SOME INTRINSIC CHARACTERIZATIONS OF MINIMAL SURFACES*

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Introduction.

Let $ds^2 = Edx^2 + 2Fdxdy + Gdy^2$ be a C^3 Riemannian metric defined in the plane. This metric will be said to satisfy the *Ricci condition* if its Gauss curvature K satisfies $K < 0$, and if the new metric

$$d\hat{s}^2 = \sqrt{-K}\, ds^2$$

is flat, i.e., its Gauss curvature $\hat{K}$ satisfies

$$\hat{K} \equiv 0.$$

It was first discovered by Ricci that every metric satisfying this condition can be realized on a minimal surface in $\mathbb{R}^3$ [1, p. 124]. Conversely, it is simple to verify that every metric on a minimal surface in $\mathbb{R}^3$ satisfies this condition away from the points where $K = 0$. M. Pinl showed, however, that in $\mathbb{R}^4$ there are minimal surfaces on which the Ricci condition fails to hold [6]. The central purpose of this paper is to classify those minimal surfaces in Euclidean n-space whose metrics do satisfy the condition, i.e., to classify those surfaces which are locally isometric to minimal surfaces in $\mathbb{R}^3$. It will be shown that all such minimal surfaces either already lie in some 3-dimensional affine subspace or they lie fully in some 6-dimensional affine subspace. (Hence, in $\mathbb{R}^5$ the Ricci condition intrinsically characterizes those minimal surfaces which lie in $\mathbb{R}^3$.) Moreover, it will be shown that the surface must be a member of a unique 2-parameter family of isometric minimal surfaces.

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The index of the surface (which can be viewed as the degree of twisting of the surface away from holomorphy) in this family is computable explicitly in terms of a differential form induced from the second order osculating planes of the surface.

The paper then considers possible generalizations of the theory. Certain properties of minimal surfaces in $\mathbb{R}^N$, having a generalized Ricci condition ($d\hat{s}^2 = (-K)^{n/(n+2)}ds^2$ is flat), are found. Finally, the analogous problem in the n-dimensional Euclidean sphere is formulated and its solution conjectured.

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1. Preliminaries.

To begin, we develop some fundamental notions. A minimal surface M in $\boldsymbol{R}^n$ shall be regarded (cf. [4]) as a conformal immersion

$$X: \mathscr{R} \to \mathbb{R}^n$$

where $\mathscr{R}$ is a Riemann surface and the components of $X = (X_1, \cdots, X_n)$ are harmonic on $\mathscr{R}$. In any local coordinate $z = x_1 + ix_2$ on $\mathscr{R}$, thc induced metric has the form $ds^2 = 2F|dz|^2$, and the coordinate functions satisfy the equation

$$\partial\bar{\partial}X_k = 0; \; k = 1, \cdots, n$$

where $\partial = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x_1} - i\dfrac{\partial}{\partial x_2}\right)$.

Hence the differentials

$$\omega_k = \phi_k dz$$

$$\phi_k(z) = \frac{\partial}{\partial z}X_k; \; k = 1, \cdots, n$$

are holomorphic on $\mathscr{R}$. Moreover, since

$$\left|\frac{\partial X}{\partial x_1}\right|^2 = \left|\frac{\partial X}{\partial x_2}\right|^2 = 2F$$

and $\left\langle \frac{\partial X}{\partial x_1}, \frac{\partial X}{\partial x_2} \right\rangle = 0$, we have that

$$\phi^2 = \Sigma\, \phi_k^2 = 0 \tag{1}$$

and

$$|\phi|^2 = \Sigma\, |\phi_k|^2 = F > 0. \tag{2}$$

We observe that the minimal surface can be recaptured from the differentials. In fact, if ω_k^*, $k = 1, \cdots, n$ are any holomorphic differentials on $\mathscr{R}$, with imaginary periods, such that $\Sigma(\omega_k^*)^2 = 0$, then the map $X^*: \mathscr{R} \to \mathbb{R}^n$, where each $X_k^* = \operatorname{Re}\left\{\int \omega_k^*\right\}$, is a (generalized) minimal surface.

