dk$$

$$= 1 + \frac{\pi q}{\alpha}(k-1).$$

Thus for $n \geqq 3$

$$\int_1^\alpha |\varphi(k)|^{-n} dk \leqq \int_1^\alpha \left(1 + \frac{\pi q}{\alpha}(k-1)\right)^{-n/2} dk$$

$$\leqq \int_1^\infty \left(1 + \frac{\pi q}{\alpha}(k-1)\right)^{-n/2} dk$$

$$= \frac{\alpha}{\pi q} \frac{2}{n-2}.$$

Hence, by Lemma 2.4,

$$|b_n| \leqq \frac{4\alpha}{\pi^2 q n(n-2)} + \frac{\alpha}{n} (\operatorname{Im} \varphi^+(\alpha))^{-n}, \qquad n \geqq 3.$$

If, for instance, we put $\alpha = 5\sqrt{2}$, then by [1, p. 324]

$$\operatorname{Im} \varphi^+(\alpha) = K'(\sqrt{0.02}) - E'(\sqrt{0.02}) \approx 2.32555.$$

Thus the sequence $n^2|b_n|$ is bounded. In particular $\Sigma_{n \text{ odd}} |b_n| < \infty$. Therefore $\Sigma_{n \text{ odd}} b_n s^n$ converges to a continuous function on $[-1, 1]$, which is real analytic in the interior of the interval. Since this function coincides with $\varphi^{-1}(s)$ in some neighbourhood of 0 and since $\varphi^{-1}(s)$ is also real analytic on $]-1, 1[$, we have

$$\varphi^{-1}(s)=\sum_{n\,\mathrm{odd}} b_n s^n, \qquad s\in[-1,1].$$

Lemma 2.7. *Let $\theta(k)=\arg(\varphi^+(k))$, $k\geqq 1$. Then $\theta(k)$ is a strictly increasing function of k, $\theta(1)=0$ and $\lim_{k\to\infty}\theta(k)=\pi/2$.*

Proof. Using

$$\theta(k)=\arctan\frac{\varphi_2(k)}{\varphi_1(k)}$$

we have by Lemma 2.5 (1)

$$\frac{d\theta(k)}{dk}=\frac{\varphi_1(k)\varphi_2'(k)-\varphi_1'(k)\varphi_2(k)}{|\varphi^+(k)|^2}=\frac{\pi}{2k\,|\varphi^+(k)|^2}>0$$

for $k>1$. Thus $\theta(k)$ is strictly increasing for $k\geqq 1$. For $k=1$, $\theta(k)=\mathrm{Arg}(1)=0$. Using that

$$E(0)=\pi/2,\quad E(1)=1,\quad \lim_{k\to 1}K(k)=+\infty$$

we have by Lemma 2.2

$$\lim_{k\to\infty}\varphi_1(k)=\frac{\pi}{2}\quad\text{and}\quad\lim_{k\to\infty}\varphi_2(k)=\infty.$$

Thus $\theta(k)\to\pi/2$ for $k\to\infty$.

Lemma 2.8. *Put $\alpha=5\sqrt{2}$ and $\theta_0=\theta(\alpha)$. For fixed $n\in\mathbf{N}$ let $p=[n\theta_0/\pi]$* (= integer part of $n\theta_0/\pi$), *and put*

$$I_r=\frac{2}{\pi n}\int_{\theta(k)=(\pi/n)(r-1)}^{\theta(k)=(\pi/n)r}|\varphi^+(k)|^{-n}\,|\sin n\theta(k)|\,dk$$

for $r=1,\dots,p$. Moreover, put

$$I'=\frac{2}{\pi n}\int_{(\pi/n)p}^{\theta_0}|\varphi^+(k)|^{-n}\,|\sin n\theta(k)|\,dk.$$

Then:

(1) $(2/\pi n)\int_1^\alpha \mathrm{Im}(\varphi^+(k)^{-n})dk=-I_1+I_2-\cdots+(-1)^pI_p+(-1)^{p+1}I'$.

