FUNCTIONS OF PROBABILITY MEASURES

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1. Introduction.

1.1. Summary of Results.

The purpose of this paper is to determine the action of analytic functions (and in certain cases non-analytic functions) on a class of measures.

Let $\mu\{\cdot\}$ be a probability measure on $(-\infty,\infty)$ and μ^{*j} be the j-fold convolution of μ with itself. Define the measure

$$\bar{\mu} = \sum_{j=0}^{\infty} c_j \mu^{*j}, \tag{1}$$

where $\{c_j; j \geqq 0\}$ is a sequence of complex numbers, and assume that

$$\phi(s) = \sum c_n s^n \tag{2}$$

is analytic in a region containing the range of the Fourier transform of μ.

(i) We will define a class of measures characterized by the rate at which their tails decay relative to the exponential. In section 2 we show that if μ is in this class, and is concentrated on the integers, then

$$\lim_{n\to\infty} \frac{\bar{\mu}_n}{\mu_n} = \text{constant}. \tag{3}$$

We prove an analogous result for the density of μ when it is absolutely continuous.

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(ii) Let $T_t = (-\infty, -t] \cup (t, \infty)$. Then, without regularity conditions on μ we show in section 3 that

$$(4) \qquad \lim_{t\to\infty} \frac{\bar{\mu}(T_t)}{\mu(T_t)} = \text{constant.}$$

The constants in (3) and (4) depend on ϕ, and are given.

(iii) For a result on the action of non-analytic functions, suppose that $v_n = O(|n|^{-\alpha})$ for some $\alpha > 2$ and that $\Sigma |c_j| \; |j|^\gamma < \infty$, for some $\gamma > \alpha$. Let $\hat{v}(\theta) = \sum_{j=-\infty}^{\infty} v_j e^{ij\theta}$, be real, $\phi(\theta) = \sum_{j=-\infty}^{\infty} c_j e^{ij\theta}$, and define d_n by

$$\phi(\hat{v}(\theta)) = \sum_{j=-\infty}^{\infty} d_j e^{ij\theta}.$$

In section 4 we show that

$$|d_n| = O(|n|^{-\alpha}).$$

If in addition $\Sigma\, v_j = 1$ and $v_j \sim cj^{-\alpha}$ as $j \to +\infty$ then

$$d_j \sim c\phi'(\hat{v}(0))j^{-\alpha} \quad \text{as } j \to +\infty.$$

Some applications of the above results to the renewal equation with a defective measure, and to the mean of a branching process are given in section 5. An appendix contains several further results related to (i).

A sequel to this paper makes essential use of the results herein to describe the asymptotic behavior of the solutions of a class of non-linear Volterra integral equations. This in turn yields some new results in the theory of branching processes. Both the present paper and the sequel are revisions of a technical report of October, 1969.

Some related results on the action of analytic functions, with different applications, have been obtained by Essen [E]; and results for non-analytic functions on weighted l_1 algebras, by Le Blanc [L].

1.2. Background Remarks.

When $c_j \equiv 1$, $\bar{\mu}$ is the well-known renewal function (see e.g. Feller, Vol. II [F]). The situation $c_j = c^j$ arises naturally in several probability contexts; e.g. the mean of a branching process, or the mean number of visits to a point in a random walk with killing probability $c < 1$. More generally, if $\{c_j\}$ is any probability function on $\{0, 1, 2, \cdots\}$, then $\bar{\mu}$ can be interpreted as the probability function of a sum of N random variables each distributed according to μ, where N is itself a random variable with distribution $\{c_j\}$.

The case when $c_j = c^j$ and $c > 1$ can be reduced to the case $c = 1$ by an exponential transformation; with the consequence that $\bar{\mu}\{[0, t]\}$ grows, at the rate $\sim e^{\alpha t}$, where $\alpha > 0$ is the so-called Malthusian parameter for (c, μ), defined by

$$c \int_0^\infty e^{-\alpha t}\, d\mu(t) = 1. \tag{5}$$

When $c > 1$ such an α of course always exists. When $c < 1$, the above remarks remain valid with $\alpha < 0$, *provided* such an α exists, i..e, provided the Laplace transform of μ can be extended sufficiently far into the left half plane (see page 362 of Feller [F] for details). This will *not* be possible, in particular, if the tail of μ decays at a slower than exponential rate, in which case the asymptotic behavior of $\bar{\mu}$ was not previously known. This is the case which will be of primary interest to us.

The appropriate generalization of the condition $c < 1$ is that $\phi(s)$ should be analytic on the range of the Fourier transform of μ. (Note that if μ is of exponential order then one cannot in general expect nice results about μ for general $\{c_j\}$. For example, even if $\bar{\mu}$ has density $p(t) = e^{-t}$, $t \geqq 0$, then

$$\bar{p}(t) = e^{-t} \sum c_j \frac{t^j}{j!},$$

and just by choosing $c_j = 0$ or 1 in different ways one can get varied decay rates.)

2. Local Limit Theorems.

Theorems 1 and 2 give *local* results for probability measures $\mu = \{\mu_n\}$, concentrated on the integers. For the measure

(1) $$\bar{\mu} = \sum_{j=0}^{\infty} c_j \mu^{*j} = \{\bar{\mu}_n\},$$

they show how the ratio $\frac{\bar{\mu}_n}{\mu_n}$ at an atom n behaves for large n. For simplicity, we have restricted μ to the non-negative integers in Theorem 1. In that case, we can expose the hypotheses on the rate of tail decay most clearly in terms of the radius of convergence $r \geqq 1$ of the *generating function*

$$\hat{\mu}(z) = \sum_n \mu_n z^n$$

of μ. Then, in Theorem 2, we apply the same methods.

Theorem 1. *For the probability measure* $\mu = \{\mu_n, n \geqq 0\}$ *assume that*

(α) $\lim_{n\to\infty} \mu_n^{*2}/\mu_n = c$ *exists* $(< \infty)$

(β) $\lim_{n\to\infty} \mu_{n+1}/\mu_n = \frac{1}{r}$ (> 0)

(γ) $\hat{\mu}(r) = d < \infty$; *that is* $\hat{\mu}$ *converges at its radius of convergence.*

Assume also that $\phi(w)$ *is a function analytic in a region containing the range of* $\hat{\mu}(z)$ *for* $|z| \leqq r$. *Then there exists a measure* $\phi(\mu) = \{\phi(\mu)_n, n \geqq 0\}$ *with*

$$\widehat{\phi(\mu)}(z) \equiv \sum_{n=0}^{\infty} \phi(\mu)_n z^n = \phi(\hat{\mu}(z)) \text{ for } |z| \leqq r,$$

and for which

(2) $$\lim_{n\to\infty} \frac{\phi(\mu)_n}{\mu_n} = \phi'(d).$$

If moreover $\phi(w) = \sum_{j=0}^{\infty} c_j w^j$ *for* $|w| \leqq 1$, *where* $\sum_{j=0}^{\infty} |c_j| < \infty$, *then we* *can identify the measure* $\phi(\mu)$ *with* $\bar{\mu} = \sum_{j=0}^{\infty} c_j \mu^{*j}$. *Finally, we must have* $c = 2d$ *in* (α).

Remark 1. The conditions of the theorem are satisfied by many concrete examples such as

(i) $\mu_n = c_\alpha r^n n^{-\alpha}$, $\alpha > 1$, $r \leqq 1$,

(ii) $\mu_n = c_{\alpha,\beta} r^n n^{-\alpha} e^{-n^\beta}$, $r \leqq 1$, α real, $0 < \beta < 1$; and

(iii) $\mu_n = cr^n e^{-n/(\log n)}$, $r \leqq 1$,

(where the constants are to normalize the μ_n so that $\sum \mu_n = 1$). However, the hypotheses are not satisfied if $\mu_n = ce^{-n}$.

In fact the following two alternative sufficient conditions (which cover most concrete cases) can be given. Namely the hypothesis (α) of Theorem 1 is satisfied if $\mu_n = r^n m_n$ with $r \leqq 1$ provided $\sum \mu_n = 1$, $\sum m_n < \infty$, and either

a) $\sup_{1 \leq l \leq n/2} \left(\frac{m_{n-l}}{m_n} \right) \leqq K$

for some constant K, or

b) $m(n) = e^{-n\psi(n)}$

where $\psi(t) \searrow 0$, $t^2 |\psi'(t)| \nearrow \infty$, and $\int_0^\infty e^{-\frac{1}{2}t^2|\psi'(t)|}\, dt < \infty$. The proof that (a) implies (α) is trivial. We will prove a continuous analog of the second assertion at the end of this section.

Proof of Theorem 1. We proceed by identifying $\bar{\mu}$ with an element of a Banach algebra of measures ν for which limits of the form (2) exist. By (α)

$$M \equiv \sup_{n \geqq 0} \frac{(\mu^{*2})_n}{\mu_n} < \infty . \tag{3}$$

Let $\mathfrak{A}$ be the collection of all sequences $\nu = \{\nu_n,\, n \geqq 0\}$ such that

$$\| \nu \| = M \sup_{n \geqq 0} \frac{|\nu_n|}{\mu_n} < \infty . \tag{4}$$

Furthermore, let $\mathfrak{A}_L$ be the subset of $\nu \in \mathfrak{A}$ for which

$$L(\nu) = \lim_{n \geqq \infty} \frac{\nu_n}{\mu_n} \tag{5}$$

exists and let $\mathfrak{A}_0$ be the subset of $\mathfrak{A}_L$ for which $L(\nu) = 0$. Note that $\mathfrak{A}$, with the norm $\| \nu \|$ of (4), is isomorphic with the Banach space l^∞ of bounded sequences, and that $\mathfrak{A}_L$ and $\mathfrak{A}_0$ are closed linear subspaces of $\mathfrak{A}$. Moreover,

$$\nu \in \mathfrak{A}_L \quad \text{iff} \quad \nu = L(\nu)\mu + \omega, \qquad \omega \in \mathfrak{A}_0. \tag{6}$$

The decomposition in (6) follows directly from (5), and is unique. On $\mathfrak{A}_L$, $L(\nu)$ is a bounded linear functional.

Lemma 1. *Under* $*$, $\mathfrak{A}$ *is a commutative Banach algebra with identity, and* $\mathfrak{A}_L$ *and* $\mathfrak{A}_0$ *are Banach subalgebras with identity. In particular, for* $\omega = \{\omega_n\} \in \mathfrak{A}_0$, *we have* $\mu * \omega \in \mathfrak{A}_L$ *with*

$$L(\mu * \omega) = \hat{\omega}(r) \tag{7}$$

where $\hat{\omega}(z) = \sum_{n=0}^{\infty} \omega_n z^n$ *is the generating function of* ω.

Proof of Lemma 1. For $\nu, \omega \in \mathfrak{A}$,

$$\frac{|(\nu * \omega)_n|}{\mu_n} = \frac{M}{\mu_n} \left| \sum_{k=0}^{n} \nu_{n-k}\omega_k \right| = \frac{M}{\mu_n} \left| \sum_{k=0}^{n} \left(\frac{\nu_{n-k}}{\mu_{n-k}}\right) \left(\frac{\omega_k}{\mu_k}\right) \mu_{n-k}\mu_k \right|$$

$$\leqq \frac{1}{M} \frac{1}{\mu_n} \sum_{k=0}^{n} M\left(\frac{|\nu_{n-k}|}{\mu_{n-k}}\right) M\left(\frac{|\omega_k|}{\mu_k}\right) \mu_{n-k}\mu_k \tag{8}$$

$$\leqq \| \nu \| \cdot \| \omega \| \frac{1}{M} \frac{(\mu^{*2})_n}{\mu_n}.$$

Taking suprema in (8) gives $\| \nu * \omega \| \leqq \| \nu \| \cdot \| \omega \|$, so $\mathfrak{A}$ is a commutative Banach algebra. The identity is $\delta_0 = \{\delta_{0n},\ n \geqq 0\}$ (Kronecker δ's).