Our main tool for studying minimal surfaces will be the generalized Gauss map as defined by Osserman and Chern [4]. This map is constructed as follows. Let $(Z_1, \cdots, Z_n)$ be a fixed system of homogeneous coordinates for $(n-1)$-dimensional complex projective space $\mathbb{P}^{n-1}$. Let $\mathbb{Q}_{n-2}$ be the hyperquadric defined by the equation $\Sigma Z_k^2 = 0$. The *generalized Gauss map* is the anti-holomorphic mapping

$$\bar{\Phi}: \mathscr{R} \to \mathbb{Q}_{n-2}$$

which in each local coordinate system is given by the functions

$$\bar{\phi}(z) = (\bar{\phi}_1(z), \cdots, \bar{\phi}_n(z)).$$

From the above discussion, it follows easily that this map is well defined.

The quadric $\mathbb{Q}_{n-2}$ may be identified with the Grassmannian

$$SO(n, \mathbb{R}) / SO(n-2, \mathbb{R}) \times SO(2, \mathbb{R})$$

of oriented planes through the origin in $\mathbb{R}^n$ as follows. Let $Z = (Z_1, \cdots, Z_n)$ represent a point $p \in \mathbb{Q}_{n-2}$. To p we associate the oriented plane in $\mathbb{R}^n$

determined by the oriented orthogonal pair of vectors $(\mathrm{Re}\{Z\}, \mathrm{Im}\{Z\})$. This correspondence is well defined and forms a diffeomorphism between the two spaces.

Introduce on $\mathbb{P}^{n-1}$ the Fubini-Study metric defined in the given homogeneous coordinates by

$$ds^2 = \frac{2|Z \wedge dZ|^2}{|Z|^4}.$$

In this metric, $\mathbb{P}^{n-1}$ has constant holomorphic curvature 2, and $\mathbb{Q}_{n-2}$ is an Einstein hypersurface. Moreover, the holomorphic isometries of $\mathbb{P}^{n-1}$ correspond to unitary transformations of the homogeneous coordinates.

The metric induced on $\mathscr{R}$ by $\overline{\Phi}$ is given by

$$d\sigma^2 = \frac{2|\phi(z) \wedge \phi'(z)|^2}{|\phi(z)|^4} |dz|^2. \tag{3}$$

A straightforward computation shows that the Gauss curvature of the minimal surface is given by

$$K = -\frac{|\phi \wedge \phi'|^2}{|\phi|^6} = -\frac{d\sigma^2}{ds^2}. \tag{4}$$

Observe that if a minimal surface in $\mathbb{R}^n$ is not a plane, then the singular points of the Gauss map (points where $K = 0$) are isolated.

For purposes of convenience, we shall consider the holomorphic Gauss map Φ, corresponding to $(\phi_1, \cdots, \phi_n)$, instead of the above. Φ describes an analytic curve in $\mathbb{Q}_{n-2}$. To each such curve is associated a class of minimal surfaces having certain common differential geometric properties. It is in this context that we shall analyse the problem posed above.

Lemma 1. *The Ricci condition will hold everywhere that $K \neq 0$ on a minimal surface in $\mathbb{R}^n$ if and only if the Gaussian image of the surface in $\mathbb{Q}_{n-2}$ has curvature 1 at all points of regularity.*

Proof. We shall work away from the isolated points where $K = 0$. Suppose the curvature of the Gauss map is $\equiv 1$. For any metric of the form $ds^2 = 2\lambda |dz|^2$ the Gauss curvature K' is given by the formula

$$K' = -\frac{1}{\lambda}\partial\bar{\partial}\log\lambda.$$

Hence, from (3) we have that

$$\begin{aligned}\frac{|\phi\wedge\phi'|^2}{|\phi|^4} &= -\partial\bar{\partial}\log\frac{|\phi\wedge\phi'|^2}{|\phi|^4} \\ &= \partial\bar{\partial}\log|\phi|^2 - \partial\bar{\partial}\log\frac{|\phi\wedge\phi'|^2}{|\phi|^2}.\end{aligned}$$

However,

$$\partial\bar{\partial}\log|\phi|^2 = \frac{|\phi|^2|\phi'|^2 - |\langle\phi,\phi'\rangle|^2}{|\phi|^4} = \frac{|\phi\wedge\phi'|^2}{|\phi|^4},$$

and therefore

$$\partial\bar{\partial}\log\frac{|\phi\wedge\phi'|^2}{|\phi|^2} = 0.$$

Hence,

$$\partial\bar{\partial}\log(-F^2K) = \partial\bar{\partial}\log|\phi|^4\frac{|\phi\wedge\phi'|^2}{|\phi|^6} = 0$$

and the Ricci condition is satisfied.

