ASYMPTOTIC VALUES AND HÖLDER CONTINUITY OF QUASICONFORMAL MAPPINGS

By

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1. Introduction

A function f defined on a set A in R^n and taking values in R^n is said to satisfy a *Hölder condition of order* $\alpha > 0$ at a point y of A if there is a constant $M > 0$ such that

$$|f(x) - f(y)| \leqq M|x - y|^\alpha \tag{1}$$

for all x in A. In the event that (1) holds with fixed M for all points x and y of A, we speak of f as *uniformly Hölder continuous with exponent* α *on* A or say that f belongs to $\mathrm{Lip}_\alpha(A)$, the *Lipschitz class with exponent* α on A. For any reasonable set A — a domain, for instance — the class $\mathrm{Lip}_\alpha(A)$ consists entirely of constant mappings when $\alpha > 1$.

Suppose that f is a quasiconformal mapping of the open unit ball $B^n = \{x \in R^n : |x| < 1\}$ into R^n. It is by now well-known that, if A is a compact subset of B^n, then $f|A$ is a member of the class $\mathrm{Lip}_\alpha(A)$ for $\alpha = K_I(f)^{1/(1-n)}$, where $K_I(f)$ designates the inner dilatation of f. Typically, however, very little can be said about the uniform Hölder continuity of f on sets which cluster at the boundary of B^n. In particular, there need not exist any $\alpha > 0$ with the property that f belongs to $\mathrm{Lip}_\alpha(B^n)$.

The goal of this paper is to describe a variety of circumstances in which it becomes possible to obtain uniform Hölder estimates for f on sets A that approach ∂B^n. The specific sets we have in mind are Stolz regions and the unit ball itself. The strategy for achieving this objective involves the study of special asymptotic values of f. Indeed, as will soon be apparent, the notion of an "asymptotic value of positive order" represents a unifying thread running through the paper.

Regarding matters of notation and terminology we conform to the usage in the book of Väisälä [10], unless otherwise stipulated.

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2. Lindelöf's theorem and Hölder continuity

Let f be a continuous mapping of B^n into R^n and let b be a boundary point of B^n. We say that f has the point b' of R^n as an *asymptotic value at b* if there is an endcut E of B^n terminating at b — meaning an arc E lying in B^n except for an endpoint at b — with the property that $f(x) \to b'$ as $x \to b$ along E. If, in addition, there exists an exponent $\alpha > 0$ for which it is true that

$$|f(x) - b'| \leqq M |x - b|^{\alpha}$$

for all x on E, where $M > 0$ is a constant, we will refer to b' as an *asymptotic value of order* α. (We stress: there is no requirement here that $\alpha \leqq 1$. Also, as a matter of convenience we are free to assume that the endcut E has its second endpoint at the origin. This can always be achieved by truncating or extending E, the latter at the possible expense of increasing the size of M.) While we have elected to deal exclusively with finite asymptotic values in this paper, there is a perfectly natural way to speak of f having ∞ as an asymptotic value of order α at b: one simply requires the mapping $I \circ f$ to have the origin as an asymptotic value of order α at b, where I denotes inversion with respect to the unit sphere. Under this definition most results in the present section admit obvious counterparts for the case of an infinite asymptotic value.

For θ in $(0, \pi/2)$ and for b in ∂B^n, the *Stolz cone* $S_\theta(b)$ with vertex at b is defined by

$$S_\theta(b) = \{x \in B^n : |x - b| < \cos\theta, (b - x) \cdot b > |b - x| \cos\theta\}.$$

It is easily demonstrated that there exists a constant c in $(0, 1)$, depending only on θ, such that

$$\text{dist}(x, \partial B^n) \geqq c |x - b| \tag{2}$$

for every x in $S_\theta(b)$.

A classical theorem of Lindelöf asserts that, if f is a bounded analytic function in the unit disk B^2 in $\mathbf{C}$ and if f has b' as an asymptotic value at a boundary point b of B^2, then $f(z) \to b'$ as $z \to b$ through any Stolz sector $S_\theta(b)$. The analogue of this result for quasiconformal mappings of B^n was established by Gehring [4]. We contribute the following observation concerning asymptotic values of positive order:

Theorem 1. *Let f be a quasiconformal mapping of B^n into R^n. Suppose that f has a point b' as an asymptotic value of order $\alpha > 0$ at a boundary point b of B^n. Then, corresponding to each θ in $(0, \pi/2)$, there exists a constant M_θ such that*

$$|f(x) - b'| \leqq M_\theta |x - b|^{\alpha}, \tag{3}$$

whenever x lies in the Stolz cone $S_\theta(b)$.