For $\nu, \omega \in \mathfrak{A}_0$, we can expand (8) further:

$$M \frac{|(\nu * \omega)_n|}{\mu_n} \leqq \frac{1}{M} \frac{1}{\mu_n} \left\{ \sum_{k=0}^{[n/2]} + \sum_{[n/2]+1}^{n} \right\} M\left(\frac{|\nu_{n-k}|}{\mu_{n-k}}\right) M\left(\frac{|\omega_k|}{\mu_k}\right) \mu_{n-k}\mu_k$$

$$\leqq \|\omega\| M \sup_{j \geqq [n/2]} \frac{|\nu_j|}{\mu_j} + \|\nu\| M \sup_{j \geqq [n/2]} \frac{|\omega_j|}{\mu_j}.$$

The suprema in (9) tend to zero as $n \to \infty$; hence $L(\nu * \omega) = 0$, and $\mathfrak{A}_0$ is a subalgebra (which contains δ_0).

Now $\mu \in \mathfrak{A}_L$, and by $(\alpha)\mu^{*2} \in \mathfrak{A}_L$ with $L(\mu^{*2}) = c$. Since $\mathfrak{A}_0$ is a subalgebra, and in view of the decomposition (6), to show that $\mathfrak{A}_L$ is a subalgebra it is sufficient to show that $\mu * \omega \in \mathfrak{A}_L$ for all $\omega \in \mathfrak{A}_0$. We show that for such ω, $L(\mu * \omega) = \sum_{k=0}^{\infty} \omega_k r^k$. For $n \geqq N$,

$$\left| \frac{(\mu * \omega)_n}{\mu_n} - \sum_{k=0}^{\infty} \omega_k r^k \right| = \left| \sum_{k=0}^{n} \frac{\mu_{n-k}}{\mu_n} \omega_k - \sum_{k=0}^{\infty} \omega_k r^k \right|$$

$$\leqq \sum_{k=0}^{N} \left| \frac{\mu_{n-k}}{\mu_n} - r^k \right| \cdot |\omega_k| + \frac{1}{\mu_n} \sum_{k=N+1}^{n} \mu_{n-k}\mu_k \left(\frac{|\omega_k|}{\mu_k}\right)$$

$$+ \sum_{k=N+1}^{\infty} |\omega_k| r^k$$

$$\leqq \sum_{k=0}^{N} \left| \frac{\mu_{n-k}}{\mu_n} - r^k \right| \cdot |\omega_k| + M \sup_{k \geqq N+1} \frac{|\omega_k|}{\mu_k} \tag{10}$$

$$+ \sum_{k=N+1}^{\infty} |\omega_k| r^k.$$

Since $\lim_{n \to \infty} \frac{|\omega_n|}{\mu_n} = 0$ and since $\sum_{0}^{\infty} |\omega_k| r^k \leqq \|\omega\| \frac{1}{M} \sum_{k=0}^{\infty} \mu_k r^k < \infty$, we can make the last two terms in (10) arbitrarily small by choosing N sufficiently large. By (β) we can make $\left| \frac{\mu_{n-k}}{\mu_n} - r^k \right|$ arbitrarily small for fixed $k \leqq N$ and sufficiently large n. Hence $L(\mu * \omega) = \sum_{k=0}^{\infty} \omega_k r^k$, and $\mathfrak{A}_L$ is a commutative

Banach algebra (containing the identity δ_0). This completes the proof of Lemma 1.

Henceforth we focus upon $\mathfrak{A}_L$. We shall construct a measure "$\phi(\mu)$" in $\mathfrak{A}_L$ and prove (2) by identifying $\bar{\mu}$ with $\phi(\mu)$. First we locate the *spectrum* of μ. Let $\mathscr{H}(\mathfrak{A}_*)$ denote the set of all homomorphisms of a particular algebra $\mathfrak{A}_*$.

Lemma 2. *We can identify $\mathscr{H}(\mathfrak{A}_0)$ with the set of all functionals of the form*

$$h_2(\omega) = \hat{\omega}(z) = \sum_{n=0}^{\infty} \omega_n z^n, \ \forall \omega = \{\omega_n\} \in \mathfrak{A}_0 \tag{11}$$

where z is any fixed element of $\{|z| \leqq r\}$.

Proof of Lemma 2. Any h_z of the form (11) is in $\mathscr{H}(\mathfrak{A}_0)$ by standard arguments. (Note that $|h_z(\omega)| \leqq \|\omega\| \frac{1}{M} \Sigma \mu_n |z|^n \leqq \|\omega\| \frac{d}{M}.$) For the converse, note that finite linear combinations of the vectors $\delta_k = \{\delta_{kn}, n \geqq 0\}$ are dense in $\mathfrak{A}_0$; and that $\delta_k = \delta_1^{*k}$ for $k \geqq 1$. That is, for $\omega = \{\omega_n, n \geqq 0\} \in \mathfrak{A}_0$

$$\left\| \omega - \sum_{k=0}^{N} \omega_k \delta_1^{*k} \right\| = M \sup_{n>N} \frac{|\omega_n|}{\mu_n} \to 0 \quad \text{as } N \to \infty.$$

Hence for $h \in \mathscr{H}(\mathfrak{A}_0)$ and any $\omega \in \mathfrak{A}_0$,

$$h(\omega) = \lim_{N\to\infty} h\left(\sum_{k=0}^{N} \omega_k \delta_1^{*k}\right) = \lim_{N\to\infty} \sum_{k=0}^{N} \omega_k [h(\delta_1)]^k = \sum_{k=0}^{\infty} \omega_n z^n \tag{12}$$

where $z = h(\delta_1)$. We complete the proof of Lemma 2 by showing that $|z| \leqq r$. Otherwise, if $z = \mathrm{Re}^{i\theta}$ with $R > r$, let $r < \rho < R$ and note that the vector $\omega^{(\rho)} = \{\omega_n^{(\rho)}, n \geqq 0\}$ with $\omega_n^{(\rho)} = \left(\frac{\rho}{R}\right)^n e^{-in\theta}\mu_n$ (for $n \geqq 0$) lies in $\mathfrak{A}_0$. Hence there must exist a finite limit

$$h(\omega^{(\rho)}) = \lim_{N\to\infty} \sum_{k=0}^{N} \omega_k^{(\rho)} z^k = \lim_{N\to\infty} \sum_{k=0}^{N} \rho^k \mu_k \tag{13}$$

—a contradiction to (β).

Lemma 3. *We can identify $\mathscr{H}(\mathfrak{A}_L)$ with the set of all functionals of the form*

$$h_z(v) = \hat{v}(z) = \sum_{n=0}^{\infty} v_n z^n \qquad \forall v = \{v_n\} \in \mathfrak{A}_L \tag{14}$$

where z is any fixed element of $\{|z| \leqq r\}$; with the addition, in case $c \neq 2d$, of the functional

$$\bar{h}(v) = (c-2d)L(v) + \hat{v}(r) \qquad \forall v = \{v_n\} \in \mathfrak{A}_L. \tag{15}$$

Proof of Lemma 3. Standard argument shows that any h_z of the form (14) must be $\in \mathscr{H}(\mathfrak{A}_L)$. The $\bar{h}$ in (15) is linear, and bounded, since $L(v)$ and $\hat{v}(r)$ are bounded linear functionals of v; and one can verify directly that $\bar{h}(v * \omega) = \bar{h}(v)\bar{h}(\omega)$ for all $v, \omega \in \mathfrak{A}_L$.

Conversely, suppose $h \in \mathscr{H}(\mathfrak{A}_L)$ and $h(\omega^0) \neq \widehat{\omega^0}(r)$ for some $\omega^0 = \{\omega_n^0\} \in \mathfrak{A}_0$. Then, restricted to $\mathfrak{A}_0$, $h \in \mathscr{H}(\mathfrak{A}_0)$. Hence

$$h(\omega) = \hat{\omega}(z_0) \qquad \forall \omega = \{\omega_n\} \in \mathfrak{A}_0 \tag{16}$$

for some z_0 with $|z_0| \leqq r$. Via (14), we have a homomorphism h_{z_0} on $\mathfrak{A}_L$ such that

$$h(\omega) = h_{z_0}(\omega) \qquad \forall \omega \in \mathfrak{A}_0. \tag{17}$$

Apply both h and h_{z_0} to the vector

$$\mu * \omega^0 = \widehat{\omega^0}(r)\mu + \omega^1 \tag{18}$$

where $\omega^1 = \{\omega_n^1\} \in \mathfrak{A}_0$. We get

$$h(\mu)h(\omega^0) + \widehat{\omega^0}(r)h(\mu) + h(\omega^1) \tag{19}$$

and

$$h_{z_0}(\mu)h_{z_0}(\omega^0) = \widehat{\omega^0}(r)h_{z_0}(\mu) + h_{z_0}(\omega^1). \tag{20}$$

Since $h_{z_0}(\omega^1) = h(\omega^1)$ and $h_{z_0}(\omega^0) = h(\omega^0) \neq \widehat{\omega^0}(r)$, we must have $h_{z_0}(\mu) = h(\mu)$ as the unique solution for the linear equation

$$x[h(\omega^0) - \widehat{\omega^0}(r)] = h(\omega^1). \tag{21}$$

But then, in view of the decomposition (6), $h \equiv h_{z_0}$ on $\mathfrak{A}_L$.

Finally, suppose that $h \in \mathfrak{A}_L$ is such that $h(\omega) = \hat{\omega}(r)$ for all $\omega = \{\omega_n\} \in \mathfrak{A}_0$. Apply h to $\mu^{*2} = c\mu + \omega^\mu$, to get

$$h^2(\mu) = ch(\mu) + \widehat{\omega^\mu}(r) = ch(\mu) + d(d-c), \tag{22}$$

since $\widehat{\omega^\mu}(r) = [\hat{\mu}(r)]^2 - c\hat{\mu}(r) = d^2 - cd$. That is, $h(\mu)$ is a solution of the equation

$$x^2 - cx + d(c-d) = (x-d)[x-(c-d)] = 0. \tag{23}$$

If $c = 2d$, the unique root of (23) is $h(\mu) = d$, and we can identify h with $h_r(v) \equiv \hat{v}(r)$. If $c \neq 2d$, then (23) has two roots $x = d$, $x = c - d$; and h can either be h_r or else must be the functional in (15). This completes the proof of Lemma 3. (Note: some such special argument is needed since in general homomorphisms of codimension-one subalgebras do not have unique extensions to homomorphisms of the full algebras.)

Note that in any case we must have $c \geqq 2d$, since for $n \geqq 2N$,

$$\frac{\mu_n^{*2}}{\mu_n} \geqq \sum_{k=0}^{N} \left(\frac{\mu_{n-k}}{\mu_n}\right)\mu_k + \sum_{k=n-N}^{n} \mu_{n-k}\left(\frac{\mu_k}{\mu_n}\right) \tag{24}$$

and, by (β), we can make each sum on the right hand side of (24) arbitrarily close to d by choosing first N and then n sufficiently large.