(2) *For $n\geqq 9$ one has $p\geqq 2$ and $I_1>I_2>\cdots>I_p>I'$.*

Proof. (1) It is clear that

$$\frac{2}{\pi n}\int_1^\alpha \operatorname{Im}(\varphi^+(k)^{-n})dk = -\frac{2}{\pi n}\int_1^\alpha |\varphi^+(k)|^{-n}\cdot\sin(n\theta(k))dk$$

$$= -I_1 + I_2 - \cdots + (-1)^p I_p + (-1)^{p+1} I'.$$

(2) Let $k=\chi(\theta)$, $\theta\in[0,\pi/2[$ be the inverse function of $\theta=\theta(k)$. Then by the formula for $d\theta(k)/dk$ derived in the proof of Lemma 2.7 we have

$$I_r = \frac{4}{\pi^2 n}\int_{(\pi/n)(r-1)}^{(\pi/n)r} \chi(\theta)\,|\varphi^+(\chi(\theta))|^{2-n}\,|\sin n\theta|\,d\theta$$

and

$$I' = \frac{4}{\pi^2 n}\int_{(\pi/n)p}^{\theta_0} \chi(\theta)\,|\varphi^+(\chi(\theta))|^{2-n}\,|\sin n\theta|\,d\theta.$$

We prove next that $k|\varphi^+(k)|^{2-n}$ is a strictly decreasing function on $[1,\alpha]$ for $n\geqq 9$:

$$\frac{d}{dk}(k|\varphi^+(k)|^{2-n}) = |\varphi^+(k)|^{2-n} + k\left(\frac{2-n}{2}\right)|\varphi^+(k)|^{-n}\frac{d}{dk}|\varphi^+(k)|^2$$

$$= |\varphi^+(k)|^{-n}(|\varphi^+(k)|^2 - (n-2)k(\varphi_1(k)\varphi_1'(k) + \varphi_2(k)\varphi_2'(k))$$

$$\leqq |\varphi^+(k)|^{-n}\left(|\varphi^+(k)|^2 - (n-2)\frac{\pi}{2}q\right)$$

$$\leqq |\varphi^+(k)|^{-n}\left(|\varphi^+(\alpha)|^2 - \frac{7\pi q}{2}\right).$$

Here we have used Lemma 2.5(2), and that $|\varphi^+(k)|^2$ is an increasing function of k (which also follows from Lemma 2.5(2)). From [1, p. 324],

$$\varphi_1(\alpha) = \varphi_1(5\sqrt{2}) = E(\sqrt{0.02}) \approx 1.56291,$$

$$\varphi_2(\alpha) = K'(\sqrt{0.02}) = E'(\sqrt{0.02}) \approx 2.32555.$$

Hence $|\varphi^+(\alpha)|^2 \approx 7.8509$ while $7\pi q/2 \approx 9.5194$. Thus $k|\varphi^+(k)|^{2-n}$ is a strictly decreasing function of k for $1\leqq k\leqq\alpha$ and $n\geqq 9$.

Using $|\sin n\theta|$ is periodic with period π/n, it now follows that

$$I_1 > I_2 > \cdots > I_p.$$

Moreover,

$$I' = \frac{4}{\pi^2 n}\int_{\pi p/n}^{\theta_0} \chi(\theta)\,|\varphi^+(\chi(\theta))|^{2-n}\,|\sin n\theta\,|\,d\theta$$

$$\leqq \frac{4}{\pi^2 n}\int_{(\pi/n)(p-1)}^{\theta_0-\pi/n} \chi(\theta)\,|\varphi^+(\chi(\theta))|^{2-n}\,|\sin n\theta\,|\,d\theta$$

$$< I_p.$$

Finally,

$$\theta_0 = \arctan\frac{\varphi_2(\alpha)}{\varphi_1(\alpha)} \geqq \arctan(1) = \pi/4.$$

Thus

$$p = \left[\frac{n\theta_0}{\pi}\right] \geqq \left[\frac{9\theta_0}{\pi}\right] \geqq 2 \qquad \text{for } n \geqq 9.$$

This completes the proof of Lemma 2.8.

LEMMA 2.9. *Let $n \geqq 9$, let I_r be as in Lemma 2.8, and let*

$$c = |\varphi^+(\sqrt{2})|\,e^{-q\theta(\sqrt{2})} \cong 1.05838.$$

Then

(1) $I_1 > (0.27/n^2)c^{-n}$,

(2) $I_2 < 0.35 \cdot I_1$.

