THE SCHWARZIAN DERIVATIVE AND UNIFORMIZATION

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1. Introduction.

Let

$$f(z, u) = 0 \tag{1.1}$$

be an irreducible algebraic or analytic equation connecting two complex variables z and u, and let $\mathfrak{R}_f$ be the associated Riemann surface. We regard $\mathfrak{R}_f$ as a branched covering surface[1] of a connected domain $\mathfrak{R}_z$, which is a subset of the extended z-plane, and write σ for the projection mapping from $\mathfrak{R}_f$ to $\mathfrak{R}_z$, so that $\mathfrak{R}_z = \sigma(\mathfrak{R}_f)$. Let $\mathfrak{B}_z$ denote a set of isolated points of $\mathfrak{R}_z$ which includes as a subset the set of images of branch-points of $\mathfrak{R}_f$ with respect to $\mathfrak{R}_z$; in most cases, $\mathfrak{B}_z$ will consist entirely of these image points. We denote by $\mathfrak{B}_z^\infty$ a subset of $\mathfrak{B}_z$ to be specified later, and put $\mathfrak{B}_z^0 = \mathfrak{B}_z - \mathfrak{B}_z^\infty$. We also write $\mathfrak{B}_f$, $\mathfrak{B}_f^\infty$ and $\mathfrak{B}_f^0$ for the sets of all points of $\mathfrak{R}_f$ which lie over $\mathfrak{B}_z$, $\mathfrak{B}_z^\infty$ and $\mathfrak{B}_z^0$, respectively, and put

$$\mathfrak{R}'_f = \mathfrak{R}_f - \mathfrak{B}_f^\infty, \quad \mathfrak{R}'_z = \mathfrak{R}_z - \mathfrak{B}_z^\infty = \mathfrak{R}_z - \sigma(\mathfrak{B}_f^\infty). \tag{1.2}$$

We denote by $\widehat{\mathfrak{R}}$ a universal (i. e. simply-connected) covering surface of $\mathfrak{R}'_f$ (and therefore of $\mathfrak{R}'_z$), which is branched over $\mathfrak{R}'_z$, and possibly also branched over $\mathfrak{R}'_f$, at points lying over points of the set $\mathfrak{B}_z^0$, but which is otherwise smooth. We shall demand further properties of $\widehat{\mathfrak{R}}$ later; it may be possible to choose $\widehat{\mathfrak{R}}$ in several different ways. Since $\widehat{\mathfrak{R}}$ is simply-connected, there is a conformal one-to-one mapping τ of $\widehat{\mathfrak{R}}$ onto a domain $\mathfrak{G}$ of the t-plane, where $\mathfrak{G}$ is either (i) the extended t-plane, (ii) the finite t-plane, or (iii) the unit circle $|t| < 1$. These three cases are called the elliptic, parabolic and hyperbolic cases, respectively.

The equation (1.1) can be uniformized by functions $z = z(t)$ and $u = u(t)$, which are defined and meromorphic on $\mathfrak{G}$. They are automorphic functions for a properly discontinuous group Γ_f of bilinear transformations S defined by

1. The terminology used is that of [12].

(1.3) $$S(t) = t_S = \frac{\alpha t + \beta}{\gamma t + \delta} \qquad (\alpha\delta - \beta\gamma = 1),$$

so that $z(t_S) = z(t)$ and $u(t_S) = u(t)$, for all $S \in \Gamma_f$ and $t \in \mathfrak{G}$; further, $S\mathfrak{G} = \mathfrak{G}$ for all $S \in \Gamma_f$.

When (1.1) is an algebraic equation of genus $p > 1$, and $\widehat{\mathfrak{R}}$ is taken to be the smooth universal covering surface of $\mathfrak{R}_f$, the functions $z(t)$ and $u(t)$ are simple automorphic functions (see [2, p. 86]) for Γ_f, which is then horocyclic (Fuchsian); in this case $\mathfrak{G}$ is hyperbolic and each fundamental region $D(\Gamma_f)$ for the group is bounded by a finite number of sides. There are no elliptic or parabolic cycles and the closure of $D(\Gamma_f)$ lies in $\mathfrak{G}$.