Tracing the above steps in the reverse direction completes the proof.

2. The main Theorem.

The problem under consideration entails describing a class of isometric minimal surfaces, at least one of which lies in $\mathbb{R}^3$. Given a minimal surface $\mathcal{M} \subset \mathbb{R}^n$, we now show how to construct a two-parameter family of minimal surfaces in $\mathbb{R}^{2n}$ which are isometric to $\mathcal{M}$. Suppose that $\mathcal{M}$ is parametrized by $X: \mathscr{R} \to \mathbb{R}^n$ where, by passing to the universal covering surface, we assume that $\mathscr{R}$ is simply-connected. Define the *conjugate minimal surface* by $\tilde{X}: \mathscr{R} \to \mathbb{R}^n$ where each component $\tilde{X}_k$ of $\tilde{X}$ is the harmonic conjugate of X_k. We then define the minimal surfaces, $\mathscr{M}_\alpha \subset \mathbb{R}^n$, by the immersions

$$X_\alpha = X \cos\alpha + \tilde{X} \sin\alpha.$$

Note that $\phi_\alpha = \partial X \cos\alpha - i\partial X \sin\alpha = e^{-i\alpha}\phi$, and therefore $\phi_\alpha^2 = 0$ and $|\phi_\alpha|^2 = |\phi|^2$. It follows that $\mathscr{M}_\alpha$ is minimal and isometric to $\mathscr{M}$.

Let $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ be an orthogonal decomposition. For each pair of real numbers (α, β), we define the minimal surface $\mathscr{M}_{\alpha,\beta} \subset \mathbb{R}^{2n}$ by the mapping

$$X_{\alpha\,\beta} = X_\alpha \cos(\beta/2) \oplus \tilde{X}_\alpha \sin(\beta/2)$$

where $\tilde{X}_\alpha$ is the immersion conjugate to X_α and $\oplus$ is direct sum with respect to the above decomposition of $\mathbb{R}^{2n}$. Evidently, $\mathscr{M}_{\alpha,\beta}$ is again isometric to $\mathscr{M}$. Note, moreover, that when $\beta \equiv \pi/2 (\mathrm{mod}\, \pi)$, the map $X_{\alpha,\beta}$ can be viewed as the holomorphic immersion $(e^{-i\alpha}/\sqrt{2})(X + i\tilde{X}): \mathscr{R} \to \mathbb{C}^n$ isometric to X.

A set $E \subset \mathbb{R}^n$ will be said to *lie fully* in $\mathbb{R}^m \subset \mathbb{R}^n$ if $E \subset \mathbb{R}^m$ and if E lies in no proper affine subspace of $\mathbb{R}^m$. We can now state the main theorem.

Theorem 1. *Let ds^2 be the metric on a minimal surface $\mathscr{M} \subset \mathbb{R}^n$, and suppose that away from the points where the Gauss curvature K vanishes the metric $d\hat{s}^2 = \sqrt{-K}\; ds^2$ is flat. Then ds^2 can be realized on a minimal surface $\mathscr{M}^* \subset \mathbb{R}^3 \subset \mathbb{R}^n$ (parametrized over the universal covering surface of $\mathscr{M}$), and up to Euclidean motions either*:

1) *$\mathscr{M} \subset \mathbb{R}^3$ and $\mathscr{M} = \mathscr{M}_\alpha^*$ for some α, or*

2) *$\mathscr{M}$ is linearly full in some $\mathbb{R}^6$, and $\mathscr{M} = \mathscr{M}_{\alpha,\beta}^*$ for constants α and β, $\beta \neq 0$ (mod π).*