PROOF. Since $\chi(\theta) \geqq 1$ for $0 \leqq \theta < \pi/2$, we have

$$I_1 \geqq \frac{4}{\pi^2 n}\int_0^{\pi/n} |\varphi^+(\chi(\theta))|^{2-n}\,|\sin n\theta\, d\theta.$$

For $\theta = \theta(k)$ (equivalently $k = \chi(\theta)$) we have

$$\frac{d}{d\theta}\log|\varphi^+(\chi(\theta))| = \frac{d}{dk}\log|\varphi^+(k)|^2\cdot\left(\frac{d\theta}{dk}\right)^{-1}$$

$$= \frac{\varphi_1(k)\varphi_1'(k) + \varphi_2(k)\varphi_2'(k)}{|\varphi^+(k)|^2}\cdot\frac{2}{\pi}\,k\,|\varphi^+(k)|^2$$

$$\geqq q$$

where $q \cong 0.86575$ is the constant defined in Lemma 2.5. Hence for $0 \leqq \theta \leqq \theta(\sqrt{2})$,

$$\log|\varphi^+(\chi(\theta))| \leqq \log|\varphi^+(\sqrt{2})| - (\theta(\sqrt{2}) - \theta)q.$$

Equivalently

$$|\varphi^+(\chi(\theta))| \leqq ce^{q\theta}, \qquad 0 \leqq \theta \leqq \theta(\sqrt{2})$$

where

$$c = |\varphi^+(\sqrt{2})|\, e^{-q\theta(\sqrt{2})}.$$

From [1, p. 324],

$$\varphi_1(\sqrt{2}) = E(1/\sqrt{2}) \approx 1.35064,$$

$$\varphi_2(\sqrt{2}) = K(1/\sqrt{2}) - E(1/\sqrt{2}) \approx 0.50343.$$

Thus

$$|\varphi^+(\sqrt{2})| \approx 1.44142,$$

$$\theta(\sqrt{2}) = \arctan(\varphi_2(\sqrt{2})/\varphi_1(\sqrt{2})) \cong 0.35678$$

and

$$c \approx 1.05838.$$

Since $\theta(\sqrt{2}) > \frac{1}{9}\pi$ (≈ 0.34907), we have for $n \geqq 9$

$$I_1 \geqq \frac{4}{\pi^2 n}\int_0^{\pi/n} (ce^{q\theta})^{2-n} \sin n\theta \, d\theta$$

$$= \frac{4c^{2-n}}{\pi^2 n^2}\int_0^{\pi} e^{-(n-2)q\theta/n} \sin\theta \, d\theta$$

$$\geqq \frac{4c^{2-n}}{\pi^2 n^2}\int_0^{\pi} e^{-q\theta} \sin\theta \, d\theta$$

$$= \left(\frac{2c}{\pi n}\right)^2 \frac{1 + e^{-q\pi}}{1 + q^2} c^{-n}.$$

Since

$$\left(\frac{2c}{\pi}\right)^2 \frac{1 + e^{-q\pi}}{1 + q^2} \approx 0.27659$$

we have proved (1).

Using

$$\varphi_1(5/\sqrt{3}) = E(\sqrt{0.12}) \approx 1.52256,$$
$$\varphi_2(5/\sqrt{3}) = K'(\sqrt{0.12}) - E'(\sqrt{0.12}) \approx 1.37189,$$

we have

$$\theta(5/\sqrt{3}) \approx 0.73339.$$

Hence

$$\theta(5/\sqrt{3}) > \frac{2\pi}{9} (\approx 0.69813).$$

Thus for $0 \leqq \theta \leqq 2\pi/9$, $\chi(\theta) \leqq 5/\sqrt{3}$. Hence for $n \geqq 9$

$$I_2 \leqq \frac{4}{\pi^2 n} \frac{5}{\sqrt{3}} \int_{\pi/n}^{2\pi/n} |\varphi^+(\chi(\theta))|^{2-n} |\sin n\theta| \, d\theta$$
$$= \frac{4}{\pi^2 n} \frac{5}{\sqrt{3}} \int_0^{\pi/n} \left|\varphi^+\left(\chi\left(\theta + \frac{\pi}{n}\right)\right)\right|^{2-n} \sin n\theta \, d\theta.$$

Since

$$\frac{d}{d\theta} \log |\varphi^+(\chi(\theta))| \geqq q,$$

it follows that

$$|\varphi^+(\chi(\theta + \pi/n))|^{2-n} \leqq e^{-(n-2)\pi q/n} |\varphi^+(\chi(\theta))|^{2-n}.$$

Thus

$$I_2 \leqq \frac{4}{\pi^2 n} \frac{5}{\sqrt{3}} e^{-(n-2)\pi q/n} \int_0^{\pi/n} |\varphi^+(\chi(\theta))|^{2-n} \sin n\theta \, d\theta$$
$$\leqq \frac{5}{\sqrt{3}} e^{-7\pi q/9} \cdot I_1.$$

But

$$\frac{5}{\sqrt{3}} e^{-7\pi q/9} \approx 0.34809.$$

This proves (2).