Proof. Choose an endcut E of B^n joining the origin to b and enjoying the property that

$$|f(y)-b'| \leqq M|y-b|^{\alpha} \tag{4}$$

for all y on E, where M is a positive constant. Then E intersects each sphere $\partial B^n(b,t)$ with $0<t<1$. Fix θ in $(0,\pi/2)$ and an arbitrary point x of $S=S_\theta(b)$. Write $r=|x-b|$. Denote by A the line segment in S with endpoints x and $x+c(b-x)/2$, where c is the constant from (2). We consider the path families

$$\Gamma_1=\Delta(A, E\cap B^n(b,r)\colon B^n), \qquad \Gamma_2=\Delta_0(A, \partial B^n\colon B^n).$$

An application of the cap-inequality [10, 10.2] immediately produces the modulus estimate

$$M(\Gamma_1)\geqq b_n \log\frac{2}{2-c}, \tag{5}$$

where $b_n>0$ is a constant depending only on the dimension n. Moreover, (2) implies that A is separated from ∂B^n by the ring $B^n(x,cr)\setminus \bar{B}^n(x,cr/2)$, with the consequence that

$$M(\Gamma_2)\leqq \frac{\omega_{n-1}}{(\log 2)^{n-1}}. \tag{6}$$

Here ω_{n-1} designates the surface area of ∂B^n.

Owing to (4), the set $f[E\cap B^n(b,r)]$ lies in the ball $B^n(b', Mr^\alpha)$. This permits us to infer that

$$d=\operatorname{dist}[b', f(A)]\leqq M_1 r^\alpha \tag{7}$$

with

$$M_1 = M\exp\left[\left(\frac{\omega_{n-1}K_O(f)}{b_n\log\dfrac{2}{2-c}}\right)^{1/(n-1)}\right],$$

where $K_O(f)$ signifies the outer dilatation of f. For, in view of (5), the alternative to (7) would result in

$$M[f(\Gamma_1)]\leqq \frac{\omega_{n-1}}{\left(\log\dfrac{d}{Mr^\alpha}\right)^{n-1}} < \frac{b_n\log[2/(2-c)]}{K_O(f)} \leqq \frac{M(\Gamma_1)}{K_O(f)},$$

which would contradict the quasiconformality of f.

If $|f(x)-b'|>d$, then in each of the sets $\partial B^n(b',s)\cap f(B^n)$ with $d<s<|f(x)-b'|$ we can choose an open spherical cap, say C_s, centered at a point of $f(A)$ and extending to the boundary of $f(B^n)$. Since $f(\Gamma_2)$ includes each of the path families $\Delta_0(f(A), \partial f(B^n): C_s)$, a second appeal to the cap-inequality yields the estimate

$$M[f(\Gamma_2)] \geqq b_n \log \frac{|f(x)-b'|}{d},$$

which, of course, holds trivially when $|f(x)-b'| = d$. In combination with (6), (7) and the inequality $M[f(\Gamma_2)] \leqq K_I(f)M(\Gamma_2)$, this leads to

$$|f(x)-b'| \leqq M_2 d \leqq M_2 M_1 |x-b|^\alpha,$$

with $M_2 = \exp[\omega_{n-1}K_I(f)/b_n(\log 2)^{n-1}]$. We have thus confirmed (3) for $M_\theta = M_2M_1$.

At how many points b of ∂B^n is it possible for a quasiconformal mapping f of B^n into R^n to have a given point b' as an asymptotic value? The answer: the set of such points b may be uncountable, although it is necessarily of conformal capacity zero. What happens if we append to the above question the phrase "of positive order"? Theorem 1 enables us to establish the following result, which will provide an answer to the modified question.