Remark. The constant β, $0 \leqq \beta < \pi$, can be computed from the formula

$$\sin\beta = -\frac{\Omega}{K ds^2} = \frac{\Omega}{d\sigma^2}$$

where Ω is a differential form induced from the second order osculating planes and given in local coordinates by

$$\Omega = 2 \frac{|\phi \wedge \bar{\phi} \wedge \phi' \wedge \bar{\phi}'|}{|\phi|^4} |dz|^2.$$

The values β, $0 \leqq \beta < \pi$, give all possible non-congruent immersions. Moreover, when $\beta = 0$ we have $\mathscr{M} \subset \mathbb{R}^3$, and when $\beta = \pi/2$, $\mathscr{M}$ is a holomorphic curve.

Proof. It follows from a theorem of E. Calabi [2] that if $\Phi: \mathscr{R} \to \mathbb{P}^{n-1}$ is a holomorphic curve having Gauss curvature $\equiv 1$ (where the Fubini-Study metric is normalized as above), then the image under Φ lies on an algebraic curve which, after a suitable holomorphic isometry of $\mathbb{P}^{n-1}$, is given by the following imbedding of $\mathbb{P}^1$ into $\mathbb{P}^{n-1}$

$$(z_0, z_1) \to (z_0^2, \sqrt{2} z_0 z_1, z_1^2, 0, \cdots, 0).$$

Hence, if we choose $p \in \mathscr{R}$ such that $\phi(p) \wedge \phi'(p) \neq 0$, then after: a) making a suitable unitary change of homogeneous coordinates, b) choosing an appropriate local coordinate, w, in a neighborhood $\mathscr{U}$ of p, and c) dividing Φ by a nonzero holomorphic function $f(w)$ defined in $\mathscr{U}$, we will obtain a map of the form $\tilde{\phi}(w) = (1, \sqrt{2}w, w^2, 0, \cdots, 0)$.

The original Gauss map was therefore of the form: $\phi(w) = f(w) U \tilde{\phi}(w)$ where U is a unitary $n \times n$ matrix. In what follows, it will be convenient to deal with the map $\Psi(w) = (1/f(w))\phi(w) = U\tilde{\phi}(w)$. Each component of Ψ is of the form $\Psi_k(w) = \alpha_{k1} + \alpha_{k2}\sqrt{2}w + \alpha_{k3}w^2$ where $\sum_k \alpha_{ki}\bar{\alpha}_{kj} = \delta_{ij}$. Moreover, since $\Psi^2 = 0$, we have that $\sum_k \alpha_{ki}\alpha_{kj} = 0$ for $i + j \neq 4$ and $\sum_k (\alpha_{k2}^2 + \alpha_{k1}\alpha_{k3}) = 0$.

From the defining equation for $\mathbb{Q}_{n-2}$, it is clear that the subgroup of holomorphic isometries of $\mathbb{P}^{n-1}$ which leaves $\mathbb{Q}_{n-2}$ invariant corresponds precisely to making real orthogonal transformations of the homogeneous coordinates. From the relation $X_k = \operatorname{Re}\left\{\int \omega_k\right\}$, it is clear that each such transformation corresponds to a rotation of $\mathscr{M}$ in $\mathbb{R}^n$.

Since we are working modulo Euclidean motions, all such changes of coordinates are admissable. It follows that we may assume Ψ to have the form

$$\begin{aligned}\Psi_1(w) &= \alpha_{11} + \alpha_{12}\sqrt{2}w + \alpha_{13}w^2\\ \Psi_2(w) &= \alpha_{21} + \alpha_{22}\sqrt{2}w + \alpha_{23}w^2\\ \Psi_3(w) &= \alpha_{32}\sqrt{2}w + \alpha_{33}w^2\\ \Psi_4(w) &= \alpha_{42}\sqrt{2}w + \alpha_{43}w^2\\ \Psi_5(w) &= \alpha_{53}w^2\\ \Psi_6(w) &= \alpha_{63}w^2,\end{aligned}$$

and $\Psi_7 = \cdots = \Psi_n = 0$ where $\mathrm{Im}(\alpha_{11}) = \mathrm{Im}(\alpha_{32}) = \mathrm{Im}(\alpha_{53}) = 0$.