When (1.1) is not algebraic, the automorphic functions need not be simple; in particular, the fundamental regions $D(\Gamma_f)$ may have infinitely many sides and the number of poles and zeros of the functions $z(t)$ and $u(t)$ may be infinite in each fundamental region.

The Schwarzian derivative $\{t, z\}$ is defined by

(1.4) $$\{t, z\} = -\frac{z'''(t)}{\{z'(t)\}^3} + \frac{3}{2}\frac{\{z''(t)\}^2}{\{z'(t)\}^4}.$$

It follows that

(1.5) $$\{t, z\} = \{t, \zeta\}\left(\frac{d\zeta}{dz}\right)^2 + \{\zeta, z\}.$$

It is easily checked from (1.4) that $\{t, z\}$ is an automorphic function for any group for which $z(t)$ is automorphic. Also $\{t, z\}$ is unaltered when t is subjected to a bilinear transformation and so does not depend upon the choice of the mapping τ.

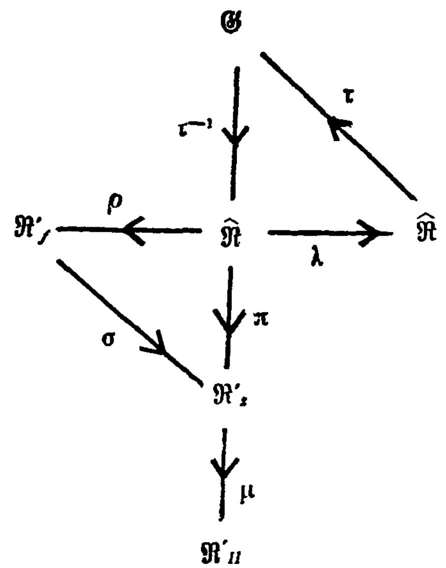
Our object is to obtain as much information as possible about the form of the Schwarzian derivative, since the explicit construction of the uniformizing functions $z(t)$ and $u(t)$ depends on this. We show, in particular, that $\{t, z\}$ can be evaluated explicitly for certain types of equation (1.1). Before this can be described the surfaces $\widehat{\mathfrak{R}}$, $\mathfrak{R}_f$ and $\mathfrak{R}_z$ must be considered in greater detail.

Points of $\widehat{\mathfrak{R}}$, $\mathfrak{R}_f$ and $\mathfrak{R}_z$ will be denoted by $\widehat{P}$, P and z, respectively. The projection mappings of $\widehat{\mathfrak{R}}$ onto $\mathfrak{R}'_f$ and $\mathfrak{R}'_z$ are denoted by ρ and π respectively, so that $\pi = \sigma\rho$ (see diagram).

Let $\widehat{\Lambda}$ denote the group of all conformal one-to-one mappings of $\widehat{\mathfrak{R}}$ onto itself, and let Λ be a subgroup of $\widehat{\Lambda}$. Thus, for $\lambda \in \Lambda$, $\tau\lambda\tau^{-1}$ is a conformal mapping of $\mathfrak{G}$ onto itself, and so

$$\tau\lambda\tau^{-1}(t) = S_\lambda(t)\,, \tag{1.6}$$

where S_λ is a bilinear transformation. Thus Λ is isomorphic to a group $\Gamma(\Lambda)$ of bilinear transformations S_λ which map $\mathfrak{G}$ onto itself. In particular, if Λ is the group Λ_f of covering transformations of $\widehat{\mathfrak{R}}$ over $\mathfrak{R}'_f$, i. e. if $\rho\{\lambda(\widehat{P})\} = \rho(\widehat{P})$ for all $\lambda \in \Lambda_f$ and $\widehat{P} \in \widehat{\mathfrak{R}}$, then $\Gamma(\Lambda_f) = \Gamma_f$.



