Exactly Solvable One-Dimensional Many-Body Problems.

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The usefulness of the Lax technique (1) to solve N-body problems has been demonstrated in specific cases by FLASCHKA (2), MANAKOV (3), MOSER (4) and CALOGERO, MARCHIORO and RAGNISCO (5). Here we extend this approach to a class of many-body problems with two-body forces (in one dimension) that includes as special cases most of those previously known.

In this letter we concentrate mainly on the classical case; the main results can be extended to the quantal case, following the procedure of CMR. The solvable character of these models is displayed exhibiting explicitly N integrals of the motion; some specific properties of the dynamical behaviour of these systems are also outlined.

A simple trick that generates novel solvable (N_1+N_2)-body problems, involving N_1 particles of one kind and N_2 particles of another (with equal masses), is introduced. It yields some integrable systems of remarkable physical interest, such as the case with the repulsive short-range two-body potential $V_e(x)=g^2a^2\sinh^{-2}(ax)$ acting between equal particles and the attractive potential $V_d(x)=-g^2a^2\cosh^{-2}(ax)$ acting between different particles.

Consider the N-body problem characterized by the Hamiltonian

$$H=\tfrac{1}{2}\sum_{j=1}^{N}p_j^2+\sum_{j>k=1}^{N}V(x_j-x_k) \tag{1}$$

with $V(-x)=V(x)$ (6). Define the matrices (7)

$$L_{jk}=\delta_{jk}p_j+(1-\delta_{jk})\alpha(x_j-x_k)\,, \tag{2}$$

$$A_{jk}=\delta_{jk}\sum_{\substack{l=1\\ l\neq j}}^{N}\beta(x_j-x_l)(1-\delta_{jk})\alpha'(x_j-x_k)\,, \tag{3}$$

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(1) P. D. LAX: *Comm. Pure Appl. Math.*, **21**, 467 (1968).

(2) H. FLASCHKA: *Phys. Rev. B*, **9**, 1924 (1974); *Prog. Theor. Phys.*, **51**, 703 (1974).

(3) S. V. MANAKOV: *Žurn. Ėksp. Teor. Fiz.*, **67**, 543 (1974).

(4) J. MOSER: *Three integral Hamiltonian systems connected with iso-spectral deformations* (preprint).

(5) F. CALOGERO, C. MARCHIORO and O. RAGNISCO: *Lett. Nuovo Cimento*, **13**, 383 (1975), hereafter referred to as CMR.

(6) We introduce here this condition for simplicity and on physical grounds; we thereby exclude models

where $\beta(-x)=\beta(x)$ and the function $\alpha(x)$ satisfies the functional equation (that also defines $\beta(x)$)

$$\alpha'(y)\alpha(z)-\alpha(y)\alpha'(z)=\alpha(y+z)[\beta(y)-\beta(z)]\,. \tag{4}$$

It is then easy to show that the Lax condition (¹)

$$\dot{L}=[A,L] \tag{5}$$

is implied by the equations of motion

$$\dot{x}_j=p_j\,,\qquad \dot{p}_j=-\sum_{\substack{k=1\\k\neq j}}^{N}V'(x_j-x_k) \tag{6}$$

with

$$V(x)=\alpha(x)\alpha(-x)+\text{const}\,. \tag{7}$$

It immediately follows that the N quantities I_n defined by setting

$$\det[L_{jk}+\lambda\delta_{jk}]=\lambda^N+\sum_{n=1}^{N}\lambda^{N-n}I_n \tag{8}$$

provide N independent constants of motion. Note that, quite generally, $I_1=\sum_{j=1}^{N}p_j$ is the total momentum of the system, and

$$I_2=\tfrac{1}{2}(I_1)^2-H\,. \tag{9}$$

Thus any solution of the functional equation (4) generates an exactly solvable many-body problem, with the two-body potential of eq. (7) [8]. Note that if $\alpha(x)$ is one such solution, $\alpha(x)\exp[ax]$ is also a solution (with the same $\beta(x)$), but it yields the same potential.

The most general solutions of eq. (4) that we have obtained are

$$\alpha_1(x)=b\,\mathrm{dn}\,(ax)/\mathrm{sn}\,(ax)\,, \tag{10a}$$

$$\alpha_2(x)=b\,\mathrm{cn}\,(ax)/\mathrm{sn}\,(ax)\,. \tag{10b}$$

They yield the same potential (up to additive constants, that we ignore hereafter)

$$V(x)=-[b^2/(e_1-e_3)]\,\wp(ax|\omega,\omega')\,. \tag{11}$$

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such as the Toda lattice (²⁻⁴).