Theorem 2. *Let f be a quasiconformal mapping of B^n into R^n and let $b_1, \ldots, b_p$ be distinct boundary points of B^n. Suppose that f has a point b' as an asymptotic value of order $\alpha_i > 0$ at b_i. Then*

$$\alpha_1^{n-1} + \cdots + \alpha_p^{n-1} \leqq 2K_O(f). \tag{8}$$

Proof. Fix θ in $(0, \pi/2)$. Invoking Theorem 1 we can choose a constant $M>0$ such that for $i=1,\ldots,p$

$$|f(x)-b'| \leqq M|x-b_i|^{\alpha_i}, \tag{9}$$

whenever x lies in $S_i = S_\theta(b_i)$. Write

$$\alpha = \max_{1\leqq i \leqq p} \alpha_i. \tag{10}$$

Fix d, with $0<d<\cos\theta$, such that the balls $B^n(b_i, d)$ are pairwise disjoint and set

$$m = \inf\{|f(x)-b'| : x \in S_i \cap \partial B^n(b_i, d), 1 \leqq i \leqq p\}. \tag{11}$$

Obviously $m>0$.

Now consider r in $(0,d)$ with the property that $Mr^\alpha < m$. For $i=1,\ldots,p$ let

$$\Gamma_i = \Delta(\partial B^n(b_i, d), \partial B^n(b_i, r^{\alpha/\alpha_i}): S_i).$$

Then

$$M(\Gamma_i) = \frac{c(\theta,n)\omega_{n-1}}{\left(\log \frac{d}{r^{\alpha/\alpha_i}}\right)^{n-1}}, \tag{12}$$

where $c(\theta, n)$ is a constant satisfying $c(\theta, n) \to \frac{1}{2}$, as $\theta \to \pi/2$. Specifically, $c(\theta, 2) = \theta/\pi$ and

$$c(\theta, n) = \frac{\omega_{n-2}}{\omega_{n-1}} \int_0^\theta (\sin t)^{n-2} dt$$

if $n > 2$. The families $f(\Gamma_i)$ lie in disjoint open sets and, in view of (9), (10) and (11), each of these families is minorized by

$$\Gamma = \Delta(\partial B^n(b', m), \partial B^n(b', Mr^\alpha): R^n).$$

It follows that

$$\sum_{i=1}^{p} M[f(\Gamma_i)] \leqq M(\Gamma) = \frac{\omega_{n-1}}{\left(\log \dfrac{m}{Mr^\alpha}\right)^{n-1}}. \tag{13}$$

Since $M(\Gamma_i) \leqq K_O(f) M[f(\Gamma_i)]$, we use (12) and (13) to conclude

$$\sum_{i=1}^{p} \frac{c(\theta, n)}{\left(\log \dfrac{d}{r^{\alpha/\alpha_i}}\right)^{n-1}} \leqq \frac{K_O(f)}{\left(\log \dfrac{m}{Mr^\alpha}\right)^{n-1}}.$$

After multiplying by $[\alpha \log(1/r)]^{n-1}$ and allowing $r \to 0$, we obtain

$$c(\theta, n) \sum_{i=1}^{p} \alpha_i^{n-1} \leqq K_O(f).$$

Inequality (8) results when we let $\theta \to \pi/2$.

We record two immediate corollaries of Theorem 2.

Corollary 1. *Let f be a quasiconformal mapping of B^n into R^n. Then*

(i) f has a given point as an asymptotic value of positive order at no more than countably many boundary points of B^n;

(ii) f has a given point as an asymptotic value of order $\alpha > 0$ at no more than $2K_O(f)/\alpha^{n-1}$ boundary points of B^n;

(iii) f has no asymptotic values of order exceeding $[2K_O(f)]^{1/(n-1)}$.

Corollary 2. *Suppose that a quasiconformal mapping f of B^n belongs to the class $\mathrm{Lip}_\alpha(B^n)$. Then at most $2K_O(f)/\alpha^{n-1}$ boundary points of B^n can have a common image under the induced extension of f to $\bar{B}^n$.*

Theorem 2 and its corollaries extend work of Aharonov and Srebro [1]. Earlier, Miniowitz [6] had established similar results for a class of quasiregular mappings of B^n under the assumption of a growth condition more restrictive than a Hölder condition. We draw attention, as well, to the paper of Essén [2], in which Corollary

2 is proved for the special case of a conformal mapping from the disk B^2 into the complex plane.

We next use Theorem 2 to retrieve topological information concerning the image of B^n under a uniformly Hölder continuous quasiconformal mapping.