Now we construct an element $\phi(\mu) \in \mathfrak{A}_L$ which we shall later identify with $\bar{\mu}$ when $\sum_{j=1}^{\infty} |c_j| < \infty$. (The construction is a standard one for proving the Wiener-Levy Theorem.) In $\mathfrak{A}_L$, the spectrum of μ, S_μ, is the set $\{h(\mu), \forall h \in \mathscr{H}(\mathfrak{A}_L)\}$. By Lemma 3, $S_\mu \subset \{\hat{\mu}(z), |z| \leqq r\} \cup \{c-d\}$. For complex $\lambda \notin S_\mu$, the $*$-inverse $[\lambda\delta_0 - \mu]^{-1}$ exists and is a vector valued analytic function of λ. Let Γ be a simple closed path in the domain of analyticity of ϕ, encircling $S_\mu \cap \{|w| \leqq d\}$ and bounded away from S_μ. There exists the vector

$$\phi(\mu) = \frac{1}{2\pi i} \oint_\Gamma [\lambda\delta_0 - \mu]^{-1} \phi(\lambda) d\lambda. \tag{25}$$

as an element of $\mathfrak{A}_L$; and for any bounded linear functional ψ on $\mathfrak{A}_L$,

$$\psi(\phi(\mu)) = \frac{1}{2\pi i} \oint_{\Gamma} \psi([\lambda\delta_0 - \mu]^{-1})\phi(\lambda)d\lambda. \tag{26}$$

If $\psi = h \in \mathscr{H}(\mathfrak{A}_L)$, then

$$h([\lambda\delta_0 - \mu]^{-1}) = [h(\lambda\delta_0 - \mu)]^{-1} = [\lambda - h(\mu)]^{-1} \tag{27}$$

since $h(\delta_0) = 1$. In particular, if $|h(\mu)| \leqq d$,

$$h(\phi(\mu)) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\phi(\lambda)}{\lambda - h(\mu)} d\lambda = \phi(h(\mu)) \tag{28}$$

by Cauchy's theorem.

To show that $\phi(\mu) = \bar{\mu}$ when $\sum_{j=0}^{\infty} |c_j| < \infty$, it suffices to show that $\phi(\mu)$ and $\bar{\mu}$ have identical characteristic functions (Fourier transforms). To this end, let $h = h_z$ in (28), where $z = e^{i\theta}$. Then

$$\sum_{n=0}^{\infty} (\phi(\mu))_n e^{in\theta} = h_{e^{i\theta}}(\phi(\mu)) = \phi(h_{e^{i\theta}}(\mu)) = \phi\left(\sum_{n=0}^{\infty} \mu_n e^{in\theta}\right). \tag{29}$$

On the other hand, since $\bar{\mu} = \sum_{j=1}^{\infty} c_j \mu^{*j}$ and $\sum_{j=1}^{\infty} |c_j| < \infty$, the characteristic function of $\bar{\mu}$

$$\sum_{n=0}^{\infty} \bar{\mu}_n e^{in\theta} = \lim_{N\to\infty} \sum_{j=0}^{N} c_j \left[\sum_{j=0}^{\infty} \mu_n e^{in\theta} \right]^j = \phi \left(\sum_{n=0}^{\infty} \mu_n e^{in\theta}\right)$$

by standard interchange of limits. Hence $\bar{\mu} = \phi(\mu) \in \mathfrak{A}_L$.

A corollary of (26):

Lemma 4. *In* (α), *c must* $= 2d$.

Proof of Lemma 4. We have noted from (24) that $c \geqq 2d$. Suppose that $c > 2d$, and choose $\phi(z) = z$. Then (25) becomes

$$\mu = \frac{1}{2\pi i} \oint_{\Gamma} [\lambda\delta_0 - \mu]^{-1} \lambda \, d\lambda. \tag{31}$$

Here we can pick the path Γ of (25)–(28) definitely separating the point $c - d$ from the rest of S_μ. As ψ in (26) pick the $\hbar$ of (15). Now, on the one hand,

$$\hbar(\mu) = c - d > 0. \tag{32}$$

On the other hand, since $[\lambda - (c-d)]^{-1}$ is analytic for $\lambda \neq c - d$, we can use (26) and (27) to get

$$\hbar(\mu) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\lambda}{\lambda - \hbar(\mu)} d\lambda = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\lambda^2}{\lambda - (c - d)} d\lambda = 0 \tag{33}$$

—a contradiction. Hence $c = 2d$, and Lemma 4 is demonstrated.

Lemma 5. $L(\mu^{*k}) = kd^{k-1}$ $\forall k \geqq 1$.

Proof of Lemma 5. $L(\mu) = 1$, and $L(\mu^{*2}) = 2d$ by Lemma 4. Suppose $L(\mu^{*j}) = jd^{j-1}$ for $j \leqq k$. Write

$$\begin{aligned}
\mu^{*(k+1)} &= \mu^{*k} * \mu = (kd^{k-1}\mu + \omega^{(k)}) * \mu && (\omega^{(k)} \in \mathfrak{A}_0) \\
&= kd^{k-1}(\mu * \mu) + \omega^{(k)} * \mu \\
&= kd^{k-1}(2d\mu + \omega^{\mu}) + \widehat{\omega^{(k)}}(r)\mu + \varpi^{(k)} && (\varpi^{(k)} \in \mathfrak{A}_0) \\
&= [2kd^k + (1 - k)d^k]\mu + (kd^{k-1}\omega^{\mu} + \varpi^{(k)}) \\
&= (k + 1)d^k\mu + \omega^{(k+1)} && (\omega^{(k+1)} \in \mathfrak{A}_0),
\end{aligned} \tag{34}$$

since $\widehat{\omega^{(k)}}(r) = [\hat{\mu}(r)]^k - kd^{k-1}\hat{\mu}(r) = (1 - k)d^k$. Thus Lemma 5 holds by induction.

Finally, we show that $L(\phi(\mu)) = \phi'(d)$. For $\lambda > \| \mu \|$, the inverse $[\lambda\delta_0 - \mu]^{-1}$ has a series expansion

$$[\lambda\delta_0 - \mu]^{-1} = \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k}\mu^{*k} \tag{35}$$

convergent in norm. Since L is a bounded linear functional,

$$L([\lambda\delta_0 - \mu]^{-1}) = \lambda^{-1} \sum_{k=0}^{\infty} \lambda^{-k}L(\mu^{*k}) = \lambda^{-1} \sum_{k=0}^{\infty} k\lambda^{-k}d^{k-1} \tag{36}$$

$$= (\lambda - d)^{-2}.$$

Now, the left hand side of (36) is a scalar function analytic in λ outside S_μ; and the right hand side is analytic for $\lambda \neq d$. Hence (36) holds for all λ outside S_μ. In particular, letting $\psi = L$ in (26), we have

$$L(\phi(\mu)) = \frac{1}{2\pi i} \oint_\Gamma L([\lambda\delta_0 - \mu]^{-1})\phi(\lambda)d\lambda = \frac{1}{2\pi i} \oint_\Gamma \frac{\phi(\lambda)}{(\lambda - d)^2} d\lambda = \phi'(d). \tag{37}$$

This completes the proof of Theorem 1.

A Corollary of the Proof. *Suppose that* $v = \{v_n,\ n = 0,1,2,\cdots\}$, *where* $v_n \sim \mu_n$ *as* $n \to \infty$. *If* ϕ *is analytic on a region containing the range of* $\hat{v}$, *then* $\lim_{n\to\infty} \frac{\bar{v}_n}{\mu_n} = \phi'(d)$.

Remark 2. If one assumes that ϕ is analytic in $|w| < r + \varepsilon$ (which is the case for many probabilistic applications), then there is a simple direct proof of Theorem 1 due to H. Kesten (see chapter IV of Athreya-Ney [A–N]).

Theorem 2. *For the probability measure* $\mu = \{\mu_n,\ n = 0, \pm 1, \pm 2, \cdots\}$ *assume that*

(α) $\lim_{|n|\to\infty} \mu_n^{*2}/\mu_n = c$ *exists* $(< \infty)$

(β) $\lim_{|n|\to\infty} \mu_{n+1}/\mu_n = 1$.

Assume also that $\phi(w)$ *is a function analytic in a region containing the* *range of* $\hat{\mu}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \mu_n e^{in\theta}$ *for* $0 \leqq \theta \leqq 2\pi$. *Then there exists a measure* $\phi(\mu) = \{\phi(\mu)_n,\ n = 0, \pm 1, \pm 2, \cdots\}$ *with*

$$\widehat{\phi(\mu)}(e^{i\theta}) \equiv \sum_{n=-\infty}^{\infty} \phi(\mu)_n e^{in\theta} = \phi(\hat{\mu}(e^{i\theta})) \quad \text{for } 0 \leqq \theta \leqq 2\pi,$$

and for which

$$\lim_{|n|\to\infty} \frac{\phi(\mu)_n}{\mu_n} = \phi'(1).$$

If moreover $\phi(w) = \sum_{j=0}^{\infty} c_j w^j$ *for* $|w| \leqq 1$ *where* $\sum_{j=0}^{\infty} |c_j| < \infty$, *then we can identify the measure* $\phi(\mu)$ *with* $\tilde{\mu} = \sum_{j=0}^{\infty} c_j \mu^{*j}$. *Finally, we must have* $c = 2$ *in* (α).

(Remark: The hypothesis (γ) of Theorem 1 is redundant here since $\sum_n \mu_n = 1$.)

Proof of Theorem 2. The proof is similar to that for Theorem 1, with the following minor modifications:

(i) Let $\mu = \sup_{-\infty<n<\infty} \mu_n^{*2}/\mu_n$.

(ii) Let $\mathfrak{A}$ be the collection of all sequences $v = \{v_n,\ n = 0, \pm 1, \pm 2, \cdots\}$ such that $\|v\| = \mu \sup_n \frac{|v_n|}{\mu_n} < \infty$; and let $\mathfrak{A}_L$ be the subset of $v \in \mathfrak{A}$ for which the $L(v) = \lim_{|n|\to\infty} \frac{v_n}{\mu_n}$ exists.

(iii) Replace r and d by 1, and replace one sided sums by two sided sums.

Examples: In Lemma 1, $L(\mu * \omega) = \hat{\omega}(1) = \sum_{n=-\infty}^{\infty} \omega_n$. In (8), replace $\sum_{k=0}^{\infty} v_{n-k}\omega_k$ by $\sum_{k=-\infty}^{\infty} v_{n-k}\omega_k$. In (10) replace $\sum_{k=0}^{N} \left|\frac{\mu_{n-k}}{\mu_n} - r^k\right| \cdot |\omega_k|$ by $\sum_{k=-N}^{N} \left|\frac{\mu_{n-k}}{\mu_n} - 1^k\right| \cdot |\omega_k|$, and replace $\sum_{k=n+1}^{\infty} |\omega_k| r^k$ by $\sum_{|k|\geqq n+1} |\omega_k| 1^k$.

(iv) The $\mathscr{H}(\mathfrak{A}_0)$ of Lemma 2 becomes all functionals of the form

$$h_{e^{i\theta}}(\omega) = \hat{\omega}(e^{i\theta}) = \sum_{n=-\infty}^{\infty} \omega_n e^{in\theta}$$

for $0 \leqq \theta \leqq 2\pi$. This is because of the double tail of μ: in the proof of Lemma 2, we can now choose vectors $\omega^{(\rho)}$ of the form $\omega_n^{(\rho)} = \left(\frac{\rho}{R}\right)^{|n|} e^{-in\theta}\mu_n$ for $n = 0, \pm 1, 2\pm, \cdots$ and force $|z| = R = 1$. Similarly for Lemma 3.

All other aspects of the proof are identical to those for Theorem 1; and we omit further details.

The following theorem is analogous to Theorem 1, and gives a local result for a one sided measure μ having a *continuous density* $m(t)$. (A similar analogue holds for Theorem 2.) For integrable functions f and g on $R^+ = [0, \infty)$, we again denote convolution by $*$:

$$(38) \qquad (f * g)(t) = \int_0^\infty f(t-s)g(s)ds = \int_0^t f(t-s)g(s)ds;$$

and we shall use "characteristic functions" with complex argument $z = x + iy$,

$$\hat{f}(z) = \int_0^\infty e^{zt} f(t)dt.$$

Let δ_t again denote the measure giving unit mass to the point t.

Theorem 3. *Let μ be a probability measure on R^+ with continuous density function $m(x) > 0$ for all x. Assume the following*:

(α') $\lim_{x\to\infty} \frac{(m*m)(x)}{m(x)} = c$ *exists* ($< \infty$).

(β') *There exists a* $\lim_{t\to\infty} \frac{m(t+s)}{m(t)} = \psi(s) > 0$ *uniformly on all compact s subintervals of R^+. It follows that $\psi(s) = e^{-\rho s}$ for some $\rho \geqq 0$ and all $s \geqq 0$.*

(γ') $\hat{m}(\rho) = d < \infty$, *where ρ is the exponent in* (β).