END OF PROOF OF THEOREM 2.1. Let $n \geqq 9$ and let $I_1, I_2, \ldots, I_p, I'$ be as in Lemma 2.8. By Lemma 2.4 with $\alpha = 5\sqrt{2}$ and Lemma 2.8, we get

$$-b_n = (I_1 - I_2 + \cdots + (-1)^{p-1} I_p + (-1)^p I') - r_n(5\sqrt{2}).$$

Since the terms in the alternating series have decreasing size by Lemma 2.8(2), and since $p \geqq 2$, one has

$$-b_n > I_1 - I_2 - r_n(5\sqrt{2}).$$

Lemma 2.9 gives

$$(I_1 - I_2) > 0.27(1 - 0.35)\frac{1}{n^2}(1.06)^{-n}$$

$$> \frac{0.175}{n^2}(1.06)^{-n}.$$

Since $\operatorname{Im} \varphi^+(5\sqrt{2}) \approx 2.32555$, we get by Lemma 2.4

$$|r_n(5\sqrt{2})| \leqq \frac{5\sqrt{2}}{n}(2.32)^{-n}.$$

Thus

$$-b_n > \frac{0.175}{n^2}(1.06)^{-n} - \frac{5\sqrt{2}}{n}(2.32)^{-n}$$

$$= \frac{0.175}{n^2} 1.06)^{-n}\left(1 - \frac{5\sqrt{2}\, n}{0.175}\left(\frac{1.06}{2.32}\right)^n\right)$$

$$> \frac{0.175}{n^2}(1.06)^{-n}(1 - 41 \cdot n \cdot 2^{-n}).$$

Since $n \cdot 2^{-n}$ is a decreasing function of $n \in \mathbf{N}$, and since $41 \cdot 9 \cdot 2^{-9} < 1$, we conclude that $b_n < 0$ for all odd $n \geqq 9$. This completes the proof of Theorem 2.1.

REMARK. In the beginning of this section we found b_1, b_3, b_5 and b_7 by solving the equation $\varphi(k) = s$ in terms of power series. Continuing this procedure, one gets

$$b_9 = -5 \cdot 2^{-14}\left(\frac{4}{\pi}\right)^9, \quad b_{11} = -15 \cdot 2^{-17}\left(\frac{4}{\pi}\right)^{11}, \quad b_{13} = -49 \cdot 2^{-20}\left(\frac{4}{\pi}\right)^{13}.$$

We doubt that it is possible to write the b_n's in a closed form. However, using the following asymptotic expressions for $\varphi_1(k)$ and $\varphi_2(k)$ for $k \to 1$ ($k \geqq 1$):

$$\varphi_1(k) = 1 - \tfrac{1}{2}(k-1)\log(k-1) + O(k-1),$$

$$\varphi_2(k) = \frac{\pi}{2}(k-1) + O(1),$$

it is not hard to prove that $I_2/I_1 \to 0$ and that $I_1 \sim 4/(n^2 \log^2 n)$ for $n \to \infty$, so that

$$b_n \sim -\frac{4}{n^2 \log^2 n} \qquad \text{for } n \to \infty \quad (n \text{ odd}).$$

§3. The main result

THEOREM 2.1. *Let $K_G^{\mathbf{C}}$ denote the complex Grothendieck constant and let φ be as in Section 2. Then*

$$K_G^{\mathbf{C}} \leqq \frac{8}{\pi(k_0+1)},$$

where k_0 is the unique solution to the equation $\varphi(k) = \frac{1}{8}\pi(k+1)$ in the interval $[0, 1]$. *One has* $8/\pi(k_0+1) \approx 1.40491$.

LEMMA 3.2. *Let $d \in \mathbf{N}$ and let $Z_1, \ldots, Z_d$ be independent complex random variables, equally distributed with density*

$$\frac{1}{\pi} e^{-|z|^2} dxdy \qquad (x = \operatorname{Re} z,\ y = \operatorname{Im} z).$$

For each $u = (u_1, \ldots, u_d) \in \mathbf{C}^d$, let

$$Z_u = \sum_{k=1}^{d} u_k Z_k$$

and let $\langle u, v\rangle = \sum_{k=1}^{d} u_k \bar{v}_k$ be the usual inner product in $\mathbf{C}^d$. If $u, v \in \mathbf{C}^d$ and $\| u \|_2 = \| v \|_2 = 1$, then

$$E(\operatorname{sign} Z_u \overline{\operatorname{sign} Z_v}) = \langle u, v\rangle \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 - |\langle u, v\rangle|^2 \sin^2 t}}\, dt.$$

