PERTURBATION THEORY FOR NULLITY, DEFICIENCY AND OTHER QUANTITIES OF LINEAR OPERATORS

By

Tosio Kato

in Tokyo, Japan

§ 1. INTRODUCTION

The present paper is concerned with some spectral properties of linear operators between (complex) Banach spaces. We are mainly interested in the stability of the nullity and deficiency and other related quantities under small perturbation and its relationship to the spectral theory. Our setting of the problem is rather general: we consider the eigenvalue problem of the form

$$Ax = \lambda Bx$$

where A, B are in general unbounded linear operators from a Banach space X to another Banach space X'. A is assumed to be closed, but its domain need not be dense in X. B is mostly assumed to be "bounded relative to A".

In this introduction, which is intended for a brief account of the problems dealt with in the sequel and the main results, we shall for simplicity restrict ourselves to bounded linear operators A, B with domain X. We denote by $N[A]$ the null space of A and by $R[A]$ the range of A. The dimensions of $N[A]$ and $X'/R[A]$ are respectively called the nullity and the deficiency of A, and will be denoted by $\alpha(A)$ and $\beta(A)$ (see § 3.1). It has been known some time that $\alpha(A)$, $\beta(A)$ have a certain kind of stability (see Atkinson [3,4],(1) Dieudonné [6], Friedman [7], Gokhberg and Krein [8], Sz.-Nagy [11,12], Yood [13]). $\alpha(A)$ and $\beta(A)$ have a difference (called the index of A) which is not changed when A is subjected to a small perturbation under certain conditions. More precisely, we have

1. Numbers in brackets refer to the Bibliography at the end of the paper. Unfortunately we have not been able to see Dieudonné [6]. We have come to notice the comprehensive work of Gokhberg and Krein [8] only after the completion of the present work. This paper, hereafter quoted as GK, contains many results in common with ours, though their methods are rather different from ours. The present paper has been revised to avoid as far as possible the repetition of the material contained in their paper.

$$\alpha(A+B)-\beta(A+B)=\alpha(A)-\beta(A)$$

provided $R[A]$ is closed, at least one of $\alpha(A)$, $\beta(A)$ is finite and $\|B\|$ is sufficiently small. $\alpha(A)$ and $\beta(A)$ themselves may decrease under such a perturbation, but they never increase as long as the perturbation is small (the first stability theorem). A different kind of perturbation is given by a completely continuous operator B, for which $\|B\|$ need not be small. Here the index $\alpha(A)-\beta(A)$ is again unchanged, but $\alpha(A)$, $\beta(A)$ may change in either direction (the second stability theorem).

Here the assumption that $R[A]$ is closed is essential. The same is true with the assumption that at least one of $\alpha(A)$, $\beta(A)$ is finite, as will be seen from an example in Remark 3 to Theorem 1, § 4.3.

The main object of the present paper is to generalize the above results in several directions. In the first place we aim at a quantitative treatment[(2)] of the problem by giving as large a limit as possible for the magnitude $\|B\|$ of the perturbation for which the first stability theorem is true. For this purpose we introduce the quantity $\gamma(A)$ which may be called the lower bound of the operator A. Consider the factor space $\tilde{X}=X/N$, $N=N[A]$, which is a Banach space since N is a closed linear manifold of X. The operator $\tilde{A}$ defined by $\tilde{A}\tilde{x}=Ax$ for each $\tilde{x}\in\tilde{X}$ is a bounded linear operator on $\tilde{X}$ to X' and the inverse $\tilde{A}^{-1}$ exists. $\gamma(A)$ is now defined by $\gamma(A)=\|\tilde{A}^{-1}\|^{-1}$ if $\tilde{A}^{-1}$ is bounded and $\gamma(A)=0$ otherwise. It is well known that $\gamma(A)>0$ if and only if $R[A]$ is closed. Also we shall show that $\gamma(A^*)=\gamma(A)$, where A^* is the adjoint of A (see § 3).

The first stability theorem is now shown to be valid if $\|B\|<\gamma(A)$, see Theorem 1, § 4.3. This result is of some interest in view of the fact that in general Banach spaces there does not exist the notion of the orthogonal complement of a given subspace. The proof of this result depends essentially on an important lemma (see Lemma 211)[(3)] proved by Bohnenblust to the effect that, given any linear manifolds M, N of X with $\dim M>\dim N$ (so that $\dim N<\infty$), there exists an $x\in M$ such that

$$d(x\,,N)=\|x\|>0$$

2. Such a quantitative result is important for applications.

3. Lemma 211 is the first lemma of § 2.1. The formulas are numbered in a similar way.

($d(x, N)$ is the distance of x from N). In many problems this lemma can make up for the lack of perpendicularity in the Banach space.

The proof of Theorem 1 also makes use of the notions of approximate nullity $\alpha'(A)$ and approximate deficiency $\beta'(A)$ of the operator A. Roughly speaking, these are defined as the multiplicity of the approximate eigenvalue 0 of A resp. A^*, see § 4.1 It is shown that $\alpha'(A)=\beta'(A)=\infty$ if and only if either $\alpha(A)=\beta(A)=\infty$ or $R[A]$ is not closed.

The introduction of the approximate nullity and deficiency enables us to formulate the second stability theorem in a somewhat different form. Instead of the completely continuous operators, we consider a class of linear operators which we shall call strictly singular operators (§ 4.5). It is easily seen that the complete continuity implies the strict singularity, but the question is open whether the converse is true except for the case in which both X and X' are Hilbert spaces (§ 4.6). The second stability theorem is now seen to hold for a strictly singular perturbation B (Theorem 2, § 4.5).

Another generalization of the stability theorem we consider is concerned with the case $\alpha(A)=\beta(A)=\infty$. As was remarked above, this is not possible without some additional condition. In order to introduce such a condition, we define a positive integer $\nu(A:B)$ for any pair A, B of linear operators. This is defined in a purely algebraic fashion in terms of two sequences M_n, N_n of linear manifolds of X (§ 5.1). In the special case in which $X=X'$ and $B=I$ (the identity operator of X), these sequences reduce to $R[A^n]$ and $N[A^n]$ respectively[4] and, in particular, $\nu(A:I)=\infty$ means that $N[A]\subset R[A^n]$ (or, equivalently, $N[A^n]\subset R[A]$) for all $n=1, 2, 3, \dots$. Some lemmas on $\nu(A:B)$ are given in § 5, and the results are applied to prove Theorem 3, § 6.1, in which the stability of $\alpha(A)$, $\beta(A)$ themselves is established, including the case $\alpha(A)=\beta(A)=\infty$, under the assumption that $R[A]$ is closed and that the perturbation B has the property $\nu(A:B)=\infty$. At the same time the stability of $\nu(A:B)$ itself and the Lipschitz-continuity of the lower bound $\gamma(A)$ is proved.

A natural question arises as to what happens when $\nu(A:B)$ is finite. § 7 is devoted to this case. Here we have a reduction theorem: the spaces X, X' are decomposed into several "finite parts" and "residual

4. This part of our theory has some contact with the work of Hukuhara [10].

parts" in such a way that, in the finite parts, the operators A, B have a canonical structure similar to the Jordan form, while in the residual part we have $\nu(A:B)=\infty$ and the situation of Theorem 3 prevails (Theorem 4, § 7.1). Unfortunately we have to assume here that, among others, at least one of $\alpha(A)$, $\beta(A)$ is finite. This theorem leads to a better understanding of the situation expressed in Theorem 1.

§ 8 is devoted to some of the conclusions that can be drawn from these theorems. The classification of complex numbers into several parts according to the spectral properties of the operator $A-\lambda B$ is discussed. In the special case in which $X=X'$ and $B=I$, this gives a finer partition of the complex plane than the usual partition into the resolvent set and the point, continuous and residual spectra. In particular it follows from our classification that a linear operator A on X to itself has a spectrum of Fredholm type provided there are at most a countable number of points λ (singular points) for which $\alpha'(A-\lambda I)=\beta'(A-\lambda I)=\infty$ (Theorem 9). This leads to a simple proof of the Riesz-Schauder theorem for a strictly singular operator, which implies the same theorem for a completely continuous operator.

As we have mentioned in the begining of this introduction, most of these theorems are actually proved for unbounded operators A, B.

§ 2. LINEAR MANIFOLDS OF A BANACH SPACE

1. Lemmas on dimensions of linear manifolds.

Let X be a Banach space[5] with the norm denoted by $\| \ \|$. For any $x \in X$ and any subset S of X, we denote by $d(x, S)$ the distance of x from S. For any two linear manifolds M, N of X,[6] we introduce the number[7]

$$(211) \qquad \delta(M, N) = \sup_{\substack{x\in M \\ \|x\|=1}} d(x, N) = \sup_{0\neq x\in M} d(x, N)/\|x\| .$$

5. In this paper we consider complex Banach spaces. But our results are valid in real Banach spaces as well with slight obvious modifications.

6. Throughout the present paper, $X, X', \ldots$ denote Banach spaces and $M, N, \ldots$ linear manifolds, unless otherwise stated positively.

7. $\operatorname{Max}[\delta(M, N), \delta(N, M)]$ is called the "opening" of M, N and employed extensively in GK. But we find it convenient to consider $\delta(M, N)$ separately.

Obviously

(212) $$0 \leqq \delta(M, N) \leqq 1 .$$

We use the symbol $\dim M$ to denote the dimension of a linear manifold M; it is defined as the greatest number of linearly independent vectors of M. Thus $\dim M$ takes only the values 0, 1, 2, ... or ∞. (Thus $\dim M > \dim N$ implies $\dim N < \infty$). We are not interested in distinguishing between different infinite cardinal numbers as the value of $\dim M$ as is done in GK, although many of our results would be valid in this finer definition of $\dim M$. Thus defined, $\dim M$ is a purely algebraic notion independent of the topology of X.

The following lemma [8] is fundamental throughout the present paper.

Lemma 211. *If $\dim M > \dim N$, there exists an $x \in M$ such that $d(x, N) = \|x\| > 0$. In particular $\delta(M, N) = 1$.*

The following is a direct consequence of this lemma.

Lemma 212. *If $\delta(M, N) < 1$, then $\dim M \leqq \dim N$.*

$\delta(M, N)$ is not symmetric with respect to M and N. In this connection the following lemma is of some interest.

Lemma 213. *If $\dim M = \dim N < \infty$, then*

(213) $$\delta(N, M) \leq \delta(M, N) [1 - \delta(M, N)]^{-1}.$$

The proof can be given on the basis of a theorem of Borsuk [9] as in the case of Lemma 211. We shall not give the details, for this lemma is not needed in the following. We only note that the inequality (213) is the best possible, as is seen from the following example. Let X be the two-dimensional Banach space consisting of vectors $x = (\xi_1, \xi_2)$, where the norm is defined by $\|x\| = |\xi_1| + |\xi_2|$. Let M, N be the one-dimensional linear manifolds defined by the equations $\xi_2 = \alpha\xi_1$ and $\xi_2 = 0$ respectively, where $|\alpha| < 1$. Then it is easily seen that the equality holds in (213).

Lemma 214. *Let M, N be closed linear manifolds of X. For any $x \in X$ and $\varepsilon > 0$, there is an $x_0 \equiv x \pmod M$ such that*

(214) $$d(x_0, N) \geqq (1 - \varepsilon) [1 - \delta(N, M)] [1 + \delta(N, M)]^{-1} \|x_0\| .$$

8. This lemma is proved in GK, Theorem 1.1, so we shall not give a proof here. It may be remarked, however, that this lemma was proved independently by H. F. Bohnenblust in 1952 in a private communication to the writer. The proof is the same as in GK.

9. See Alexandroff and Hopf [2], p. 485.

Proof. If $x \in M$, we need only to take $x_0 = 0$. We assume therefore that $x \notin M$. Then there exists an $x_0 \equiv x \pmod{M}$ such that

$$d(x_0, M) \geqq (1-\varepsilon)\,\|x_0\| > 0 .$$

For this x_0 there is a $y \in N$ such that $\|x_0 - y\| \leqq d + \varepsilon\,\|x_0\|$, where $d = d(x_0, N)$. In particular $\|y\| \leqq (1+\varepsilon)\,\|x_0\| + d$. On the other hand we have $d(y, M) \leqq \delta\,\|y\|$, where $\delta = \delta(N, M)$. Hence

$$(1-\varepsilon)\,\|x_0\| \leqq d(x_0, M) \leqq \|x_0 - y\| + d(y, M)$$
$$\leqq d + \varepsilon\,\|x_0\| + \delta\,[(1+\varepsilon)\,\|x_0\| + d]$$

and

$$d(x_0, N) = d \geqq [(1-\varepsilon-\delta)(1+\delta)^{-1} - \varepsilon]\,\|x_0\| .$$

This is equivalent to (214) since $\varepsilon > 0$ is arbitrary. (Note that (214) is trivial if $\delta = 1$).

From Lemma 214 follows immediately

Lemma 215. *Let $M \subset M_1$ and let M not be dense in M_1. Then*

$$(215) \qquad \delta(M_1, N) \geqq [1 - \delta(N, M)]\,[1 + \delta(N, M)]^{-1} .$$

Remark. If N is finite-dimensional, we have a stronger result to the effect that $\delta(N, M) < 1$ implies $\delta(M_1, N) = 1$. This follows directly from Lemma 212. The same is true even for infinite-dimensional N provided X is a Hilbert space (see Lemma 222). We have not been able to obtain the same result in the general case.

Lemma 216. *Let M, N be closed linear manifolds of X, and for every $y \in N$ let there exist a $\varphi(y) \in M$ such that $\|y - \varphi(y)\| \leqq \delta\,\|y\|$, where $\delta < 1$ is independent of y. If*

$$(216) \qquad \delta(M, N) < (1-\delta)(1+\delta)^{-1},$$

then the closed linear manifold M_0 spanned by all $\varphi(y)$, $y \in N$, coincides with M.

Remark. If $\dim N < \infty$, (216) can be replaced by a weaker condition $\delta(M, N) < 1$. The same is true if X is a Hilbert space. See the remark after Lemma 215.

Proof. For every $y \in N$ we have $d(y, M_0) \leqq \|y - \varphi(y)\| \leqq \delta\,\|y\|$ so that $\delta(N, M_0) \leqq \delta$. If M_0 were a proper subset of M, we must have $\delta(M, N) \geqq (1-\delta)(1+\delta)^{-1}$ by Lemma 215.

2. The special case of Hilbert space. If X is a Hilbert space, we can sharpen some of the lemmas stated in the preceding paragraph.

Lemma 221. *Let M, N be closed linear manifolds in a Hilbert space X with the corresponding (perpendicular) projections E, F. If $\delta(N, M) < 1$, there are two possibilties. Either*

i) $\delta(M, N) = \delta(N, M) = \|E - F\| < 1$; E maps N onto M and F maps M onto N, both mappings being one-to-one. Or

ii) $\delta(M, N) = 1$; E maps N onto an $M_0 \subset M$ one-to-one, the image M_0 being a closed linear manifold and a proper subset of M, and $\delta(M_0, N) = \delta(N, M_0) = \delta(N, M) = \|E_0 - F\| < 1$, where E_0 is the projection on M_0; there exists an $x \in M$ such that $d(x, N) = \|x\| > 0$.

Proof. For each $y \in N$, we have $\|y - Ey\| = d(y, M) \leqq \delta \|y\|$, where $\delta = \delta(N, M)$. Hence $\|Ey\| \geqq (1 - \delta)\|y\|$ and the map $N \to EN \subset M$ is one-to-one and bicontinuous by $\delta < 1$. Set $M_0 = EN$; M_0 is thus a closed linear submanifold of M.

Each $x \in M_0$ has the form $x = Ey$, $y \in N$. $x \neq 0$ implies $y \neq 0$ and a simple calculation gives

$$\|x - \|y\|^{-2}(x, y)y\|^{-2} = \|x\|^2 \|y\|^{-2} \|y - Ey\|^2 = \|x\|^2 \|y\|^{-2} d(y, M)^2 \leqq \delta^2 \|x\|^2 .$$

This shows that $d(x, N) \leqq \delta \|x\|$. Since this is true for every $x \in M_0$, we have $\delta(M_0, N) \leqq \delta$. On the other hand it is obvious that

$$\delta(N, M_0) = \delta(N, M) = \delta .$$

It follows from a known theorem (see Achieser and Glasmann [1], § 34) that $\|E_0 - F\| = \delta < 1$ and in turn this implies that F maps M_0 onto N one-to-one. Thus we conclude as above that

$$\delta = \delta(N, M_0) \leqq \delta(M_0, N) .$$

Combined with the above inequality, this proves that

$$\delta(M_0, N) = \delta = \delta(N, M) .$$

We now distinguish between two cases: $M_0 = M$ and $M_0 \neq M$. In the former case, the proposition i) follows immediately. In the latter case, it remains to prove the last statement of ii); this implies in particular $\delta(M, N) = 1$. Take an $x \in M$ which does not belong to M_0. Since

$Fx \in N$, there is an $x_0 \in M_0$ such that $Fx_0 = Fx$ because $FM_0 = N$. Thus $F(x - x_0) = 0$ and so $d(x - x_0, N) = \|x - x_0\|$. Since $x - x_0 \neq 0$, we obtain the required result by writing x in place of $x - x_0$.

Lemma 222. Let M, N be closed linear manifolds of a Hilbert space such that $\delta(N, M) < 1$, and let M_1 be a linear manifold such that $M_1 \supset M$, $M_1 \neq M$. Then there is an $x \in M_1$ such that $d(x, N) = \|x\| > 0$. In particular $\delta(M_1, N) = 1$.

Proof. We have two possibilities i), ii) of Lemma 221. If the case ii) happens, the required x exists already in M. If i) happens, we choose an $x \in M_1$, $x \notin M$, and apply the same argument as in the proof of Lemma 221 to find an $x_0 \in M$ such that $d(x - x_0, N) = \|x - x_0\| > 0$.

3. Dual lemmas.

We now consider the lemmas which are dual of the foregoing ones. Let X^* be the adjoint space of X consisting of all bounded linear functionals on X. For any subset S of X, we denote by $S^\perp$ the totality of $f \in X^*$ such that $f \perp x$ (that is, $f(x) = 0$) for all $x \in S$. $S^\perp$ is a regularly closed[10] linear manifold of X^*.

If M is a linear manifold of X and $f \in X^*$, we have

$$\sup_{\substack{x \in M \\ \|x\|=1}} |f(x)| = \|f\|_M = d(f, M^\perp), \tag{231}$$

where $\|f\|_M$ is the least upper bound of the restriction of the linear functional f on M. (231) is an easy consequence of the Hahn-Banach extension theorem, and is dual to the following relationship:

$$\sup_{\substack{f \in M^\perp \\ \|f\|=1}} |f(x)| = d(x, M). \tag{232}$$

From these two formulas we conclude easily[11]

Lemma 231. For two linear manifolds M, N of X, we have

$$\delta(M, N) = \delta(N^\perp, M^\perp). \tag{233}$$

For any linear manifold M of X, we denote by codim M the dimension of the factor space $\tilde{X} = X/M$; thus codim M is defined in a purely algebraic

10. See Banach [5], Chapter 8.
11. Cf. GK, Theorem 6.2.

fashion. If M is a closed linear manifold of a Banach space X, we have

$$\operatorname{codim} M = \dim M^{\perp}. \tag{234}$$

Applying Lemma 211 to the pair $M^{\perp}, N^{\perp}$ and noting (231) and (234), we obtain

Lemma 232. Let M, N be two closed linear manifolds of X such that $\operatorname{codim} M > \operatorname{codim} N$. Then there is an $f \in M^{\perp}$ such that $\|f\| = \|f\|_N > 0$.

Similarly Lemma 213 gives

Lemma 233. Let M, N be two closed linear manifolds such that $\operatorname{codim} M = \operatorname{codim} N < \infty$. If $\|f\|_N \leqq \delta \|f\|$ for every $f \in M^{\perp}$, then $\|g\|_M \leqq \delta(1-\delta)^{-1} \|g\|$ for every $g \in N^{\perp}$.

4. Factor space.

In what follows frequent use will be made of the factor space $\tilde{X} = X/N$, where N is a closed linear manifold of X. As is well known (see, for example, Hausdorff [9]), $\tilde{X}$ is a Banach space with the norm defined by

$$\|\tilde{x}\| = \inf_{x \in \tilde{x}} \|x\| = \inf_{z \in N} \|x - z\| = d(x, N). \tag{241}$$

In particular

$$\|\tilde{x}\| \leqq \|x\|. \tag{242}$$

Lemma 211 can now be given the following form.

Lemma 241. If $\dim M > \dim N$, there is an $x \in M$ such that $\|\tilde{x}\| = \|x\| > 0$, where $\tilde{x} \in \tilde{X} = X/N$. ($N$ is closed by $\dim N < \infty$).

Lemma 242. If $\|\tilde{x}\| < \|x\|$ for every $x \in M$, $x \neq 0$, where $\tilde{x} \in \tilde{X} = X/N$, then $\dim M \leqq \dim N$.