(⁷) The primes indicate differentiation with respect to the argument; the dot (see below) indicates (total) differentiation with respect to time.

(⁸) A. ERDÉLYI (Editor): *Higher Transcendental Functions*, Vol. **3** (New York, N. Y., 1955). We use hereafter, whenever referring to the properties and parameters of elliptic functions, the notation of this book.

Here $\wp(z|\omega, \omega')$ is the Weierstrass function

$$(12) \qquad \wp(z|\omega, \omega') = z^{-2} + \sum_{-\infty}^{\infty}{}' [(z + 2m\omega + 2n\omega')^{-2} - (2m\omega + 2n\omega')^{-2}],$$

and e_1, e_3 are its values for $z = \omega$ and $z = \omega'$ (8).

The constants a and b and the 2 half-periods ω and ω' of the Weierstrass function can be chosen arbitrarily; of course, a physically interesting model obtains only if $V(x)$, eq. (11), is real. Previously known cases correspond to special choices of the half-periods: for $\omega = -i\omega' = \infty$, $V(x) = g^2/x^2$ (9); for $\omega' = i\infty$ (and $\omega = \pi/2$), $V(x) = g^2 \cdot a^2/\sin^2(ax)$ (10); for $\omega = \infty$ (and $\omega' = i\pi/2$), $V(x) = g^2a^2/\sinh^2(ax)$ (11).

The potential of eq. (11) (see also below) has remarkable properties also in the context of the quantal two-body problem (12); and it may play a special role in Lie group theory, just as the potentials $g^2a^2/\sin^2(ax)$ and $g^2a^2/\sinh^2(ax)$ do (13). It should also be mentioned that SUTHERLAND has been able to obtain explicitly the ground-state properties of the one-dimensional quantal many-body problem with a two-body potential that also involves elliptic functions (14).

Set formally $x_j = y_j$ for $1 \leqslant j \leqslant N_1$, $x_j = z_j + d$ for $N_1 < j \leqslant N_1 + N_2 = N$, and interpret the co-ordinates y_j as those of the particles of one type and the co-ordinates z_j as those of the particles of another type. Then the solvability of any N-body model with the pair potential $V(x)$ implies the solvability of the model with N_1 particles of one type and N_2 particles of another type, with the potential $V_e(x) = V(x)$ acting between equal particles and the potential $V_d(x) = V(x \pm d)$ acting between different particles.

Clearly this trick can yield a physically reasonable potential only if $V(x)$ is periodic with period $2d$. The reality and periodicity properties of the Weierstrass function (15) imply that the following choices in eq. (11) produce interesting potentials:

i) Let $e_1 > e_2 > e_3$, $e_1 > 0$, $e_3 < 0$, $\Delta > 0$, a and ω real, b and ω' imaginary, $d = \omega'/a$; then both potentials are real and periodic with period $2\omega/a$, and, as x goes from 0 to ω/a, $V_e(x)$ decreases from $+\infty$ to 0 (16), while $V_d(x)$ increases from $-|b|^2 \cdot (e_2 - e_3)/(e_1 - e_3)$ to 0 (17).

ii) Let $e_1 = e_3^*$, $\Delta < 0$, $\omega' = \omega^*$, a real, $b^2/(e_1 - e_3)$ real and negative, $d = (\omega - \omega')/a$; then both $V_e(x)$ and $V_d(x)$ are real and periodic with period $2(\omega + \omega')/a$, and, as x

(9) This case was first solved in the quantal case: for $N = 3$ by C. MARCHIORO: *Journ. Math. Phys.*, **11**, 2193 (1970), and for arbitrary N by F. CALOGERO: *Journ. Math. Phys.*, **12**, 419 (1971). In the classical case it was solved for arbitrary N by J. MOSER: ref. (4).

(10) This case was first solved in the quantal case by B. SUTHERLAND: *Phys. Rev. A*, **5**, 1375 (1972); then, in the classical case, by J. MOSER: ref. (4). The matrix L used by MOSER corresponds to the solution $\alpha_2(x)$.

(11) This case was solved by CMR; the matrix L used corresponds to the solution $\alpha_2(x)$; that corresponding to the solution $\alpha_1(x) - iga/\sinh(ax)$ is actually more convenient, because the nondiagonal elements then vanish rather than becoming constant, when all particles separate.

(12) B. SUTHERLAND: *Phys. Rev. A*, **8**, 2514 (1973). The potential discussed by SUTHERLAND is actually a special case of that acting between different particles in case i) below.