A straightforward calculation now shows that:

$$\begin{aligned}\Psi(w) = (1 + \cos(\beta)w^2, i(1 - \cos(\beta)w^2), 2\sin(\beta/2)w,\\ 2i\cos(\beta/2)w, \sin\beta w^2, i\sin\beta w^2, 0, \cdots, 0)\end{aligned} \tag{5}$$

for some real constant β.

It follows immediately that $\mathcal{M} \subset \mathbb{R}^6$. Moreover, the surface $\mathcal{M}_{0,\pi/2}$ must lie in $\mathbb{R}^6$, and can be viewed as a holomorphic curve in $\mathbb{C}^3$. Each fixed rea. part (i.e., projection of the surface onto a 3-dimensional, real subspace of $\mathbb{C}^3$) is a minimal surface in $\mathbb{R}^3$ which is isometric to $(\sqrt{2}/2)\mathcal{M}$. It is possible to find such a surface $\hat{\mathcal{M}}$ having its Gauss map over $\mathcal{U}$ in the form: $\hat{\phi} = (f(w)(1 + w^2),\ if(w)(1 - w^2),\ 2if(w)w)$. Direct computation then shows that, up to rotations, $\mathcal{M} = \hat{\mathcal{M}}_{0,\beta}$.

We now suppose that $\mathcal{M}^*$ is the surface given by the Ricci Theorem over a domain $\mathcal{V}$, $p \in \mathcal{V} \subset \mathcal{U}$. If $\phi^* = (\phi_1^*, \phi_2^*, \phi_3^*)$ is the associated Gauss map, we have that

$$|\phi^*|^2 \equiv |\hat{\phi}|^2 \text{ on } \mathcal{V}. \tag{6}$$

It follows from Calabi [2, Thm 2] or by dividing (6) by $|f|^2$, taking $\partial\bar{\partial}$ and using an induction argument, that there is a unitary 3×3 matrix, U, such that $\phi^* = U\hat{\phi}$. An easy argument now shows that modulo elements in $O(3, \mathbb{R})$,

$U = e^{-i\alpha} \times$ (identity). Hence, $\hat{\mathscr{M}}_\alpha$ is a minimal surface in $\mathbb{R}^3$ which coincides with $\mathscr{M}^*$ over $\mathscr{V}$.

The formula for β follows directly from (5) and the proof is complete.

There are several immediate consequences of Theorem 1.

Corollary 1. *A minimal surface in $\mathbb{R}^5$ satisfies the Ricci condition if and only if it lies in some 3-dimensional affine subspace.*

Hence, the Ricci condition intrinsically characterizes those minimal surfaces in $\mathbb{R}^5$ which lie in 3–dimensional hyperplanes.

Corollary 2. *If $\mathscr{M}$ is a minimal surface in $\mathbb{R}^3$, then the family $\mathscr{M}_{\alpha,\beta}$ for $(\alpha,\beta) \in S^1 \times S^1$ constitutes* all *the minimal surfaces lying in Euclidean space and isometric to $\mathscr{M}$.*

In particular, the family $\mathscr{M}_{\alpha,0}$, the classical family of associate surfaces, gives all the minimal surfaces in $\mathbb{R}^3$ isometric to $\mathscr{M}$. One example of this phenomenon is the well known family of surfaces joining the catenoid and the helicoid.

We observe that any minimal surface in $\mathbb{R}^n$ satisfying the Ricci condition can be obtained locally from the generalized Weierstrass representation:

$$\phi = f(1 + \cos\beta g^2, i(1 - \cos\beta g^2), 2\sin(\beta/2)g,$$
$$2i\cos(\beta/2)g, \sin\beta g^2, i\sin\beta g^2, 0, \cdots, 0)$$

where f and g are arbitrary holomorphic functions.

There is an immediate generalization of the above theory in light of the Calabi theory of holomorphic curves of constant curvature in $\mathbb{P}^{n-1}$.