Recall that a domain D is said to be *finitely connected* at a boundary point b if b has arbitrarily small neighborhoods U such that $U \cap D$ consists of finitely many components. A domain D is said to be *m-connected* at b if m is the smallest integer for which there exist arbitrarily small neighborhoods U of b such that $U \cap D$ consists of exactly m components. If D is finitely connected at each boundary point, then D is said to be *finitely connected on the boundary.* Should there be an integer M with the property that D is m-connected at each point b of ∂D for some $m = m(b)$ not exceeding M, D is said to be *boundedly connected on the boundary.*

Now let f be a quasiconformal mapping of B^n onto a bounded domain D. Then f is uniformly continuous if and only if D is finitely connected on the boundary [7], [10]. When such is the case, D is m-connected at a point b of ∂D if and only if exactly m points of ∂B^n are mapped to b under the induced boundary extension of f; otherwise infinitely many points of ∂B^n are mapped to b [7].

In view of the above and Corollary 2 we have:

Corollary 3. *Let f be a quasiconformal mapping of B^n onto a domain D. If f belongs to* $\mathrm{Lip}_\alpha(B^n)$ *for some $\alpha > 0$, then D is boundedly connected on the boundary.*

A further implication of Theorem 1 is a result on uniform Hölder continuity in Stolz cones.

Theorem 3. *Let f be a quasiconformal mapping of B^n into R^n. Suppose that f has some point of R^n as an asymptotic value of order $\alpha > 0$ at a boundary point b of B^n. Then for each θ in $(0, \pi/2)$ the restriction of f to the Stolz cone $S = S_\theta(b)$ belongs to* $\mathrm{Lip}_\beta(S)$ *for* $\beta = \min\{\alpha, K_I(f)^{1/(1-n)}\}$.

Proof. Let b' be the asymptotic value of f referred to in the statement of the theorem. Fix θ in $(0, \pi/2)$. Theorem 1 insures the existence of a positive constant M_θ such that

$$|f(x) - b'| \leqq M_\theta |x - b|^\alpha, \tag{14}$$

whenever x lies in $S = S_\theta(b)$.

Consider an arbitrary pair of points x and y in S. We distinguish two cases. Suppose first that

$$\mathrm{dist}(x, \partial B^n) \leqq |x - y|.$$

In this event

$$|b - x| \leqq \frac{1}{c}|x - y|, \quad |b - y| \leqq \frac{2}{c}|x - y|. \tag{15}$$

Here c is the constant corresponding to θ in (2). It follows by using (14), (15) and the triangle inequality that

(16) $$|f(x)-f(y)| \leqq (1+2^\alpha)c^{-\alpha}M_\theta |x-y|^\alpha \leqq 3c^{-\alpha}M_\theta |x-y|^\alpha.$$

The remaining case has

$$\operatorname{dist}(x, \partial B^n) > |x-y|.$$

Let R be the ring domain $B^n(x,d)\backslash \bar{B}^n(x,r)$, where we have set $d = \operatorname{dist}(x, \partial B^n)$ and $r = |x-y|$. If Γ is the family of paths joining the boundary components of R through R, then

(17) $$M(\Gamma) = \frac{\omega_{n-1}}{\left(\log \frac{d}{r}\right)^{n-1}}.$$

On the other hand, the Teichmüller modulus estimate gives

(18) $$M[f(\Gamma)] \geqq \frac{\omega_{n-1}}{\left[\log \lambda_n\left(1+\frac{|b'-f(x)|}{|f(x)-f(y)|}\right)\right]^{n-1}},$$

with a constant $\lambda_n \geqq 1$ depending only on n. (See [3].) Since $|x-y|<|b-x|$, and thus $|b-y|<2|b-x|$, we can combine (2) and (14) with the triangle inequality to arrive at

(19) $$|f(x)-f(y)| \leqq (1+2^\alpha)M_\theta |b-x|^\alpha \leqq 3M_\theta (d/c)^\alpha.$$

The validity of (2), (14) and (19) allows us to manipulate the lower bound in (18) so as to secure the estimate

(20) $$M[f(\Gamma)] \geqq \frac{\omega_{n-1}}{\left[\log \frac{4\lambda_n M_\theta (d/c)^\alpha}{|f(x)-f(y)|}\right]^{n-1}}.$$

Because $M[f(\Gamma)] \leqq K_I(f)M(\Gamma)$, (17) and (20) in tandem give rise to the bound

$$|f(x)-f(y)| \leqq 4\lambda_n M_\theta c^{-\alpha} r^{\alpha'} d^{\alpha-\alpha'},$$

where we have inserted α' for $K_I(f)^{1/(1-n)}$. If $\alpha < \alpha'$, then, recalling that $|x-y| = r < d < 1$, we are led to conclude