Assume also that $\phi(w)$ is a function analytic in a region containing the *range of* $\hat{m}(z)$ *for* $\operatorname{Re}(z) \leqq \rho$. *Then there exists a measure* $\phi(\mu)$ *on* R^+, *with continuous density* $\phi(m)(x)$ *on* $0 < x < \infty$, *such that*

$$\widehat{\phi(m)}(z) \equiv \int_0^\infty e^{zt}\phi(m)(t)dt = \phi(\hat{m}(z)) \text{ for } \operatorname{Re}(z) \leqq \rho$$

and such that

$$\lim_{t\to\infty} \frac{\phi(m)(t)}{m(t)} = \phi'(d). \tag{39}$$

If moreover $\phi(w) = \sum_{j=0}^\infty c_j w^j$ *for* $|w| \leqq 1$, *where* $\sum_{j=0}^\infty |c_j| < \infty$, *we can identify the measure* $\phi(\mu)$ *with* $\bar{\mu} = \sum_{j=0}^\infty c_j \mu^{*j}$, *where* $\bar{\mu} - c_0\delta_0$ *has density*

$$\bar{m}(t) = \sum_{j=1}^\infty c_j m^{*j}(t). \tag{40}$$

Furthermore we must have $c = 2d$ *in* (α').

Proof. The proof again follows lines similar to that of Theorem 1, with modifications to take the density into account.

(i) Let $M = \sup_{t\geqq 0} \dfrac{(m*m)(t)}{m(t)}$

(ii) Let $\mathfrak{A}$ be the collection of all Radon measures ν on R^+ of the form

$$\nu = e_\nu\delta_0 + \nu', \tag{41}$$

where e_ν is a scalar and ν' has a continuous density $f_\nu(t)$ with $\sup_{t\geqq 0} \dfrac{|f_\nu(t)|}{m(t)} < \infty$. Let

$$\|\nu\| = |e_\nu| + M \sup_{t\geqq 0} \frac{|f_\nu(t)|}{m(t)}. \tag{42}$$

Let $\mathfrak{A}_L$ be the subset of $\mathfrak{A}$ for which there exists the limit

$$L(\nu) = \lim_{t\to\infty} \frac{f_\nu(t)}{m(t)}, \tag{43}$$

and let $\mathfrak{A}_0$ be the subset of $\mathfrak{A}_L$ for which $L(\nu) = 0$ and also $e_\nu = 0$.

Replacing the sums in the proofs for Theorem 1 now by integrals, we can show that $\mathfrak{A}$ is a Banach algebra of measures relative to convolution and to the norm in (42). Furthermore, $\mathfrak{A}_L$ and $\mathfrak{A}_0$ are closed subalgebras; and

$$L(\mu * \nu) = \int_0^\infty e^{\rho t} f_\nu(t)dt = \hat{f}_\nu(\rho) \tag{44}$$

for all ν in $\mathfrak{A}_0$.

Note also that for a given subinterval $[a,b]$ of R^+ and for measures $\nu \in \mathfrak{A}_0$ with densities f_ν having support in $[a,b]$, the norm $\|\nu\|$ of (42) is equivalent to the supremum norm $\sup_{a\leqq t\leqq b} |f_\nu(t)|$.

(iii) The main step is to characterize all homomorphisms in $\mathscr{H}(\mathfrak{A}_L)$ as having the form

$$h_u(\nu) = e_\nu + \int_0^\infty e^{ut} f_\nu(t)dt \quad \text{for all } \nu \in \mathfrak{A}_L \tag{45}$$

where u is some complex number with $\mathrm{Re}(u) \leqq \rho$.

As in Theorem 1, we can establish (45) for $\mathfrak{A}_L$ by first demonstrating it for ν's in $\mathfrak{A}_0$; and indeed for the dense subset of ν's in $\mathfrak{A}_0$ whose densities f_ν have compact support. To this end, for any $x > 0$, we approximate the unit mass δ_x be measures $N_{x,\sigma}$ with corresponding density functions $n_{x,\sigma}(t)$ $(0 < \sigma < x)$ which are piecewise linear, with

$$n_{x,\sigma}(t) = \begin{cases} 0 & \text{for } |t-x| \geqq \sigma \\ \sigma^{-1} & \text{for } t = x. \end{cases}$$

For any function f on R^+, and $s \geqq 0$, let

$$(\tau_s f)(t) = \begin{cases} 0 & \text{for } 0 \leqq t < s \\ f(t-s) & \text{for } t \geqq s \end{cases}.$$

If f is the density of a measure ν, let $\tau_s\nu$ denote the shifted measure corresponding to $\tau_s f$. With this notation, $\tau_s n_{x,\sigma} = n_{x+s,\sigma}$ for $0 < \sigma < x$ and any $s \geqq 0$. Moreover, if f is any continuous function with support in an interval $[0, b]$, then

$$(46)\qquad (n_{x,\sigma} * f)(t) = \int_0^b n_{x,\sigma}(t-s)f(s)ds = \int_0^b \tau_s n_{x,\sigma}(t)f(s)ds$$

$$= \int_0^b n_{x+s,\sigma}(t)f(s)ds$$

$$= \lim_{\|\pi\| \to 0} \sum_{s_k \in \pi} n_{x+s_k}(t)f(s_k)[s_k - s_{k-1}].$$

The limit of sums in (46) is for partitions $\pi = \{s_k\}$ of $[0, b]$ with meshes $\|\pi\|$, and is uniform in t on R^+. Let f now be the density of a measure ν. Since all the t-functions indicated in (46) have support in $[0, x + b + \sigma]$, we can conclude that

$$(47)\qquad \lim_{\|\pi\| \to 0} \Big\| N_{x,\sigma} * \nu - \sum_{s_k \in \pi} N_{x+s_k} f(s_k)[s_k - s_{k-1}] \Big\| = 0.$$

Furthermore, if $f(0) = 0$ and f has (say) a continuous derivative on R^+, then

$$(48)\qquad \lim_{\sigma \downarrow 0} \| N_{x,\sigma} * \nu - \tau_x \nu | = 0 \qquad \text{for } x > 0;$$

$$(49)\qquad \lim_{\sigma \downarrow 0} \| N_{x+\sigma,\sigma} * \nu - \tau_x \nu \| = 0 \quad \text{for } x \geqq 0;$$

and in the same norm,

$$(50)\qquad \lim_{\sigma \downarrow 0} N_{x+\sigma,\sigma} * N_{y+\sigma,\sigma} * \nu = \lim_{\sigma \downarrow 0} N_{x+y+\sigma,\sigma} * \nu = \tau_{x+y}\nu$$

for all $x, y \geqq 0$.

Now to identify $\mathscr{H}(\mathfrak{A}_0)$, note that for any $h \in \mathscr{H}(\mathfrak{A}_0)$ ($h \not\equiv 0$) and any $s \geqq 0$, there exists a $\nu_s \in \mathfrak{A}_0$, whose density f_s has all of the above properties, for which $h(\tau_s v_s) \neq 0$. (To find v_s for $s > 0$, pick some measure ω_s with support contained in an interval $[a, b]$ where $0 < a < b \leqq s$, and for which $h(\omega_s) \neq 0$. For some N, ω_s^{*N} will have support in (s, ∞), and we can let v_s be the measure resulting from a leftward shift of ω_s^{*N} by the amount s. There must exist such an ω_s with $h(\omega_s) \neq 0$; otherwise, h would induce a nontrivial linear functional, bounded in the supremum norm on $[0, s]$, which would vanish on all continuous functions with compact supports not including the origin.) By (49),

$$\lim_{\sigma \downarrow 0} h(N_{s+\sigma,\sigma} * v_s) = h(v_s) \lim_{\sigma \downarrow 0} h(N_{s+\sigma,\sigma}) = h(\tau_s v_s) \neq 0. \tag{51}$$

Hence there exist the limit

$$\lim_{\sigma \downarrow 0} h(N_{s+\sigma,\sigma}) = \psi(s) \text{ for each } s \geqq 0. \tag{52}$$

By (50) and (51), $\psi(s) \not\equiv 0$, and

$$\psi(s + t) = \psi(s)\psi(t) \text{ for all } s, t \geqq 0;$$

that is, $\psi(s) = e^{zs}$ for some complex number z. Next pick any $v \in \mathfrak{A}_0$, with support in $[0, b]$ and density f. Let $x = \sigma$ in (47), and apply h to get

$$h(N_{\sigma,\sigma})h(v) = \int_0^b h(N_{s+\sigma,\sigma})f(s)ds.$$

Then let $\sigma \to 0$ as in (52), for the representation

$$h(v) = \int_0^\infty e^{zt}f(t)dt. \tag{53}$$

That $\operatorname{Re} z$ must $\leqq \rho$ follows from the hypothesis (γ') as in the proof of Theorem 1. All other steps are likewise similar, and we omit further details.

Sufficient condition for α'.

In remark 1 we indicated (without proof) a sufficient condition for (α). We now prove the analogous statement for the density case. (We treat the analog of the case $r = 1$; and the case $r < 1$ then follows.)

Assume that $m(t) \geqq 0$, $\int_0^\infty m(t)dt = 1$ and

$$m(t) = e^{-t\psi(t)},$$

with $\psi(t)$ as given in Remark 1. Then write

$$I = \frac{1}{m(t)} \int_A^{t/2} p(x)p(t-x)dx = \int_A^{t/2} e^{-x[\psi(x)-\psi(t-x)]} e^{+t[\psi(t)-\psi(t-x)]}dx.$$

But $\psi(a) - \psi(b) = \int_a^b (-\psi'(t))t^2 \frac{dt}{t^2} \geqq a^2(-\psi'(a))\left[\frac{1}{a} - \frac{1}{b}\right]$, and hence

$$\psi(x) - \psi(t-x) \geqq \frac{1}{2}x(-\psi'(x)) \text{ for } x \leqq \frac{t}{3},$$

and

$$\psi(t-x) - \psi(t) \geqq \frac{1}{2}(t-x)^2(-\psi'(t-x))t^{-1} \text{ for } \frac{t}{3} \leqq x \leqq \frac{t}{2}.$$

Thus

$$I \leqq \int_A^{t/3} e^{-x[\psi(x)-\psi(t-x)]}dx + \int_{t/3}^{t/2} e^{+t[\psi(t)-\psi(t-x)]}dx$$

$$\leqq \int_A^{t/3} e^{-\frac{1}{2}x^2(-\psi'|(x))}dx + \int_{t/3}^{t/2} e^{-\frac{1}{2}(t-x)^2(-\psi'(t-x))}\, dx$$

$$= o_A(1) + o_t(1).$$

By (β)

$$\int_0^A + \int_A^t \frac{p(x)p(t-x)}{p(t)} = 2 + o_A(1),$$

implying (α).

3. A Global Theorem.

Theorem 4 gives a "global" result in that it compares the *cumulative* "tails" of the measure

$$\bar{\mu} = \sum_{j=0}^{\infty} c_j \mu^{*j}$$

with those of μ. As such it is weaker than the "local" Theorems 1, 2, and 3. However we need not restrict μ here to be discrete or absolutely continuous.

By way of notation let $|\nu|$ denote the "absolute value" measure associated with a (Radon) measure ν defined for the Borel sets of $R = (-\infty, \infty)$, and let $\nu_\tau(A) = \nu(A-\tau)$, the measure translated by any real τ. Also let $T_t = (-\infty, -t] \cup (t, \infty)$ for $t > 0$ and let $T_0 = R$.

Theorem 4. *Let μ be a probability measure on the Borel set of R, with $\mu(T_t) > 0$ for all $t \geqq 0$, and satisfying*

(α'') $\lim_{t\to\infty} \mu^{*2}(T_t)/\mu(T_t) = c$ *exists* $(< \infty)$

and

(β'') $\lim_{t\to\infty} |\mu - \mu_\tau|(T_t)/\mu(T_t) = 0$ *for all real* τ.