We now depart from the completely general situation just described and make certain simplifying assumptions. We recall that a smooth covering manifold $\mathfrak{M}^*$ of a manifold $\mathfrak{M}$ is said to be regular if its group of covering transformations is transitive, i. e. if there is a covering transformation mapping P_1^* onto P_2^*, where P_1^* and P_2^* are any two points of $\mathfrak{M}^*$ lying over some point P of $\mathfrak{M}$. We shall say that a branched covering manifold $\mathfrak{M}^*$ of a manifold $\mathfrak{M}$ is regular, if $\mathfrak{M}^* - \mathfrak{B}^*$ is regular over $\mathfrak{M} - \mathfrak{B}$, where $\mathfrak{B}^*$ is the set of all points of $\mathfrak{M}^*$ which lie over the set $\mathfrak{B}$ of projections of branch-points on $\mathfrak{M}$; any covering transformation of $\mathfrak{M}^* - \mathfrak{B}^*$ over $\mathfrak{M} - \mathfrak{B}$ can then be extended uniquely to a covering transformation of $\mathfrak{M}^*$ over $\mathfrak{M}$. We now assume that the equation (1.1) is such that $\mathfrak{R}_f$ is an unlimited branched covering surface of $\mathfrak{R}_z$, and that $\widehat{\mathfrak{R}}$ is a regular covering surface of $\mathfrak{R}'_z$; $\widehat{\mathfrak{R}}$ is then automatically a regular covering surface of $\mathfrak{R}'_f$, but $\mathfrak{R}_f$ will not in general be regular over $\mathfrak{R}_z$. Such a surface $\widehat{\mathfrak{R}}$ can usually be chosen in several different ways by making different choices of $\mathfrak{B}_z$ and its subset $\mathfrak{B}_z^\infty$. For example, if we take $\mathfrak{B}_z^\infty = \mathfrak{B}_z$, then $\widehat{\mathfrak{R}}$ is merely the smooth universal covering surface of $\mathfrak{R}'_z$. On the other hand, if $\mathfrak{R}_f$ is regular over $\mathfrak{R}_z$, we can take $\mathfrak{B}_z^0 = \mathfrak{B}_z$ and take $\widehat{\mathfrak{R}}$ to be the smooth universal covering surface of $\mathfrak{R}_f$. In these two cases $\mathfrak{B}_z$ may be chosen to consist entirely of images of branch-points of $\mathfrak{R}_f$ with respect to $\mathfrak{R}_z$.

In other cases, where $\mathfrak{R}_f$ is not regular over $\mathfrak{R}_z$, it may be possible to construct $\widehat{\mathfrak{R}}$ as a regular branched surface over $\mathfrak{R}'_z$ without taking $\mathfrak{B}_z^\infty$ to be the whole of $\mathfrak{B}_z$. To each $P \in \mathfrak{R}_f$ there corresponds a unique finite positive integer $q(P)$ such that P is a branch-point of order $q(P) - 1$ of $\mathfrak{R}_f$ with respect to $\mathfrak{R}_z$; if $q(P) = 1$, P is a regular point of $\mathfrak{R}_f$ with respect to $\mathfrak{R}_z$. We now assign, by some rule, a positive integer $r(P)$ such that the product $q(P)\, r(P)$ is the same for all points P lying over the same point z of $\mathfrak{R}_z$. Thus

$$q(P)\, r(P) = p(z)\,, \tag{1.7}$$

say, for all $P \in \mathfrak{R}_f$ such that $\sigma(P) = z$. We make the restriction that the set of points P of $\mathfrak{R}_f$ at which $r(P) > 1$ is isolated. On the other hand, we allow $r(P)$ to be infinite, in which case $p(z)$ is also infinite; it may not, in fact, be possible to satisfy (1.7) for finite $r(P)$. It follows that $r(P)$ is either finite for all points P lying over a given $z \in \mathfrak{R}_z$, or $r(P)$ is infinite for all such points. We take $\mathfrak{B}_z^\infty$ to be the set of points z for which $p(z)$ is infinite and $\mathfrak{B}_z^0$ to be the set of points for which $p(z)$ is finite and greater than unity. This defines $\mathfrak{R}'_f$ and $\mathfrak{R}'_z$, and the regular covering surface $\widehat{\mathfrak{R}}$ is to be constructed (2) so that all the branch-points $\widehat{P}$ lying over P on $\mathfrak{R}'_f$ are of order $r(P) - 1$, while all the branch-points $\widehat{P}$ lying over z on $\mathfrak{R}'_z$ have order $p(z) - 1$.