(21) $$|f(x)-f(y)| \leqq 4\lambda_n M_\theta c^{-\alpha} |x-y|^\alpha.$$

In case $\alpha \geqq \alpha'$, we can at least salvage the inequality

(22) $$|f(x)-f(y)| \leqq 4\lambda_n M_\theta c^{-\alpha} |x-y|^{\alpha'}.$$

Together, inequalities (16), (21) and (22) show that

$$|f(x)-f(y)| \leqq M|x-y|^\beta,$$

with $\beta = \min\{\alpha, K_I(f)^{1/(1-n)}\}$ and $M = 4\lambda_n M_0 c^{-\alpha}$. Accordingly, $f \mid S$ belongs to $\mathrm{Lip}_\beta(S)$, as asserted.

3. Theorems of Hardy–Littlewood type

In this section we intend to shift our focus from the microscopic to the macroscopic. The prior discussions have been concerned primarily with the behavior of a quasiconformal mapping of B^n near a specific boundary point of B^n. We turn now to an examination of the global consequences for such a mapping of the requirement that its induced boundary function satisfies a uniform Hölder condition. Here we interpret the notion of boundary values of a function in the sense of radial limits. Recall that a quasiconformal mapping f of B^n into R^n possesses a finite radial limit at almost every point of ∂B^n. We use A_f to designate the set of points at which such a limit exists and we set

$$\dot{f}(x) = \lim_{t \to 1^-} f(tx)$$

for x in A_f. It is the function $\dot{f}$ that we have in mind when referring to the boundary function associated with f.

Let us review the situation in classical function theory. If f is a function analytic and bounded in the unit disk B^2 in the complex plane, then Fatou's theorem insures that f has finite radial limits almost everywhere on the unit circle. Assume, adopting the notation introduced above, that the boundary function $\dot{f}$ belongs to the class $\mathrm{Lip}_\alpha(A_f)$. Then a theorem of Hardy and Littlewood, in conjunction with standard results from H^p-theory, shows that f must be a member of $\mathrm{Lip}_\alpha(B^2)$. The next theorem transports this result to the setting of quasiconformal mappings in n-space.

Theorem 4. *Let f be a quasiconformal mapping of B^n into R^n. Assume that its associated boundary function $\dot{f}$ is a member of the class $\mathrm{Lip}_\alpha(A_f)$. Then f belongs to $\mathrm{Lip}_\beta(B^n)$ for $\beta = \min\{\alpha, K_I(f)^{1/(1-n)}\}$.*

Proof. As a member of the class $\mathrm{Lip}_\alpha(A_f)$, the function $\dot{f}$ is uniformly continuous on A_f. Since the set A_f is dense in ∂B^n, it follows that $\dot{f}$ has a unique extension to a function continuous on ∂B^n. This fact permits us to invoke Corollary 7.23 in [8] and to conclude that the mapping f admits an extension to a continuous mapping of $\bar{B}^n$, for which extension we retain the notation f. Because $f = \dot{f}$ on A_f, it is manifest that $f \mid \partial B^n$ belongs to $\mathrm{Lip}_\alpha(\partial B^n)$. Let M be a Lipschitz constant for $f \mid \partial B^n$ corresponding to the Hölder exponent α.

We next show that there is a constant M_0 such that

$$|f(x) - f(b)| \leqq M_0 |x - b|^\alpha, \tag{23}$$

whenever b lies on ∂B^n and x lies on the radial segment of B^n terminating at b. Fix points b and x satisfying the preceding requirements. The argument which establishes an estimate of the type (23) is virtually identical to that used in the proof of Theorem 1 to demonstrate (3). For this reason we feel justified in glossing over certain details of the proof.