§3. NULLITY, DEFICIENCY AND LOWER BOUND OF A LINEAR OPERATOR

1. Nullity and deficiency. In this paragraph we consider linear operators from an arbitrary vector space X into another vector space X'. By this we mean that the operator A under consideration has its domain $D[A]$ in X and range $R[A]$ in X', where $D[A]$ need not be the whole space X. For convenience we express this by saying that A is a linear operator $(X \to X')$. If X_0, X'_0 are linear manifolds of X, X' respectively, we can define a linear operator A_0 $(X_0 \to X'_0)$ by setting $A_0 x = Ax$ for every $x \in X_0$ such that $x \in D[A]$ and $Ax \in X'_0$. We shall say that A_0 is induced by A in the pair X_0, X'_0. If in particular $X'_0 = X'$, A_0 will be called a restriction of A to X_0. In case $D[A] \subset X_0$, the restriction of A to X_0 will be said to be trivial; in such a case A_0 is not essentially different from A.

The null space of A, denoted by $N[A]$, is the linear manifold consisting of all $x \in D[A]$ such that $Ax = 0$. We set[12]

(311) $$\alpha(A) = \dim N[A], \qquad \beta(A) = \operatorname{codim} R[A] = \dim X'/R[A].$$

$\alpha(A)$ is called the nullity of A and $\beta(A)$ the deficiency of A. It should be noted that none of $\alpha(A)$ and $\beta(A)$ is changed when A is replaced by its trivial restriction. On the other hand $\beta(A)$, but not $\alpha(A)$, will be changed if A is replaced by A_0 induced by A in the pair X, X'_0, where $R[A] \subset X'_0 \subset X'$.

For any subset S of X, we denote by AS the image under A of the set $S \cap D[A]$. In other words, AS is the set of Ax for all $x \in S \cap D[A]$. In particular $AX = R[A]$, and we ahall often write AX in place of $R[A]$. Also we use the symbol $A^{-1}S'$ to denote the inverse image under A of a subset S' of X', even when the inverse operator A^{-1} does not exist. If A^{-1} exists, the set $A^{-1}S'$ coincides with what is defined above as the direct image of S' under the map A^{-1}. Obviously we have

$$N[A] = A^{-1}\{0\}.$$

It is easy to verify that, for any $S \subset X$,

(312) $$A(BS) = (AB)S,$$

where B is a linear operator $(X \to X')$, A is a linear operator $(X' \to X'')$; the product AB is defined as usual as $(AB)x = A(Bx)$ with domain $D[AB]$ consisting of all $x \in X$ such that Bx and $A(Bx)$ make sense.

The following lemmas are easily proved, though not quite trivial because of the circumstance that the domain of the operator A need not be the whole space X.

Lemma 311. *If M, N are linear manifolds of X such*

12. Thus our definition of $\alpha(A)$ and $\beta(A)$ is again purely algebraic, and slightly differs from that employed in GK.

that $N \subset D[A]$, then[13]

$$A(M+N) = AM + AN. \tag{313}$$

Lemma 312. For any linear manifold M' of X', we have

$$\dim A^{-1}M' \leqq \dim M' + \alpha(A) \tag{314}$$

where the equality holds if $M' \subset R[A]$, and

$$\operatorname{codim} A^{-1}M' \leqq \operatorname{codim} M' + \operatorname{codim} D[A] \tag{315}$$

where the equality holds if $R[A] + M' = X'$.

Lemma 313. For any linear manifold M of X, we have

$$\operatorname{codim} AM \leqq \operatorname{codim} M + \beta(A) \tag{316}$$

where the equality holds if $N[A] \subset M$.

We now define the operator $\tilde{A}$ which will be used extensively in the following. Let $N = N[A]$ and consider the factor space $\tilde{X} = X/N$. Since $N \subset D = D[A]$, a coset $\tilde{x} \in \tilde{X}$ which contains at least one $x \in D$ consists entirely of elements of D. The totality of such $\tilde{x}$ will be denoted by $\tilde{D}$. On setting

$$\tilde{A}\tilde{x} = Ax \qquad \text{for} \quad \tilde{x} \in \tilde{D}, \tag{317}$$

we define a linear operator $\tilde{A}(\tilde{X} \to X')$. We have

$$D[\tilde{A}] = \tilde{D}, \quad R[\tilde{A}] = R[A], \quad N[\tilde{A}] = \{0\}. \tag{318}$$

The inverse $\tilde{A}^{-1}$ exists and is a linear operator $(X' \to \tilde{X})$.

In general, for any subset S of X, we denote by $\tilde{S}$ the totality of $\tilde{x} \in \tilde{X}$ containing at least one $x \in S$. Then it can easily be proved that

$$\tilde{A}\tilde{S} = AS. \tag{319}$$

2. Lower bound of a closed linear operator. We now consider a closed linear operator $A(X \to X')$ where X and X' are Banach spaces. The null space $N = N[A]$ is then closed, so that the factor space $\tilde{X} = X/N$ is also a Banach space (see § 2.4).

Lemma 321. $\tilde{A}$ is a closed linear operator $(\tilde{X} \to X')$.

The proof is simple and may be omitted.

We now introduce a quantity $\gamma(A)$, which may be called the lower bound of A; it is defined as the least upper bound of numbers $\gamma \geqq 0$

13. As usual we denote by $M + N$ the set of all $x + y$ with $x \in M$, $y \in N$. If $M \cap N = \{0\}$ in addition, we write $M \oplus N$ for $M + N$.

such that

$$\|Ax\| \geqq \gamma \|\tilde{x}\| = \gamma d(x, N) \quad \text{for all} \quad x \in D[A]. \tag{321}$$

It should be noted that $\gamma(A) = \infty$ if and only if $Ax = 0$ for every $x \in D[A]$. In other cases $\gamma(A)$ is actually the largest number γ with the property (321). Obviously we have

$$\gamma(A) = \gamma(\tilde{A}). \tag{322}$$

We note that $\gamma(A)$ is unchanged by a trivial restriction (see § 3.1) of A.

Lemma 322. *If A is a closed linear operator, $R[A]$ is closed if and only if $\gamma(A) > 0$.*

Proof. Since $R[A] = R[\tilde{A}]$ and $\gamma(A) = \gamma(\tilde{A})$, it is sufficient to prove the lemma for $\tilde{A}$ instead of A. Since $\tilde{A}^{-1}$ exists and is closed, the lemma is a direct consequence of the closed graph theorem.

When A is a bounded linear operator $(X \to X')$ with $D[A] = X$, it is known that $\gamma(A^*) = \gamma(A)$ where A^* is the adjoint of A and is a bounded linear operator $(X'^* \to X^*)$. For general closed linear operators A we are dealing with, the adjoint A^* need not exist. But we can introduce an operator A^+ which is essentially equivalent to the adjoint. Let X_0 be the closure of $D[A]$ and let A_0 be the trivial restriction of A to X_0. X_0 is a Banach space and A_0 is a closed linear operator $(X_0 \to X')$ with domain dense in X_0, so that the adjoint A_0^* exists. We define A^+ by

$$A^+ = A_0^*. \tag{323}$$

Thus A^+ is a closed linear operator $(X'^* \to X_0^*)$.

The domain of A^+ is not necessarily dense in X'^*. However, it is weakly dense in the sense that $x' \in X'$ and $f'(x') = 0$ for all $f' \in D[A^+]$ imply $x' = 0$. This can be proved by considering the graphs of the operators A and A^+, but we shall not give the proof here since we do not need this fact in the following.

$A^+ f' = 0$ is equivalent to $f'(Ax) = f'(A_0 x) = 0$ for every $x \in D[A]$. Hence

$$N[A^+] = R[A]^+. \tag{324}$$

Thus it follows from (231) that $d(f', N[A^+]) = \|f'\|_{R'}$, where $R' = R[A]$. This proves

Lemma 323. *$\gamma(A^+)$ is the least upper bound of numbers $\gamma \geqq 0$ such that $(R' = R[A])$*

(325) $$\|A^+f'\| \geqq \gamma \|f'\|_{R'} \qquad \text{for all} \quad f' \in D[A^+].$$

We shall now prove the following lemma, which is basic in the present work.

Lemma 324. *For any closed linear operator A, we have*

(326) $$\gamma(A^+) = \gamma(A).$$

In particular A has a closed range if and only if A^+ has a closed range.

Proof. For simplicity we write $\gamma = \gamma(A)$, $\gamma^+ = \gamma(A^+)$. We first show that $\gamma^+ \geqq \gamma$. If $\gamma = \infty$, we have $Ax = 0$ for all $x \in D[A]$ and so $D[A^+] = X'^*$, $A^+f' = 0$ for all $f' \in X'^*$, hence $\gamma^+ = \infty$. We may therefore assume that $0 < \gamma < \infty$, for $\gamma^+ \geqq \gamma$ is trivial if $\gamma = 0$. Thus $R[A]$ is closed by Lemma 322. Let $f' \in D[A^+]$. For any $\varepsilon > 0$, there is an $x' \in R' = R[A]$ such that $x' \neq 0$ and $|f'(x')| \geqq (1-\varepsilon) \|f'\|_{R'} \|x'\|$. x' has a form $x' = Ax$, $x \in D = D[A]$, so that

$$|f'(Ax)| \geqq (1-\varepsilon) \|f'\|_{R'} \|Ax\| \geqq (1-\varepsilon)\gamma \|f'\|_{R'} \|\tilde{x}\|,$$

where

$$\tilde{x} \in \tilde{X} = X/N, \quad N = N[A] \quad \text{and} \quad \tilde{x} \neq 0$$

because $\tilde{A}\tilde{x} = Ax = x' \neq 0$. Since

$$|f'(Ax)| = |A^+f'(x)| \leqq \|A^+f'\| \|x\|,$$

we have

$$\|x\| \|A^+f'\| \geqq (1-\varepsilon)\gamma \|f'\|_{R'} \|\tilde{x}\|.$$

Here $\|x\|$ can be arbitrarily near to $\|\tilde{x}\|$, since the only requirement on x is that $x \in \tilde{x}$. Thus we have

$$\|A^+f'\| \geqq (1-\varepsilon)\gamma \|f'\|_{R'}.$$

In view of Lemma 323, this shows that $\gamma^+ \geqq \gamma$.

We next prove[14] the opposite inequality $\gamma^+ \leqq \gamma$. Since this is trivial if $\gamma^+ = 0$, we shall assume that $\gamma^+ > 0$. We shall now show that, denoting by S the unit sphere $\|x\| \leqq 1$ of X, the closure of AS contains the sphere $\|x'\| < \gamma^+$ of the subspace $[R']$ which is the closure of R'; then a standard argument (see, for instance, Hausdorff [9]) shows that AS itself contains the latter sphere and that $\|Ax\| \geqq \gamma^+ \|\tilde{x}\|$, leading to the desired result

14. The proof given below is an adaptation of a proof given by Yosida [14] n the case in which A is bounded with $D[A] = X$ and A^* has a bounded inverse.

$\gamma \geqq \gamma^+$. It should be noted that this argument is valid if A is closed, even if it is not bounded.

Let y' be a point of $[R']$ with a positive distance from AS. Since AS is a convex set, there is a $g' \in [R']^*$ such that[15] $\operatorname{Re} g'(y') > \operatorname{Re} g'(Ax)$ for all $x \in D \cap S$. This g' can be extended to an element, again denoted by g', of X'^* with $\|g'\| = \|g'\|_{R'} > 0$. By replacing x by $e^{i\vartheta} x$ with a suitable real ϑ, it is easy to see that

$$|g'(Ax)| \leqq \operatorname{Re} g'(y') \leqq |g'(y')|$$

for all $x \in D \cap S$. This implies that $x \to g'(Ax)$ is a bounded linear functional defined for $x \in D$. Hence $A^+ g'$ exists and

$$\|A^+ g'\| \leqq |g'(y')| \leqq \|g'\| \, \|y'\| \leqq \|g'\|_{R'} \|y'\|.$$

On the other hand we have $\|A^+ g'\| \geqq \gamma' \|g'\|_{R'}$ for any $\gamma' < \gamma^+$ by Lemma 323. Thus we must have $\|y'\| \geqq \gamma^+$, which shows that the closure of AS contains the sphere $\|x'\| \leqq \gamma^+$ of $[R']$.

3. Closed linear operator with closed range. Let X and X' be Banach spaces. We are particularly interested in a closed linear operator $A(X \to X')$ with closed range $R[A]$. As we have shown above, this is equivalent to the condition that $\gamma(A) > 0$, and A has closed range if and only if A^+ has closed range.

Lemma 331. Let A be a closed linear operator $(X \to X')$ with closed range. For a (not necessarily closed) linear manifold M of X, AM is closed if $M + N[A]$ is closed.

Proof. Construct the factor space $\tilde{X} = X/N$, $N = N[A]$, and the linear manifold $\tilde{M}$ of $\tilde{X}$ as in § 3.1. We have $AM = \tilde{A}\tilde{M}$ by (319). $\tilde{A}$ has a bounded inverse $\tilde{A}^{-1}$ with bound $\gamma(A)^{-1}$, so that $\tilde{A}\tilde{M}$ is closed if $\tilde{M}$ is closed, for it is the inverse image of $\tilde{M}$ under the continuous map $\tilde{A}^{-1}$. It remains to show that $\tilde{M}$ is closed if $M + N$ is closed.

Suppose that there is a sequence $\tilde{x}_n \in \tilde{M}$ such that $\tilde{x}_n \to \tilde{x} \in \tilde{X}$. This implies that $d(x_n - x, N) \to 0$ and so there is a sequence $z_n \in N$ such that $x_n - x - z_n \to 0$. Since we may assume that $x_n \in M$, this shows that $x \in M + N$ if $M + N$ is closed. Thus we have $\tilde{x} \in \tilde{M}$ and $\tilde{M}$ is closed.

15. We denote by Re the real part of a complex number.

Lemma 332. If A is a closed linear operator $(X \to X')$ with $\beta(A) < \infty$, then A has closed range.

Proof. Let $\beta(A) = m < \infty$. Take an m-dimensional Banach space Y with a basis $y_1, \dots, y_m$ and construct the direct sum $Z = X \oplus Y$. Z may be regarded as a Banach space. Since $\operatorname{codim} R[A] = m < \infty$, we can find m elements $x'_1, \dots, x'_m$ of X' which are linearly independent modulo $R[A]$ and which span the linear manifold X' with $R[A]$. Define a linear operator $B(Z \to X')$ by setting $Bx = Ax$ for $x \in D[A]$ and $By_k = x'_k$ for $k = 1, \dots, m$. B is a closed linear operator, for its graph has a finite dimension modulo the graph of A, which is closed because A is closed. The range of B is the whole space X' and hence closed. Furthermore, it is easily seen that $N[B] = N[A] \subset X$. Hence $X + N[B] = X$ is closed in Z. It follows from Lemma 331 that $R[A] = AX = BX$ is closed.

Lemma 333. Let A be a closed linear operator $(X \to X')$ and let M be a closed linear manifold of X with $\operatorname{codim} M < \infty$. Let A_1 be the restriction of A to M. Then A_1 is a closed linear operator $(M \to X')$. A_1 has a closed range if and only if A has a closed range.

Proof. That A_1 is closed is obvious. Since $\operatorname{codim} M < \infty$, $M + N[A]$ is closed. Hence $R[A_1] = A_1 M = AM$ is closed by Lemma 331, if $R[A]$ is closed. Conversely, if $R[A_1]$ is closed, $R[A] = AX$ is closed because $\dim(AX/AM) \leq \operatorname{codim} M < \infty$.

Lemma 334. Let A be a closed linear operator $(X \to X')$ with closed range. For any $f \in N[A]^{\perp}$, there is an $f' \in X'^{*}$ such that $f'(Ax) = f(x)$ for all $x \in D[A]$. Here f' can further satisfy one of the following conditions.

i) $\|f'\| \leqq \gamma(A)^{-1} \|f\|$;

ii) $f' \in M'^{\perp}$, where M' is a closed linear manifold of X' such that $M' + R[A]$ is closed and $f \in (A^{-1} M')^{\perp}$; the last condition is satisfied if $M' \cap R[A] = \{0\}$.

Proof. $Ax = 0$ implies that $x \in N[A]$, $f(x) = 0$. We can therefore define a linear functional f' by setting $f'(Ax) = f(x)$; f' is thereby defined on $R' = R[A]$ and is bounded. In fact, we have

$$|f'(Ax)| = |f(x)| \leqq \|f\| \, \|x\|;$$

but as x may be any element of $\tilde{x} \in \tilde{X} = X/N$, $N = N[A]$, containing x, it follows that

$$|f'(Ax)| \leqq \|f\| \, \|\tilde{x}\| \leqq \|f\| \, \gamma^{-1} \|Ax\|, \qquad \gamma = \gamma(A).$$

f' can now be extended on the whole space X' without changing its bound $\gamma^{-1}\|f\|$; this proves i).

To prove ii), we set $L' = M' \cap R'$. L' is a closed linear manifold of X' and $f'(x') = 0$ if $x' \in L'$. In fact, $x' \in L'$ implies that $x' = Ax$ for some $x \in D[A]$ and $Ax \in M'$; hence $x \in A^{-1}M'$ and $f'(x') = f'(Ax) = f(x) = 0$ by hypothesis.

We now introduce the factor space $\tilde{X}' = X'/L'$ and its subspaces $\tilde{M}' = M'/L'$, $\tilde{R}' = R'/L'$. Obviously we have $\tilde{M}' \cap \tilde{R}' = \{0\}$ and

$$\tilde{X}'_0 = \tilde{M}' \oplus \tilde{R}' = (M' + R')/L'$$

is closed because $X'_0 = M' + R'$ is closed by hypothesis. Thus $\tilde{X}'_0$ is a Banach space and there is defined a bounded projection E from $\tilde{X}'_0$ onto $\tilde{R}'$ along $\tilde{M}'$.

The linear functional f' defined on R' induces a linear functional $\tilde{f}'$ on $\tilde{R}'$ defined by $\tilde{f}'(\tilde{x}') = f'(x')$; the definition is justified because $\tilde{x}' = 0$ implies $x' \in L'$ and so $f'(x') = 0$ by what is proved above. Furthermore, $\tilde{f}'$ is bounded. In fact, we have

$$|\tilde{f}'(\tilde{x}')| = |f'(x')| \leqq \gamma^{-1} \|f\| \, \|x'\|$$

and hence

$$|\tilde{f}'(\tilde{x}')| \leqq \gamma^{-1} \|f\| \, \|\tilde{x}'\|$$

because x' can be replaced by any other element of $\tilde{x}'$.

We can now extend the linear functional f' to a linear functional, again denoted by f', defined on X'_0. We set $f'(x') = \tilde{f}'(E\tilde{x}')$ for $x' \in X'_0$. It is easy to see that f' is a bounded linear extension of the original f' with bound not exceeding $\gamma^{-1}\|E\| \, \|f\|$, and that $f'(x') = 0$ for $x' \in M'$. This f' can finally be extended to an $f' \in X'^*$ by the Hahn-Banach theorem.

Remark. $f'(Ax) = f(x)$ implies that $A^+ f' = f_0$, where $f_0 \in X_0^*$ is the restriction of the functional f on X_0, X_0 being the closure of $D[A]$.

Lemma 335. *If A is a closed linear operator $(X \to X')$ with closed range, we have*

(331) $\quad N[A^+] = R[A]^\perp, \quad R[A^+] = N[A]^\perp, \quad \alpha(A^+) = \beta(A).$

Here $N[A]^\perp$ is to be constructed from $N[A]$ regarded as a subset of the space X_0 which is the closure of $D[A]$, so that $N[A]^{\perp}\subset X_0^*$.

Proof. The first equality has already been proved, see (324). Hence

$$\alpha(A^+) = \dim N[A^+] = \operatorname{codim} R[A] = \beta(A)$$

in virtue of the closure of $R[A]$. The second equality of (331) follows from the remark after Lemma 334, which implies that $N[A]^{\perp}\subset R[A^+]$, while the opposite inclusion is obvious.

4. Product of closed linear operators.

We shall prove a lemma concerning the closure of the products of two closed linear operators.

Lemma 341. Let X, X', X'' be Banach spaces. Let A be a closed linear operator $(X' \to X'')$ with a closed range and with $\alpha(A)<\infty$, and let B be a closed linear operator $(X \to X')$. Then AB is a closed linear operator $(X \to X'')$. If in addition B has a closed range, AB has a closed range.

Proof. Suppose that there is a sequence $x_n \in D[AB]$ such that $x_n \to x \in X$, $ABx_n \to x'' \in X''$, $n \to \infty$. We have to show that

$$x \in D[AB] \quad \text{and} \quad ABx = x''.$$

Consider the factor space $\tilde{X}' = X'/N'$, $N' = N[A]$, and the operator $\tilde{A}(\tilde{X}' \to X'')$ defined by $\tilde{A}\tilde{y}' = Ay'$. On setting $x'_n = Bx_n$, we have $\tilde{A}\tilde{x}'_n = Ax'_n = ABx_n \to x'' \in R[A]$ since $R[A]$ is closed. $\tilde{A}$ has a bounded inverse and so we have $\tilde{x}'_n \to \tilde{A}^{-1}x''$. Set $\tilde{x}' = \tilde{A}^{-1}x''$; then

$$x'' = \tilde{A}\tilde{x}' = Ax', \quad x' \in D[A].$$

Since $\tilde{x}'_n \to \tilde{x}'$, there is a sequence $z'_n \in N'$ such that $x'_n - z'_n \to x'$.