The covering transformations λ or $\widehat{\mathfrak{R}}$ over $\mathfrak{R}'_z$ form a subgroup Λ_z of $\widehat{\Lambda}$ which contains Λ_f as a subgroup. If $\Gamma_z = \Gamma(\Lambda_z)$, then $z(t)$ is an automorphic function for Γ_z, which is properly discontinuous. (3) Any other automorphic function $\varphi(t)$ for Γ_z is uniquely determined by the value of $z(t)$ and so is a single-valued function of $z = z(t)$. In particular, $\{t, z\}$,

2. For constructions of regular branched covering surfaces of closed Riemann surfaces see Fourès [4] and Rankin [11]; see also Fourès [3].

3. This may be proved as in Theorem 9—5 of [12]; the neighbourhood $\widehat{N}$ can only cover S a finite number of times p.

which is an automorphic function for Γ_z, is a single-valued meromorphic function of z on $\mathfrak{R}'_z$, and we may therefore write

$$\varphi(t) = \{t, z(t)\} = 2R(z) \qquad (z \in \mathfrak{R}'_z). \tag{1.8}$$

The coefficient 2 stems from the fact that t can be expressed as the quotient of two linearly independent solutions of the differential equation

$$\frac{d^2\eta}{dz^2} + R(z)\eta = 0;$$

if, therefore, $R(z)$ can be determined explicitly and this equation can be solved, Γ_z can be found and the uniformizing functions can be set up explicitly.

We now make one further assumption regarding the form of equation (1.1). Let G denote a properly discontinuous group of bilinear transformations T for which $T\mathfrak{R}_z = \mathfrak{R}_z$. Usually G will be finite or Fuchsian. We write

$$\zeta = Tz = \frac{az+b}{cz+d} \qquad (ad - bc = 1). \tag{1.9}$$

We suppose that there exists a group Ω_G of conformal transformations λ_T $(T \in G)$ of $\mathfrak{R}_f$ onto itself such that

$$\sigma\{\lambda_T(P)\} = \zeta = Tz = T\sigma(P) \qquad (P \in \mathfrak{R}_f). \tag{1.10}$$

When this happens we say that the equation (1.1) is *invariant under the group* G. Every equation is invariant, in particular, under the group consisting of the identity transformation only. This situation arises, for example, when the variables z and u can be replaced, by means of a birational transformation, by variables $\zeta = Tz$ and w such that

$$f(\zeta, w) = \rho(z) f(z, u), \tag{1.11}$$

where $\rho(z)$ is a meromorphic function.

A fundamental region for G in $\mathfrak{R}_z$ with boundary points identified can be regarded as a manifold $\mathfrak{R}_G$ for which $\mathfrak{R}_z$ is a regular covering manifold. If G contains no elliptic transformations, $\mathfrak{R}_z$ is a smooth covering manifold of $\mathfrak{R}_z$. If, however, z_0 is an elliptic fixed point of G of period k, then z_0 is a branch point of order $k-1$ with respect to $\mathfrak{R}_G$.[4] We denote by $\mathfrak{R}'_G$ the projection of $\mathfrak{R}'_z$ on $\mathfrak{R}_G$. We suppose that $\hat{\mathfrak{R}}$ is constructed so that it is also a regular covering manifold of $\mathfrak{R}'_G$. For example, in (1.7)

4. Coverings of this type for Fuchsian groups with elliptic transformations have been considered by Fourès [5].

we have $q(\lambda_T P) = q(P)$, and must take $r(\lambda_T P) = r(P)$ for all $\lambda_T \in \Omega_G$. The group of transformations Ω_G on $\mathfrak{R}_f$ can then be extended to $\widehat{\mathfrak{R}}$ to form a group Λ_G which contains $\widehat{\Lambda}$ and Λ_z as normal subgroups, and we write $\Gamma_G = \Gamma(\Lambda_G)$.

For any $T \in G$ take $S \in \Gamma_G$ such that $z(St) = Tz(t)$. Then, by (1.8), (1.5) and (1.9),

$$
\begin{aligned}
2R(Tz) &= 2R\{z(St)\} = \varphi(St) = \{St\,,\, z(St)\} \\
&= \{t\,,\, z(St)\} = \{t\,,\, Tz(t)\} = (cz+d)^4 \{t\,,\, z(t)\} \\
&= 2\,(cz+d)^4 R(z)\,.
\end{aligned} \tag{1.12}
$$

It follows that $R(z)$ is an automorphic form of dimension -4 for G (with multiplier system 1).