Let $r = |x - b|$ and let $d = \operatorname{dist}[f(b), f(A)]$, where A denotes the line segment with endpoints x and $(x + b)/2$. We here consider the path families

$$\Gamma_1 = \Delta_0(A, \partial B^n \cap B^n(b, r): B^n), \quad \Gamma_2 = \Delta_0(A, \partial B^n: B^n).$$

Since $M(\Gamma_1) \geqq b_n \log 2$ and since $f[\partial B^n \cap B^n(b, r)]$ is situated inside $B^n(f(b), Mr^\alpha)$, we conclude as in (7) that

$$d \leqq M_1 r^\alpha, \tag{24}$$

with $M_1 = M \exp[(\omega_{n-1} K_O(f)/b_n \log 2)^{1/(n-1)}]$. Assuming that $|f(x) - f(b)| > d$, we employ the modulus estimates

$$M(\Gamma_2) \leqq \frac{\omega_{n-1}}{(\log 2)^{n-1}}$$

and

$$M[f(\Gamma_2)] \geqq b_n \log \frac{|f(x) - f(b)|}{d},$$

in conjunction with (24), to insure that

$$|f(x) - f(b)| \leqq M_2 d \leqq M_2 M_1 |x - b|^\alpha, \tag{25}$$

with $M_2 = \exp[\omega_{n-1} K_I(f)/b_n (\log 2)^{n-1}]$. Inequality (25) is trivial when $|f(x) - f(b)| = d$. Consequently, (23) obtains for $M_0 = M_2 M_1$.

Finally, fix an arbitrary pair of points x in B^n and y in ∂B^n. Let b be a point of ∂B^n closest to x. The obvious inequalities $|b - x| \leqq |y - x|$ and $|b - y| \leqq 2|y - x|$, combined with the triangle inequality, with (23) and with the estimate $|f(y) - f(b)| \leqq M|y - b|^\alpha$, allow us to conclude that

$$|f(x) - f(y)| \leqq (M_0 + 2^\alpha M)|x - y|^\alpha.$$

We have thus maneuvered into a situation covered by Theorem 1 in [9], a result which justifies the assertion that f belongs to $\operatorname{Lip}_\beta(B^n)$ for $\beta = \min\{\alpha, K_I(f)^{1/(1-n)}\}$.

As a special instance of the preceding result we record:

Corollary 4. *Let f be a continuous mapping of $\bar{B}^n$ into R^n which is quasiconformal in B^n. Assume that $f \mid \partial B^n$ is a member of the class* $\operatorname{Lip}_\alpha(\partial B^n)$. *Then f belongs to* $\operatorname{Lip}_\beta(B^n)$ *for $\beta = \min\{\alpha, K_I(f)^{1/(1-n)}\}$.*

The Hölder exponent β obtained in Theorem 4 and Corollary 4 is not, in general, subject to improvement. To see this fix β, $0 < \beta \leqq 1$, and consider the radial stretching of $\bar{B}^n$ onto itself defined by $f(x) = |x|^{\beta-1}x$ for $x \neq 0$ and $f(0) = 0$. Then f is a homeomorphism which is quasiconformal in B^n, with $K_I(f) = \beta^{1-n}$. Since f reduces to the identity mapping on ∂B^n, $f \mid \partial B^n$ belongs to $\mathrm{Lip}_\alpha(\partial B^n)$ for $\alpha = 1$. On the other hand, the optimal Hölder exponent for f at the origin is quite clearly $\beta = \min\{1, K_I(f)^{1/(1-n)}\}$.

The next theorem describes one set of circumstances in which the effect of the dilatation on a Hölder exponent is muted.

Theorem 5. *Let f be a quasiconformal mapping of B^n into R^n and let $\alpha > 0$. Assume the existence of a constant M such that*

(26) $$|f(x) - f(y)| \leqq M|x-y|^\alpha,$$

whenever x and y lie on the same sphere centered at the origin. Then f belongs to $\mathrm{Lip}_\alpha(B^n)$.

Proof. Fix a pair of points x and y in B^n. We may assume that $|x|$ and $|y|$ differ — say $|x| < |y|$ — for otherwise (26) is at our disposal. We consider two cases.

If x and y lie on the same radius of B^n, then the argument used in the proof of Theorem 4 to establish (23), but applied to the mapping $f \mid B^n(|y|)$, shows that

(27) $$|f(x) - f(y)| \leqq M_0|x-y|^\alpha,$$

where $M_0 = M \exp[(\omega_{n-1}K_O(f)/b_n \log 2)^{1/(n-1)} + (\omega_{n-1}K_I(f)/b_n [\log 2]^{n-1})]$. In the other case, meaning x and y are not co-radial, let y_0 be the point of B^n which is located on the radius of B^n passing through x and which satisfies $|y_0| = |y|$. Since $|x - y_0| \leqq |x-y|$ and $|y_0 - y| \leqq 2|x-y|$, we employ (26), (27) and the triangle inequality to infer

$$|f(x) - f(y)| \leqq M_0|x-y_0|^\alpha + M|y_0 - y|^\alpha \leqq (M_0 + 2^\alpha M)|x-y|^\alpha.$$

All possible configurations of x and y have now been accounted for and f is seen to be a member of $\mathrm{Lip}_\alpha(B^n)$.