Assume first that the sequence z'_n is bounded. In virtue of the assumption that $\dim N' = \alpha(A) < \infty$, we may assume that $z'_n \to z'$ for some $z' \in N'$; otherwise we need only to replace the sequence z'_n by its suitable subsequence. Thus $Bx_n = x'_n = (x'_n - z'_n) + z'_n \to x' + z'$. Since $x_n \to x$ and B is closed, we see that $x \in D[B]$ and $Bx = x' + z'$. Hence $Bx \in D[A]$ and $ABx = A(x' + z') = Ax' = x''$, as we wished to show.

Assume now that the sequence z'_n is not bounded. We may assume as above that $\zeta_n = \|z'_n\| \to \infty$. Set $u'_n = \zeta_n^{-1} z'_n$. Then $\|u'_n\| = 1$ and $\zeta_n^{-1} x'_n - u'_n = \zeta_n^{-1}(x'_n - z'_n) \to 0$. Again, choosing a subsequence if necessary, we may assume that $u'_n \to u' \in N'$. Thus

$$B(\zeta_n^{-1} x_n) = \zeta_n^{-1} x'_n \to u'.$$

Since $\zeta_n^{-1} x_n \to 0$ and B is closed, we have $u' = B0 = 0$, which contradicts the fact that $u'_n \to u'$, $\|u'_n\| = 1$. This excludes the possibility that the sequence z'_n is unbounded, and completes the proof of the closure of AB.

If B has closed range, $R[AB] = ABX = A(BX)$ is closed by Lemma 331, since $BX + N[A]$ is closed in virtue of $\dim N[A] < \infty$.

§ 4. APPROXIMATE EIGENVALUE AND SOME STABILITY THEOREMS

1. Approximate nullity and deficiency. Let X, X' be Banach spaces and let A be a closed linear operator $(X \to X')$. $\alpha(A) = \dim N[A]$ is equal to the multiplicity of the eigenvalue 0 of A. (If $\alpha(A) = 0$, 0 is usually not called an eigenvalue of A, but we may regard 0 as an eigenvalue with multiplicity zero). We now introduce the multiplicity $\alpha'(A)$ of the *approximate eigenvalue* 0 of A.

Let there be a number $m = 0, 1, 2, \dots$ or ∞ with the property that, given any $\varepsilon > 0$, there is an m-dimensional closed linear manifold $N_\varepsilon \subset D[A]$ such that

(411) $$\|Ax\| \leqq \varepsilon \|x\| \qquad \text{for every } x \in N_\varepsilon,$$

while this is not true if m is replaced by a larger number. In such a case we set $\alpha'(A) = m$ by definition. $\alpha'(A)$ takes the values 0, 1, 2, ... or ∞ as well as $\alpha(A)$. $\alpha'(A)$ will be called the *approximate nullity* of A. We define the *approximate deficiency* of A by

(412) $$\beta'(A) = \alpha'(A^+).$$

It is by no means obvious that $\alpha'(A)$ is defined for every closed linear operator A; it would be a priori possible that there is no finite m with the required property while the condition is not satisfied by $m = \infty$. However, the following lemma shows that actually $\alpha'(A)$, and hence also $\beta'(A)$, is defined for every A.

Lemma 411. *Assume that for each finite m and $\varepsilon > 0$, there is an $N_\varepsilon \subset D[A]$ with $\dim N_\varepsilon = m$ and with the property (411). Then we have $\alpha'(A) = \infty$.*

Proof. Let M be a closed linear manifold of X with $\operatorname{codim} M < \infty$. Then there is a linear manifold N_ε with $\dim N_\varepsilon > \operatorname{codim} M$ and with the property (411). The linear manifold $M \cap N_\varepsilon$ has a positive dimension, and each $x \in M \cap N_\varepsilon$ satisfies the inequality $\|Ax\| \leqq \varepsilon \|x\|$. Thus Lemma 411 is reduced to the following

Lemma 412. *Assume that, for any $\varepsilon > 0$ and any closed linear manifold M of X with $\operatorname{codim} M < \infty$, there is an $x \in M \cap D[A]$ such that $\|x\| = 1$, $\|Ax\| \leqq \varepsilon$. Then $\alpha'(A) = \infty$.*

Proof. We have to show that, for each $\varepsilon > 0$, there is a closed linear manifold $N_\varepsilon \subset D[A]$ with $\dim N_\varepsilon = \infty$ and with the property (411). For this purpose, we construct two sequences $x_n \in D[A]$ and $f_n \in X^*$ with the following properties:

$$(413) \qquad \begin{aligned} &\|x_n\| = 1, \quad \|f_n\| = 1, \quad f_n(x_n) = 1, \\ &f_k(x_n) = 0, \quad k = 1, 2, \ldots, n-1, \\ &\|Ax_n\| \leqq 3^{-n}\varepsilon, \quad n = 1, 2, 3, \ldots . \end{aligned}$$

Supposing that x_k, f_k have been constructed for $k = 1, \ldots, n-1$, x_n and f_n can be found in the following way. Let $M \subset X$ be the set of all x such that $f_k(x) = 0$, $k = 1, \ldots, n-1$. Since M is a closed linear manifold of X with $\operatorname{codim} M < \infty$, there is an $x_n \in M \cap D[A]$ such that $\|x_n\| = 1$, $\|Ax_n\| \leqq 3^{-n}\varepsilon$. For this x_n there is an $f_h \in X^*$ such that $\|f_n\| = 1$ and $f_n(x_n) = 1$.

Obviously x_n are linearly independent, so that the linear hull L_ε of the set x_n is of infinite dimension. Each $x \in L_\varepsilon$ has the form

$$(414) \qquad x = \xi_1 x_1 + \ldots + \xi_n x_n .$$

Hence for $k = 1, 2, \ldots, n$,

$$(415) \qquad f_k(x) = \xi_1 f_k(x_1) + \ldots + \xi_{k-1} f_k(x_{k-1}) + \xi_k ,$$

by (413), and we have

$$(416) \qquad |\xi_k| \leqq 2^{k-1} \|x\|, \quad k = 1, 2, \ldots, n .$$

This can be proved by induction. For $k = 1$ this is obvious from (415) and (413). Assuming that (416) has been proved for $k < r$, we see from (415) that

$$\begin{aligned} |\xi_r| &\leqq |f_r(x)| + |\xi_1| |f_r(x_1)| + \ldots + |\xi_{r-1}| |f_r(x_{r-1})| \\ &\leqq \|x\| + \|x\| + \ldots + 2^{r-2}\|x\| = 2^{r-1}\|x\| . \end{aligned}$$

It follows from (413), (414) and (416) that

$$\begin{aligned} \|Ax\| &\leqq |\xi_1| \|Ax_1\| + \ldots + |\xi_n| \|Ax_n\| \\ &\leqq (1/3 + 2/3^2 + \ldots + 2^{n-1}/3^n)\varepsilon \|x\| \leqq \varepsilon \|x\| . \end{aligned}$$

Thus A is defined everywhere in L_ε and is bounded with bound not exceeding ε. Since A is closed, A is also defined and bounded with the same bound in the closure N_ε of L_ε. This proves that (411) is true with $\dim N_\varepsilon = \infty$.

2. Closed linear operator with non-closed range. It is obvious that

$$\alpha'(A) \geqq \alpha(A). \tag{421}$$

Lemma 421. *If A is a closed linear operator with a closed range* (that is, $\gamma(A)>0$), *then*

$$\alpha'(A) = \alpha(A), \quad \beta'(A) = \beta(A). \tag{422}$$

Proof. $\gamma(A)>0$ implies $\gamma(A^+)>0$ and $\alpha(A^+) = \beta(A)$, see Lemmas 324 and 335. In view of (412), it is therefore sufficient to show that $\alpha'(A) = \alpha(A)$. Suppose that there is a closed linear manifold N_ε with $\dim N_\varepsilon > \alpha(A) = \dim N[A]$ and with the property (411). According to Lemma 241, there is an $x \in N_\varepsilon$ such that $\|\tilde{x}\| = \|x\| = 1$, where $\tilde{x} \in \tilde{X} = X/N[A]$. Hence we have $\|Ax\| \geqq \gamma = \gamma(A)$ on the one hand and $\|Ax\| \leqq \varepsilon$ on the other, leading to the inequality $\varepsilon \geqq \gamma$. In other words, there is no N_ε with the above property for $\varepsilon < \gamma$. This proves that $\alpha'(A) \leqq \alpha(A)$. Combined with (421), the lemma is proved.

Lemma 422. *If A is a closed linear operator with non-closed range* (that is, $\gamma(A) = 0$), *then*

$$\alpha'(A) = \beta'(A) = \infty. \tag{423}$$

Proof. Since $\gamma(A) = \gamma(A^+)$ and $\beta'(A)$ is given by (412), it is sufficient to prove that $\gamma(A) = 0$ implies $\alpha'(A) = \infty$. Let M be any closed linear manifold of X with $\operatorname{codim} M < \infty$, and let $A_1(M \to X')$ be the restriction of A to M. Then $R[A_1]$ is not closed by Lemma 333. Thus $\gamma(A_1) = 0$ and there exists, for any $\varepsilon > 0$, an $x \in M \cap D[A]$ such that $\|x\| = 1$ and $\|Ax\| \leqq \varepsilon \|\tilde{x}\| \leqq \varepsilon \|x\| = \varepsilon$, where $\tilde{x} \in \tilde{X} = X/N[A]$. This shows that the assumption of Lemma 412 is satisfied.

We note that Lemmas 421 and 422 imply the following inequality corresponding to (421):

$$\beta'(A) \geqq \beta(A). \tag{424}$$

3. A stability theorem.

We now prove our first theorem on the stability of the property of a linear operator having a closed range.

Theorem 1. Let X and X' be Banach spaces and let A be a closed linear operator $(X \to X')$ with closed range (so that $\gamma(A) > 0$) and with at least one of $\alpha(A)$ and $\beta(A)$ finite. Let B be a bounded linear operator $(X \to X')$ such that $D[B] \supset D[A]$ and $\|B\| < \gamma(A)$. Then the linear opertor $A+B$ is closed and has closed range $(\gamma(A+B) > 0)$. Furthermore, we have

$$\alpha(A+B) \leqq \alpha(A), \quad \beta(A+B) \leqq \beta(A). \tag{431}$$

Remark 1. Also we have

$$\alpha(A+B) - \beta(A+B) = \alpha(A) - \beta(A). \tag{432}$$

The proof of (432) will be given later (Theorem 6).

Proof. That $A+B$ defined with $D[A+B] = D[A]$ is a closed linear operator $(X \to X')$ is obvious. To show that $A+B$ has closed range and that (431) holds, it is sufficient to show that

$$\alpha'(A+B) \leqq \alpha(A), \quad \beta'(A+B) \leqq \beta(A). \tag{433}$$

In fact, (433) implies (431) by (421) and (424). On the other hand, (433) implies that at least one of $\alpha'(A+B)$ and $\beta'(A+B)$ is finite; Lemma 422 then shows that $A+B$ has closed range.

To prove (433), suppose that there is a closed linear manifold M_ε such that $\|(A+B)u\| \leqq \varepsilon \|u\|$ for every $u \in M_\varepsilon$. Then

$$(\|B\| + \varepsilon)\|u\| \geqq \|Bu\| + \|(A+B)u\| \geqq \|Au\| \geqq \gamma \|\tilde{u}\|,$$

where $\gamma = \gamma(A)$ and $\tilde{u} \in \tilde{X} = X/N$, $N = N[A]$. If ε is so small that $0 < \varepsilon < \gamma - \|B\|$, it follows that $d(u, N) = \|\tilde{u}\| < \|u\|$ if $0 \neq u \in M_\varepsilon$. Hence we have $\dim M_\varepsilon \leqq \dim N = \alpha(A)$ by Lemma 242, which means that $\alpha'(A+B) \leqq \alpha(A)$.

To prove the second inequality of (433), we may assume that $D[B] = D[A]$, for $A+B$ is not changed when we restrict B on the domain $D[A]$. Then both A^+ and B^+ are linear operators $(X'^* \to X_0^*)$ where X_0 is the closure of $D[A]$, and we have the obvious relationship $(A+B)^+ = A^+ + B^+$. Also it is obvious that $\|B^+\| \leqq \|B\|$ because

$B^+ = B_0^*$ where B_0 is the operator $(X_0 \to X')$ which is the restriction of B to X_0. Since $\gamma(A^+) = \gamma(A)$ by Lemma 324, we have $\|B^+\| < \gamma(A^+)$. Applying to the pair A^+, B^+ what has been proved above, we obtain

$$\beta'(A+B) = \alpha'((A+B)^+) = \alpha'(A^+ + B^+) \leqq \alpha(A^+) = \beta(A),$$

where we have used (412) and (331).

Remark 2. Theorem 1 gives rise to two questions. The first is whether we can give some estimate of $\gamma(A+B)$ in terms of $\gamma(A)$ and $\|B\|$. Since Theorem 1 implies that $\gamma(A+B) > 0$ if $\|B\| < \gamma(A)$, one might suspect that something like the inequality $\gamma(A+B) \geqq \gamma(A) - \|B\|$ holds. But this is not correct. By its very nature $\gamma(A)$ does not have such a lower semi-continuity. $\gamma(A)$ is defined as $\inf \|Ax\| / \|\tilde{x}\|$ where $\tilde{x} \in \tilde{X} = X/N[A]$; but $N[A]$ may change quite discontinuously with A, leading to lower discontinuity of $\gamma(A)$. In fact, let A be a linear operator in a finite-dimensional Banach space X and let $\alpha(A) > 0$. Then it is easily seen that $\gamma(A - \lambda I) \leqq |\lambda|$ for sufficiently small $\lambda \neq 0$, whereas $\gamma(A) > 0$.

We shall see later, however, that there is a case in which we have the inequality $\gamma(A+B) \geqq \gamma(A) - \|B\|$ (see Theorem 3).

Remark 3. The second question is whether Theorem 1 remains true when $\alpha(A) = \beta(A) = \infty$. The answer is again in the negative as is seen from the following example. Let X be a Hilbert space with a complete orthonormal set $x_1, x_2, \dots, y_1, y_2, \dots$. Define an operator A by

(434) $$Ax_n = 0, \quad Ay_n = \alpha_n x_n + \beta_n y_n, \quad n = 1, 2, \dots,$$

where α_n, β_n are constants such that

(435) $$|\alpha_n|^2 + |\beta_n|^2 = 1, \quad \beta_n \neq 0.$$

Assume further that the set β_n is dense in the unit circle of the complex plane.

It is easily seen that A can be extended uniquely to a bounded linear operator defined everywhere in X, which will be again denoted by A. It is also easy to see that A is a partially isometric operator with $\alpha(A) = \beta(A) = \infty$, $\gamma(A) = 1$. On the other hand, it can be verified without difficulty that $\alpha(A - \lambda I) = 0$ for λ which is different from 0 and all β_n, whereas $\alpha(A - \beta_n I) = 1$, $N[A - \beta_n I]$ being spanned by the single vector Ay_n.

Set $u_m = \alpha_m x_m + \lambda y_m$. Then

$$(436)\qquad \frac{\|(A-\lambda I)\,u_m\|}{\|u_m\|} = \frac{|\lambda|\;|\beta_m-\lambda|}{|\alpha_m|^2+|\lambda|^2} \leqq \frac{|\beta_m-\lambda|}{|\lambda|}.$$

If $\lambda \neq 0$ and $\lambda \neq \beta_n$ for every n, we have $\|\tilde{u}_m\| = \|u_m\|$ because $\alpha(A-\lambda I)=0$, where $\tilde{u} \in \tilde{X} = X/N[A-\lambda I]$. If $\lambda = \beta_n$, we have still $\|\tilde{u}_m\| = \|u_m\|$ for $m \neq n$ because $(u_m, u_n) = 0$ and u_n coincides with Ay_n which spans $N[A-\lambda I]$. In each case (436) shows that

$$\gamma(A-\lambda I) = \inf_u \|(A-\lambda I)\,u\| / \|\tilde{u}\| \leqq \inf_{m\neq n} \|(A-\lambda I)\,u_m\| / \|u_m\| = 0$$

provided $|\lambda| \leqq 1$, for the set β_n is dense in the unit circle. Thus $\lambda = 0$ is the only point in the unit circle for which $A-\lambda I$ has closed range.

4. Extension to unbounded perturbation. Theorem 1 can be extended to unbounded B in the following fashion.

Theorem 1a. *Theorem 1 is true if B is a linear operator $(X \to X')$ such that $D[B] \supset D[A]$ and*

$$(441)\qquad \|Bx\| \leqq \sigma\,\|x\| + \tau\,\|Ax\| \quad \text{for all } x \in D[A],$$

where σ and τ are non-negative constants such that

$$(442)\qquad \sigma + \tau\gamma(A) < \gamma(A).$$

Remark 1. As is well known, such constants σ, τ exist if $D[B] \supset D[A]$ and B has a closed extension.

Remark 2. Again the equality (432) is true, as will be proved later (Theorem 6).

Proof. First we show that $A+B$ is a closed linear operator. We note that $D[A+B] = D[A] = D$ and that (441) implies

$$(443)\qquad \|(A+B)\,x\| \geqq -\sigma\,\|x\| + (1-\tau)\,\|Ax\|, \quad x \in D,$$

where $1-\tau > 0$ by (442) and $\gamma(A) > 0$. Suppose there is a sequence $x_n \in D$ such that $x_n \to x \in X$ and $(A+B)\,x_n \to x' \in X'$. Then x_n and $(A+B)\,x_n$ are Cauchy sequences. Applying (443) to $x_n - x_m$ in place of x and noting that $1-\tau > 0$, we see that Ax_n is also a Cauchy sequence. Since A is closed, it follows that $x \in D$ and $Ax_n \to Ax$. Applying (441) to $x_n - x$ in place of x, we see then that $Bx_n \to Bx$. Thus we obtain $x' = (A+B)\,x$. This proves the closure of $A+B$. It should be noted that so far we have only used the assumption that $\tau < 1$.

We now introduce in D a new norm by

$$(444)\qquad |||x||| = (\sigma+\varepsilon)\,\|x\| + (\tau+\varepsilon)\,\|Ax\| \geqq \varepsilon\,\|x\|,$$

where ε is an arbitrary but fixed positive constant. In view of the closure of A, it is easy to see that D becomes a Banach space by the new norm, which will be denoted by $\widehat{X}$. We can now regard A and B as linear operators $(\widehat{X} \to X')$; in this new interpretation they will be denoted by $\widehat{A}$ and $\widehat{B}$ respectively. $\widehat{A}$ and $\widehat{B}$ are defined everywhere in $\widehat{X}$ and bounded as

$$(445) \qquad \|\widehat{A}\| \leqq (\tau+\varepsilon)^{-1}, \qquad \|\widehat{B}\| \leqq 1 .$$

It is obvious that $R[\widehat{A}] = R[A]$ is closed and that

$$(446) \qquad \begin{aligned} \alpha(\widehat{A}) &= \alpha(A), & \beta(\widehat{A}) &= \beta(A), \\ \alpha(\widehat{A}+\widehat{B}) &= \alpha(A+B), & \beta(\widehat{A}+\widehat{B}) &= \beta(A+B), \end{aligned}$$

so that at least one of $\alpha(\widehat{A})$ and $\beta(\widehat{A})$ is finite.

Let us now see how $\gamma(\widehat{A})$ is related to $\gamma(A)$. By definition

$$\gamma(\widehat{A}) = \inf \|\widehat{A}x\| / |||\tilde{x}||| = \inf \|Ax\| / |||\tilde{x}|||$$

where $\tilde{x} \in \tilde{X} = X/N$, $N = N[A]$. But

$$(447) \qquad |||\tilde{x}||| = \inf_{z \in N} |||x-z||| = \inf_{z \in N} [(\sigma+\varepsilon)\|x-z\| + (\tau+\varepsilon)\|A(x-z)\|]$$
$$= (\sigma+\varepsilon)\|\tilde{x}\| + (\tau+\varepsilon)\|Ax\|$$

(note that $Az = 0$). Hence

$$(448) \qquad \gamma(\widehat{A}) = \inf_{x \in D} \frac{\|Ax\|}{(\sigma+\varepsilon)\|\tilde{x}\| + (\tau+\varepsilon)\|Ax\|} = \frac{\gamma(A)}{\sigma+\varepsilon+(\tau+\varepsilon)\gamma(A)}$$

in virtue of the definition $\gamma(A) = \inf \|Ax\| / \|\tilde{x}\|$.

In view of the assumption (442), we can make $\gamma(\widehat{A}) > 1$ by taking ε small enough. Since $\|\widehat{B}\| \leqq 1$ by (445), we can thus apply Theorem 1 to the pair $\widehat{A}, \widehat{B}$, with the result that $R[\widehat{A}+\widehat{B}] = R[A+B]$ is closed and that (431) holds for A, B replaced by $\widehat{A}, \widehat{B}$. Considering (446), this proves Theorem 1a.