Assume that $\phi(w) = \sum_{j=0}^{\infty} c_j w^j$ is analytic in $|w| < 1 + \varepsilon$ for some $\varepsilon > 0$. Then with $\bar{\mu}$ defined as above,

$$\lim_{t\to\infty} \bar{\mu}(T_t)/\mu(T_t) = \phi'(1). \tag{1}$$

Furthermore we must have $c = 2$ in (α').

Remarks. The condition (β) of Theorem 2 implies (β'') in Theorem 4. Moreover, (β'') implies that

(β''') $\lim_{t\to\infty} \mu(T_{t-\tau})/\mu(T_t) = 1$ for all real τ.

Hence

$$\lim_{t\to\infty} \frac{e^{s(t-\tau)}\mu(T_{t-\tau})}{e^{st}\mu(T_t)} = e^{-s\tau} \text{ for all } s > 0 \text{ and } \tau > 0.$$

From this it follows that

(γ'') $\lim_{t\to\infty} e^{st}\mu(T_t) = \infty$ for all $s > 0$.

With the generality allowed for μ in the above hypotheses, we shall not be able to determine the spectrum S_μ exactly in the algebra $\mathfrak{A}_L$ to be constructed; hence the requirement that ϕ be analytic on an entire disc $|w| < 1 + \varepsilon$. If we excluded the possibility of a continuous singular component in μ, then we could argue in closer analogy to the proofs of Theorems 1 and 2, using only the condition that ϕ be analytic on a region containing the range of the characteristic function

$$\hat{\mu}(v) = \int_{-\infty}^{\infty} e^{ivx} d\mu(x) \quad \text{for } v \in (-\infty, \infty).$$

(See also Theorem 4′, which follows as a corollary of Theorem 3.)

Proof of Theorem 4. By (α''), $M = \sup_{t \geqq 0} \mu^{*2}(T_t)/\mu(T_t) < \infty$. Let $\mathfrak{A}$ be the collection of all Radon measures ν on R such that

$$\|\nu\| = M \sup_{t>0} \frac{|\nu|(T_t)}{\mu(T_t)} < \infty \tag{2}$$

where $|\nu|$ denotes the total variation measure of ν. By standard methods one can check that $\mathfrak{A}$, with the norm (2), is a Banach space. Let $\mathfrak{A}_0$ be the subset of $\nu \in \mathfrak{A}$ for which

$$\lim_{t\to\infty} \frac{|\nu|(T_t)}{\mu(T_t)} = 0. \tag{3}$$

One can check that $\mathfrak{A}_0$ is a closed subspace of $\mathfrak{A}$. Let $\mathfrak{A}_L$ be the set of all $\nu \in \mathfrak{A}$ of the form

$$\nu = a\mu + \omega, \quad a = \text{scalar}, \quad \omega \in \mathfrak{A}_0. \tag{4}$$

From (3) it follows that the representation (4) is unique for each $\nu \in \mathfrak{A}_L$. Moreover, on $\mathfrak{A}_L$ the limit

$$L(\nu) = L(a\mu + \omega) = \lim_{t\to\infty} \frac{\nu(T_t)}{\mu(T_t)} = a \tag{5}$$

is a bounded linear functional. Moreover, $\mathfrak{A}_L$ is a closed subspace of $\mathfrak{A}$.

Lemma 1. *Under* $*$, *$\mathfrak{A}$ is a commutative Banach algebra with identity, and $\mathfrak{A}_L$ and $\mathfrak{A}_0$ are subalgebras with identity. In particular, for $\omega \in \mathfrak{A}_0$, we have $\mu * \omega \in \mathfrak{A}_L$ with*

$$L(\mu * \omega) = \omega(R). \tag{6}$$

Proof of Lemma 1. δ_0, the measure having unit mass at 0, is again the identity. For $\nu, \omega \in \mathfrak{A}$,

$$M\frac{|\nu*\omega|(T_t)}{\mu(T_t)} \leqq \frac{M}{\mu(T_t)}\int_{-\infty}^{\infty} \frac{|\nu|(T_t-x)}{\mu(T_t-x)}\mu(T_t-x)|\omega|(dx) \leqq \frac{\|\nu\|}{\mu(T_t)}\int_{-\infty}^{\infty} \mu(T_t-x)(\omega)dx$$

$$= \|\nu\| \frac{1}{M\mu(T_t)} M \int_{-\infty}^{\infty} \frac{|\omega|(T_t-x)}{\mu(T_t-x)}\mu(T_t-x)\mu(dx) \tag{7}$$

$$\leqq \|\nu\| \cdot \|\omega\| \frac{1}{M}\frac{\mu^{*2}(T_t)}{\mu(T_t)}.$$

Taking suprema in (7) gives $\|\nu*\omega\| \leqq \|\nu\|\cdot\|\omega\|$. Thus $\mathfrak{A}$ is a Banach algebra.

For $\nu, \omega \in \mathfrak{A}_0$, and for fixed $N > 0$, we have

$$M\frac{|\nu*\omega|(T_t)}{\mu(T_t)} \leqq \left\{ \int_{-N}^{N} + \int_{N}^{\infty} + \int_{-\infty}^{-N} \right\} \frac{M}{\mu(T_t)} \frac{|\nu|(T_t-x)}{\mu(T_t-x)}\mu(T_{t-x})|\omega|(dx). \tag{8}$$

We can dominate the first integral in (8) by

$$M \sup_{|x|\leqq N}\left[\frac{|\nu|(T_t-x)}{\mu(T_t-x)}\right]\frac{(|\omega|*\mu)(T_t)}{\mu(T_t)}$$

$$= M \sup_{|x|\leqq N}\left[\frac{|\nu|(T_t-x)}{\mu(T_t-x)}\right]\frac{1}{\mu(T_t)}\int_{-\infty}^{\infty}\frac{|\omega|(T_t-y)}{\mu(T_t-y)}\mu(T_t-y)\mu(dy)$$

$$\leqq \|\omega\| \sup_{|x|\leqq N}\left[\frac{|\nu|(T_t-x)}{\mu(T_t-x)}\right]\frac{\mu^{*2}(T_t)}{\mu(T_t)},$$

which becomes arbitrarily small as $t\to\infty$, since $\nu\in\mathfrak{A}_0$. We can dominate the second integral in (8) as follows—using integration by parts:

$$\|\nu\|\frac{1}{\mu(T_t)}\int_{N}^{\infty}\mu(T_t-x)|\omega|(dx)$$

$$\leqq \|\nu\|\frac{\mu(T_t-N)}{\mu(T_t)}|\omega|(T_N) + \|\nu\|\frac{1}{\mu(T_t)}\int_{N}^{\infty}|\omega|(T_x)d\mu(T_t-x) \tag{10}$$

$$\leqq \|\nu\|\frac{\mu(T_t-N)}{\mu(T_t)}|\omega|(T_N) + \|\nu\|\sup_{x\geqq N}\left[\frac{|\omega|(T_x)}{\mu(T_x)}\right]\frac{\mu^{*2}(T_t)}{\mu(T_t)}.$$

(Here we may interpret the integrals as limits (under partition refinement) of Stieltjes sums. The integrand and integrator functions are suitably left- and right-hand continuous, respectively, to justify existence and integration by parts. Cf. Hildebrandt [Hi].)

We can make the last term in (10) arbitrarily small by choosing N sufficiently large, since $\omega\in\mathfrak{A}_0$. So also with $|\omega|(T_N)$. Note that $\lim_{t\to\infty}\frac{\mu(T_{t-N})}{\mu(T_t)} = 1$, by (β'''). Hence we can make the second integral in (8) arbitrarily small by choosing first N and then t large. A similar result holds for the third integral too. Thus $\nu * \omega \in \mathfrak{A}_0$.

By (α''), $\mu^{*2} \in \mathfrak{A}_L$ with $L(\mu^{*2}) = c$. Thus, as in Lemma 1 of Theorem 1, to show that $\mathfrak{A}_L$ is an algebra, it suffices to show that $\mu * \omega \in \mathfrak{A}_L$ for all $\omega \in \mathfrak{A}_0$. Again, we shall also show that $L(\mu * \omega) = \omega(R)$. For $t \geqq N$,

$$\begin{aligned} \frac{|(\mu * \omega) - \omega(R)\mu|(T_t)}{\mu(T_t)} &\leqq \frac{1}{\mu(T_t)} \int_{-\infty}^{\infty} |\mu_x - \mu|(T_t)|\omega|(dx) \\ &\leqq \int_{-N}^{N} \left[\frac{|\mu_x - \mu|(T_t)}{\mu(T_t)}\right]|\omega|(dx) + \left\{\int_{N}^{\infty} + \int_{-\infty}^{-N}\right\} \frac{\mu(T_t - x)}{\mu(T_t)}|\omega|(dx) \\ &\qquad + |\omega|(T_N). \end{aligned} \tag{11}$$

As $t \to \infty$, the integral $\int_{-N}^{N}$ tends to zero by (β'') (dominated convergence). We can make the integrals $\int_{N}^{\infty} + \int_{-\infty}^{-N}$ arbitrarily small by choosing first N and then t large just as for (8), since $\omega \in \mathfrak{A}_0$.

Also $|\omega|(T_N) \to 0$ as $N \to \infty$. Hence $\mu * \omega - \omega(R)\mu \in \mathfrak{A}_0$ and $\mu * \omega \in \mathfrak{A}_L$ with $L(\mu * \omega) = \omega(R)$. This completes the proof of Lemma 1.

At this point, our method diverges from that in the proof of Theorems 1 and 2 (because of the possible continuous singular component in μ). We shall compare the homomorphisms $\mathscr{H}(\mathfrak{A}_0)$ and $\mathscr{H}(\mathfrak{A}_L)$ with $\mathscr{H}(\mathscr{M})$ for the standard algebra $\mathscr{M}$ of Radon measures ν on R, under $*$, and with total-variation norm, $\mathrm{Var}(\nu)$.

Lemmas 2 and 3. *The spectrum S_μ of μ in $\mathfrak{A}_L$ is contained in the set* $\{w: |w| \leqq 1\} \cup \{c-2\}$.