Simple examples illustrate that the quasiconformality of f is an essential hypothesis in Theorem 5. The next result might be looked at as a "uniform version" of Theorem 3.

Theorem 6. *Let f be a quasiconformal mapping of B^n into R^n and let $\alpha > 0$. Assume the existence of a constant M such that the boundary function $\dot{f}$ associated with f satisfies*

(28) $$|f(x) - \dot{f}(b)| \leqq M|x-b|^\alpha,$$

whenever b lies in A_f and x is a point on the radius of B^n terminating at b. Then f belongs to $\mathrm{Lip}_\beta(B^n)$ *for* $\beta = \min\{\alpha, K_I(f)^{1/(1-n)}\}$.

Proof. We show that $\dot{f}$ is a member of the class $\mathrm{Lip}_\alpha(A_f)$ and, in this way, derive Theorem 6 as a consequence of Theorem 4. For this, fix a pair of points b_1 and b_2 in A_f and choose points x_1 and x_2 on the radii of B^n terminating at b_1 and b_2, respectively, such that

$$|x_1 - b_1| = |x_2 - b_2| = 2|x_1 - x_2|.$$

For the path family $\Gamma = \Delta_0(A, \partial B^n : B^n)$, where A is the line segment joining x_1 and x_2, we have the estimate

$$M(\Gamma) \leqq \frac{\omega_{n-1}}{\left(\log \dfrac{|x_1 - b_1|}{|x_1 - x_2|}\right)^{n-1}} = \frac{\omega_{n-1}}{(\log 2)^{n-1}}.$$

On the other hand,

$$M[f(\Gamma)] \geqq b_n \log \frac{|f(x_1) - f(x_2)|}{|f(x_1) - \dot{f}(b_1)|}.$$

This is a consequence of the cap-inequality when $|f(x_1) - \dot{f}(b_1)| < |f(x_1) - f(x_2)|$ and holds trivially otherwise. It follows from the above modulus estimates and from the quasiconformality of f that

$$|f(x_1) - f(x_2)| \leqq M_0 |f(x_1) - \dot{f}(b_1)|, \tag{29}$$

where $M_0 = \exp[\omega_{n-1} K_I(f)/b_n (\log 2)^{n-1}]$. Since $|x_1 - b_1| \leqq 2|b_1 - b_2|$, we can utilize (28) and (29) to conclude that

$$\begin{aligned} |\dot{f}(b_1) - \dot{f}(b_2)| &\leqq |\dot{f}(b_1) - f(x_1)| + |f(x_1) - f(x_2)| + |f(x_2) - \dot{f}(b_2)| \\ &\leqq (2M + MM_0)|x_1 - b_1|^\alpha \\ &\leqq 2^\alpha M(2 + M_0)|b_1 - b_2|^\alpha. \end{aligned} \tag{30}$$

Hence $\dot{f}$ belongs to $\mathrm{Lip}_\alpha(A_f)$, as desired.

The radial analogue of Theorem 5 is supplied by

Theorem 7. *Let f be a quasiconformal mapping of B^n into R^n and let $\alpha > 0$. Assume the existence of a constant M such that*

$$|f(x) - f(y)| \leqq M|x - y|^\alpha,$$

whenever x and y lie on the same radius of B^n. Then f belongs to $\mathrm{Lip}_\alpha(B^n)$.

Proof. Fix points x and y in B^n with $|x| = |y|$. We consider the restriction of f to the ball $B^n(|x|)$. The identical method used to derive inequality (30) in the proof of Theorem 6 can be applied to the mapping $f \mid B^n(|x|)$. The result is the estimate

$$|f(x) - f(y)| \leqq 2^\alpha M(2 + M_0)|x - y|^\alpha,$$

where $M_0 = \exp[\omega_{n-1}K_I(f)/b_n(\log 2)^{n-1}]$. An appeal to Theorem 5 completes the proof.

We remark that alternative proofs for Theorems 5, 6 and 7 can be based on results of Gehring and Martio to be found in [5]. Theorem 6, for instance, is a straightforward consequence of Theorems 3.5 and 2.24 in their paper.

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