5. Second stability theorem.

For convenience a closed, bounded linear operator B $(X \to X')$ will be said to be **strictly singular** if there is no linear manifold $M \subset D[B]$ with $\dim M = \infty$ such that the map $M \to BM$ is a homeomorphism.[16] In other words, a closed, bounded linear operator B is stricly singular if

16. Since B is bounded, it is sufficient to consider only closed M.

and only if the existence of a $\gamma > 0$ such that $\|Bx\| \geqq \gamma \|x\|$ for all $x \in M$ implies that dim $M < \infty$.

As is easily seen, a completely continuous operator is strictly singular. The converse is true if both X and X' are Hilbert spaces, as will be shown below. But the question is open whether the same is true in the general case. Also we do not know whether the strict singularity of B implies the same for B^+.

We can now prove our second stability theorem.

Theorem 2. *Let X, X' be Banach spaces. Let A be a closed linear operator $(X \to X')$ with closed range and with $\alpha(A) < \infty$ and let B be a strictly singular operator $(X \to X')$. Then $A+B$ is a closed linear operator with closed range and $\alpha(A+B) < \infty$.*

Proof. It is easy to verify that $A+B$ is a closed linear operator; note that $D[A+B] = D[A] \cap D[B]$ and $D[B]$ is closed because B is bounded and closed.

Set $N = N[A]$. Since dim $N < \infty$, there is a closed linear manifold X_0 of X such that $X = X_0 \oplus N$. Let A_0, B_0 be respectively the restrictions of A, B to X_0. Then $\alpha(A_0) = 0$ and, since $R[A_0] = R[A]$ is closed, we have $\|Ax\| = \|A_0 x\| \geqq \gamma_0 \|x\|$ for every $x \in D[A_0] = D[A] \cap X_0$, where $\gamma_0 = \gamma(A_0) > 0$.

Suppose that there is a closed linear manifold $M_\varepsilon \subset D[A_0 + B_0]$ such that $\|(A_0 + B_0) u\| \leqq \varepsilon \|u\|$ for every $u \in M_\varepsilon$. Then

$$\|Bu\| = \|B_0 u\| \geqq \|A_0 u\| - \|(A_0 + B_0) u\| \geqq (\gamma_0 - \varepsilon) \|u\| .$$

If ε is so small that $\gamma_0 - \varepsilon > 0$, this shows that dim $M_\varepsilon < \infty$ because B is strictly singular. Thus we have proved that $\alpha'(A_0 + B_0) < \infty$. This implies on the one hand that $R[A_0 + B_0]$ is closed (see Lemma 422); then $R[A+B]$ is also closed by $\dim R[A+B]/R[A_0 + B_0] \leqq \dim N < \infty$. On the other hand we have $\alpha(A_0 + B_0) < \infty$ and hence

$$\alpha(A+B) \leqq \alpha(A_0 + B_0) + \dim N < \infty .$$

Theorem 2 can be extended to unbounded B. We shall say that a closed linear operator B $(X \to X')$ is *strictly singular relative to a closed linear operator* A $(X \to X')$ if $D[B] \supset D[A]$ and there is no linear manifold $M \subset D[B]$ with dim $M = \infty$ such that

$$\|Bx\| / (\|x\| + \|Ax\|) \geqq \gamma > 0$$

for all $x \in M$. If we introduce the norm $|||x||| = \|x\| + \|Ax\|$ in $D = D[A]$, D becomes a Banach space $\hat{X}$. Let $\hat{A}$, $\hat{B}$ be the operators A, B regarded as operators $(\hat{X} \rightarrow X')$. As is easily seen, $\hat{B}$ is strictly singular.

We can now prove

Theorem 2a. *Let X, X' be Banach spaces. Let A be a closed linear operator $(X \rightarrow X')$ with closed range and with $\alpha(A) < \infty$. Let B be a closed linear operator $(X \rightarrow X')$ strictly singular relative to A. Then $A+B$ is a closed linear operator $(X \rightarrow X')$ with closed range and $\alpha(A+B) < \infty$.*

Proof. Let $\hat{X}$, $\hat{A}$, $\hat{B}$ be as above. Both $\hat{A}$ and $\hat{B}$ are bounded and defined everywhere in $\hat{X}$. Since $\hat{B}$ is strictly singular, $\alpha(\hat{A}) = \alpha(A) < \infty$ and $R[\hat{A}] = R[A]$ is closed, Theorem 2 is applicable to the pair $\hat{A}$, $\hat{B}$, with the result that $R[A+B] = R[\hat{A} + \hat{B}]$ is closed and

$$\alpha(A+B) = \alpha(\hat{A} + \hat{B}) < \infty .$$

It only remains to show that $A+B$ is a closed operator. To see this, it is convenient to apply Lemma 341. Let E be the operator which maps every $x \in D$ onto the same element $x \in \hat{X}$. The inverse E^{-1} is defined on $\hat{X}$ and bounded by $\|E^{-1}\| \leqq 1$. Hence E is a closed linear operator $(X \rightarrow \hat{X})$. Since $A + B = (\hat{A} + \hat{B})E$, it follows from Lemma 341 that $A+B$ is closed.

6. Properties of strictly singular operators. This paragraph is devoted to proving some lemmas on strictly singular operators introduced in the preceding paragraph.

Lemma 461. *If B is a strictly singular operator, the same is true for AB and BC whenever the products are significant and A, C are closed, bounded linear operators.*

Proof. Both AB and BC are bounded, closed linear operators. Suppose that we have a linear manifold $M \subset D[AB]$ such that $\|ABx\| \geqq \gamma \|x\|$, $\gamma > 0$, for all $x \in M$. Then we have $\|Bx\| \geqq \gamma \|A\|^{-1} \|x\|$ so that $\dim M < \infty$, which shows that AB is strictly singular. Suppose again that $\|BCx\| \geqq \gamma \|x\|$, $\gamma > 0$, for all $x \in M \subset D[BC]$. Then $\|BCx\| \geqq \gamma \|C\|^{-1} \|Cx\|$ and so we must have $\dim CM < \infty$. But since $Cx = 0$ and $x \in M$ imply $x = 0$, we have $\dim M = \dim CM < \infty$. This shows that BC is strictly singular.

Lemma 462. *If B_1, B_2 are strictly singular operators $(X \to X')$, their linear combination $\lambda_1 B_1 + \lambda_2 B_2$ is also strictly singular.*

Proof. It is obvious that we may assume that $\lambda_1 = \lambda_2 = 1$. Set $B = B_1 + B_2$ and suppose that there is an $M \subset D[B] = D[B_1] \cap D[B_2]$ such that $\|Bx\| \geqq \gamma \|x\|$, $\gamma > 0$, for every $x \in M$. We have to show that $\dim M < \infty$. As remarked above, we may assume that M is a closed linear manifold.

Let B^0, B^0_1, B^0_2 be respectively the restrictions of B, B_1, B_2 to M. B^0_1, B^0_2 are strictly singular. $\|B^0 x\| \geqq \gamma \|x\|$ shows that B^0 has closed range with $\alpha(B^0) = 0$. It follows from Theorem 2 that $B^0 - B^0_1 = B^0_2$ has closed range with $\alpha(B^0_2) < \infty$. Let $N = N[B^0_2]$. Since $\dim N < \infty$, there is a closed linear manifold L such that $M = L \oplus N$. Let B_0 be the restriction of B^0_2 to L. Then $\alpha(B_0) = 0$ and $R[B_0] = R[B^0_2]$ is closed. Thus we have $\|B^0_2 x\| = \|B_0 x\| \geqq \gamma_0 \|x\|$, $\gamma_0 = \gamma(B_0) > 0$, for all $x \in L$. Since B^0_2 is strictly singular, we must have $\dim L < \infty$. Hence $\dim M < \infty$ as we wished to show.

Lemma 463. *Let B_n, $n = 1, 2, 3, \ldots$, be a sequence of strictly singular operators $(X \to X')$ with common domain D such that $\|B_n - B\| \to 0$, $n \to \infty$, for some bounded linear operator B with domain D. Then B is strictly singular.*

Proof. Let $M \subset D$ be a closed linear manifold such that $\|Bx\| \geqq \gamma \|x\|$, $\gamma > 0$, for $x \in M$. We have to show that $\dim M < \infty$. Let n be so large that $\|B_n - B\| = \varepsilon < \gamma$. Then $\|B_n x\| \geqq \|Bx\| - \|(B_n - B) x\| \geqq (\gamma - \varepsilon) \|x\|$ for $x \in M$. Since B_n is strictly singular, we must have $\dim M < \infty$.

Remark. The above lemmas show that the set of all strictly singular operators $(X \to X)$ with domain X is a closed two-sided ideal of the ring of all bounded linear operators. From this and the fact that the identity operator I of X is not strictly singular if $\dim X = \infty$, it follows that the strict singularity implies complete continuity if $X' = X$ and X is a Hilbert space. But this is true even if $X' \neq X$ provided X and X' are Hilbert spaces. A simple proof of this fact is given by considering the operator $|B| = (B^* B)^{1/2}$ which is a self-adjoint operator on X. (Note that here we use B^* in the sense of the "Hilbert adjoint", deviating from the general usage in the present paper of the adjoint operator in the sense of "Banach adjoint", see Zaanen [15].) As is well known, there is a partially isometric operator $C\,(X \to X')$ such that $B = C\,|B|$, $|B| = C^* B$. Thus, by Lemma 461, B is strictly singular if and only if $|B|$ is strictly singular. But it is easily seen that a self-adjoint, strictly singular operator is completely continuous. Thus we see that $|B|$ and hence $B = C\,|B|$ are completely continuous.

§5. $\nu(A : B)$

1. Definition.

In this paragraph we consider two vector spaces X, X' and two linear operators A, B $(X \to X')$. We define by induction two sequences of linear manifolds of X

$$M_n = M_n[A : B], \quad N_n = N_n[A : B], \qquad n = 0, 1, 2, ..$$

in the following way.

$$M_0 = X, \quad M_n = B^{-1}(AM_{n-1}), \qquad n = 1, 2, ... \tag{511}$$

$$N_0 = \{0\}, \quad N_n = A^{-1}(BN_{n-1}), \qquad n = 1, 2, ... \tag{512}$$

Here $A^{-1}S'$ denotes the inverse image of S' under the map A, see §3.1.

It is easy to see by induction that

$$\begin{aligned} X = M_0 \supset D[B] \supset M_1 \supset M_2 \supset ... \supset N[B], \\ \{0\} = N_0 \subset N_1 \subset N_2 \subset ... \subset D[A]. \end{aligned} \tag{513}$$

It should be noted that

$$N_1 = N_1[A : B] = N[A]. \tag{514}$$

In the particular case in which $X' = X$ and B is the identity operator I of X, we have

$$M_n[A : I] = R[A^n], \quad N_n[A : I] = N[A^n]. \tag{515}$$

The following inequalities can be proved easily by the definition (511), (512) and the inequalities (314), (315), (316),

$$\dim N_n \leqq n\,\alpha(A), \quad \dim BN_n \leqq n\,\alpha(A). \tag{516}$$

$$\begin{aligned} \operatorname{codim} M_n \leqq n\,\beta(A) + n \operatorname{codim} D[B], \\ \operatorname{codim} AM_n \leqq (n+1)\,\beta[A] + n \operatorname{codim} D[B]. \end{aligned} \tag{517}$$

The following lemma is basic in our work.

Lemma 511. Let n be a positive integer. The following $n+1$ conditions are equivalent to one another.

$$(1)\ N_1 \subset M_n,\quad (2)\ N_2 \subset M_{n-1},\ \dots,\ (n)\ N_n \subset M_1,$$
$$(*)\ AN_{k+1} = BN_k,\quad N_k \subset D[B],\quad k = 1, 2, \dots, n.$$

When these equivalent conditions are satisfied, we have

$$\dim N_{k+1} = \dim BN_k + \alpha(A), \tag{518}$$
$$\operatorname{codim} AM_k = \operatorname{codim} M_k + \beta(A), \qquad k = 0, 1, \dots, n. \tag{519}$$

Proof. First we prove the equivalence of the n conditions (1) to (n). For each $r = 1, 2, \dots, n-1$, (r) implies ($r+1$). In fact, if $N_r \subset M_{n-r+1}$, then[17] by (513) and (514)

$$N_{r+1} = A^{-1}(BN_r) \subset A^{-1}(BM_{n-r+1}) \subset A^{-1}(AM_{n-r}) \subset M_{n-r} + N_1$$
$$\subset M_{n-r} + N_r \subset M_{n-r} + M_{n-r+1} = M_{n-r}.$$

Conversely ($r+1$) implies (r). In fact, if $N_{n+r} \subset M_{n-r}$, then

$$N_r \subset N_{r+1} \subset M_{n-r} = B^{-1}(AM_{n-r-1})$$

so that each $x \in N_r$ has the property that $Bx = Ay$ with a $y \in M_{n-r-1}$. Then $y \in A^{-1}(BN_r) = N_{r+1} \subset M_{n-r}$ and $x \in B^{-1}(AM_{n-r}) = M_{n-r+1}$. This proves $N_r \subset M_{n-r+1}$.

Next we prove that (*) is equivalent to other conditions. Suppose that (n) is satisfied. This implies that $N_k \subset N_n \subset M_1 = B^{-1}(AX) \subset D[B]$ for $k \leq n$, so that each $x \in N_k$ has the property that $Bx = Ay$ for some $y \in X$. Then $y \in A^{-1}(BN_k) = N_{k+1}$ and so $Bx \in AN_{k+1}$. This shows that $BN_k \subset AN_{k+1}$. But the definition (512) of N_{k+1} implies the opposite inclusion. Thus (n) implies (*).

Conversely (*) implies $BN_n = AN_{n+1}$ and hence

$$N_n \subset B^{-1}(AN_{n+1}) \subset B^{-1}(AX) = M_1,$$

that is, (*) implies (n).

(518) now follows from Lemma 312 because $N_{k+1} = A^{-1}(BN_k)$ and $BN_k = AN_{k+1} \subset R[A]$. Similarly (519) follows from Lemma 313 because $N[A] = N_1 \subset M_k$.

Since M_n is a non-increasing sequence by (513), the conditions of

17. Note that, for any linear manifolds $M \subset X$, $M' \subset X'$ and any linear operator $A\ (X \to X')$, we have

$$A(A^{-1}M') \subset M',\quad A^{-1}(AM) \subset M + N[A].$$

Lemma 511 are satisfied for all $n' \leqq n$ if they are satisfied for n. The smallest number n for which these conditions are not satisfied will be denoted by $\nu(A:B)$. If there is no such n, we set $\nu(A:B) = \infty$. Obviously we have $\nu(A:B) \geqq 1$ for any pair A, B. It should be noted that $\nu(A:B) = \infty$ if either $\alpha(A) = 0$ or $\beta(A) = 0$ and $D[B] = X$ (see (516) and (517)).

If $X' = X$, $\nu(A:I)$ is the smallest n for which $N[A]$ is not a subset of $A^n X$ or, what is the same thing, for which $N[A^n]$ is not a subset of $R[A]$.

2. Lemmas on $\nu(A:B)$.

The following lemmas are required later.

Lemma 521. Let S be a non-empty subset of $N[A]$ and define S_k, $k = 1, 2, \ldots$, successively by $S_1 = S$, $S_{k+1} = A^{-1}(BS_k)$. Then S_k is not empty for $k \leqq \nu = \nu(A:B)$. Let L_k be the linear hull of S_k. Then

$$S_k \subset L_k \subset N_k = N_k[A:B], \qquad k = 1, 2, \ldots, \nu, \tag{521}$$

$$\left.\begin{array}{ll} N[A] \subset L_k = A^{-1}(BL_{k-1}), & (522) \\ AS_k = BS_{k-1}, \quad AL_k = BL_{k-1}, & (523) \end{array}\right\} \quad k = 2, 3, \ldots, \nu.$$

Proof. It is obvious that $S_k \subset N_k$, compare the definition of S_k and N_k. Thus the argument used to deduce (*) from (n) in the proof of Lemma 511 is applicable to show that $S_k = BS_{k-1}$ as long as $2 \leqq k \leqq \nu$. In particular this shows successively that $S_1, \ldots, S_\nu$ are non-empty. (Note that $S_{k-1} \subset N_{k-1} \subset M_1 \subset D[B]$ for $k \leqq \nu$ and that $S_k \subset D[A]$ for all k.) Also it follows that $AL_k = BL_{k-1}$. This proves (523).

(523) implies that $A^{-1}(BL_{k-1}) = A^{-1}(AL_k) = L_k + N[A]$. Thus (522) will be proved if we only show that $N = N[A] \subset L_k$ for $2 \leqq k \leqq \nu$. But this is an easy consequence of the fact that S_k is not empty. In fact, let $x \in S_k$. If $k \geqq 2$, we have $A(x+z) = Ax \in BS_{k-1}$ for any $z \in N$. Thus $x + z \in A^{-1}(BS_{k-1}) = S_k$ and so $z = (x+z) - x \in L_k$, which shows that $N \subset L_k$.

Lemma 522. Let S be a subset of $N[A]$ and, for each $z \in S$ let there exist a sequence $z_1, \ldots, z_n$ such that $z_1 = z$ and $Az_{k+1} = Bz_k$, $k = 1, 2, \ldots, n-1$, where n is independent of z. If the linear hull of S is identical with $N[A]$, we have

$\nu(A:B)\geq n$. If S_k, L_k are defined as in Lemma 521, we have $L_k = N_k(A:B)$ for $k=1, 2, \dots n$.

Proof. $Az_n = Bz_{n-1}$ implies that $z_{n-1}\in B^{-1}(AX) = M_1$. Then $Az_{n-1}=Bz_{n-2}$ implies that $z_{n-2}\in B^{-1}(AM_1)=M_2$, and so on. In this way we see that $z=z_1\in M_{n-1}$. Thus $S\subset M_{n-1}$ and so $N_1 = N[A]\subset M_{n-1}$ by hypothesis, which proves that $\nu(A:B)\geq n$. Then (522) shows successively that $L_k = N_k$, $k=1, 2, \dots, n$ because $L_1 = N_1$.

3. Restriction of operators.

The definition of $\nu(A:B)$ depends, at least formally, on the underlying spaces X and X', for this is certainly the case with the linear manifolds $M_n[A:B]$ and $N_n[A:B]$. Let us now inquire how these are changed when A and B are restricted. For later application we need to consider rather a simple case.

Lemma 531. Let A, B be linear operators $(X\to X')$ and let X_0 be a linear manifold of X such that $D[A]\subset X_0\subset D[B]$. Let A_0, B_0 $(X_0\to X')$ be respectively the restrictions of A, B to X_0. Then

(531) $$M_n[A_0:B_0] = M_n[A:B]\cap X_0,$$

(532) $$N_n[A_0:B_0] = N_n[A:B]\subset X_0,$$

(533) $$\nu(A_0:B_0) = \nu(A:B).$$

Proof. For simplicity we write $M_n = M_n[A, B]$, $M_n^0 = M_n[A_0:B_0]$ and similarly for N_n. First we note that $D[A_0] = D[A]\subset X_0$ and $D[B_0] = X_0$. It is also obvious that

$$N_n\subset D[A]\subset X_0,\quad N_n^0\subset D[A_0] = D[A]\subset X_0.$$

Since $M_0^0 = X_0 = M_0\cap X_0$ and $N_0^0 = \{0\} = N_0$, we employ induction to prove (531) and (532). Suppose that (531) has been proved for $n-1$ instead of n. Then $A_0M_{n-1}^0 =$
$A_0(M_{n-1}^0\cap D[A_0]) = A_0(M_{n-1}\cap X_0\cap D[A_0]) = A(M_{n-1}\cap D[A]) = AM_{n-1}$, hence $M_n^0 = B_0^{-1}(A_0M_{n-1}^0) = B_0^{-1}(AM_{n-1}) = B^{-1}(AM_{n-1})\cap D[B_0] = M_n\cap X_0$. Again suppose that (532) has been proved for $n-1$ instead of n. Then $B_0N_{n-1}^0 = BN_{n-1}$ and

$$N_n^0 = A_0^{-1}(B_0N_{n-1}^0) = A^{-1}(BN_{n-1})\cap D[A_0] = N_n\cap D[A] = N_n\subset X_0.$$

This completes the induction.

Now it can be seen that the conditions of Lemma 511 are equivalent whether applied to the pair A, B or to the pair A_0, B_0. To see this it is sufficient to observe that $N_1 \subset M_n$ is equivalent to $N_1 \subset M_n \cap X_0$, since we know that $N_1 \subset X_0$. But the latter condition is exactly $N_1^0 \subset M_n^0$ by (531) and (532). This leads immediately to (533).

4. Closed linear operators.

The linear manifolds $M_n = M_n[A:B]$, $N_n = N_n[A:B]$ and the number $\nu = \nu(A:B)$ have been defined in a purely algebraic fashion without reference to any topology. We shall now consider their relation to the topology of the spaces and the closure of the operators A, B.