An equation which is invariant under G is invariant under any subgroup H of G, and there exist corresponding manifolds $\mathfrak{R}_H$ and $\mathfrak{R}'_H$. We write Λ_H for the group of covering transformations of $\widehat{\mathfrak{R}}$ over $\mathfrak{R}'_H$ and put $\Gamma_H = \Gamma(\Lambda_H)$.

The algebraic equations considered in [10] are invariant under certain cyclic or polyhedral groups. For them $\mathfrak{R}_f$ is regular over $\mathfrak{R}_z$, so that $\widehat{\mathfrak{R}}$ could be chosen to be the universal covering surface of $\mathfrak{R}_f$. The group Γ_f is then a subgroup of a group of Riemann-Schwarz triangle functions and $R(z)$ is a rational function. The theorems which we prove in the following sections enable us to deal with equations of a more general type and with different choices of groups Γ_f. They also make clear what was not apparent from the *ad hoc* arguments used in [10], namely that, for the majority of the algebraic equations considered there, the explicit determination of the Schwarzian derivative was possible because the points of $\mathfrak{B}_z$ were elliptic fixed points of G.

2. Boundary points.

We assume in this section that the equation (1.1) is invariant under a properly discontinuous group G, and that $\widehat{\mathfrak{R}}$ is a simply-connected regular covering surface of $\mathfrak{R}'_z$ and $\mathfrak{R}'_f$ of the kind described in § 1. By transforming z by a suitable bilinear transformation, which merely replaces G by a conjugate group, we can arrange that either (i) the point at infinity is not in the closure[5] of $\mathfrak{R}'_z$, or, if this is not possible, that (ii) $z = \infty$

5. Here and elsewhere $\mathfrak{R}'_z$ is regarded as a subset of the extended z-plane, i. e. the Riemann-Neumann sphere, with the appropriate topology.

is a point of $\mathfrak{R}'_z$, but does not belong to $\mathfrak{B}_z$ and is not a fixed point of G.

We now define an accessible boundary point. The definition differs from that given by Nevanlinna ([8, p. 184], [6]) in that we make incisions in $\widehat{\mathfrak{R}}$ rather than $\mathfrak{R}'_f$. This difference is necessitated by the fact that $\widehat{\mathfrak{R}}$ may be branched over $\mathfrak{R}'_f$, but for a smooth covering surface $\widehat{\mathfrak{R}}$ the two definitions are equivalent. Also the homotopic properties required in Nevanlinna's definition can be omitted since $\widehat{\mathfrak{R}}$ is simply-connected. See also Ohtsuka [9] and Kuramochi [7].

Let z_0 be a point on the boundary of $\mathfrak{R}'_z$. We define an *incision* $\widehat{\gamma}$ in $\widehat{\mathfrak{R}}$ over z_0 to be a continuous mapping $\widehat{\gamma}(x)$ of the interval $0 \leq x < 1$ into $\widehat{\mathfrak{R}}$ which is such that

$$\gamma_z(x) = \pi\{\widehat{\gamma}(x)\} \rightarrow z_0 \quad \text{as} \quad x \rightarrow 1-.$$

It follows, in particular, that $\widehat{\mathfrak{R}}$ contains no interior point which is a point of accumulation of $\widehat{\gamma}(x)$ as $x \rightarrow 1-$. Two incisions $\widehat{\gamma}$ and $\widehat{\gamma}'$ in $\widehat{\mathfrak{R}}$ are said to be *equivalent* if (i) they are incisions over the same boundary point z_0, and if (ii), for every non-negative $x < 1$, there exists a curve α_x in $\widehat{\mathfrak{R}}$ joining $\widehat{\gamma}(x)$ to $\widehat{\gamma}'(x)$, such that $\pi(\alpha_x)$ tends uniformly to z_0 as $x \rightarrow 1-$. The first of these two conditions is a consequence of the second. An *accessible boundary point* of $\widehat{\mathfrak{R}}$ lying over z_0 is then defined to be a class $\mathfrak{L}(z_0)$ of equivalent incisions $\widehat{\gamma}$ in $\widehat{\mathfrak{R}}$ over z_0. There is no loss of generality in assuming that no incision $\widehat{\gamma}$ passes through a branch-point, i.e. a point lying over a point of $\mathfrak{B}^0_z$, since this can always be achieved by a suitable deformation. This assumption is useful when covering transformations are applied.