Proof. For $\omega \in \mathfrak{A}_0$ and any Borel subset B of R let ω_B denote the measure ω cut down to the set B. Note that with respect to the norm $\|\ \|$ of (2), we have

(12) $$\omega = \lim_{n\to\infty} \omega_{(-n,n]}.$$

Moreover, for fixed $n \geqq 1$, we can write

(13) $$\omega_{(-n,n]} = \sum_{k=-n}^{n-1} \delta_k * \omega^{(k)},$$

where δ_k is the measure with unit mass at k, and where

(14) $$\omega^{(k)} = \delta_{-k} * \omega_{(k,k+1]}$$

has support in $(0,1]$ and has

(15) $$\mathrm{Var}(\omega^{(k)}) = \mathrm{Var}(\omega_{(k,k+1]}).$$

Now pick any $h_0 \in \mathscr{H}(\mathfrak{A}_0)$. For $k \geqq 1$,

(16) $$h_0(\delta_k) = h_0(\delta_1^{*k}) = [h_0(\delta_1)]^k.$$

We must have $|h_0(\delta_1)| \leqq 1$. Otherwise, exactly as in the proof of Lemma 2 of Theorem 1 (with $r = 1$), we could construct a discrete measure $\omega^{(\rho)}$ lying in $\mathfrak{A}_0$, for which there existed a finite limit

(17) $$h_0(\omega^{(\rho)}) = \lim_{N\to\infty} \sum_{k=0}^{N} \rho^k \mu((0,k])$$

where $\rho = e^s > 1$. Such a limit would contradict (γ'') above. Similarly for $k < 0$ we can get

(18) $$|h_0(\delta_k)| = |h_0(\delta_{-1}^{-k}| = |h_0(\delta_{-1})|^{-k} \leqq 1.$$

And $h_0(\delta_0) = 1$, since δ_0 is the identity of $\mathfrak{A}_0$. Thus we can apply h_0 to the ω's in (12) and (13) and find

(19) $$h_0(\omega) = \lim_{n\to\infty} \sum_{k=-n}^{n-1} h_0(\delta_k) h_0(\omega^{(k)}),$$

and

(20) $$|h_0(\omega)| \leqq \lim_{n\to\infty} \sum_{k=-n}^{n-1} |h_0(\omega^{(k)})|.$$

Since the support of $\omega^{(k)}$ is contained in $(0,1]$, it follows from the definition (2) that

$$\|\omega^{(k)}\| \leqq M \frac{\operatorname{Var}(\omega^{(k)})}{\mu(T_1)}. \tag{21}$$

Hence

$$\begin{aligned} |h_0(\omega)| &\leqq \frac{M}{\mu(T_1)} \lim_{n\to\infty} \sum_{k=-n}^{n-1} \operatorname{Var}(\omega^{(k)}) \\ &= \frac{M}{\mu(T_1)} \lim_{n\to\infty} \sum_{k=-n}^{n-1} \operatorname{Var}(\omega_{(k,k+1]}) \\ &= \frac{M}{\mu(T_1)} \lim_{n\to\infty} \operatorname{Var}(\omega_{(-n,n]}) = \frac{M}{\mu(T_1)} \operatorname{Var}(\omega). \end{aligned} \tag{22}$$

The elements ω of $\mathfrak{A}_0$ form a subalgebra of the standard algebra $\mathscr{M}$, dense in the *variation norm.* From (22) we can conclude that the homomorphism h_0 is continuous in the variation norm. Thus h_0 can be extended to all of $\mathscr{M}$ as a unique element $\tilde{h}_0$ of $\mathscr{H}(\mathscr{M})$.

Now pick any $h \in \mathscr{H}(\mathfrak{A}_L)$. Let h_0 denote its restriction to $\mathfrak{A}_0$ and let $\tilde{h}_0$ denote the extension of h_0 in $\mathscr{H}(\mathscr{M})$. If $h_0(\omega^0) \neq \omega^0(R)$ for some $\omega^0 \in \mathfrak{A}_0$, write

$$\mu * \omega^0 = \omega^0(R)\mu + \omega^1 \text{ for some } \omega^1 \in \mathfrak{A}_0. \tag{23}$$

If $h_0(\omega) = \omega(R)$ for all $\omega \in \mathfrak{A}_0$, write

$$\mu^{*2} = c\mu + \omega^u \qquad (\omega^u \in \mathfrak{A}_0). \tag{24}$$

We can consider (23) and (24) as equations in both $\mathscr{M}$ and $\mathfrak{A}_L$; and we can apply $\tilde{h}_0$ and h to them, to find (as in Lemma 3 of Theorem 1) that the values $\tilde{h}_0(\mu)$ and $h(\mu)$ both satisfy the linear equation

$$x[h_0(\omega^0) - \omega^0(R)] = h_0(\omega^1) \tag{25}$$

or both satisfy the equation

$$x^2 = cx + 1 - c. \tag{26}$$

Thus, again as in Lemma 3 of Theorem 1, we can conduct that $h(\mu) = \tilde{h}_0(\mu)$ or else $h(\mu) = c-1$. Since $|\tilde{h}(\mu)| \leqq \mathrm{Var}(\mu) = 1$, the present lemmas are proved.

The rest of the proof of Theorem 4 may be carried out exactly as was the corresponding part of Theorem 1, with replacement of summation $(\sum_n \omega_n)$ by integration $(\int \omega(dx))$.

If we restrict ourselves to absolutely continuous measures μ, then we obtain global results such as the following direct corollary of Theorem 3 (via l'Hospital's rule):

Theorem 4′. *Let μ be a probability measure on the Borel sets of R^+ with continuous density satisfying the hypothesis of Theorem 3, and let c, ρ, d, and $\phi(w)$ be as in that theorem. Then the measure $\phi(\mu)$ of Theorem 3 has tail behavior*

$$\lim_{t\to\infty} \frac{\phi(\mu)(T_t)}{\mu(T_t)} = \phi'(d).$$

4. Non-Analytic Functions.

Non-analytic functions can be shown to act on many algebras of the types considered in sections 2 and 3. In this section we develop a more "classical" technique to demonstrate this fact.

The general question as to which functions operate on which algebras is very difficult. Here we limit ourselves to convolution algebras of complex measures

$$v = \{v_n,\ n = 0, \pm 1, \pm 2, \cdots\}$$

satisfying

$$|v_n| = O(|n|^{-\alpha}), \quad \alpha > 2$$

and

$$\hat{v}(\theta) = \sum_{n=-\infty}^{\infty} v_n e^{in\theta} \text{ real.}$$

We also consider some Banach sub-Algebras of these.

While it is not too difficult to show that some non-analytic functions "operate" on the present algebras precise results are difficult. Here we offer the following theorem.

Theorem 5. *Let*

$$\hat{v}(\theta) = \sum_{j=-\infty}^{\infty} v_j e^{ij\theta}$$

be a real valued function with period 2π, *where*

$$|v_j| = O\{1/|j|^\alpha\}, \quad \alpha > 2.$$

Let

$$\phi(x) = \sum_{j=-\infty}^{\infty} c_j e^{ijx},$$

and assume that

$$\sum_{j=-\infty}^{\infty} |c_j| \, |j|^r < \infty$$

for some $r > \alpha$.

Let

$$d_n = \int_{-\pi}^{\pi} e^{in\theta} \phi(\hat{v}(\theta)) d\theta.$$

Then

$$|d_n| = O\left(\frac{1}{|n|^\alpha}\right).$$

Remark. One could use our methods to obtain results for some other algebras. In particular we could obtain results of the type above for $1 < \alpha \leqq 2$, with a much messier, more restrictive hypothesis on r. We could also simplify our proof if r were assumed sufficiently large. It would be easy to conclude less about d_n, for example $d_n = O\left(\frac{1}{|n|^{\alpha-1}}\right)$ can be shown trivially.

We also prove that if v_j is asymptotic to $cj^{-\alpha}$ (as $j \to +\infty$), then the d_j's are asymptotic to $j^{-\alpha}$, namely

Theorem 6. *Let $\hat{v}$ and ϕ satisfy the hypothesis of Theorem 5. In addition assume that $\sum_{j=-\infty}^{\infty} v_j = 1$, and that*

$$v_j \sim cj^{-\alpha} \quad as \ j \to +\infty.$$

Then

$$d_j \sim c\phi'(\hat{v}(0))j^{-\alpha} \ as \ j \to +\infty$$

Proof of Theorem 5. We shall consider $n > 0$. A similar proof works if $n < 0$.

We begin by outlinig the method of proof. We have

$$d_n = \int_{-\pi}^{\pi} e^{-in\theta}\phi(\hat{v}(\theta))d\theta.$$

Thus

$$d_n = \sum_{l=-\infty}^{\infty} c_l \int_{-\pi}^{\pi} e^{-in\theta}e^{il\hat{v}(\theta)}d\theta. \tag{1}$$

Note that if we could change the contour of integration $[-\pi, \pi]$ to a new contour on which $\theta = \sigma - i\tau$, with τ fixed and psoitive, then $|e^{-in\theta}| = e^{-n\tau}$. Thus

$$\int_{-\pi}^{\pi} e^{-in\theta} e^{il\hat{v}(\theta)} d\theta$$

would tend to zero exponentially fast. However, this can not be done since $\hat{v}(\theta)$ is not analytic. Hence we try to replace $\hat{v}(\theta)$ by

$$\hat{v}_m(\theta) = \sum_{|j| \leqq m} v_j e^{ij\theta},$$

and compute the error we make. Also we would like to keep l small in (1), and m small so that $\exp(il\hat{v}_m(\theta))$ does not get too large when $\operatorname{Im}\theta = -\tau$. Actually most lines of the proof are used to show that terms in the sum (1) with l large (in a sense to be made precise later), and the error made by replacing $\hat{v}$ by $\hat{v}_m$, are not too large. We also remark that the actual choice of τ depends on n.

We return now to consider the expression (1) for d_n. We take $\delta > 0$ small (to be chosen more precisely later) and let $m = [n^{1-\delta}]$. Let

$$\hat{v}_m(\theta) = \sum_{j=-m}^{m} v_j e^{ij\theta}. \tag{2}$$

Then from (1) we have

$$d_n = I_n + II_n \tag{3}$$

where

$$I_n = \sum_{l=-\infty}^{\infty} c_l \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}_m(\theta)), \tag{4}$$

and

$$II_n = \sum_{l=-\infty}^{\infty} c_l \int_{-\pi}^{\pi} e^{-in\theta}(\exp(il\hat{v}(\theta)) - \exp(il\hat{v}_m(\theta))d\theta. \tag{5}$$

We consider II_n first.

$$II_n = \sum_{-\infty}^{\infty} c_l \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}(\theta)) \{1 - \exp il(\hat{v}_m(\theta) - \hat{v}(\theta))\} d\theta$$

$$= \sum_{-\infty}^{\infty} ilc_l \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}(\theta)) \{\hat{v}_m(\theta) - \hat{v}(\theta)\} d\theta$$

$$+ O\left\{ \sum_{-\infty}^{\infty} l^2 |c_l| \int_{-\pi}^{\pi} |\hat{v}_m(\theta) - \hat{v}(\theta)|^2 d\theta \right\}.$$

By Parseval's identity the second term above is $O\left(\frac{1}{|n|^{(1-\delta)(2\alpha-1)}}\right) = O\left(\frac{1}{|n|^{\alpha}}\right)$ if δ is sufficiently small. Hence we can write

$$II_n = A_n + B_n + O(n^{-\alpha}) \tag{6}$$

where

$$A_n = \sum_{-\infty}^{\infty} ilc_l \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}(\theta)) \{\hat{v}_m(\theta) - \hat{v}_{\rho n}(\theta)\} d\theta \tag{7}$$

and

$$B_n = \sum_{-\infty}^{\infty} il_{cl} \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}(\theta)) \{\hat{v}_{\rho n}(\theta) - \hat{v}(\theta)\} d\theta, \tag{8}$$

where ρ is a small positive number to be chosen later. Now

$$B_n = -i \sum_{|j|>\rho n} v_j \int_{-\pi}^{\pi} e^{-i(n-j)\theta} \sum_{l=-\infty}^{\infty} c_l l \exp(il\hat{v}(\theta))d\theta$$

$$= - \sum_{|j|>\rho n} v_j \int_{-\pi}^{\pi} \phi'(\hat{v}(\theta))e^{-i(n-j)\theta} d\theta .$$

Since ϕ' and $\hat{v}$ are continuously differentiable, the same is true for $\phi'(f)$. Hence $\phi'(\hat{v})$ has an absolutely convergent Fourier Series. See Zygmund Vol. I [Z]. Thus

$$|B_n| \leqq \max_{|j|>\rho n}|v_j| \sum_{-\infty}^{\infty} \left| \int_{-\infty}^{\pi} \phi'(\hat{v}(\theta))e^{-i(n-j)\theta}d\theta\right|$$

$$\leqq K \max_{|j|>\rho n} |v_j| .$$

Hence

$$(9) \qquad |B_n| = O\left(\frac{1}{n^{\alpha}}\right).$$

We turn now to A_n. We divide the sum for A_n in (7) into two parts $\sum_{|l|\geqq n^{1-\varepsilon}} \quad \sum_{|l|>n^{1-\varepsilon}}$. ($\varepsilon$ is a sufficiently small positive number.) In considering the second sum we use the Schwarz inequality and Parseval's relation to obtain

$$\left| \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}(\theta))\{\hat{v}_m(\theta) - \hat{v}_{\rho n}(\theta)\}d\theta \right.$$

$$\leqq K\left\{\int_{-\pi}^{\pi} \left|\hat{v}_m(\theta) - \hat{v}_{\rho n}(\theta)\right|^2 d\theta\right\}^{1/2} = \mathcal{O}(m^{-\alpha+1/2}).$$