Let X, X' be Banach spaces and let A, B be closed linear operators $(X \to X')$. Then $N_1 = N[A]$ is a closed linear manifold of X, but other linear manifolds $M_n, N_n, n \geqq 1$, are not necessarily closed without further assumption. Our first lemma in this connection is

Lemma 541. *Let A be a closed linear operator $(X \to X')$ with closed range $(\gamma(A) > 0)$ and with $\alpha(A) < \infty$. Let B be a closed, bounded linear operator $(X \to X')$. Then M_n, AM_n and N_n are closed linear manifolds of X and $\dim N_n < \infty$ for all n.*

Proof. $\dim N_n < \infty$ follows from (516), so that N_n are closed. To prove the closure of M_n and AM_n, it is sufficient to show that AM and $B^{-1}(AM)$ are closed whenever M is closed. Since $\dim N[A] = \alpha(A) < \infty$, the linear manifold $M + N[A]$ is closed with M, and this implies that AM is closed (Lemma 331). Since B is a continuous function, the inverse image $B^{-1}(AM)$ of the closed set AM is closed in $D[B]$, which is closed in X because B is closed and bounded. Thus $B^{-1}(AM)$ is closed in X.

Lemma 542. *Let A be a closed linear operator $(X \to X')$ with $\beta(A) < \infty$. Let B be a closed, bounded linear operator $(X \to X')$ with $\operatorname{codim} D[B] < \infty$. Then M_n, AM_n are closed and $\operatorname{codim} M_n < \infty$, $\operatorname{codim} AM_n < \infty$ for all n (in particular $R(A)$ is closed).*

Proof. $R(A)$ is closed by Lemma 332. That $\operatorname{codim} M_n$ and $\operatorname{codim} AM_n$ are finite follows from (517). The closure of M_n and AM_n is obvious for $n = 0$. Suppose that this has been proved for n. Then $M_{n+1} = B^{-1}(AM_n)$

is closed as the inverse image of AM_n as in the preceding lemma. Since codim M_{n+1} is finite, it follows that $M_{n+1}+N[A]$ is closed. Hence AM_{n+1} is closed by Lemma 331. This completes the induction.

In the special case in which $X'=X$ and $B=I$, these lemmas take the following form by (515).

Lemma 543. Let A be a closed linear operator $(X\to X)$ with closed range and with $\alpha(A)<\infty$. Then, for every $n=1,2,\dots$, A^n is a closed linear operator $(X\to X)$ and $N[A^n]$, $R[A^n]$ are closed with dim $N[A^n]<\infty$.

Proof. Only the proposition that A^n is closed needs a proof. This can be proved by successive application of Lemma 341.

Lemma 544. Let A be a closed linear operator $(X\to X)$ with $\beta(A)<\infty$. Then, for every $n=1,2,\dots$, $R[A^n]$ is closed and codim $R[A^n]<\infty$.

Remark. There is not a complete duality between Lemmas 543 and 544. It should be noted that much less can be asserted in Lemma 544 than in Lemma 543. (A similar relation exists between Lemmas 541 and 542). In Lemma 544 $N[A^n]$ need not be closed if $n\geqq 2$ and, consequently, A^n need not be closed. This can be seen from the following example in which X is a Hilbert space and $N[A^2]$ is not closed.

Let X be a Hilbert space with a complete orthonormal set

$$x_j,\ j=0,\ \pm 1,\ \pm 2,\ \dots .$$

Define the operator $A(X\to X)$ by

$$A\sum_{j=-\infty}^{\infty}\xi_j x_j=\sum_{j=1}^{\infty}\xi_{2j}x_j+\sum_{j=1}^{\infty}j\,\xi_{2j-1}x_{1-j}. \tag{541}$$

The domain of A is by definition the set of all $x=\Sigma\xi_j x_j$ of X such that $\Sigma j^2|\xi_{2j-1}|^2<\infty$.

In particular we have

$$\begin{aligned} &Ax_j=0 \quad \text{for } j=0,\ -1,\ -2,\dots,\\ &Ax_1=x_0,\ Ax_3=2x_{-1},\ Ax_5=3x_{-2},\ \dots,\\ &Ax_2=x_1,\ Ax_4=x_2,\ Ax_6=x_3,\ \dots . \end{aligned} \tag{542}$$

Let N, X_1, X_2 be the mutually orthogonal closed linear manifolds of X spanned respectively by $\{x_0,x_{-1},x_{-2},\dots\}$, $\{x_1,x_3,\dots\}$, $\{x_2,x_4,\dots\}$. Obviously $X=N\oplus X_1\oplus X_2$ and $N[A]=N$. As is easily seen, we have

$$AX_1 = N, AX_2 = X_1 \oplus X_2, A(X_1 \oplus X_2) = X$$

and the restriction of A to $X_1 \oplus X_2$ has a bounded inverse (with bound 1) with domain X. Thus A is closed, has closed range X and $\beta(A) = 0$, $\gamma(A) = 1$. Now $N[A^2]$ is not closed. In fact, it is easy to see that

$$N[A^2] = N \oplus (X_1 \cap D[A]),$$

which is not closed because $X_1 \cap D[A]$ is not closed.

In this example we have $\alpha(A^*) = \beta(A) = 0$, and A^* has closed range by Lemma 324 (actually we have $R[A^*] = X_1 \oplus X_2$). Thus $(A^*)^n$ are closed linear operators by Lemma 543. That A^2 is not closed implies, therefore, that $A^2 \neq ((A^*)^2)^*$. But this is not surprising since A is not bounded.

5. An estimate of $\nu(A:B)$.

The following lemma is, in a certain sense, a generalization of Lemma 522.

Lemma 551. *Let A be a closed linear operator $(X \to X')$ with closed range. Let B be a bounded, closed linear operator $(X \to X')$. Let S be a subset of $N = N[A]$ such that, for each $z \in S$, there is a sequence $z_1, \ldots, z_n$ with the properties $z_1 = z$, $Az_{k+1} = Bz_k$, $k = 1, 2, \ldots, n-1$, where n is independent of z. Further assume that the linear hull L of S is dense in N. Then we have $\nu(A:B) \geqq n$ and L_k is dense in $N_k = N_k[A:B]$ for $k = 1, 2, \ldots, n$, where L_k are defined as in Lemma 521.*

Proof. We prove the lemma by induction. The proposition is obvious for $n = 1$. Suppose that the proposition has been proved for n replaced by $n-1$ and assume that the assumptions are satisfied for n.

By induction hypothesis it follows immediately that $\nu = \nu(A:B) \geqq n-1$ (which is equivalent to $N_{n-2} \subset M_1$) and that L_k is dense in N_k for $k = 1, 2, \ldots, n-1$. We have to prove that $N_{n-1} \subset M_1$ and that L_n is dense in N_n.

Let $u_{n-1} \in S_{n-1}$, where S_k are defined as in Lemma 521. By definition $u_{n-1} \in A^{-1}(BS_{n-2})$ so that there is a $u_{n-2} \in S_{n-2}$ such that $Au_{n-1} = Bu_{n-2}$. Proceeding in this way, we see that there is a sequence $u_1, u_2, \ldots, u_{n-1}$ such that

$$Au_2 = Bu_1, \; Au_3 = Bu_2, \ldots, \; Au_{n-1} = Bu_{n-2}$$

where $u_1 \in S = S_1$. Then it follows from the assumption that there is a sequence $z_1, \ldots, z_n$ such that $z_1 = u_1$, $Az_2 = Bz_1$, $Az_3 = Bz_2, \ldots, Az_n = Bz_{n-1}$.

On setting $v_k = u_{k+1} - z_{k+1}$, $k = 1, 2, \dots, n-2$, we see that

$$v_1 \in N,\ Av_2 = Bv_1,\ Av_3 = Bv_2, \dots, Av_{n-2} = Bv_{n-3}.$$

Thus we have successively $v_2 \in A^{-1}(BN) = N_2, \dots, v_{n-2} \in A^{-1}(BN_{n-3}) = N_{n-2}$. Since we know that $N_{n-2} \in M_1$ as noted above, we see that $v_{n-2} \in M_1$. Also we have $z_{n-1} \in B^{-1}(AX) = M_1$. Hence $u_{n-1} = v_{n-2} + z_{n-1} \in M_1$. Since u_{n-1} was an arbitrary element of S_{n-1}, we conclude that $S_{n-1} \in M_1$ and hence $L_{n-1} \subset M_1$ because L_{n-1} is the linear hull of S_{n-1} (see Lemma 521). But L_{n-1} is dense in N_{n-1} as we remarked above and M_1 is closed as the inverse image of $R[A]$ under the map B. Hence we have that $N_{n-1} \subset M_1$ and so $\nu \geqq n$.

It remains to show that L_n is dense in N_n. Let $x \in N_n$ be arbitrary. Since $N_n = A^{-1}(BN_{n-1})$, there is a $y \in N_{n-1}$ such that $Ax = By$. Since L_{n-1} is dense in N_{n-1}, there is a sequence $y_k \in L_{n-1}$ such that $y_k \to y$, $k \to \infty$. We may assume without loss of generality that

$$\| y_k - y_{k-1} \| \leqq 2^{-k},\ k = 1, 2, 3, \dots;$$

otherwise we need only to replace the sequence y_k by its suitable subsequence. Setting $w_k = y_k - y_{k-1}$, we can write

$$y = y_0 + \sum_{k=1}^{\infty} w_k,\ \| w_k \| \leqq 2^{-k},\ w_k \in L_{n-1}.$$

Since $L_{n-1} \subset N_{n-1} \subset M_1 = B^{-1}(AX)$ as just proved, there is for each k an $x_k \in D[A]$ such that $Bw_k = Ax_k$. We have

$$\| B \|\ \| w_k \| \geqq \| Bw_k \| = \| Ax_k \| \geqq \gamma \| \tilde{x}_k \|,$$

where $\tilde{x}_k \in \tilde{X} = X/N$ and $\gamma = \gamma(A) > 0$ by hypothesis. Obviously we may choose x_k in such a way that $\| x_k \| \leqq (1+\varepsilon) \| \tilde{x}_k \|$, where ε is any given positive number. Then we have

$$\| x_k \| \leqq (1+\varepsilon)\gamma^{-1} \| B \|\ \| w_k \| \leqq (1+\varepsilon)\gamma^{-1} \| B \| 2^{-k}$$

and the series Σx_k is absolutely convergent.

Since $y_0 \in L_{n-1} \subset M_1$, we can take an $x_0 \in D[A]$ such that $Ax_0 = By_0$. Now we set

$$x' = x_0 + \sum_{k=1}^{\infty} x_k$$

Then

$$Ax' = Ax_0 + \Sigma Ax_k = By_0 + \Sigma Bw_k = By;$$

the term-by-term application of A can be justified easily by considering the closure of A and B and the boundedness of B.

Since $w_k \in L_{n-1}$, x_k belongs to $A^{-1}(BL_{n-1}) = L_n$ by (522); note that we have already proved that $\nu \geqq n$. Similarly we have $x_0 \in L_n$. It follows that x' belongs to the closure $[L_n]$ of L_n. On the other hand, we have $Ax' = By = Ax$ and so $x - x' \in N \subset L_n$ by (522). Thus we see that $x \in L_n + [L_n] = [L_n]$. Since x was an arbitrary element of N_n, we conclude that L_n is dense in N_n, thereby completing the induction.

6. Adjoint operators.

Let A, B be closed linear operators $(X \to X')$ with domains $D[A]$, $D[B]$ dense in X so that the adjoints A^*, B^* exist and are closed linear operators $(X'^* \to X^*)$. We consider the sequences

(561) $$M^*_n = M_n[A^* : B^*],\ N^*_n = N_n[A^* : B^*]$$

defined for the pair A^*, B^*. M^*_n, N^*_n are linear manifolds of X'^*.

Lemma 561. *Under the above conditions we have*

(562) $$M^*_n \subset (BN_n)^\perp,\quad N^*_{n+1} \subset (AM_n)^\perp,\quad n = 0, 1, 2, \ldots, \ldots$$

Proof. (562) is true for $n = 0$: the first inclusion is trivial by $M^*_0 = X'^*$ and $N_0 = \{0\}$, and the second inclusion follows from

$$N^*_1 = N[A^*] = R[A]^\perp = (AM_0)^\perp \text{ (see (324))}.$$

We shall therefore assume that (562) has been proved for $n-1$ in place of n and prove it for n. Let $f' \in M^*_n$ and $x \in D[B] \cap N_n$. By definition $f' \in (B^*)^{-1}(A^* M^*_{n-1})$ and $x \in A^{-1}(BN_{n-1})$. Hence there are a $g' \in M^*_{n-1}$ and a $y \in N_{n-1}$ such that $B^* f' = A^* g'$ and $Ax = By$. Consequently

$$f'(Bx) = B^* f'(x) = A^* g'(x) = g'(Ax) = g'(By) = 0$$

because $M^*_{n-1} \subset (BN_{n-1})^\perp$ by induction hypothesis. This proves the first inclusion of (562). The second inclusion can be proved in the same way.

It should be observed that in general the inclusion $\subset$ in (562) cannot be replaced by the equality except for the case $n = 0$. The condition under which the equality holds in (562) does not seem to be simple and we shall not consider it in detail. We only note the following

Lemma 562. *Let A be a closed linear operator $(X \to X')$ with domain dense in X and with closed range. Let B be a bounded linear operator $(X \to X')$ with $D[B] = X$. Then*

(563) $$M^*_1 = (BN_1)^{\perp}.$$

(564) $$\nu(A^* : B^*) = \nu(A : B).$$

Proof. Let $f' \in (BN_1)^{\perp}$ and $f = B^* f'$. Then $f \in N_1^{\perp} = N[A]^{\perp}$ and so $f \in R[A^*]$ by Lemma 335. It follows that $f' \in (B^*)^{-1}(A^* X'^*) = M_1^*$. This shows that $(BN_1)^{\perp} \subset M_1^*$. Since the opposite inclusion holds by (562), this proves (563).

According to the definition of $\nu = \nu(A:B)$, we have $N_1 \subset M_n$ for any $n < \nu$. Since $M_n = B^{-1}(AM_{n-1})$, we have $BN_1 \subset BM_n \subset AM_{n-1}$ and hence $(AM_{n-1})^{\perp} \subset (BN_1)^{\perp}$. In virtue of (562) and (563), it follows that $N_n^* \subset M_1^*$. This implies that $\nu^* = \nu(A^* : B^*) > n$, and proves that $\nu^* \geqq \nu$.

To prove the opposite inequality, let $n < \nu^*$. Then we have $N_1^* \subset M_n^*$ and this implies $(AX)^{\perp} \subset (BN_n)^{\perp}$ by (324) and (562). Since AX is closed, this implies in turn that $BN_n \subset AX$. Since $D[B] = X$, this finally implies that

$$N_n \subset B^{-1}(AX) = M_1.$$

This shows that $\nu > n$ and hence that $\nu \geqq \nu^*$.

§6. STABILITY THEOREM IN THE CASE $\nu(A:B) = \infty$

1. Theorem.

In what follows we are interested in the eigenvalue problem of the form

(611) $$Ax = \lambda Bx$$

where A, B are linear operators $(X \rightarrow X')$ and also in the associated equation

(612) $$A^* f' = \lambda B^* f'$$

when the adjoints A^*, B^* exist. (611) means that $x \in N[A - \lambda B]$ and (612) means that $f' \in N[A^* - \lambda B^*] = R[A - \lambda B]^{\perp}$. We are thus interested in the behavior of the linear manifolds $N[A - \lambda B]$, $R(A - \lambda B]^{\perp}$.

The main purpose of the present section is to prove the following theorem.

Theorem 3. Let X, X' be Banach spaces, let A be a closed linear operator $(X \rightarrow X')$ with closed range not identical with $\{0\}$ (so that $0 < \gamma(A) < \infty$) and let B be a linear operator $(X \rightarrow X')$ such that $D[B] \supset D[A]$ and

(613) $$\| Bx \| \leqq \sigma \| x \| + \tau \| Ax \| \quad \text{for every } x \in D[A],$$

where σ, τ are non-negative constants. Furthermore let

$\nu(A:B)=\infty$. Then there is a constant $\rho>0$ such that, for $|\lambda|<\rho$, $A-\lambda B$ is a closed linear operator ($X \to X'$) with closed range, $\alpha(A-\lambda B)$ and $\beta(A-\lambda B)$ are constant,

$$\nu(A-\lambda B:B)=\infty$$

and $\gamma(A-\lambda B)$ is finite, positive and Lipschitz-continuous. The closed linear manifolds $N(A-\lambda B)\subset X$ and $R[A-\lambda B]^{\perp}\subset X'^{*}$ depend on λ analytically: by this we mean that each λ_0, $|\lambda_0|<\rho$, has a neighborhood with the following property[18]. For each $z\in N[A-\lambda_0 B]$, there is a regular analytic function $\lambda\to z(\lambda)\in N[A-\lambda B]$ defined in this neighborhood such that $z(\lambda_0)=z$ and the closed linear hull of all these $z(\lambda)$ for fixed λ coincides with $N[A-\lambda B]$. Similar proposition holds for $R[A-\lambda B]^{\perp}$.

For the constant ρ we can take in general

$$\rho=\gamma/(3\sigma+\tau\gamma) \qquad (\gamma=\gamma(A)>0) \tag{614}$$

and the Lipschitz continuity of $\gamma(\lambda)=\gamma(A-\lambda B)$ can be expressed by

$$|(d/d\lambda)\log[3\sigma+\tau\gamma(\lambda)]|\leqq\tau/(1-\tau|\lambda|), \tag{615}$$

$$\gamma-(3\sigma+\tau\gamma)|\lambda|\leqq\gamma(\lambda)\leqq(\gamma+3\sigma|\lambda|)/(1-\tau|\lambda|), \tag{616}$$

where $|d/d\lambda|$ denotes the upper derivative. If $\tau=0$, (615) should be replaced by

$$|d\gamma(\lambda)/d\lambda|\leqq 3\sigma. \tag{617}$$

The number 3σ in (614), (615), (616) and (617) can be replaced by σ if at least one of the following conditions is satisfied: (i) $\alpha(A)<\infty$, (ii) $\beta(A)<\infty$, (iii) X is a Hilbert space, (iv) X' is a Hilbert space.

Before going into the proof of this theorem, we note that $A-\lambda B$ is a closed linear operator for $|\lambda|<\tau^{-1}$, as is seen from the proof of Theorem 1a, § 4.4. In what follows we consider only such λ.

For brevity we use the notations

18. If $\alpha(A)<\infty$, it follows that there is a basis of $N[A-\lambda B]$ consisting of $\alpha(A)$ vectors $z_i(\lambda)$, $i=1,\dots,\alpha(A)$, which are regular analytic in the neighborhood of λ_0. Similar result holds for $R[A-\lambda B]^{\perp}$ if $\beta(A)<\infty$.

$$
\begin{aligned}
&D = D[A] = D[A-\lambda B], && \tilde{X} = X/N[A],\\
&N(\lambda) = N[A-\lambda B], && N(0) = N[A] = N\\
&R(\lambda) = R[A-\lambda B], && R(0) = R[A] = R',\\
&\alpha(\lambda) = \alpha(A-\lambda B), && \alpha(0) = \alpha(A) = \alpha,\\
&\beta(\lambda) = \beta(A-\lambda B), && \beta(0) = \beta(A) = \beta,\\
&\gamma(\lambda) = \gamma(A-\lambda B), && \gamma(0) = \gamma(A) = \gamma,\\
&\nu(\lambda) = \nu(A-\lambda B:B), && \nu(0) = \nu(A:B) = \infty.
\end{aligned}
$$

Note that $N(\lambda)$ is a closed linear manifold of X as the null space of the closed linear operator $A-\lambda B$.

Also we note that $\gamma(\lambda) < \infty$ for $|\lambda| < \rho$. In fact, $\gamma(\lambda) = \infty$ implies that $(A-\lambda B)x = 0$ for all $x \in D$. Hence

$$\|Ax\| = |\lambda| \, \|Bx\| \leqq |\lambda| (\sigma \|x\| + \tau \|Ax\|)$$

and so

$$\gamma \|\tilde{x}\| \leqq \|Ax\| \leqq \|x\| \sigma |\lambda| / (1-\tau|\lambda|),$$

where $\tilde{x} \in \tilde{X}$. Since x can vary freely in $\tilde{x}$, it follows that $\gamma \leqq \sigma|\lambda|/(1-\tau|\lambda|)$ and hence that $|\lambda| \geqq \gamma/(\sigma+\tau\gamma) \geqq \rho$. Here we made use of the fact that there is at least one $\tilde{x} \neq 0$ because $R[A] \neq \{0\}$ by hypothesis.

2. Stability of α.

The proof of Theorem 3 will be given in several steps. We shall first show that

$$\delta(N(\lambda), N) \leqq \sigma|\lambda| / (1-\tau|\lambda|)\gamma. \tag{621}$$

Let $u \in N(\lambda)$. Then $Au = \lambda Bu$ and the calculation given at the end of the preceding paragraph shows that $\gamma \|\tilde{u}\| \leqq \|u\| \sigma |\lambda| / (1-\tau|\lambda|)$. Recalling the definition (211) of $\delta(M, N)$ and that $\|\tilde{u}\| = d(u, N)$, this proves (621).