PROOF. The sign of a complex number z is

$$\operatorname{sign} z = \begin{cases} z/|z|, & z \neq 0, \\ 0, & z = 0. \end{cases}$$

It is elementary to check that for any $z \in \mathbf{C}$,

$$(*) \qquad \operatorname{sign} z = \frac{1}{4}\int_0^{2\pi} \operatorname{sign}(\operatorname{Re}(e^{-i\theta}z))e^{i\theta}d\theta.$$

Put $X_{2k-1} = \sqrt{2}\operatorname{Re}(Z_k)$ and $X_{2k} = \sqrt{2}\operatorname{Im}(Z_k)$, $k = 1, \dots, d$. Then $(X_i)_{i=1}^{2d}$ is a set of independent real valued random variables each normally distributed with $E(X_i) = 0$ and $E(X_i^2) = 1$. For $a \in \mathbf{R}^{2d}$, put

$$X_a = \sum_{i=1}^{2d} a_i X_i.$$

For $a, b \in \mathbf{R}^{2d}$, $\| a \|_2 = \| b \|_2 = 1$, (X_a, X_b) form a joint normal distribution, $E(X_a) = E(X_b) = 0$, $E(X_a^2) = E(X_b^2) = 1$ and $E(X_a X_b) = \langle a, b \rangle$. Thus by [7, proof of lemma 1].

$$E(\operatorname{sign} X_a \operatorname{sign} X_b) = \frac{2}{\pi}\operatorname{Arcsin}\langle a, b\rangle.$$

For $u, v \in \mathbf{C}^d$,

$$\operatorname{Re} Z_u = \frac{1}{\sqrt{2}} X_a, \quad \operatorname{Re} Z_v = \frac{1}{\sqrt{2}} X_b$$

where $a_{2k-1} = \operatorname{Re} u_k$, $a_{2k} = -\operatorname{Im} u_k$, $b_{2k-1} = \operatorname{Re} v_k$, and $a_{2k} = -\operatorname{Im} v_k$, $k = 1, \dots, d$. Hence

$$\begin{aligned} E(\operatorname{sign}(\operatorname{Re} Z_u)\operatorname{sign}(\operatorname{Re} Z_v)) &= \frac{2}{\pi}\operatorname{Arcsin}\langle a, b\rangle \\ &= \frac{2}{\pi}\operatorname{Arcsin}\operatorname{Re}\langle u, v\rangle, \end{aligned}$$

so by formula $(*)$

$$E(\operatorname{sign} Z_u \overline{\operatorname{sign} Z_v}) = \frac{1}{16}\int_0^{2\pi}\int_0^{2\pi} \frac{2}{\pi}\operatorname{Arcsin}\operatorname{Re}\langle e^{-i\theta}u, e^{-i\varphi}v\rangle e^{i(\theta-\varphi)}d\theta d\varphi.$$

Assume now that $\langle u, v\rangle \in \mathbf{R}$. Then the integral is equal to

$$\frac{1}{8\pi}\int_0^{2\pi}\int_0^{2\pi}\arcsin(\cos(\theta-\varphi)\langle u,v\rangle)e^{i(\theta-\varphi)}d\theta d\varphi$$

$$=\frac{1}{4}\int_0^{2\pi}\operatorname{Arcsin}(\cos t\langle u,v\rangle)e^{it}dt.$$

Since $\operatorname{Arcsin}(\cos t\langle u,v\rangle)$ is an even function of period 2π, the imaginary part of the integral vanishes. Thus

$$E(\operatorname{sign} Z_u\overline{\operatorname{sign} Z_v})=\frac{1}{4}\int_0^{2\pi}\operatorname{Arcsin}(\cos t\langle u,v\rangle)\cos t\,dt.$$

The integral is the sum of four integrals, namely the integrals over $[0,\pi/2]$, $[\pi/2,\pi]$, $[\pi,3\pi/2]$ and $[3\pi/2,2\pi]$, and these integrals are equal. By substituting t with $\pi/2-t$ in the integral from 0 to $\pi/2$ we thus have

$$E(\operatorname{sign} Z_u\overline{\operatorname{sign} Z_v})=\int_0^{\pi/2}\operatorname{Arcsin}(\langle u,v\rangle\sin t)\sin t\,dt.$$