Thus

$$\left| \sum_{|l|>n^{1-\varepsilon}} \right|$$

$$\leqq K\ m^{-\alpha+1/2} \sum_{|l|>n^{1-\varepsilon}} |l|\ |c_l|$$

$$\leqq K m^{-\alpha+1} (n^{1-\varepsilon})^{r-1} \sum_{-\infty}^{\infty} |l|^r |c_l|$$

$$= O\left(\frac{1}{n^{(\alpha-1/2)(1-\delta)} n^{(1-\varepsilon)(r-1)}}\right)$$

$= O\left(\frac{1}{n^\alpha}\right)$ if ε and δ are sufficiently small. Hence we have

$$A_n = \sum_{|l| \leqq n^{1-\varepsilon}} lc_l \int_{-\pi}^{\pi} e^{-in\theta} \exp\{il\hat{v}(\theta)\} \{\hat{v}_m(\theta) - \hat{v}_{\rho n}(\theta)\} d\theta + O\left(\frac{1}{n^\alpha}\right). \tag{10}$$

In (10) we again replace $\hat{v}(\theta)$ by $\hat{v}_m(\theta)$, and write

$$A_n = D_n + E_n \tag{11}$$

where

$$D_n = \sum_{|l| \leqq n^{1-\varepsilon}} lc_l \int_{-\pi}^{\pi} e^{-in\theta} \exp(il\hat{v}_m(\theta)) \{\hat{v}_m(\theta) - \hat{v}_{\rho n}(\theta)\} d\theta, \tag{12}$$

and

$$E_n = O(n^{-\alpha}) + O\left\{ \sum_{l \leqq n^{1-\varepsilon}} l^2 c_l \int_{-\pi}^{\pi} |\hat{v}(\theta) - \hat{v}_m(\theta)|\ |\hat{v}_m(\theta) - \hat{v}_{\rho n}(\theta)|\ d\theta \right\}. \tag{13}$$

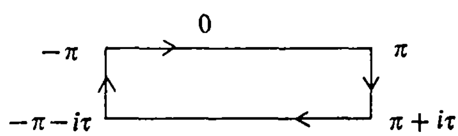
Using the Schwarz inequality and the Parseval relation, we can conclude that the integral in (13) is $O\left(\frac{1}{m^{2\alpha}}\right) = O\left(\frac{1}{n^\alpha}\right)$ if δ is sufficiently small. Thus

(14) $$|E_n| = O(n^{-\alpha}).$$

We now turn to D_n. We consider the integral in (12) as a complex integral and move the contour of integration to the line

$$\theta = \sigma - i\tau, \quad \tau = \frac{1}{n}\log n^{\beta}, \quad -\pi \leqq \sigma \leqq \pi$$

with β to be chosen later. (The contributions from portions of the contour parallel to the imaginary axis cancel, when we use Cauchy's theorem on the rectangle in the diagram.)



For sufficiently large n, $m\tau < 1$; thus we have

(15) $$|\hat{v}_m(\sigma - i\tau) - \hat{v}_m(\sigma)| = O(\tau),$$

since

$$|\hat{v}_m(\sigma - i\tau) - \hat{v}_m(\sigma)| \leqq \sum_{|j| \leqq m} |v_j|(1 - e^{j\tau})$$
$$= O(\tau).$$

Hence from (15) and the fact that $\hat{v}_m(\sigma)$ is real, we have

(16) $$|\operatorname{Im}\hat{v}_m(\sigma - i\tau)| = O(\tau).$$

Now since $|l| \leqq n^{1-\varepsilon}$, (16) gives

(17) $$|\exp il\,\hat{v}_m(\sigma - i\tau)| = O(1).$$

One also sees easily that

(18) $$|\hat{v}_m(\sigma - i\tau) - \hat{v}_{\rho n}(\sigma - i\tau)| = O(e^{\rho n \tau})$$
$$= O(n^{\rho\beta})$$

and

$$(19) \qquad \left| e^{in(\sigma - i\tau)} \right| = O(n^{-\beta}).$$

Choosing β large, and using (17), (18), (19) and the indicated change of contour of integration, we see

$$(20) \qquad |D_n| = O(n^{-\alpha}).$$

Equations (20), (14), and (11) thus give

$$(21) \qquad |A_n| = O(n^{-\alpha}).$$

Then (21), (19), and (6) give

$$(22) \qquad |II_n| = O(n^{-\alpha}).$$

Hence via (3), we can finish the proof by proving

$$(23) \qquad |I_n| = O(n^{-\alpha}).$$

In the expression for I_n we divide the sum into two parts, $\sum_{|l| < n^{1-\varepsilon}} + \sum_{l \geqq n^{1-\varepsilon}}$. To estimate the second sum we use the trivial estimate

$$\left| \sum_{|l| \geqq n^{1-\varepsilon}} \right| \leqq \sum_{|l| \geqq n^{1-\varepsilon}} |c_l|$$

$$\leqq (n^{1-\varepsilon})^{-r} \sum_{-\infty}^{\infty} |l|^r |c_l|$$

$$= O\left(\frac{1}{n^{\alpha}}\right)$$

if ε is sufficiently small. The first sum, $\sum_{|l| < n^{1-\varepsilon}}$, can be shown to be $O(1/n^{\alpha})$ exactly in the manner for D_n above. Thus (23) is proved. This completes the proof of Theorem 5.

The proof of Theorem 6 rests upon two lemmas. We begin by making a few comments preparatory to the first of these lemmas.

We let Λ_α be the Banach Algebra of doubly infinite sequences $v = \{v_n\}$ with $v_n = \bar{v}_{-n}$ (so that $\hat{v}(\theta) = \sum_{-\infty}^{\infty} v_n e^{in\theta}$ is real) and $|v_n| = O\left(\frac{1}{|n|^\alpha}\right)$, with norm $\| \ \|$ defined by

$$\|v\| = \sup\{|v_0|, |n|^\alpha |v_n|, \ n = \pm 1, \pm 2, \cdots\}.$$

(The fact that Λ_α is a Banach Algebra is essentially contained in section 2.) We also consider the Banach space of sequences $c = \{c_0, c_{\pm 1}, c_{\pm 2}, \cdots\}$

$$V_r = \left\{c \ \Big| \ \sum_{-\infty}^{\infty} |c_n| \, |n|^r < \infty\right\}$$

with norm $||| \ \ |||$, where

$$|||c||| = \sum_{n=-\infty}^{\infty} |c_n|(1 + |n|^r).$$

Theorem 5 asserts that if $r > \alpha$ V_r operates on Λ_α in the following way: Let c be in in V_r and $v \in \Lambda_\alpha$. Then $c(v) = \left\{\int_{-\pi}^{\pi} e^{-in\theta} \hat{c}(\hat{v}(\theta)) d\theta, \ n = 0, \pm 1, \pm 2, \cdots\right\}$ is in Λ_α, where

$$\hat{c}(x) = \phi(x) = \sum c_n e^{inx}$$

in the terminology of Theorem 5. We are now ready for the first of our lemmas.

Lemma 1. *Let $r > \alpha$. For each fixed $v \in \Lambda_\alpha$, the transformation*

$$T: V_r \to \Lambda_\alpha$$

given by

$$T(c) = c(v) = \{d_n\}$$

is continuous from V_r to Λ_α.

Proof of Lemma 1. The proof of Lemma 1 is almost the same as the proof of Theorem 5, since in that proof the estimates for all terms in the decomposition of d (in (3), (6), and (11)), except for B_n (see (8)) were in fact less than or equal to $K|n|^{-\alpha}\|v\|\,|||c|||$ for a suitable constant K. Hence it suffices to obtain an estimate of the same type for B_n, in particular it suffices to show

$$|B_n| \leqq K(v)|n|^{-\alpha}|||c|||. \tag{24}$$

Proceeding as in the proof of Theorem 5, we can conclude that

$$|B_n| \leqq \max_{j>\rho n}|v_j| \cdot \sum_{j=-\infty}^{\infty}\left|\int_{-\pi}^{\pi}\phi'(\hat{v}(\theta))e^{-i(n-j)\theta}d\theta\right|,$$

or

$$|B_n| \leqq Kn^{-\alpha}\sum_{j=-\infty}^{\infty}|e_j| \tag{25}$$

where

$$e_j = \int_{-\pi}^{\pi}\phi'(\hat{v}(\theta))e^{-ij\theta}d\theta.$$

In the proof of Theorem 5, we said $\Sigma|e_j| < \infty$ since $\phi'(\hat{v})$ is continuously differentiable. The theorem we quote says that if

$$\psi(\theta) = \Sigma f_n e^{in\theta}$$

and

$$|\psi(\theta + h) - \psi(\theta)| \leqq Mh^{1/2+\theta}$$

then

$$\Sigma|f_n| < \infty.$$

The proof given in Zygmund [Z] p. 240 actually shows that

$$\sum_{n=-\infty}^{\infty} |f_n| \leqq K(M + |f_0|).$$

Thus

$$\sum_{j=-\infty}^{\infty} |e_j| \leqq \max_{\theta} |\phi'(\hat{v}(\theta)) + \max_{\theta} \left|\frac{d}{d\theta}(\phi'(\theta))\right|$$

$$\leqq \sum_{l=-\infty}^{\infty} |l| \; |c_l| + \max_{\theta} |\phi''(\hat{v}(\theta))\hat{v}'(\theta)| \leqq K \sum_{l=-\infty}^{\infty} |l|^2 |c_l|$$

$$\leqq K |||c|||.$$

Thus 25 gives 24, and the lemma is proved. We now come to the second lemma:

Lemma 2. *Theorem 6 is true for ϕ's which are trigonometric polynomials.*

In order to prove Lemma 2 we first note that if v satisfies the hypothesis of Theorem 6, then $v^{*k}(n) \sim kv(n)$. (Proved as for Theorem 1.)

Proof of Lemma 2.

We let $A_\alpha = \left\{v \in \Lambda_\alpha \,\middle|\, \lim_{n \to +\infty} \frac{v_n}{n^\alpha} \text{ exists}\right\}$. We define L on A_α, by

$$L(v) = \lim_{n \to +\infty} v_n n^{-\alpha}.$$

Clearly L is continuous on A_α in the topology of Λ_α. To prove Lemma 2, it suffices to consider

$$\phi(x) = e^{inx}$$

for a fixed n. Let $v \in A_\alpha$, with $\sum_{n=-\infty}^{\infty} v_n = 1$.

$$e^{in\hat{v}(\theta)} = \sum_{j=0}^{\infty} \frac{(in)^j}{j!}[\hat{v}(\theta)]^j$$

$$= \sum_{k=-\infty}^{\infty} e^{ik\theta} \sum_{j=0}^{\infty} \frac{(v^{*j})_k (in)^j}{j!}.$$

Hence

$$\phi(v) = \left\{ \sum_{k=0}^{\infty} \frac{(v^{*j})_k (in)^j}{j!} \right\}_{k=0,\pm 1,\pm 2,\cdots}.$$

Since $\| v^{*j} \| \leqq c^j$ for some constant c, it is clear that

$$\sum_{j=0}^{R} \frac{v^{*j}(in)^j}{j!}$$

converge in the topology of Λ_α to $\phi(v)$. Hence $\phi(v)$ is in A_α and

$$\begin{aligned} L(\phi(v)) &= \sum_{j=0}^{\infty} \frac{(in)^j}{j!} L(v^{*j}) \\ &= \sum_{j=0}^{\infty} \frac{(in)^j j L(v)}{j!} \\ &= (in)L(v) \sum_{j=0}^{\infty} \frac{(in)^j}{j!} \\ &= (in)L(v) e^{in\hat{v}(0)} \\ &= L(v)\,\phi'(\hat{v}(0)) \end{aligned}$$

and the lemma is proved.