The right side of (621) is smaller than one if $|\lambda| < \gamma/(\sigma+\tau\gamma)$. Thus Lemma 212 shows that

$$\alpha(\lambda) \leqq \alpha \quad \text{for} \quad |\lambda| < \gamma/(\sigma+\tau\gamma). \tag{622}$$

We next prove the opposite inequality to (622) by an explicit construction. Let us consider a sequence z_k with the following properties:

$$
\begin{aligned}
&z_k \in N_k = N_k[A:B], \quad Az_{k+1} = Bz_k,\\
&\omega \|z_{k+1}\| \leqq \|Az_{k+1}\|, \quad k = 1, 2, \ldots,
\end{aligned}
\tag{623}
$$

where ω is a positive constant. For brevity such a sequence z_k will be called an ω-s e q u e n c e.

We shall show that, for each $z \in N$ and each $\omega < \gamma$, there is an ω-sequence such that $z_1 = z$. We set $z_1 = z$ and construct z_k by induction. Suppose that $z_1, \dots, z_k$ have been constructed with the properties (623). Since $z_k \in N_k \subset M_1 = M_1[A:B] = B^{-1}(AX)$ in virtue of $\nu(A:B) = \infty$, there is a $z_{k+1} \in D$ such that $Az_{k+1} = Bz_k$. This implies that $\gamma \|\tilde{z}_{k+1}\| \leqq \|Az_{k+1}\|$ and that z_{k+1} may be replaced by any other element of $\tilde{z}_{k+1}$. Thus we can choose z_{k+1} in such a way that $\omega \| z_{k+1} \| \leqq \| Az_{k+1} \|$. On the other hand we see that $z_{k+1} \in A^{-1}(BN_k) = N_{k+1}$. This completes the induction.

Each ω-sequence z_k satisfies the following inequalities:

$$\text{(624)} \qquad \begin{aligned} &\| Bz_k \| = \| Az_{k+1} \| \leqq \sigma\, \omega^{-(k-1)} (\sigma + \tau\, \omega)^{k-1} \| z_1 \|, \\ &\| z_{k+1} \| \leqq \sigma\, \omega^{-k} (\sigma + \tau\, \omega)^{k-1} \| z_1 \|, \quad k = 1, 2, \dots . \end{aligned}$$

To see this, we note that $\|Az_{k+1}\| = \|Bz_k\| \leqq \sigma \|z_k\| + \tau \|Az_k\|$. For $k = 1$ this gives $\| Az_2 \| \leqq \sigma \| z_1 \|$ in virtue of $Az_1 = 0$. For $k \geqq 2$ we have by (623) $\| Az_{k+1} \| \leqq (\sigma\, \omega^{-1} + \tau) \| Az_k \|$. This leads immediately to the first inequality of (624), and the second follows by another use of (623).

For each ω-sequence z_k we can now construct the series

$$\text{(625)} \qquad u(\lambda) = \sum_{k=1}^{\infty} \lambda^{k-1} z_k .$$

In virtue of (624) this series is absolutely convergent for $|\lambda| < \omega/(\sigma + \tau\, \omega)$. The same is true for the series obtained from (625) by term-by-term application of A and B. In view of the closure of A and the inequality (613), it follows easily that $u(\lambda) \in D$ and by (623) that $Au(\lambda) = \lambda Bu(\lambda)$. This shows that $u(\lambda) \in N(\lambda)$. Furthermore,

$$\| u(\lambda) - z_1 \| \leqq \sum_{k=2}^{\infty} |\lambda|^{k-1} \| z_k \| \leqq \delta_\omega(\lambda) \| z_1 \|,$$

where

$$\text{(626)} \qquad \delta_\omega(\lambda) = \sigma |\lambda| / [\omega - (\sigma + \tau\, \omega) |\lambda|].$$

Since there is such a $u(\lambda) \in N(\lambda)$ for each $z = z_1 \in N$ and $\omega < \gamma$, we conclude that

$$\text{(627)} \qquad \delta(N, N(\lambda)) \leqq \delta(\lambda) = \sigma |\lambda| / [\gamma - (\sigma + \tau\, \gamma) |\lambda|].$$

The right side is smaller than one if $|\lambda| < \gamma/(2\sigma + \tau\, \gamma)$. Then we have $\alpha \leqq \alpha(\lambda)$ by Lemma 212. Combined with (622), we have thus proved that

$$\text{(628)} \qquad \alpha(\lambda) = \alpha \quad \text{for} \quad |\lambda| < \gamma/(2\sigma + \tau\, \gamma).$$

Although the vector $u(\lambda)$ given by (625) belongs to $N(\lambda)$, it is not clear whether every vector of $N(\lambda)$ can be expressed in this form by means of an ω-sequence. (This is true if $\alpha<\infty$, at least for small $|\lambda|$.) In any case let $S^{\omega}(\lambda)$ be the set of these $u(\lambda)$ and let $L^{\omega}(\lambda)$ be the linear hull of $S^{\omega}(\lambda)$. We shall show that $L^{\omega}(\lambda)$ is dense in $N(\lambda)$ for

(629) $$|\lambda|<\gamma/(3\sigma+\tau\gamma)$$

provided ω is sufficiently near to γ. A simple calculation shows that (629) implies that (see (621))

$$\delta(N(\lambda), N)\leqq\sigma|\lambda|/(1-\tau|\lambda|)\gamma<[1-\delta(\lambda)]/[1+\delta(\lambda)].$$

The same inequality holds with $\delta(\lambda)$ replaced by $\delta_{\omega}(\lambda)$ if ω is sufficiently near to γ. Since there is a $u(\lambda)\in L^{\omega}(\lambda)\subset N(\lambda)$ such that

$$\|z-u(\lambda)\|\leqq\delta_{\omega}(\lambda)\|z\|$$

for each $z\in N$, it follows from Lemma 216 that $L^{\omega}(\lambda)$ is dense in $N(\lambda)$. This proves the analytic dependence of $N(\lambda)$ on λ at the particular point $\lambda=0$.

3. Continuity of γ.

According to Theorem 1a, § 4.4, $R(\lambda)$ is closed (that is, $\gamma(\lambda)>0$) if $|\lambda|<\gamma/(\sigma+\tau\gamma)$ and at least one of α, β is finite. But there we could not give any estimate of $\gamma(\lambda)$, see Remark 2 after Theorem 1. We shall now show that we can give an estimate of $\gamma(\lambda)$ under the present assumption $\nu(A:B)=\infty$.

According to Lemma 214 there is, for each $x\in X$ and $\varepsilon>0$, a $u\in N(\lambda)$ such that, for $y=x-u$,

(631) $$\|\tilde{y}\|=d(y,N)\geqq\frac{1-\delta(N, N(\lambda))}{1+\delta(N, N(\lambda))}(1-\varepsilon)\|y\|.$$

Suppose that $x\in D$. Since $(A-\lambda B)u=0$, we have

(632) $$\begin{aligned}\|(A-\lambda B)x\|=\|(A-\lambda B)y\|&\geqq\|Ay\|-|\lambda|\,\|By\|\\ &\geqq(1-\tau|\lambda|)\|Ay\|-\sigma|\lambda|\,\|y\|\\ &\geqq(1-\tau|\lambda|)\gamma\|\tilde{y}\|-\sigma|\lambda|\,\|y\|\end{aligned}$$

and, by making use of (631) and (627), obtain the inequality

(633) $$\|A-\lambda B)x\|\geqq[(\gamma-(2\sigma+\tau\gamma)|\lambda|)(1-\varepsilon)-\sigma|\lambda|]\,\|y\|.$$

We now consider the factor space $\overline{X}=X/N(\lambda)$, whose elements will be denoted by $\bar{x}, \bar{y}, \ldots$. Since $x-y=u \in N(\lambda)$, we have $\|y\| \geqq \|\bar{y}\|=\|\bar{x}\|$ so that the $\|y\|$ on the right of the inequality (633) can be replaced by $\|\bar{x}\|$. Recalling the definition of $\gamma(\lambda)=\gamma(A-\lambda B)$, we see that $\gamma(\lambda)$ is not smaller than the number in the brackets of (633). On letting $\varepsilon \to 0$, we thus arrive at the inequality

$$\gamma(\lambda) \geqq \gamma-(3\sigma+\tau\gamma)|\lambda|. \tag{634}$$

This shows that $\gamma(\lambda)>0$ and hence that $R(\lambda)$ is closed for $|\lambda|<\gamma/(3\sigma+\tau\gamma)$. It should be noticed that this is true even if $\alpha=\beta=\infty$.

As we shall see below, (634) is the most important one among various inequalities derived above. We shall now show that it can be replaced by a sharper estimate

$$\gamma(\lambda) \geqq \gamma-(\sigma+\tau\gamma)|\lambda| \tag{635}$$

at least for $|\lambda|<\gamma/(2\sigma+\tau\gamma)$ in the special cases in which either (i) $\alpha<\infty$ or (iii) X is a Hilbert space.

For this purpose, we note that (631) can be improved as follows: for each $x \in X$ and $|\lambda|<\gamma/(2\sigma+\tau\gamma)$, there is a $u \in N(\lambda)$ such that $y=x-u$ has the property

$$\|\tilde{y}\|=d(y, N)=\|y\|. \tag{636}$$

This is obvious if $x \in N(\lambda)$, for then we need only to set $u=x$ so that $y=0$. So we may assume that $x \notin N(\lambda)$. Let $N^0(\lambda)$ be the linear hull of x and $N(\lambda)$. Then there is a $y \in N^0(\lambda)$ such that $\|\tilde{y}\|=\|y\|>0$. In fact, this is a direct consequence of Lemma 222 in the case (iii), for we have $\delta(N, N(\lambda))<1$ for $|\lambda|<\gamma/(2\sigma+\tau\gamma)$, see (627). Also this is true in the case (i) in virtue of Lemma 211, for we have $\dim N^0(\lambda)=\alpha \dot{+} 1>\alpha=\dim N$. Now this y cannot belong to $N(\lambda)$ because $\delta(N(\lambda), N)<1$ by (621). Hence we may assume that y has the form $y=x-u$, $u \in N(\lambda)$. This proves (636).

As is easily seen, the use of (636) instead of (631) in the calculation (632) to (633) leads to the improved result (635). We shall show later that (635) is true for $|\lambda|<\gamma/(\sigma+\tau\gamma)$.

4. Stability of ν and β.

We now turn to the proof of $\nu(\lambda)=\infty$. For this proof it is convenient to consider first the case in which B is bounded (that is, $\tau=0$). Since nothing changes when B is replaced by its closure, we may assume that B is bounded and closed (note Lemma 531).

We have proved above that $R(\lambda)$ is closed for $|\lambda|<\gamma/3\sigma=\gamma/3\,\|B\|$. Therefore, we can apply Lemma 551 to the pair $A-\lambda B$, B. Let $S=S^{\omega}(\lambda)$ be the set of $u(\lambda)\in N(\lambda)$ given by (625). As we have seen above, the linear hull $L^{\omega}(\lambda)$ of $S^{\omega}(\lambda)$ is dense in $N(\lambda)$ if $|\lambda|<\gamma/3\,\|B\|$ (see (629)) and ω is sufficiently close to γ. To apply Lemma 551, it is therefore sufficient to construct, for each $u(\lambda)\in S^{\omega}(\lambda)$, a sequence $u_k(\lambda)$, $k=1,2,\dots$, such that $u_1(\lambda)=u(\lambda)$ and

$$(641)\qquad (A-\lambda B)\,u_{k+1}(\lambda)=Bu_k(\lambda),\quad k=1,2,\dots .$$

Such a sequence is given by

$$(642)\qquad u_k(\lambda)=\sum_{i=0}^{\infty}\binom{k+i-1}{k-1}\lambda^i z_{i+k}.$$

In fact, these series are convergent for $|\lambda|<\omega/\sigma=\omega/\|B\|$ as well as the series of $u(\lambda)$ and it is obvious that $u_1(\lambda)=u(\lambda)$. The term-by-term application of A to these series is justified as before and leads to (641) after a simple calculation. This proves that

$$(643)\qquad \nu(\lambda)=\infty\quad\text{for}\quad |\lambda|<\gamma/3\,\|B\|$$

provided B is bounded.

The general case can be reduced to the above case by the device used in the proof of Theorem 1a. We introduce[19] the new norm $|||\ \ |||$ by (444), thereby converting D into a Banach space $\widehat{X}$ and the operators A, B into bounded linear operators $\widehat{A}$, $\widehat{B}$ on $\widehat{X}$ to $\widehat{X}'$. We note that (see (446))

$$(644)\qquad \begin{aligned} \widehat{\alpha}(\lambda)&=\alpha(\widehat{A}-\lambda\widehat{B})=\alpha(A-\lambda B)=\alpha(\lambda),\\ \widehat{\beta}(\lambda)&=\beta(\widehat{A}-\lambda\widehat{B})=\beta(A-\lambda B)=\beta(\lambda),\\ \widehat{\nu}(\lambda)&=\nu(\widehat{A}-\lambda\widehat{B}:\widehat{B})=\nu(A-\lambda B:B)=\nu(\lambda); \end{aligned}$$

for the last equality see Lemma 531 and note that $\nu(A:B)$ is defined in a purely algebraic fashion.

Since $R[\widehat{A}]=R[A]$ is closed and $\nu(\widehat{A}:\widehat{B})=\nu(A:B)=\infty$, the assumptions of Theorem 3 are satisfied for A, B replaced by $\widehat{A}$, $\widehat{B}$. By what is

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19. We could have done this in earlier stages of our proof, but we have avoided it because it appears that this device leads to less sharp estimates of $\gamma(\lambda)$ then those obtained in the preceding paragraph by a more direct method. In particular, there is the inconvenience that $\widehat{X}$ is in general not a Hilbert space even if X is.

proved above, it follows that $\nu(\lambda)=\widehat{\nu}(\lambda)=\infty$ for $|\lambda|<\widehat{\gamma}/3\|\widehat{B}\|$. Here we have $\|\widehat{B}\|\leqq 1$ by (445) and $\widehat{\gamma}$ is given by (448). Since $\varepsilon>0$ is arbitrary, we conclude that

(645) $$\nu(\lambda)=\infty \quad \text{for } |\lambda|<\gamma/3(\sigma+\tau\gamma).$$

Next we show that $\beta(\lambda)=\beta$. For this proof we again make use of the operators $\widehat{A}, \widehat{B}$. Since these are bounded and defined everywhere on $\widehat{X}$, their adjoints $\widehat{A}^*, \widehat{B}^*$ exist and are bounded linear operators on X'^* to $\widehat{X}^*$. Since $R[\widehat{A}^*]$ is closed by Lemma 324 and $\nu(\widehat{A}^*:\widehat{B}^*)=\infty$ by Lemma 562, the assumptions of Theorem 3 are also satisfied by the pair $\widehat{A}^*, \widehat{B}^*$. Hence we see that $\alpha(\widehat{A}^*-\lambda\widehat{B}^*)=\alpha(\widehat{A}^*)$ for $|\lambda|<\gamma(\widehat{A}^*)/2\|\widehat{B}^*\|$, see (628). Here $\|\widehat{B}^*\|=\|\widehat{B}\|\leqq 1$ as noted above and $\gamma(\widehat{A}^*)=\gamma(\widehat{A})=\widehat{\gamma}$ by (326), where $\widehat{\gamma}$ is given by (448). Also we have $\alpha(\widehat{A}^*-\lambda\widehat{B}^*)=\beta(\widehat{A}-\lambda\widehat{B})=\beta(\lambda)$ by (331) for $|\lambda|<\gamma/(3\sigma+\tau\gamma)$, since $R(\lambda)$ is then closed. Thus we have proved that

(646) $$\beta(\lambda)=\beta \quad \text{for } |\lambda|<\gamma/3(\sigma+\tau\gamma),$$

Also we see that $R(\lambda)^{\perp}=R[\widehat{A}-\lambda\widehat{B}]^{\perp}=N[\widehat{A}^*-\lambda\widehat{B}^*]$ depends on analytically at least at $\lambda=0$.

5. Special cases (ii), (iv).

We shall show that the estimate of $\gamma(\lambda)$ can also be sharpened to (635) in the special cases in which either (ii) $\beta<\infty$ or (iv) X' is a Hilbert space. Since $\alpha(\widehat{A}^*)=\beta<\infty$ in case (ii) and X'^* is a Hilbert space in case (iv), we see that (636) is applicable to the pair $\widehat{A}^*, \widehat{B}^*$. Thus we see that, for each $f'\in X'^*$ and $|\lambda|<\gamma(\widehat{A}^*)/2\|\widehat{B}^*\|$, there is an

$$h'\in N(\widehat{A}^*-\lambda\widehat{B}^*]=R[\widehat{A}-\lambda\widehat{B}]^{\perp}=R(\lambda)^{\perp}$$

such that $g'=f'-h'$ has the property that $d(g', N[\widehat{A}^*])=\|g'\|$. Since $N[\widehat{A}^*]=R[A]^{\perp}=R'^{\perp}$, we have by (231)

(651) $$\|g'\|_{R'}=d(g', R'^{\perp})=\|g'\|.$$

Noting that $\|\widehat{B}^*\|\leqq 1$ and $\gamma(\widehat{A}^*)=\gamma$ as above, we see that (651) is true at least if $|\lambda|<\gamma/2(\sigma+\tau\gamma)$.

For every $x\in D$ we have $h'((A-\lambda B)x)=0$ so that

(652) $$f'((A-\lambda B)x)=g'((A-\lambda B)x=g'(Ax)-\lambda g'(Bx).$$

Since $\|g'\|_{R'}$ is the least upper bound of $|g'(x')|/\|x'\|$ for $x'\in R'=R[A]$, there is for each $\varepsilon>0$ an $x\in D$ such that $|g'(Ax)|\geqq(1-\varepsilon)\|g'\|_{R'}\|Ax\|$

and $Ax \neq 0$ (note that $R' \neq \{0\}$ by hypothesis). In view of (651), (652) and (613), we thus obtain

$$|f'((A-\lambda B)x)| \geqq (1-\varepsilon) \|g'\|_{R'} \|Ax\| - |\lambda| \|g'\| \|Bx\|$$
$$\geqq \|g'\| [(1-\varepsilon-\tau|\lambda|) \|Ax\| - \sigma|\lambda| \|x\|].$$

Here we may assume that $\|Ax\| \geqq \gamma \|\tilde{x}\| \geqq (1-\varepsilon) \gamma \|x\|$, for x may be replaced by any other element of $\tilde{x}$. Also we may assume that $x \neq 0$ because $\tilde{x} \neq 0$ by $Ax \neq 0$. On the other hand, $f' - g' = h' \in R(\lambda)^{\perp}$ implies that $\|f'\|_{R(\lambda)} \leqq \|g'\|$. Furthermore, we have $|f'((A-\lambda B)x)| \leqq \|(A-\lambda B)^{+} f'\| \|x\|$ provided $f' \in D[(A-\lambda B)^{+}]$. Hence we obtain

(653) $$\|(A-\lambda B)^{+} f'\| \geqq [(1-\varepsilon-\tau|\lambda|)(1-\varepsilon)\gamma - \sigma|\lambda|] \|f'\|_{R(\lambda)}.$$

Recalling the characterization of $\gamma^{+}(\lambda) = \gamma((A-\lambda B)^{+})$ by Lemma 323, we see that $\gamma^{+}(\lambda)$ is not smaller than the number in the brackets on the right of (653). Since $\gamma(\lambda) = \gamma^{+}(\lambda)$ by Lemma 324, we obtain, on letting $\varepsilon \to 0$,

(654) $$\gamma(\lambda) \geqq \gamma - (\sigma + \tau\gamma)|\lambda|$$

which is identical with (635). We note that (654) has been proved for $|\lambda| < \gamma/2(\sigma+\tau\gamma)$.

6. Completion of the proof.

We have so far proved that $\alpha(\lambda) = \alpha$, $\beta(\lambda) = \beta$, $\nu(\lambda) = \infty$ and that $\gamma(\lambda)$ satisfies the inequality (634) in the general case and (635) in the special cases (i) to (iv), at least in a certain region $|\lambda| < r$. For the radius r of this circle we can safely take $r = \gamma/3(\sigma+\tau\gamma)$, as is seen by considering various conditions imposed on λ for the inequalities proved above. We shall now show that this circle can be extended to the larger circle $|\lambda| < \rho$, where ρ is as given in Theorem 3, namely $\rho = \gamma/3(\sigma+\tau\gamma)$ in the general case and $\rho = \gamma/(\sigma+\tau\gamma)$ in the special cases. The value of ρ is given as the value of $|\lambda|$ for which the right side of (634) resp. (635) becomes zero.

We shall give the proof only for the general case; the special cases can be dealt with quite in the same way. Let ρ_0 be the radius of the largest circle $|\lambda| < \rho_0$ in which the above results are valid; we have to show that $\rho_0 \geqq \rho$.