Finally, using

$$\frac{d}{dx}\operatorname{Arcsin}(x)=(1-x^2)^{-1/2},$$

one gets by partial integration

$$E(\operatorname{sign} Z_u\overline{\operatorname{sign} Z_v})=\langle u,v\rangle\int_0^{\pi/2}\frac{\cos^2 t}{\sqrt{1-\langle u,v\rangle^2\sin^2 t}}dt.$$

If $\langle u,v\rangle\notin\mathbf{R}$, we can choose $c\in\mathbf{C}$, $|c|=1$, such that $c\langle u,v\rangle=|\langle u,v\rangle|$. Since $\langle cu,v\rangle\in\mathbf{R}$ and since $\operatorname{sign} Z_{cu}=c\cdot\operatorname{sign} Z_u$, we have

$$\begin{aligned}E(\operatorname{sign} Z_u\overline{\operatorname{sign} Z_v})&=\bar{c}E(\operatorname{sign} Z_{cu}\overline{\operatorname{sign} Z_v})\\&=\bar{c}\,|\langle u,v\rangle|\int_0^{\pi/2}\frac{\cos^2 t}{\sqrt{1-|\langle u,v\rangle|^2\sin^2 t}}dt\\&=\langle u,v\rangle\int_0^{\pi/2}\frac{\cos^2 t}{\sqrt{1-|\langle u,v\rangle|^2\sin^2 t}}dt.\end{aligned}$$

This completes the proof of Lemma 3.2.

The following definition can essentially be found in Krivine's paper [7, pp. 23–24], but we will phrase it differently, so that we can take Lindenstrauss' and

Pelczynski's matrix formulation of Grothendieck's inequality [8] as starting point.

On the set of complex $n \times n$ matrices $M_n(\mathbf{C})$ we consider two norms $\| \ \|_\otimes$ and $\| \ \|_*$ defined by:

$$\| A \|_\otimes \leqq 1 \Leftrightarrow A \in \overline{\mathrm{conv}}\{(s_i\bar{t}_j)_{i,j=1,\dots,n} \mid |s_i| \leqq 1,\ |t_i| \leqq 1\}$$

$$\| A \|_* \leqq 1 \Leftrightarrow \begin{cases} A \text{ is in the closed convex hull of the} \\ \text{matrices of the form } \langle x_i, y_j \rangle, \text{ where} \\ x_1, \dots, x_n, y_1, \dots, y_n \text{ are vectors in the} \\ \text{unit ball of some complex Hilbert space.} \end{cases}$$

It is clear that $\| A \|_* \leqq \| A \|_\otimes$. A straightforward dualization of [8, Theorem 2.1] gives

LEMMA 3.3. *For every $n \in \mathbf{N}$ and every $A \in M_n(\mathbf{C})$,*

$$\| A \|_\otimes \leqq K_G^{\mathbf{C}} \| A \|_*.$$

Moreover, $K_G^{\mathbf{C}}$ is the smallest constant for which this inequality holds for all $n \in \mathbf{N}$ and all $A \in M_n(\mathbf{C})$.

LEMMA 3.4. (1) *Let $A \in M_n(\mathbf{C})$. Then $\| A \|_* \leqq 1$ if and only if there exist unit vectors $x_1, \dots, x_n, y_1, \dots, y_n$ in a Hilbert space $\mathcal{H}$, such that*

$$A_{ij} = \langle x_i, y_j \rangle, \qquad i, j = 1, \dots, n.$$

(2) *Let $A \circ B$ denote the Schur product of $A, B \in M_n(\mathbf{C})$:*

$$(A \circ B)_{ij} = A_{ij} B_{ij}, \qquad i, j = 1, \dots, n.$$

Then $\| A \circ B \|_* \leqq \| A \|_* \| B \|_*$.

PROOF. Let $\mathcal{D}_n$ be the set of $n \times n$ matrices of the form $\langle x_i, y_j \rangle_{i,j=1}^n$, where x_i, y_j are unit vectors in some Hilbert space $\mathcal{H}$. Since the unit ball of a Hilbert space is the convex hull of the unit sphere, we have

$$\| A \|_* \leqq 1 \Leftrightarrow A \in \overline{\mathrm{conv}}(\mathcal{D}_n).$$