The proof of Theorem 6 now follows by Lemma 1, Lemma 2, the continuity of L and the fact that the trigonometric polynomials are dense in V_r in the $|||\ |||$ norm.

5. Application to the Renewal Equation and the Mean of a Branching Process.

As an application of the previous results we will look at asymptotic properties of solutions of equations of the form

$$H(t) = \xi(t) + \gamma \int_0^t H(t-y)dM(y), \tag{1}$$

where $M(\cdot)$ is a distribution on $[0,\infty)$, $0<\gamma$, and $\xi(t)$ is a given function. When $\gamma = 1$, then (1) is the renewal equation. The renewal measure is defined by

$$U(t) = \sum_{n=0}^{\infty} M_n(t), \tag{2}$$

where $M_n(\cdot)$ is the n-fold convolution of $M(\cdot)$. The solution of (1) with $\gamma = 1$ can be expressed in the form

$$H(t) = \int_0^t \xi(t-y)dU(y), \tag{3}$$

and if ξ is directly Riemann integrable (see chapter XI of W. Feller, Vol. II [F]) then

$$\lim_{t\to\infty} H(t) = \frac{1}{\mu}\int_0^\infty \xi(t)dt, \tag{4}$$

where $\mu = \int_0^\infty tdM(t)$.

If $\gamma \neq 1$, then there may exist a number $\alpha(\gamma, M)$, called the *Malthusian parameter*, such that

$$\gamma \int_0^\infty e^{-\alpha t}dM(t) = 1. \tag{5}$$

Such an α can always be found if $\gamma > 1$, and may or may not exist if $\gamma < 1$. When it does, then one can multiply (1) through by if $e^{-\alpha t}$, and if $e^{-\alpha t}\ \xi(t)$ is directly R-integrable then a little manipulation yields

$$H(t) \sim e^{\alpha t}\ \frac{\int_0^\infty e^{-\alpha t}\xi(t)dt}{\gamma\int_0^\infty te^{-\alpha t}dM(t)}. \tag{6}$$

Note that $\alpha > 0$ if $\gamma > 1$ and $\alpha < 0$ if $\gamma < 1$.

Remark. At first sight one might guess that the Malthusian parameter exists whenever $1 - M(t)$ is of exponential order. To see that this is not so suppose M has density m satisfying (α'), (β') and (γ') in Theorem 3. Then the Malthusian parameter for (γ, M) will exist if and only if $\hat{m}(\rho) \geqq \gamma^{-1}$. Thus a density function of the form $ce^{-\lambda x}x^{-k}$ for some $k > 0$ will only have such a parameter for γ sufficiently close to 1. Observe also that under the hypothesis of Theorem 3 on the analyticity of ϕ, the Malthusian parameter can never exist. In fact given any density $m(t)$ satisfying α' and β', we can assert that (γ, m) fails to have Malthusian parameter if and only if $(1 - \gamma w)^{-1}$ is analytic on the range of $\hat{m}$, in which case the conclusion of the theorem holds for $c_j = \gamma^j$.

When $\gamma < 1$ and the distribution M satisfies the hypotheses (α'') and (β'') of Theorem 3 then the Malthusian parameter will not exist, but the behavior of H can still be given. We illustrate this when $\xi(t) \sim 1 - M(t)$, since this is case of interest in the application which follows, but similar results can clearly be derived depending on the asymptotic behavior of ξ.

Example 1. If M satisfies (α'') and (β'') and $\xi(t)$ is measurable and $\sim 1 - M(t)$ then

$$H(t) \sim \frac{1 - M(t)}{1 - \gamma} \text{ as } t \to \infty. \tag{7}$$

Proof. It is easy to verify that

$$H(t) = \int_0^t \xi(t - y) dU_\gamma(y), \tag{8}$$

where

$$U_\gamma(t) = \sum_{n=0}^{\infty} \gamma^n M_n(t).$$

A truncation argument shows that

$$(9) \qquad H(t) \sim \int_0^t [1 - M(t-y)]\, dU_\gamma(y)$$

$$= \frac{1-\gamma}{\gamma}\left[\frac{1}{1-\gamma} - U_\gamma(t)\right].$$

But by Theorem 4

$$\frac{1}{1-\gamma} - U_\gamma(t) \sim \frac{\gamma}{(1-\gamma)^2}[1 - M(t)],$$

and this with (9) implies (7).

Example 2. If $M(\cdot)$ satisfies the hypothesis of Theorem 4′ then a calculation analogous to that above shows that

$$(10) \qquad H(t) \sim \frac{1-\gamma}{(1-d\gamma)^2}[1 - M(t)]$$

where $0 < d < 1$ is the constant specified in Theorem 3 in terms of M, and where the hypothesis forces $0 < d\gamma < 1$. Thus the behavior $H(t) \sim$ const. $[1 - M(t)]$ is not limited to the case of measures whose tails are of exponential order.

The mean of a Branching Process.

Consider an age dependent branching process with mean particle production (per particle) equal to γ, and lifetime distribution M. (See chapter VI of Harris [Ha] or chapter IV of Athreya, Ney [A-N] for definitions.) The mean number of particles $H(t)$ at time t satisfies (1) with $\xi(t) = 1 - M(t)$. Its asymptotic behavior is thus given by (6), (7) or (10), depending on the behavior of M.

6. Appendix.

The methods of Section 4 can also be used to prove results of the type in Section 1; and we state some below. Our proofs are lengthy, and are contained in a University of Wisconsin technical report.

L. Carleson has informed us of a very simple approach, which has been exploited by Essén [E]. However, our method enables one to compute bounds explicitly, while the Carleson-Essén technique does not.

Suppose that $\nu\{\cdot\}$ is an absolutely continuous complex measure on R with density $m(\cdot)$, let $\hat{m}$ be the Fourier transform of m and consider a function $M(t)$ satisfying

(1.1) $M(t)$ is positive and monotone decreasing as $t \to +\infty$.

(1.2) $M(t) = M(-t)$.

(1.3) $\displaystyle\int_0^\infty M(t)dt < \infty$.

(1.4) $\omega(t) \equiv -\frac{1}{t}\log[tM(t)\}$ decreases to zero as $t \to +\infty$.

(1.5) $|\omega'(t)| \geqq \alpha\omega(t)/t$ for some $\alpha > 0$.

One can easily verify that e.g. $(1 + t^2)^{-\alpha/2}$, $\alpha > 1$ and e^{-t^α}, $0 < \alpha < 1$ satisfy these conditions.

Theorem A-1. *Assume that ϕ is analytic on an open set containing the closure of the range of $\hat{m}(\cdot)$ and $\phi(0) = 0$. If*

$$|m(t)| \leqq KM(t), \quad -\infty < t < \infty$$

then there exists a function $\bar{m}$ such that

$$|\bar{m}(t)| \leqq K_1M(t), \quad -\infty < t < \infty$$

and

$$\phi(\hat{m}(t)) = \widehat{(\bar{m}(t))}.$$

For applications to probability we observe the following

Remark. Let $\phi(s) = \sum_{n=1}^{\infty} c_n s^n$ define a function regular in the disc $|s| < r$ and vanishing at the origin. Then if $|\hat{m}(t)| < r$ we may take the $\bar{m}(t)$ of Theorem A–1 to be

$$\bar{m}(t) = \sum_{n=1}^{\infty} c_n m^{*n}(t)$$

where the infinite series converges in L_1.

The above conclusions do not hold for a general ϕ.

There is also an analogue of Theorem A–1 for Fourier series (i.e., when v is a discrete complex valued measure $\{v_n, \ -\infty < n < \infty\}$).

Theorem A-2. *Assume that $\phi(s)$ is regular on an open set containing the range of $\hat{v}(s) = \sum_{j=-\infty}^{\infty} e^{ijs} v_j$. Then if*

$$|v_j| \leqq KM(j), \quad -\infty < j < \infty,$$

there exists a sequence $\{\bar{v}_j\}$ such that

$$|\bar{v}_j| \leqq K_1 M(j).$$

and

$$\phi(\hat{v}(s)) = \sum_{j=-\infty}^{\infty} \bar{v}_j e^{ijs}.$$

Theorem A–2 can be used to characterize the maximal ideals in certain Banach algebras.

Then if $|v_n| \leqq KM(n)$, and $\phi(s) = s^2$, we have

$$\{v_n\}^{*2} \leqq K_1 M(n).$$

Hence there is a constant K_3 such that all sequences $v = \{v_n\}$ with $|v_n| \leqq KM(n)$ form a Banach algebra B_M with

$$\| v \| = K_3 \sup_n |v_n / M(n)|.$$

Corollary. *Each homomorphism, h, of the Banach algebra B_M is of the form*

$$h[\{v_n\}] = \sum_{n=-\infty}^{\infty} v_n e^{in\theta}, \quad -\pi \leqq \theta \leqq \pi.$$

Idea of Proof of Theorem A-1. We must show

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} e^{-ixy} e^{-\varepsilon|x|} \phi(\hat{m}(x))dx = O(M(y)).$$

Note that if

$$\operatorname{Im} x = -v = +\frac{1}{y}\log yM(y),$$

then

$$\left| e^{-ixy} \right| = yM(y).$$

Hence we would be almost finished if we could consider the above integral as an integral in the complex plane, and move the contour of integration down to $\operatorname{Im} x = -v$. There are two obstacles to this:

1) $e^{-\varepsilon|x|}$ is not analytic;
2) $\hat{m}(x)$ is not analytic.

The first obstacle can be dealt with by splitting the integral into $\int_{-\infty}^{0} + \int_{0}^{\infty}$.

The fact that $\hat{m}(x)$ is not analytic leads to greater difficulty. By a somewhat subtle argument we replace $\hat{m}(x)$ by $\hat{m}_y(x)$ where

$$m_y(x) = \begin{cases} m(x) & \text{if } |x| \leqq y \\ 0 & \text{if } |x > y. \end{cases}$$

We still cannot push the integral down to $\operatorname{Im} x = -v$, however it actually suffices to put

$$\operatorname{Im} x = -\tau = \frac{1}{y}\log \frac{yM(y)}{\delta}$$

for an arbitrarily small δ; and we can in fact justify this change o fcontour.

Of course we still need to gain a factor of $1/y$, and we do this by writing

$$e^{-ixy} = \frac{i}{y} de^{-ixy}$$

and integrating by parts. This in turn leads to certain difficulties arising from $\frac{d}{dx}\hat{m}(x-i\tau)$, which, though not quite trivial, can be dealt with.

Theorem A-3. *Let $M(t)$ be as in Theorem A–1 and be convex, and such that $\frac{M'(t)}{M(t)\omega(t)}$ is bounded for $t \geqq t_0 \geqq 0$. Assume ϕ is analytic on an open set containing the range of $\hat{m}$ and $\phi(0) = 0$. If $|\hat{m}(x)| \leqq kM(x)$ and $\hat{m}(x) \cong M(x)$ as $x \to +\infty$, then there is a function $\tilde{m}(x)$ such that*

$$\hat{\tilde{m}} = \phi(\hat{m}),$$

and

$$\tilde{m}(x) \sim \phi'(\hat{m}(0))M(x) \text{ as } x \to +\infty.$$

Remark. If $\phi(s)$ is defined by $\phi(s) = \sum_{n=1}^{\infty} c_n s^n$ for $|s| < \rho$ and if the range of $\hat{m}$ is contained in the disc $|x| < \rho$, then the function $\tilde{m}$ of Theorem 5 is given by

$$\tilde{m} = \sum_{n=1} c_n m^{*n}. \tag{1}$$

where the sum converges in L_1.

As before, there is a discrete analogue to the above result for Fourier series when $m(t)$ is replaced by the discrete measure $\{v_n, -\infty < n < \infty\}$. Of course here $\phi(0)$ need not be 0.

The proof of Theorem A–3 is similar to the proof of Theorem A–1, though the details become more complicated.

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