Suppose that $\rho_0 < \rho$. Then (634) shows that $\gamma(\lambda)$ has a positive lower bound γ_0 in the circle $|\lambda| < \rho_0$. For each λ of this circle, the assumptions of Theorem 3 are satisfied for the pair A, B replaced by the pair

$A-\lambda B$, B; only the inequality (613) should be replaced by

(661) $$\|Bx\| \leq [\sigma\|x\| + \tau\|(A-\lambda B)x\|]/(1-\tau|\lambda|)$$

which is itself a direct consequence of (613). The application of what has been proved to the pair $A-\lambda B$, B leads to the result that

$$\alpha(\lambda+\lambda') = \alpha(\lambda) = \alpha, \quad \beta(\lambda+\lambda') = \beta(\lambda) = \beta, \quad \nu(\lambda+\lambda') = \nu(\lambda) = \infty$$

and that

(662) $$\gamma(\lambda+\lambda') \geqq \gamma(\lambda) - (1-\tau|\lambda|)^{-1}[3\sigma+\tau\gamma(\lambda)]|\lambda'|$$

at least for

(663) $$|\lambda'| < (1-\tau|\lambda|)\,\gamma(\lambda)/3[\sigma+\tau\gamma(\lambda)].$$

(According to (661) we have to multiply the constants σ, τ by the factor $(1-\tau|\lambda|)^{-1}$ when we apply Theorem 3 to the pair $A-\lambda B$, B).

If we substitute (634) for $\gamma(\lambda)$ of (662), we obtain after a simple calculation

(664) $$\gamma(\lambda+\lambda') \geqq \gamma - (3\sigma+\tau\gamma)(|\lambda|+|\lambda'|) = \gamma - (3\sigma+\tau\gamma)|\lambda+\lambda'|$$

provided λ'/λ is real and positive. It should be noticed that (664) has the same form as (634). On the other hand, the right side of (663) has a positive lower bound for $|\lambda|<\rho_0$ because $\gamma(\lambda)\geqq\gamma_0>0$. Thus it follows that $|\lambda|<\rho_0$ is not the largest circle with the property described above, in contradiction to the definition of ρ_0. This gives the desired result $\rho_0\geqq\rho$.

The inequality (615) or (617) expressing the Lipschitz-continuity of $\gamma(\lambda)$ can be deduced easily from (662). Then (616) is obtained by integrating (615). The calculation is a simple exercise and may be omitted.

The analytic dependence of $N(\lambda)$ and $R(\lambda)^{\perp}$ on λ now follows at each point λ of the circle $|\lambda|<\rho$, since this was already proved for $\lambda=0$. This completes the proof of Theorem 3.

§7. FINITE $\nu(A:B)$. REDUCTION THEOREM

1. Reduction theorem.

We now proceed to the consideration of two operators A, B for which $\nu(A:B)$ is finite. The main problem here is the "reduction" of the spaces X, X'. This means the decomposition of the domain space X and the range space X' into subspaces in such a way that A and B can be regarded as operators between the corresponding subspaces. Our result is summarized in the following theorem.

Theorem 4. Let X and X' be Banach spaces. Let A be a closed linear operator $(X \to X')$ with closed range and with at least one of $\alpha(A)$ and $\beta(A)$ finite. Let B be a bounded, closed linear operator $(X \to X')$ with $D[B] \supset D[A]$. Furthermore, let $\nu(A:B) = \nu$ be finite. Then the spaces X, X' can be decomposed in the following manner.

(711) $X = X_1 \oplus \dots \oplus X_r \oplus X_0, \quad X' = X'_1 \oplus \dots \oplus X'_r \oplus X'_0.$

Here all X_i, X'_i, $i = 0, 1, \dots, r$, are closed linear manifolds and

(712) $X_i \subset D[A], \quad \dim X_i = \dim X'_i = m_i < \infty, \qquad i = 1, 2, \dots, r.$

For each $i = 0, 1, \dots, r$, X_i and X'_i form an "invariant pair" in the sense that $AX_i \subset X'_i$, $BX_i \subset X'_i$. Thus we can define the linear operators A_i, B_i $(X_i \to X'_i)$ induced by A, B in the pair X_i, X'_i of subspaces. For $i \geqq 1$, there exist bases $\{x_{i1}, x_{i2}, \dots, x_{i,m_i}\}$ and $\{x'_{i1}, x'_{i2}, \dots, x'_{i,m_i}\}$ of X_i and X'_i respectively related to each other in the following way.

(713) $$Bx_{ik} = x'_{ik}, \quad Ax_{ik} = x'_{i,k-1},$$
$$k = 1, 2, \dots, m_i, \quad i = 1, 2, \dots, r,$$

where we set $x'_{i0} = 0$ by convention. Thus B_i, $i \geqq 1$, is an isomorphism of X_i onto X'_i, whereas

(714) $\alpha(A_i) = \beta(A_i) = 1, \quad \nu(A_i : B_i) = m_i, \qquad i = 1, 2, \dots, r.$

The pair X_i, X'_i does not permit any further decomposition in the above sense. The "residual part" A_0 of A is a closed linear operator $(X_0 \to X'_0)$ with closed range, and B_0 is a bounded, closed linear operator $(X_0 \to X'_0)$ with $D[B_0] \supset D[A_0]$. Furthermore

(715) $\alpha(A_0) = \alpha(A) - r, \quad \beta(A_0) = \beta(A) - r, \quad \nu(A_0 : B_0) = \infty,$

so that at least one of $\alpha(A_0)$ and $\beta(A_0)$ is finite.

For each $n = 1, 2, \dots$ let r_n be the number of the subspaces X_i, $i \geqq 1$, such that $m_i = n$. Then

(716) $$r = r_1 + r_2 + \dots,$$

(717) $$\dim N_n = \dim BN_{n-1} + \alpha(A_0) + r_n + r_{n+1} + \dots,$$
$$\operatorname{codim} AM_{n-1} = \operatorname{codim} M_{n-1} + \beta(A_0) + r_n + r_{n+1} + \dots,$$
$$n = 1, 2, 3, \dots,$$

where $M_n = M_n[A:B]$ and $N_n = N_n[A:B]$. These equalities determine $r_1, r_2, \dots$ completely.

Remark. The decomposition (711) with the properties described above is in general not unique, though it is unique up to isomorphism. Even the decomposition of the spaces into the "finite parts" $X_1 \oplus \dots \oplus X_r$ and $X'_1 \oplus \dots \oplus X'_r$ on the one hand and the "residual parts" X_0, X'_0 on the other is in general not unique. See § 7.5.

Before going into the proof of Theorem 4, we note once for all that we may assume that $D[B] = X$. Otherwise $D[B] = X^0$ is a closed linear manifold of X. Let A^0, B^0 be the trivial restrictions of A, B to X^0. Then the assumptions of the theorem are satisfied for A^0, B^0 (see Lemma 531). If the theorem is proved for the pair A^0, B^0 with the decompositions $X^0 = X_1^0 \oplus \dots \oplus X_r^0 \oplus X_0^0$ and $X' = X'_1 \oplus \dots \oplus X'_r \oplus X'_0$, it is easy to find an $X_0 \supset X_0^0$ such that $X = X_1^0 \oplus \dots \oplus X_r^0 \oplus X_0$. We have only to set $X_i = X_i^0$, $i \geqq 1$, to obtain the decomposition of the theorem.

2. Subspaces L_k.

We now give the proof of the theorem. We introduce the linear manifolds $M_n = M_n[A:B]$ and $N_n = N_n[A:B]$. That $\nu[A:B] = \nu < \infty$ means that $N = N_1 \subset M_{\nu-1}$ is true but $N \subset M_\nu$ is false. Set

(721) $$N^0 = N \cap M_\nu;$$

then we have

(722) $$0 < \dim N/N^0 < \infty,$$

the second inequality being true because either $\dim N = \alpha(A) < \infty$ or $\operatorname{codim} M_\nu < \infty$ in virtue of $\beta(A) < \infty$ (see (517) and the remark at the end of the preceding paragraph, according to which we assume that $D[B] = X$). Hence there exists a finite-dimensional linear manifold L_1 such that

(723) $$N = N^0 \oplus L_1, \quad L_1 \cap M_\nu = \{0\}, \quad \{0\} \neq L_1 \subset N \subset M_{\nu-1}.$$

We shall now construct a sequence of linear manifolds $L_1, L_2, \dots, L_\nu$ with the following properties:

$$L_k \subset N_k \subset M_{\nu-k}, \quad L_k \cap M_{\nu-k+1} = \{0\},$$
(724) $$AL_k = BL_{k-1}, \ (L_0 = \{0\}), \quad k = 1, 2, \ldots, \nu,$$
$$0 < \dim L_1 = \dim L_2 = \ldots = \dim L_\nu = r < \infty$$

and the mappings $L_k \to AL_k$, $k = 2, 3, \ldots, \nu$, and $L_k \to BL_k$, $k = 1, 2, \ldots, \nu$, are one-to-one.

We have already constructed L_1. So we assume that $L_1, \ldots, L_k$ have been constructed and construct L_{k+1} as follows. $L_k \subset M_{\nu-k}$ implies that $L_k \subset B^{-1}(AM_{\nu-k-1})$ and hence that $BL_k \subset AM_{\nu-k-1}$. Since $\dim L_k < \infty$, there is a linear manifold $L_{k+1} \subset M_{\nu-k-1}$ which is mapped by A onto BL_k in a one-to-one fashion: $AL_{k+1} = BL_k$. This shows that

$$L_{k+1} \subset A^{-1}(BL_k) \subset A^{-1}(BN_k) = N_{k+1} \subset M_{\nu-k-1},$$

the last inclusion being true since $k + 1 \leq \nu$. If $x \in L_{k+1} \cap M_{\nu-k}$, there is a $y \in L_k$ such that $Ax = By$ and $y \in B^{-1}(AM_{\nu-k}) = M_{\nu-k+1}$. Thus $y \in L_k \cap M_{\nu-k+1}$ and hence $y = 0$ by (724). Then $Ax = 0$ and so $x = 0$ by the one-to-one property of the map $L_{k+1} \to AL_{k+1}$. This shows that $L_{k+1} \cap M_{\nu-k} = \{0\}$. Finally we have to show that also the map $L_{k+1} \to BL_{k+1}$ is one-to-one. $Bx = 0$ implies $x \in N[B] \subset M_{\nu-k}$ by (513). Hence $Bx = 0$ for an $x \in L_{k+1}$ implies that $x \in L_{k+1} \cap M_{\nu-k} = \{0\}$. This one-to-one property of the mappings finally shows that

$$\dim L_{k+1} = \dim AL_{k+1} = \dim BL_k = \dim L_k.$$

This completes the induction.

We note that, for each $k = 1, 2, \ldots, \nu$, the linear manifolds $L_k, L_{k+1}, \ldots, L_\nu$ and $M_{\nu-k+1}$ are linearly independent. This follows immediately from the fact that $L_i \subset M_{\nu-i}$ but $L_i \cap M_{\nu-i+1} = \{0\}$ and that $X = M_0 \supset M_1 \supset M_2 \supset \ldots$.

Similarly the linear manifolds $BL_k, BL_{k+1}, \ldots, BL_\nu$ and $AM_{\nu-k}$ are linearly independent. To see this, suppose that

$$By_k + By_{k+1} + \ldots + By_\nu + y' = 0$$

with $y_i \in L_i$ and $y' \in AM_{\nu-k}$. Then we have

$$y_k + \ldots + y_\nu \in B^{-1}(AM_{\nu-k}) = M_{\nu-k+1}.$$

By the linear independence of $L_k, \ldots, L_\nu$ and $M_{\nu-k+1}$ noted above, it follows that $y_k = \ldots = y_\nu = 0$ and $y' = 0$.

3. Construction of bases.

So far our arguments have been purely algebraic. For the construction to follow, we make use of the fact that M_n and AM_n, $n = 0, 1, 2, \ldots$, are all closed by Lemmas 541 and 542.

We now introduce bases of the linear manifolds $L_1, \ldots, L_\nu$. We take an arbitrary basis $x_{11}, x_{21}, \ldots, x_{r1}$ of L_1. In virtue of the relations $AL_k = BL_{k-1}$ and the one-to-one property of these mappings, we can then choose the bases of L_k, $\{x_{1k}, x_{2k}, \ldots, x_{rk}\}$, for $k \geqq 2$ in such a way that the following relations hold:

$$(731) \qquad Ax_{ik} = Bx_{i,k-1}, \quad x_{ik} \in L_k \subset N_k \subset M_{\nu-k}, \qquad i = 1, 2, \ldots, r, \quad k = 1, 2, \ldots, \nu,$$

where we set $x_{i0} = 0$ by convention. According to the remarks above, the $(\nu - k + 1)r$ elements $x_{ik}, \ldots, x_{i\nu}$ $(i = 1, \ldots, r)$ of X are linearly independent modulo $M_{\nu-k+1}$, and the $(\nu - k + 1)r$ elements $Bx_{ik}, \ldots, Bx_{i\nu}$ $(i = 1, 2, \ldots, r)$ of X' are linearly independent modulo $AM_{\nu-k}$.

In particular $Bx_{i\nu}$, $i = 1, \ldots, r$, are linearly independent modulo AX. Since AX is closed, there exist r linear functionals $f'_{j\nu} \in X'^*$, $j = 1, \ldots, r$, such that

$$(732) \qquad f'_{j\nu}(Bx_{i\nu}) = \delta_{ij}, \quad f'_{j\nu} \in (AX)^{\perp}.$$

Here δ_{ij} is the Kronecker symbol.

We now construct by induction $r\nu$ elements $f'_{jl} \in X'^*$, $j = 1, \ldots, r$, $l = 1, \ldots, \nu$, with the following properties:

$$(733) \qquad \begin{aligned} &f'_{jl}(Ax) = f'_{j,l+1}(Bx) = B^* f'_{j,l+1}(x), \quad x \in D[A], \\ &f'_{jl} \in (BL_\nu)^{\perp} \cap (AM_{\nu-l})^{\perp}, \qquad l = 1, \ldots, \nu - 1. \end{aligned}$$

We shall show how to find $f'_{j,l-1}$, $j = 1, \ldots, r$, when already $f'_{j\nu}, \ldots, f'_{jl}$, $j = 1, \ldots, r$, have been constructed, where $l \geqq 2$. Since $N \subset M_{\nu-1} \subset M_{\nu-l+1}$, we have $BN \subset AM_{\nu-l}$, $(AM_{\nu-l})^{\perp} \subset (BN)^{\perp}$. Hence $f'_{jl} \in (BN)^{\perp}$ by (732) or (733), and this implies that $B^* f'_{jl} \in N^{\perp}$. Furthermore, BL_ν is linearly independent of AX as remarked above. Thus $BL_\nu \cap AX = \{0\}$ and $BL_\nu + AX$ is closed because AX is closed and BL_ν is finite-dimensional. Thus Lemma 334 is applicable with $f = B^* f'_{jl}$ and $M' = BL_\nu$, with the result that there exists an $f'_{j,l-1} \in X'^*$ such that

$$f'_{j,l-1}(Ax) = B^* f'_{jl}(x) = f'_{jl}(Bx) \quad \text{and} \quad f'_{j,l-1} \in (BL_\nu)^{\perp}.$$

Furthermore, for any $x \in D[A] \cap M_{\nu-l+1}$ we have $Bx \in BM_{\nu-l+1} \subset AM_{\nu-l}$

and $f'_{j,l-1}(Ax) = f'_{jl}(Bx) = 0$ because $f'_{jl} \in (AM_{\nu-l})^{\perp}$. This shows that $f'_{j,l-1} \in (AM_{\nu-l+1})^{\perp}$. Thus we see that $f'_{j,l-1}$ satisfies all the requirements of (733), and the induction is complete.

The $r\nu$ elements $Bx_{ik} \in X'$ and the $r\nu$ functionals f'_{jl} form a biorthogonal set:

(734) $$f'_{jl}(Bx_{ik}) = \delta_{ij}\delta_{kl}, \quad i, j = 1, \ldots, r, \quad k, l = 1, \ldots, \nu.$$

For $k = \nu$ this has already been proved, see (732), (733) and note that $x_{i\nu} \in L_\nu$. This is also true for $k < l$ because then

$$Bx_{ik} \in BL_k \subset BM_{\nu-k} \subset AM_{\nu-k-1} \subset AM_{\nu-l},$$

see (724) and (733). If $\nu > k \geq l$, (734) follows from

$$f'_{jl}(Bx_{ik}) = f'_{jl}(Ax_{i,k+1}) = f'_{j,l+1}(Bx_{i,k+1}) = \ldots$$
$$= f'_{j,l+\nu-k}(Bx_{i\nu}) = \delta_{ij}\delta_{kl}$$

by (731), (732) and (733).

4. Construction of projections.

We define $r\nu$ operators E_{ik} by

(741) $$E_{ik}\,x = f'_{ik}(Bx)\,x_{ik}, \quad i = 1, \ldots, r, \quad k = 1, \ldots, \nu.$$

E_{ik} are bounded linear operators $(X \to X)$ defined everywhere in X. It follows easily from (734) that

(742) $$E_{jl}E_{ik} = \delta_{ij}\delta_{kl}E_{ik}.$$

This shows that E_{ik} are mutually orthogonal projections in X.

Similarly we define $r\nu$ operators E'_{ik} by

(741') $$E'_{ik}\,x' = f'_{ik}(x')\,Bx_{ik}, \quad i = 1, \ldots, r, \quad k = 1, \ldots, \nu.$$

E'_{ik} are bounded linear operatots $(X' \to X')$ defined everywhere in X'. Again it follows from (734) that

(742') $$E'_{jl}E'_{ik} = \delta_{ij}\delta_{kl}E'_{ik}.$$

Thus E'_{ik} are mutually orthogonal projections in X'.

It follows immediately from the definition that

(743) $$E'_{ik}B = BE_{ik}, \quad i = 1, \ldots, r; \quad k = 1, \ldots, \nu.$$

Also we have by (731)

$$AE_{ik}\,x = f'_{ik}(Bx)\,Ax_{ik} = f'_{ik}(Bx)\,Bx_{i,k-1}$$

for all $x \in X$, and by (733) or (732)

$$E'_{ik} Ax = f'_{ik}(Ax) Bx_{ik} = f'_{i,k+1}(Bx) Bx_{ik} \quad (f'_{i,\nu+1} = 0)$$

for all $x \in D[A]$. These equalities show that

(744) $$E'_{ik} A \subset AE_{i,k+1}, \quad i = 1, \ldots, r; \quad k = 0, 1, \ldots, \nu,$$

where we set $E'_{i0} = 0$ and $E_{i,\nu+1} = 0$ by convention.

We now set

(745) $$E_i = \sum_{k=1}^{\nu} E_{ik}, \quad E'_i = \sum_{k=1}^{\nu} E'_{ik}, \quad E = \sum_{i=1}^{r} E_i, \quad E' = \sum_{i=1}^{r} E'_i.$$

It follows from (743) and (744) that

(746) $$E'_i B = BE_i, \quad E'B = BE, \quad E'_i A \subset AE_i, \quad E'A \subset AE.$$

Let the ranges of E_i and E'_i be denoted respectively by X_i and X'_i. These are ν-dimensional linear manifolds of X and X' and spanned by the bases $\{x_{i1}, \ldots, x_{i\nu}\}$ and $\{x'_{i1}, \ldots, x'_{i\nu}\}$ respectively, where

(747) $$x'_{ik} = Bx_{ik}.$$

Note that x'_{ik} are linearly independent, as is seen from the fact that x'_{ik} and f'_{ik} form a biorthogonal set. Each of X, X' is decomposed into the direct sum of $r+1$ closed linear manifolds:

$$X = X_1 \oplus \ldots \oplus X_r \oplus X_0, \quad X' = X'_1 \oplus \ldots \oplus X'_r \oplus X'_0,$$

where X_0, X'_0 are the ranges of $I - E$, $I' - E'$ respectively (I, I' are respectively the identity operators of X, X').

The bases of the subspaces X_i, X'_i are connected with each other by (747) and

(748) $$Ax_{ik} = x'_{i,k-1} \qquad (x'_{i0} = 0).$$

Thus B induces an isomorphism of X_i onto X'_i, whereas A acts in X_i in a way analogous to the Jordan canonical form.

The "residual spaces" X_0, X'_0 are also invariant with respect to A, B in the sense that $AX_0 \subset X'_0$ and $BX_0 \subset X'_0$. This is a direct consequence of the fact that $(I'-E')A \subset A(I-E)$ and $(I'-E')B = B(I-E)$.

Let A_i, B_i be the linear operators $(X_i \to X'_i)$ induced by A, B respectively, $i = 0, 1, 2, \ldots, r$. For $i \geq 1$, B_i has the inverse B_i^{-1} and the operator $B_i^{-1} A_i$ has exactly the Jordan canonical form with respect to the basis $\{x_{i1}, \ldots, x_{i\nu}\}$ of X_i. From this it is easily seen that

$$\alpha(A_i) = \beta(A_i) = 1, \quad \nu(A_i : B_i) = \nu, \quad i = 1, \ldots, r.$$

Hence we must have for the residual part A_0

(749) $$\alpha(A_0) = \alpha(A) - r, \quad \beta(A_0) = \beta(A) - r.$$

5. Completion of the reduction.

Let us now consider the residual parts A_0, B_0 of the operators A, B respectively. It is obvious that A_0 is a closed linear operator with closed range, for the same is true with A and X_0 is a closed linear manifold of X with codim $X_0 = r\nu < \infty$ (see Lemma 333). Furthermore, it follows from (749) that at least one of $\alpha(A_0)$, $\beta(A_0)$ is finite. Therefore, we can apply the reduction procedure described above to the pair A_0, B_0, provided $\nu(A_0 : B_0)$ is finite. This leads to a further reduction of the numbers $\alpha(A_{00})$, $\beta(A_{00})$ for the residual parts A_{00}, B_{00} of A_0, B_0. This process can be continued only a finite number of times because at least one of $\alpha(A)$, $\beta(A)$ was finite, and must come to an end with the final residual parts A^0, B^0 such that $\nu(A^0 : B^0) = \infty$. With a slight change of notations, this proves Theorem 4 except for the last equalities (717).