In the definition of $\mathcal{D}_n$ we can put $\mathcal{H} = \mathbf{C}^{2n}$, because $x_1, \dots, x_n, y_1, \dots, y_n$ span a subspace of $\mathcal{H}$ of dimension at most $2n$. This shows that $\mathcal{D}_n$ is a compact subset of $M_n(\mathbf{C})$, and therefore

$$\overline{\mathrm{conv}}(\mathcal{D}_n) = \mathrm{conv}(\mathcal{D}_n).$$

Next, let $A, B \in \mathscr{D}_n$ and $\lambda \in [0, 1]$. Choose Hilbert spaces $\mathscr{H}$, $\mathscr{K}$ and unit vectors $x_i, y_j \in \mathscr{H}$, $z_i, w_j \in \mathscr{K}$, such that

$$A_{ij} = \langle x_i, y_j \rangle, \quad B_{ij} = \langle z_i, w_j \rangle.$$

Then

$$(1-\lambda)A_{ij} + \lambda B_{ij} = \langle (1-\lambda)^{1/2} x_i \oplus \lambda^{1/2} z_i, (1-\lambda)^{1/2} y_j \oplus \lambda^{1/2} w_j \rangle$$

where the last inner product is taken in $\mathscr{H} \oplus \mathscr{K}$. Thus $(1-\lambda)A + \lambda B \in \mathscr{D}_n$. Hence $\mathrm{conv}(\mathscr{D}_n) = \mathscr{D}_n$. This proves (1). To prove (2) it is sufficient to show that $\| A \circ B \|_* \leqq 1$ whenever $\| A \|_* \leqq 1$ and $\| B \|_* \leqq 1$. By (1) we have

$$A_{ij} = \langle x_i, y_j \rangle, \quad B_{ij} = \langle z_i, w_j \rangle$$

where $x_i, y_j \in \mathrm{Unitsphere}(\mathscr{H})$, $z_i, w_j \in \mathrm{Unitsphere}(\mathscr{K})$ for a pair of Hilbert space $\mathscr{H}$ and $\mathscr{K}$. Hence

$$A_{ij} B_{ij} = \langle x_i \otimes z_i, y_j \otimes w_j \rangle$$

and $\| x_i \otimes z_i \| = \| y_j \otimes w_j \| = 1$ where norms and scalar products are computed in the Hilbert tensorproduct $\mathscr{H} \otimes \mathscr{K}$. Hence $\| A \circ B \|_* \leqq 1$.

LEMMA 3.5. (1) *The function*

$$\Phi(k) = k \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 - |k|^2 \sin^2 t}} \, dt$$

is a homeomorphism of the closed unit disc $\bar{D} = \{ z \in \mathbf{C} \mid |z| \leqq 1 \}$ *onto itself.*

(2) *Let* $A \in M_n(\mathbf{C})$. *Then*

$$\| A \|_* \leqq 1 \Rightarrow \| \Phi(A) \|_{\otimes} \leqq 1$$

where $\Phi(A)$ *is the matrix with elements* $\Phi(A)_{ij} = \Phi(A_{ij})$.

PROOF. Let φ be as in Section 2. Then

$$|\Phi(k)| = \varphi(|k|),$$

$$\arg \Phi(k) = \arg k, \quad k \neq 0$$

so (1) follows from the fact that φ is a homeomorphism of $[0, 1]$ onto itself and $\varphi(0) = 0$. If $A \in M_n(\mathbf{C})$ and $\| A \|_* \leqq 1$, there exist by Lemma 3.4 (2) a Hilbert space $\mathscr{H}$ and unit vectors $u_1, \ldots, u_n, v_1, \ldots, v_n \in \mathscr{H}$, such that

$$A_{ij} = \langle u_i, v_j \rangle.$$

Since $\operatorname{span}\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ is finite dimensional, we can assume that $\mathscr{H} = \mathbf{C}^d$ for some $d \in \mathbf{N}$. Let $Z_1, \ldots, Z_d$ be complex random variables on a probability space $(\Omega, d\omega)$ satisfying the conditions of Lemma 3.1, and put

$$Z_u = \sum_{k=1}^{d} u_k Z_k, \qquad u \in \mathbf{C}^d.$$

By Lemma 3.1,

$$\Phi(\langle u_i, v_j \rangle) = \int_\Omega \operatorname{sign} Z_{u_i}(\omega) \overline{\operatorname{sign} Z_{v_j}(\omega)}\, d\omega.$$