Theorem 2. *Let ds^2 be the metric on a minimal surface $\mathscr{M} \subset \mathbb{R}^N$, and suppose that away from the (isolated) points where the Gauss curvature $K = 0$, the new metric defined by $d\hat{s}^2 = [-K]^{n/(n+2)}ds^2$; $n \geqq 2$ is flat. Then $\mathscr{M}$ must lie in some $\mathbb{R}^{2n+2}$ and cannot lie in any $\mathbb{R}^n$. Moreover, if $\mathscr{M}$ is complete then the total curvature C satisfies*

$$C = -2\pi n k$$

where $k = 1, 2, \cdots, \infty$.

Proof. The condition that $d\hat{s}^2$ is flat translates into the condition that the Gauss map have constant curvature, $2/n$. By Calabi [2], the Gauss image lies on an algebraic curve, $\mathscr{C}_n$, which, up to unitary transformations, is given by

$$(z_0, z_1) \to (z_0^n, \sqrt{n} z_0^{n-1} z_1, \cdots, \sqrt{\binom{n}{k}} z_0^{n-k} z_1^k, \cdots, z_1^n, 0, \cdots, 0).$$

The first part of the theorem follows as above from the fact that we are working modulo real orthogonal transformations.

If $\mathscr{M}$ is complete and the total curvature is bounded, then the Riemann surface $\mathscr{R}$ is conformally equivalent to a compact surface $\mathscr{R}_0$, punctured at a finite number of points. Moreover, the functions ω_l/ω_k extend to meromorphic functions on $\mathscr{R}_0$ [4]. Hence, the map Φ: $\mathscr{R}_0 \to \mathscr{C}_n \subset \mathbb{Q}_{n-2}$ is a finite branched covering, and

$$\begin{aligned} C(S) &= -\text{Area of the Gaussian image surface} \\ &= -k \times \text{Area of } \mathscr{C}_n \\ &= -2\pi n k \end{aligned}$$

where k is a positive integer.

3. Conjecture for S^3.

The theorem characterizing metrics on minimal surfaces in $\mathbb{R}^3$ can be extended to an intrinsic characterization of the metrics on surfaces of constant mean curvature in a 3-dimensional space form. A particular case of this is the following *spherical Ricci condition*. If the Gauss curvature K of a C^3-Riemannian metric ds^2, defined in the plane, satisfies $K < 1/r^2$ for some constant $r > 0$, and if the metric $d\hat{s}^2 = \sqrt{\frac{1}{r^2} - K}\, ds^2$ is everywhere flat, then ds^2 can be locally realized on a continuous 1-parameter family of minimal surfaces in the Euclidean 3-sphere, $S^3(r)$, of curvature $1/r^2$ [5]. This family can be viewed as a continuous family of immersions ψ_θ: $U \to S^3(r)$, for $0 \leqq \theta \leqq \pi$, where U is some domain in the plane. Generally speaking, the immersions

ψ_θ will be mutually non-congruent. Moreover, if one chooses any numbers $\theta_1 < \theta_2 < \cdots < \theta_m$ between 0 and π and any numbers $\alpha_1, \cdots, \alpha_m$ with $\sum \alpha_k^2 = 1$, and if, further, one views each map ψ_{θ_k} as an $\mathbb{R}^4$-valued function with $|\psi_{\theta_k}|^2 \equiv r^2$, then one can construct a minimal immersion $\Psi: U \to S^{4m-1} \subset \mathbb{R}^{4m}$ by setting

$$\Psi = \alpha_1 \psi_{\theta_1} \oplus \cdots \oplus \alpha_m \psi_{\theta} \ . \tag{7}$$

(Again the symbol $\oplus$ implies orthogonal direct sum.) Evidently Ψ induces the metric ds^2 over U. Furthermore, Ψ is almost always linearly full in $\mathbb{R}^{4m}$. (Consider, for example, the metric $ds^2 = dx^2 + dy^2$.) The author conjectures that every minimal surface in $S^n(r)$ whose metric satisfies the spherical Ricci condition away from the points where the Gauss curvature $= 1/r^2$ must be of the form of (7).

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