Hereafter we use the notations of Theorem 4. To prove (717), we note that for $i \geqq 1$

$$\begin{aligned} &\dim N_n[A_i : B_i] = \dim N_{n-1}[A_i : B_i] + \eta_{n,m_i}, \\ &\operatorname{codim} M_n[A_i : B_i] = \operatorname{codim} M_{n-1}[A_i : B_i] + \eta_{n,m_i} \end{aligned} \tag{751}$$

where $\eta_{n,m} = 1$ for $n \leqq m$ and $\eta_{n,m} = 0$ for $n > m$. This can be seen most easily from the fact that $M_n[A_i : B_i]$ and $N_n[A_i : B_i]$ are respectively the range and the null space of the operator $(B_i^{-1} A_i)^n$, where $B_i^{-1} A_i$ is a linear operator in the m_i dimensional space X_i with a Jordan canonical form with the eigenvalue 0.

Since $M_n[A_i : B_i] = B_i^{-1}(A_i M_{n-1}[A_i : B_i])$ and B_i is an isomorphism of X_i onto X'_i, (751) can be written as

$$\begin{aligned} &\dim N_n[A_i : B_i] = \dim B_i N_{n-1}[A_i : B_i] + \eta_{n,m_i}, \\ &\operatorname{codim} A_i M_{n-1}[A_i : B_i] = \operatorname{codim} M_{n-1}[A_i : B_i] + \eta_{n,m_i}. \end{aligned} \tag{752}$$

On the other hand, $\nu(A_0 : B_0) = \infty$ implies by Lemma 511 that

$$\begin{aligned} &\dim N_n[A_0 : B_0] = \dim B_0 N_{n-1}[A_0 : B_0] + \alpha(A_0), \\ &\operatorname{codim} A_0 M_{n-1}[A_0 : B_0] = \operatorname{codim} M_{n-1}[A_0 : B_0] + \beta(A_0). \end{aligned} \tag{753}$$

But it is obvious that

$$\dim N_n[A : B] = \sum_{i=0}^{r} \dim N_n[A_i : B_i]$$

and similar relation holds also for $\dim BN_n[A : B]$, $\operatorname{codim} M_n[A : B]$

and codim $AM_n[A : B]$. The addition of the corresponding equalities of (752) and (753) thus leads to the required results (717).

Since at least one of $\alpha(A)$ and $\beta(A)$ is finite, (717) determines the numbers $r_1, r_2, \ldots$. For example if $\alpha(A) < \infty$, we have $\alpha(A_0) < \infty$ and hence

$$r_n = (\dim N_n - \dim BN_{n-1}) - (\dim N_{n+1} - \dim BN_n)$$

where all terms on the right are finite.

The fact that r_n are determined uniquely shows that the decomposition of X and X' in the manner described in Theorem 4 is unique up to isomorphism. This is obvious for the finite parts X_i, X'_i, $i \geqq 1$. Then the structure of X_0, X'_0 with A_0, B_0 operating between them is also determined as isomorphic to the factor spaces $X/(X_1 \oplus \ldots \oplus X_r)$, $X'/(X'_1 \oplus \ldots \oplus X'_r)$ respectively.

However, even the finite part $X_1 \oplus \ldots \oplus X_r$ and the residual part X_0 need not be determined uniquely. For example let X be a Hilbert space with a complete orthonormal set $\{x_0, x_1, x_2, \ldots\}$. Set $B = I$ and let A be a bounded linear operator $(X \to X)$ with $D[A] = X$ such that

$$Ax_0 = 0, \quad Ax_1 = 0, \quad Ax_2 = x_1, \quad Ax_3 = x_2, \ldots.$$

Then A is partially isometric (hence has closed range) and $\alpha(A) = 2$, $\beta(A) = 1$. A decomposition of the kind stated in Theorem 4 is given by $X = X_1 \oplus X_0$, where X_0 is the closed linear manifold spanned by $\{x_1, x_2, \ldots\}$ and X_1 is a one-dimensional linear manifold spanned by $x_0 + \xi x_1$, where ξ may be any number. Thus X_1 is not uniquely determined.

In this example the adjoint A^* of A is given by

$$A^* x_0 = 0, \quad A^* x_1 = x_2, \quad A^* x_2 = x_3, \ldots,$$

(here A^* is the Hilbert adjoint of A, again deviating from the general usage of this paper). A^* is again partially isometric with $\alpha(A^*) = 1$, $\beta(A^*) = 2$. A decomposition of Theorem 4 is given by $X = X_1 \oplus X_0$, where X_1 is the one-dimensional linear manifold spanned by x_0 and X_0 is a closed linear manifold spanned by $x_1 + \xi x_0, x_2, x_3, \ldots$, where ξ is an arbitrary number. Thus X_0 is not uniquely determined.

In this connection it will be remarked that the finite parts $X_1 \oplus \ldots \oplus X_r$, $X'_1 \oplus \ldots \oplus X'_r$ are uniquely determined if $\alpha(A_0) = 0$. Similarly the residual parts X_0, X'_0 are uniquely determined if $\beta(A_0) = 0$

and $D[B]=X$. This follows from the fact that we have

$$X_1 \oplus \dots \oplus X_r = N_n[A:B]$$

for sufficiently large n provided $\alpha(A_0)=0$, and we have $X_0 = M_n[A:B]$, $X'_0 = AX_0$ for sufficiently large n provided $\beta(A_0)=0$ and $D[B]=X$ (see (516) and (517)).

6. Instability of finite ν.

We now prove a theorem which shows that the property $\nu(A:B)<\infty$ is unstable or exceptional in a certain sense.

Theorem 5. *Let X, X' be Banach spaces, let A be a closed linear operator $(X \to X')$ with closed range with at least one of $\alpha(A)$ and $\beta(A)$ finite. Let B be a linear operator $(X \to X')$ such that $D[B] \supset D[A]$ and*

(761) $$\|Bx\| \leq \sigma\|x\| + \tau\|Ax\|, \qquad x \in D[A].$$

Furthermore, let $\nu(A:B)<\infty$. Then there is a constant $\rho>0$ and a positive integer r such that, for $0<|\lambda|<\rho$, $A-\lambda B$ is a closed linear operator $(X \to X')$ with closed range and

(762) $$\alpha(A-\lambda B) = \alpha(A) - r, \quad \beta(A-\lambda B) = \beta(A) - r,$$

(763) $$\nu(A-\lambda B : B) = \infty.$$

Proof. We may assume that B is bounded with $D[B]=X$. The general case can be reduced to this case by the device used in the proof of Theorem 3; we need only to note (644).

Thus Theorem 4 is applicable, and we shall use the notations of this theorem. For each pair A_i, B_i, $i \geq 1$, we have

(764) $$\alpha(A_i - \lambda B_i) = \beta(A_i - \lambda B_i) = 0,$$

(765) $$\nu(A_i - \lambda B_i : B_i) = \infty,$$

for every complex number $\lambda \neq 0$. This is a direct consequence of the fact that the operator $B_i^{-1}A_i$ in the finite-dimensional space X_i has the only eigenvalue 0. On the other hand, Theorem 3 is applicable to the pair A_0, B_0, for A_0 is a closed linear operator with closed range ($\gamma(A_0)>0$) and with $\nu(A_0:B_0)=\infty$. Thus we see that $A_0 - \lambda B_0$ is a closed linear operator with closed range and

(766) $$\alpha(A_0 - \lambda B_0) = \alpha(A_0), \quad \beta(A_0 - \lambda B_0) = \beta(A_0),$$

(767) $$\nu(A_0 - \lambda B_0 : B_0) = \infty.$$

These equalities are true for $|\lambda|<\gamma(A_0)\,\|B_0\|^{-1}$ because at least one of $\alpha(A_0)$ and $\beta(A_0)$ is finite.

It follows from (764) and (766) that

$$\alpha(A-\lambda B)=\sum_{i=0}^{r}\alpha(A_i-\lambda B_i)=\alpha(A_0)=\alpha(A)-r$$

by (715). This proves the first equality of (762). The second equality can be proved similarly. On the other hand, it is obvious that (765) and (767) imply (763).

7. Extension of stability theorem.

In Theorem 5 it is difficult to give an estimate of the constant ρ. It is true that we can take $\rho=\gamma(A_0)\,\|B_0\|^{-1}\geqq\gamma(A_0)\,\|B\|^{-1}$ if B is bounded, as was mentioned in the proof; but there is no simple relationship between $\gamma(A_0)$ and $\gamma(A)$. Note that A_0 is not necessarily uniquely determined by A as was remarked above.

In this connection the following theorem is of interest. It may be regarded as an extension of Theorem 1a and, in part, of Theorem 3.

Theorem 6. Let X, X' be Banach spaces. Let A be a closed linear operator $(X\to X')$ with closed range and with at least one of $\alpha(A)$ and $\beta(A)$ finite. Let B be a linear operator $(X\to X')$ such that $D[B]\supset D[A]$ and (761) holds. Then, for

$$|\lambda|<\gamma(A)\,[\sigma+\tau\gamma(A)]^{-1}, \tag{771}$$

$A-\lambda B$ is a closed linear operator with closed range and both $\alpha(A-\lambda B)$ and $\beta(A-\lambda B)$ are constant except at a countable number of values of λ which have no accumulation point inside of (771). Let p, q be these constant values of α, β respectively. Then, for each of these exceptional points λ, there is a finite, positive integer $r(\lambda)$ such that

$$\alpha(A-\lambda B)=p+r(\lambda),\quad \beta(A-\lambda B)=q+r(\lambda) \tag{772}$$

$$0<r(\lambda)\leqq r(0). \tag{773}$$

There are no such exceptional points in the circle (771) unless $\lambda=0$ is itself such an exceptional point.

Proof. That $A-\lambda B$ is a closed linear operator with closed range for (771) has been proved in Theorem 1a. Thus we can apply either Theorem 3 or Theorem 5 to the pair $A-\lambda_0 B$, B for each λ_0 satisfying (771). The result is that there is a neighborhood of λ_0 in which $\alpha(\lambda)=\alpha(A-\lambda B)$ and $\beta(\lambda)=\beta(A-\lambda B)$ are constant at least except for $\lambda=\lambda_0$. It follows that these constant values must be the same throughout the circle (771), and the theorem follows immediately. That $r(\lambda)\leqq r(0)$ is a direct consequence of Theorem 1a, according to which $\alpha(\lambda)\leqq\alpha(0)$, $\beta(\lambda)\leqq\beta(0)$. In particular there can be no λ for which $r(\lambda)>0$ unless $r(0)>0$.

We note that (772) gives the proof of (432).

§ 8. CLASSIFICATION OF COMPLEX NUMBERS ACCORDING TO SPECTRAL PROPERTIES

1. Classification of complex numbers.

This section is devoted to some results that can be deduced from foregoing theorems.

Let X, X' be Banach spaces, let A be a closed linear operator $(X\to X')$ and let B be a linear operator $(X\to X')$ such that $D[B]\supset D[A]$ and

$$\|Bx\|\leqq\sigma\|x\|+\tau\|Ax\|,\qquad x\in D[A], \tag{811}$$

where σ, τ are non-negative constants. Then the operator $A-\lambda B$ is a closed linear operator $(X\to X')$ for

$$|\lambda|<\tau^{-1}. \tag{812}$$

This follows from the proof of Theorem 1a as we have remarked there.

We shall denote by Λ the set of λ satisfying (812). If B is bounded, we can take $\tau=0$ and Λ coincides with the whole complex plane. In what follows we consider only λ belonging to Λ unless otherwise stated explicitly. For brevity we write $\alpha(\lambda)=\alpha(A-\lambda B)$, $N(\lambda)=N[A-\lambda B]$ and similarly define $R(\lambda)$, $\beta(\lambda)$, $\alpha'(\lambda)$, $\beta'(\lambda)$, $\gamma(\lambda)$ and $\nu(\lambda)=\nu(A-\lambda B:B)$.

The set of all λ such that $A-\lambda B$ has closed range and at least one of $\alpha(\lambda)$, $\beta(\lambda)$ is finite will be denoted by Δ. According to Theorem 1a or 6, Δ *is an open set*. The components of Δ will be denoted by Δ_1, Δ_2, ... These are connected open sets.

The subset of Δ consisting of all points λ with $\nu(\lambda)<\infty$ [resp.

$\nu(\lambda) = \infty$] will be denoted by Π [resp. Δ']. According to Theorem 3 and 6, Δ' is an open set and Π is an isolated set consisting only of internal points of Δ. Hence each $\lambda \in \Pi$ is an internal point of some Δ_i. We denote by Π_i the set of all $\lambda \in \Pi$ belonging to Δ_i. Also it follows that $\Delta'_i = \Delta' \cap \Delta_i$, $i = 1, 2, \ldots$, are exactly the components of Δ'.

Theorem 3 shows that $\alpha(\lambda)$ and $\beta(\lambda)$ are constant in each Δ'_i, these constant values will be called the indices of Δ'_i (and also of Δ_i) and denoted by α_i, β_i. By definition at least one of α_i, β_i is finite. Theorem 3 shows further that $\gamma(\lambda)$ is continuous in Δ' and hence also in each Δ'_i, and that $N(\lambda)$, $R(\lambda)^{\perp}$ depend analitically on λ for $\lambda \in \Delta'$.

Theorem 6 shows that, for each $\lambda \in \Pi_i$, there is an integer $r(\lambda)$ such that

$$(813) \qquad 0 < r(\lambda) < \infty, \quad \alpha(\lambda) = \alpha_i + r(\lambda), \quad \beta(\lambda) = \beta_i + r(\lambda).$$

For this reason, each $\lambda \in \Pi$ will be called a discrete eigenvalue (of A with respect to B) and $r(\lambda)$ the multiplicity of this discrete eigenvalue.

The complement of Δ in Λ will be denoted by Σ; Σ is relatively closed in Λ. The points of Σ will be called singular points (of A with respect to B). It follows from the definition that, for $\lambda \in \Sigma$, either the range of $A - \lambda B$ is not closed ($\gamma(\lambda) = 0$) or

$$\alpha(\lambda) = \beta(\lambda) = \infty .$$

Lemmas 421 and 422 show that λ is a singular point if and only if $\alpha'(\lambda) = \beta'(\lambda) = \infty$.

Σ can be divided into two parts Σ' and Σ''. Σ' is the set of all λ such that $A - \lambda B$ has closed range and $\alpha(\lambda) = \beta(\lambda) = \nu(\lambda) = \infty$. Theorem 3 shows that Σ' is an open set and $\gamma(\lambda)$ is continuous for $\lambda \in \Sigma'$. Also $N(\lambda)$ and $R(\lambda)^{\perp}$ depend analytically on λ for $\lambda \in \Sigma'$. Σ'' is the set of all λ such that either the range of $A - \lambda B$ is not closed or

$$\alpha(\lambda) = \beta(\lambda) = \infty > \nu(\lambda) .$$

Σ'' is relatively closed in Λ.

It is easily seen that the boundaries of Δ_i and Σ' are subsets of Σ''.

We denote by $\rho(\lambda)$ the distance of a point $\lambda \in \Pi$ from Σ''. In other words, $\rho(\lambda)$ is the distance of λ from the boundary of Δ_i to which λ belongs. Theorem 6 shows that

$$\rho(\lambda) \geqq \gamma(\lambda) / [\sigma + \tau\gamma(\lambda)] > 0, \qquad \lambda \in \Pi. \tag{814}$$

For each $\lambda \in \Delta'$ let $\rho'(\lambda)$ be the distance of λ from $\Pi \cup \Sigma''$. In other words, $\rho'(\lambda)$ is the distance of λ from the boundary of Δ'_i to which λ belongs. Theorem 3 shows that

$$\rho'(\lambda) \geqq \gamma(\lambda) / [\sigma + \tau\gamma(\lambda)] > 0, \qquad \lambda \in \Delta'. \tag{815}$$

The two inequalities (814) and (815) have the same form, but it should be noted that the meaning of $\rho'(\lambda)$ is somewhat different from that of $\rho(\lambda)$. This is an expression of the discontinuity of $\gamma(\lambda)$ at a discrete eigenvalue λ_i.

We can also define the distance $\rho'(\lambda)$ of a $\lambda \in \Sigma'$ from Σ'' (or, equivalently, from the boundary of Σ'). It follows from Theorem 3 that

$$\rho'(\lambda) \geqq \gamma(\lambda) / [3\sigma + \tau\gamma(\lambda)] > 0, \qquad \lambda \in \Sigma'. \tag{816}$$

The factor 3 can be dropped if either X or X' is a Hilbert space.

2. The case of countable singular points.

The number of the components Δ_i of Δ may be 0, 1, 2, ..., or ∞. If there are more than one components, the boundary of Δ must have the power of continuum. This is a simple consequence of the fact that Δ is an open set. But the boundary of Δ consists only of singular points. Consequently, there is one and only one component of Δ if there are at most a countable number of singular points. Thus we have proved the following theorem.

Theorem 7. *Let A, B be as above. Assume that there are at most a countable number of points λ* (singular points) *such that $\alpha'(A-\lambda B) = \beta'(A-\lambda B) = \infty$. Then the closed linear operator $A-\lambda B$ has closed range for all $\lambda \in \Lambda$ except possibly for the singular points, and $\alpha(A-\lambda B)$, $\beta(A-\lambda B)$ are constant for all $\lambda \in \Lambda$ except for the singular points and the discrete eigenvalues. There are at most a countable number of discrete eigenvalues λ, for which we have*

$$\alpha(A-\lambda B) = \alpha + r(\lambda), \quad \beta(A-\lambda B) = \beta + r(\lambda), \tag{721}$$
$$0 < r(\lambda) < \infty$$

where α, β are the constant values stated above and at least one of them is finite.

The simplest situation from our point of view is the case in which there is no singular point. The following theorem is concerned with such a case.

Theorem 8. Let X, X' be Banach spaces. Let A be a closed linear operator $(X \to X')$ with closed range and with $\alpha(A) < \infty$. Let B be a strictly singular operator relative to A (see §4.5). Then $A - \lambda B$ is a closed linear operator with closed range for every complex number λ. $\alpha(A - \lambda B) = \alpha < \infty$ and $\beta(A - \lambda B) = \beta$ are constant except for the (at most countable) discrete eigenvalues λ for which we have (721).

Proof. There is no singular point as is easily seen from Theorem 2a, §4.5. Hence the theorem follows from Theorem 7.

3. Fredholm spectrum and Riesz-Schauder theorem.

The foregoing theorems can be applied to the case in which $X' = X$ and $B = I$ (the identity operator of X). In this case Λ is the whole complex plane. The components Δ'_i for which $\alpha_i = \beta_i = 0$ are exactly the components of the resolvent set of A. If in particular A is bounded with domain X, its resolvent set is not empty. Hence we must have $\alpha = \beta = 0$ in Theorem 7. This gives

Theorem 9. Let X be a Banach space and let A be a bounded linear operator $(X \to X)$ with domain X. If there is only a countable number of points λ (singular points) for which $\alpha'(A - \lambda I) = \beta'(A - \lambda I) = \infty$, A has a Fredholm spectrum except for these singular points.

By saying that A has a Fredholm spectrum in a set Γ of the complex plane, we mean that $A - \lambda I$ has closed range for all $\lambda \in \Gamma$ and that $\alpha(A - \lambda I) = \beta(A - \lambda I) = 0$ except for at most countable number of discrete eigenvalues λ for which

$$0 < \alpha(A - \lambda I) = \beta(A - \lambda I) = r(\lambda) < \infty. \tag{731}$$

Suppose in particular that $\lambda = 0$ is the only possible singular point in Theorem 9. Then A has a Fredholm spectrum with the single exception of the point $\lambda = 0$. In other words, the Riesz-Schauder theorem holds for A. Thus we obtain

Theorem 10. *Let X be a Banach space and let A be a bounded linear operator $(X \to X)$ with domain X. The Riesz-Schauder theorem holds for A if and only if $\lambda = 0$ is the only possible singular point, that is, if and only if $\alpha'(A - \lambda I) = \beta'(A - \lambda I) = \infty$ is not true for any $\lambda \neq 0$.*

Remark. This theorem shows that the Riesz-Schauder theorem holds for a strictly singular operator B, for there is no singular point μ for the operator of the form $I - \mu B$ as is seen from the proof of Theorem 8.

Since the complete continuity implies the strict singularity (see § 4.5), our theory furnishes an independent proof of the Riesz-Schauder theorem for a completely continuous operator. In this connection it should be remarked that, in proving Theorems 7 to 10, we do not need the fact that Σ' is an open set or various estimates for the quantity $\gamma(A - \lambda B)$, to which a greater part of the present paper is devoted.

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Department of Physics
University of Tokyo
Tokyo, Japan

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