(13) M. A. OLSHANETZKY and A. M. PERELOMOV: *J. Phys. A* (to be submitted to).

(14) B. SUTHERLAND: preprint ITP-SB-75-2 (to be published). It will be interesting to apply the trick mentioned below to this potential.

(15) See, for instance, Subsect. **13**.15 of ref. (8).

(16) Here and in the following we take advantage of the possibility to add arbitrary constants to $V_e(x)$ and/or $V_d(x)$.

(17) A qualitatively similar behaviour obtains for a and ω imaginary, b and ω' real.

goes from 0 to $(\omega + \omega')/a$, $V_e(x)$ decreases from $+\infty$ to 0, while $V_d(x)$ increases from 0 to $+\infty$ [18].

It should be noted that in all cases the potential $V_e(x)$ is singular at zero separation, and is therefore better chosen repulsive in that region [19]; the potential $V_d(x)$ is instead always finite at $x = 0$, and its derivative has the opposite sign to that of $V_e(x)$ (it yields therefore generally attractive forces at short distance). The potentials of type ii) resemble to the Pösch-Teller potential [20]; case i) is perhaps more interesting, since in this case $V_d(x)$ is finite-valued. It will be interesting to analyse explicitly the trajectories of the particles; one conjectures that all orbits in phase space are closed, at least if only equal particles are present (in case i), if different particles are present and the energy is sufficiently large, the trajectories could be periodic only mod $(2\omega/a)$ in the co-ordinates). The quantal problem is also very appealing, especially in case i), when the presence of even only one different particle (« impurity ») should originate a band spectrum [21]. Also the thermodynamical limit will be interesting, especially when different particles are present, in view of the attractive nature of their interaction.

Space does not permit here a more detailed discussion of these potentials and of the properties of the corresponding many-body systems [22]. There are, however, two cases which deserve special mention. They obtain from cases i) and ii) in the limit $\omega \to \infty$ respectively $\mathrm{Re}\,\omega \to \infty$; more directly they can be obtained by applying the trick described above directly to the potentials discussed by CMR or by SUTHERLAND [10] and MOSER [5], with $d = i\pi/2$ respectively $d = \pi/2$. The second possibility implies that all the results obtained by these authors can be directly taken over [23], except for the conclusions connected with the impenetrability of the particles (that apply now only to equal particles) and the absence of bound states (in the first case) [24]. However the physical properties of the systems when particles of different kinds are present differ substantially from those of systems with only equal particles, due to the attractive and nonsingular nature at short range of the interaction between different particles.

To justify the last statement, we discuss now tersely, in the classical case, the physically more interesting system, namely that with two kinds of particles interacting via the potentials $V_e(x) = g^2a^2/\sinh^2(ax)$ and $V_d(x) = -g^2a^2/\cosh^2(ax)$ (obtained from the CMR model with $d = i\pi/2$) [25]. In this case the c.m. ground-state energy in the

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[18] A qualitatively similar behaviour obtains for a imaginary, $b^2/(e_1 - e_3)$ real and positive, $d = (\omega + \omega')/a$.

[19] This is actually not necessary in the classical case; indeed an explicit treatment of this case is possible (although it leads generally to collapse in a finite time), and should be particularly interesting in the degenerate case $V(x) = -g^2/x^2$, especially if this potential could be supplemented by a hard core and the limit of large N could be considered (for $N = 3$ the problem with a hard core is certainly tractable, since the classical problem has been explicitly integrated in closed form: C. MARCHIORO: unpublished; D. C. KHANDEKAR and S. V. LAWANDE: *Amer. Journ. Phys.*, **40**, 458 (1972)).

[20] See problem 38 in S. FLÜGGE: *Practical Quantum Mechanics*, Vol. **1** (Berlin, 1971).

[21] See however, in this connection, ref. [12].

[22] Actually the doubly periodic nature of the Weierstrass function implies the possibility to introduce models with 4 different kinds of particles and 4 different potentials, all of them reasonable, *i.e.* even.

[23] Thus one can obtain directly, from the results of SUTHERLAND, ref. [10], the complete energy spectrum for the second system. This opens the possibility to evaluate a number of nontrivial multiple integrals, following G. GALLAVOTTI and C. MARCHIORO: *Journ. Math. Anal. Appl.*, **44**, 661 (1973).