If $\mathfrak{L}(z_0)$ is an accessible boundary point of $\widehat{\mathfrak{R}}$ lying over z_0, we shall say that $\mathfrak{L}(z_0)$ is a *simple boundary point* of $\widehat{\mathfrak{R}}$ when $\tau\{\widehat{\gamma}(x)\}$ tends to a point t_0 as $x \rightarrow 1-$; such a point t_0 is necessarily on the boundary of $\mathfrak{G}$.

For any covering transformation λ of Λ_z, $\lambda(\widehat{\gamma})$ is an incision which lies over the same boundary point z_0 as $\widehat{\gamma}$ does, and we denote by $\lambda\mathfrak{L}(z_0)$ the set of all incisions $\lambda(\widehat{\gamma})$ for $\widehat{\gamma} \in \mathfrak{L}(z_0)$. It is clear that $\lambda\mathfrak{L}(z_0)$ is also a class of equivalent incisions and that $\lambda\mathfrak{L}(z_0)$ and $\mathfrak{L}(z_0)$ are either identical or disjoint. When $\mathfrak{L}(z_0)$ is a simple boundary point so is $\lambda\mathfrak{L}(z_0)$, the corresponding boundary point of $\mathfrak{G}$ being $S_\lambda(t_0)$, where S_λ is given by (1.6).

If z_0 is an isolated boundary point of $\mathfrak{R}'_z$, the accessible boundary points of $\widehat{\mathfrak{R}}$ which lie over it are all of the form $\lambda\mathfrak{L}(z_0)$, for $\lambda \in \Lambda_s$, where $\mathfrak{L}(z_0)$ is any one such accessible boundary point. It follows that these 'points' are either all simple boundary points or none of them are; in the former case we say that z_0 is a *simple isolated boundary point* of $\mathfrak{R}'_s$, simplicity being measured with respect to $\widehat{\mathfrak{R}}$.

If z_0 is a simple isolated boundary point of $\mathfrak{R}'_s$ and t_0 is any point on the boundary of $\mathfrak{G}$ which is associated with it, then t_0 is a fixed point of a parabolic transformation S of Γ_s [6, p. 50], and we can construct a fundamental region $D(\Gamma_s)$ for Γ_s which has sides δ and δ' terminating at t_0 for which $\delta' = S(\delta)$. We can do this, for example, by taking $D(\Gamma_s)$ to be the image in $\mathfrak{G}$ of a copy of $\mathfrak{R}'_s$ which has been slit to make it simply-connected. One such slit will emanate from z_0 and its two sides map into δ and δ'. This holds, in particular, when $z_0 \in \mathfrak{B}_s^\infty$ and is simple.

We now suppose that z_0 is a parabolic fixed point for a group G under which (1.1) is invariant. Let $D(G)$ be a fundamental region for G having z_0 as a cusp. It may happen that $D(G)$ has *two* cusps at z_0, but we then regard these two cusps as different and confine our attention to one of them only. We may suppose that the sides γ ond γ' of $D(G)$ which meet at the cusp z_0 are arcs of circles which touch at z_0 and are orthogonal there to the line l which joins their centres. Let K_ρ be a circle of radius ρ which touches l at z_0 and lies on the same side of l as the cusp in question. If ρ is sufficiently small, $\rho < \rho_0$ say, the part Δ_ρ of K_ρ which lies between γ and γ' will be a curvilinear triangle forming a subset of $D(G)$. Let P be the parabolic transformation of G for which

$$\gamma' = P\gamma \quad \text{and} \quad z_0 = P(z_0). \tag{2.1}$$

Then K_ρ is the union of the sets $P^r(\Delta_\rho)$ for $r = 0, \pm 1, \pm 2, \ldots$.