Hence, by the definition of $\| \ \|_\otimes$, we have

$$\| \Phi(\langle u_i, v_j \rangle)_{i,j=1}^n \|_\otimes \leqq 1.$$

PROOF OF THEOREM 3.1. Let $\Phi^{-1}: \bar{D} \to \bar{D}$ be the inverse function of Φ (cf. Lemma 3.5). Since $\Phi(k) = \operatorname{sign}(k)\varphi(|k|)$, we have

$$\Phi^{-1}(s) = \operatorname{sign}(s)\varphi^{-1}(|s|).$$

Thus, by Theorem 2.1,

$$\Phi^{-1}(s) = \operatorname{sign}(s) \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} b_n |s|^n$$

where $b_1 = 4/\pi$ and $\Sigma_{n \text{ odd}} |b_n| < \infty$. Since

$$s \to \sum_{n \text{ odd}} |b_n| s^n$$

is a strictly increasing continuous function on $[0, 1]$ and since $\Sigma_{n \text{ odd}} |b_n| \geqq 4/\pi > 1$, there is a unique $\beta_0 \in]0, 1[$ for which

$$\sum_{n \text{ odd}} |b_n| \beta_0^n = 1.$$

Next, we show that $K_G^{\mathbf{C}} \leqq 1/\beta_0$: Let $A \in M_n(\mathbf{C})$, and assume that $\| A \|_* \leqq 1$. For $s \in \bar{D}$,

$$\Phi^{-1}(\beta_0 s) = \sum_{k=0}^{\infty} b_{2k+1} \beta_0^{2k+1} \operatorname{sign}(s) |s|^{2k+1}$$

$$= \sum_{k=0}^{\infty} b_{2k+1} \beta_0^{2k+1} s^{k+1} \bar{s}^k.$$

Since $(M_n(\mathbf{C}), 0)$ is a Banach algebra with norm $\| \ \|_*$ and since clearly $\| \bar{C} \|_* = \| C \|_*$ for all $C \in M_n(\mathbf{C})$, we have

$$\| \Phi^{-1}(\beta_0 A) \|_* \leqq \sum_{k=0}^{\infty} |b_{2k+1}| \beta_0^{2k+1} = 1,$$

where $\Phi^{-1}(\beta_0 A)$ denotes the matrix with elements $\Phi^{-1}(\beta_0 A_{ij})$, $i, j = 1, \ldots, n$. Hence by Lemma 3.5

$$\| \beta_0 A \|_\otimes = \| \Phi \circ \Phi^{-1}(\beta_0 A) \|_\otimes \leqq 1.$$

Thus we have proved that for any $n \times n$ matrix

$$\| A \|_* \leqq 1 \Rightarrow \| A \|_\otimes \leqq 1/\beta_0,$$

so, by Lemma 3.3, $K_G^{\mathbf{C}} \leqq 1/\beta_0$. To compute β_0, we use that $b_1 = 4/\pi$ and $b_n \leqq 0$ for $n \geqq 3$ (cf. Theorem 2.1). This gives

$$\sum_{n \text{ odd}} |b_n| \beta_0^n = \frac{8}{\pi} \beta_0 - \varphi^{-1}(\beta_0).$$

Hence β_0 is a solution to the equation

$$\varphi^{-1}(\beta_0) = \frac{8}{\pi} \beta_0 - 1.$$

Now, put $k_0 = \varphi^{-1}(\beta_0)$. Then $k_0 \in [0, 1]$ and

$$\varphi(k_0) = \beta_0 = \frac{\pi}{8}(k_0 + 1).$$

From the Taylor expansion of $\varphi(k)$ it follows that $\varphi'(k) \geqq \pi/4$ for $0 \leqq k < 1$, so that

$$\varphi(k) - \tfrac{1}{8}\pi(k+1)$$

is an increasing function on $[0, 1]$. Therefore k_0 is the unique solution to the equation

$$\varphi(k) = \frac{\pi}{8}(k+1)$$

in the interval $[0, 1]$. Recall from Section 2 that

$$\varphi(k)=\frac{1}{k}(E(k)-(1-k^2)K(k)), \qquad 0<k<1.$$

By second-order interpolation in the tables [1, p. 324] one finds

k^2	$E(k)$	$K(k)$	$\varphi(k)$	$\varphi(k)-\frac{1}{8}\pi(k+1)$
0.6601	1.264955	2.020408	0.711684	− 0.000070
0.6602	1.264898	2.020537	0.711754	− 0.000023
0.6603	1.264841	2.020666	0.711825	+ 0.000023
0.6604	1.264784	2.020795	0.711895	+ 0.000069

(slight adjustments in the last decimal place have been carried out by use of more accurate tables [10]). Hence

$$k_0 \approx (0.66025)^{1/2} \approx 0.81256$$

which gives

$$K_G^{\mathbf{C}} \leqq \frac{1}{\beta_0}=\frac{8}{\pi(k_0+1)} \approx 1.40491.$$

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