[24] The proof by CMR that $\{I_n, I_m\} = 0$ exploits the asymptotic separation of the particles, but, since this result is of a purely algebraic nature, it continues to hold even in the more general case with different particles (and also, by analytic continuation in a, for the Sutherland model and its generalized version with different particles). One conjectures that such a result holds also in the more general case, with the potential of eq. (7).

[25] In the quantal case the potential $V_d(x)$ is, in the two-body case, exactly solvable, and it has peculiar, but nontrivial, properties: see, for instance, problem 39 in ref. [20]. One conjectures that an analysis such as that given, for the N-body problem with attractive delta-function potentials, by C. N. YANG (*Phys. Rev. Lett.*, **19**, 1312 (1967); *Phys. Rev.*, **168**, 1920 (1968)) is also applicable in this case, at least

two-body problem with different particles is clearly [26] -1. Remarkably, exactly the same value obtains as the minimum of the potential energy for the $(2+1)$-body problem, in the symmetrical configuration with the different particle at the centre and the two equal particles at a distance arcsinh $(2^{-\frac{1}{2}})$ from it; it can moreover be proved that this is a position of stable equilibrium for the system. Thus the classical $(2+1)$-body can be bound; indeed it is actually easy to evaluate explicitly, in closed form, the trajectories of the particles for any motion that preserves the symmetry of the system [27]. An interesting question is whether the ground state of the (N_1+N_2)-body system is also bound, for $N=N_1+N_2>3$, and in particular for $N\to\infty$. For small N the answer can be easily obtained with the help of a computer; for large N, one conjectures that the system is not bound, but rather tends to break into clusters, due to the positive definiteness of $V_e(x)+V_d(x)$. This conjecture is supported by the following remark:

Let $w_N(x)$ be the potential energy per particle in an equally spaced configuration of N particles of alternating type (with nearest-neighbour distance x); then clearly there always exists a finite value x_N such that, for any $x>x_N$, $w_N(x)<0$; yet for all finite values of x, the energy per particle $w_\infty(x)$ in such an infinite lattice is positive. Indeed

$$w_\infty(x)=\frac{1}{3}+4(2K^2-1)K^2(k)/(3\pi^2)=\frac{1}{3}-\frac{1}{3}(1-2K^2)\left[\frac{2}{\pi}K(k)\right]^2, \tag{13}$$

where $K(k)$ is the complete elliptic integral and its argument k^2 is related to x through

$$x=(\pi/2)K(k')/K(k)\,,\qquad k'=\sqrt{1-k^2}\,, \tag{14}$$

so that $w_\infty(x)$ decreases monotonically from $+\infty$ to 0 as x varies from 0 to ∞. Note however that $\min_x[w_\infty(x)]$ need not coincide with the $N\to\infty$ limit of the energy per particle of the ground state of the $(N/2+N/2)$-body system.

As for scattering states, it is clear from the results of CMR that (both in the classical and quantal cases) an initial incoming state with N separated particles of momenta $p_j(-\infty)$ goes into a final state of N separated particles of momenta $p_j(+\infty)$, the set $\{p_j(+\infty)\}$ coinciding with the set $\{p_j(-\infty)\}$; however, if different particles are present, the distribution of the final momenta between the particles need not be uniquely determined (indeed, in the quantal case different final states will generally occur with finite probabilities).

This also implies that in the 3-body case with different particles an initial configuration with, say, $p_1(-\infty)=-2p$, $p_2(-\infty)=p_3(-\infty)=p$, $x_2(-\infty)=x_3(-\infty)$ (particle 2 being the different one), goes definitely into the final state with $p_1(+\infty)=p_2(+\infty)=p$, $p_3(+\infty)=-2p$, $x_1(+\infty)=x_2(+\infty)$. Thus an initial state with one of the equal particles incoming from one side alone and the other two from the other side as a completely bound pair (with no internal kinetic energy) goes definitely into a final state having again one single particle and a completely bound pair; but while, of course,

for the special values of the coupling constant $g^2=n(n-1)$, $n=2,3,\ldots$, in which cases the two-body potential is transparent (vanishing reflection coefficient), but it has $n-1$ bound states (and a zero-energy resonance); in these cases also the group-theoretical structure underlying the N-body quantal problem is particularly transparent [13].

[26] We use hereafter, as units of energy and length, g^2a^2 and $1/a$.

[27] The existence of a stable bound state in the classical case need not imply the existence of a bound three-body state for the quantal problem, in which case a more detailed analysis is needed.

the two equal particles always go back in the direction they came from (they cannot overcome each other), the different particle always exchanges partner.

Additional results will be published elsewhere, together with the detailed proofs of all those reported here.

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