We now make the additional assumption that the parabolic fixed point z_0, is a boundary point of $\mathfrak{R}'_s$ (it cannot be an interior point) and that, for some positive $\rho_1 < \rho_0$, the interior of $\Delta_{\rho'}$ is contained in $\mathfrak{R}'_s$ and contains no points of $\mathfrak{B}_s$. The same is then true of the interior of K_{ρ_1}. Let $\mathfrak{L}(z_0)$ be a class of incisions $\widehat{\gamma}$ in $\widehat{\mathfrak{R}}$ whose projections γ_s in $\mathfrak{R}'_s$ are entirely contained in K_{ρ_1}. These incisions are all equivalent and define an accessible boundary point over z_0. As before, each $\lambda \in \Lambda_s$ defines a different or the same accessible boundary point $\lambda\mathfrak{L}(z_0)$, and these accessible boundary points are all simple whenever one of them is. When they are simple, we shall say that z_0 is a simple parabolic fixed point of G (with respect to $\hat{\mathfrak{R}}$).

It may be noted that this situation is essentially the same as for a simple isolated boundary point, since z_0 lies over an isolated boundary point of $\mathfrak{R}'_H$, where H is the subgroup of G generated by P. For we can find a bilinear transformation T, not necessarily unique or in G, such that $\infty = Tz_0$, and

$$P = T^{-1}UT, \tag{2.2}$$

where

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Write

$$\zeta = Tz, \quad w = \mu(z) = e^{2\pi i\zeta}, \tag{2.3}$$

so that $\mu(Pz) = \mu(z)$. We may suppose T chosen so that $\mu(z) \to 0$ as $z \to z_0$ through Δ_ρ. If we define $\mathfrak{R}'_H$ to be the subset of the w-plane given by

$$\mathfrak{R}'_H = \{w : w = \mu(z), z \in \mathfrak{R}'_z\},$$

then $\mathfrak{R}'_z$ is a covering manifold of $\mathfrak{R}'_H$ and projects into $\mathfrak{R}'_H$ under the projection mapping μ. The region Δ_ρ projects onto a deleted circular neighbourhood of $w = 0$. When z_0 is a simple parabolic fixed point, the point $w = 0$ is a simple isolated boundary point of $\mathfrak{R}'_H$. As before, each of the associated boundary points t_0 of $\mathfrak{S}$ is a parabolic fixed point for a parabolic transformation S_P of Γ_H (and also of Γ_G).

A condition for a boundary point to be simple has been given by Ohtsuka [9, p. 105] generalizing work of Nevanlinna [8] and Kaila [6]. Suppose that $\hat{\mathfrak{R}}$ is a covering surface of a Riemann surface $\mathfrak{Z}$ such that the closure of the projection $\overline{\mathfrak{R}}$ of $\hat{\mathfrak{R}}$ in $\mathfrak{Z}$ is compact. Then, in the hyperbolic case, accessible boundary points of $\hat{\mathfrak{R}}$ are simple, except possibly when $\overline{\mathfrak{R}}$ is a closed surface of genus (i) zero or (ii) one, and $\mathfrak{Z} - \overline{\mathfrak{R}}$ contains (i) less than three, or (ii) no points, respectively. This result, which is slightly more general than that given by Ohtsuka since $\hat{\mathfrak{R}}$ may be branched, is most simply applied when $\overline{\mathfrak{R}}$ and $\mathfrak{Z}$ can be taken to be $\mathfrak{R}'_z$ and the extended z-plane, respectively.

3. The Schwarzian derivative as an automorphic form for G.

Under the assumptions made, the Schwarzian derivative $\{t, z\}$ is holomorphic on $\mathfrak{R}_z$, except possibly at the points of $\mathfrak{B}_z$. On a deleted neighbourhood of a point $z_0 \in \mathfrak{B}_z^0$ we can write

$$R(z) = \tfrac{1}{2}\{t, z\} = \Phi(z)\,(z - z_0)^{-2}, \tag{3.1}$$

where $\Phi(z)$ is holomorphic at z_0 and

$$\Phi(z_0) = \frac{1}{4}\left(1 - \frac{1}{p^2}\right). \tag{3.2}$$

Here $p - 1 = p(z_0) - 1$ and is the (positive) order of the branch-points of $\widehat{\mathfrak{R}}$ which lie over z_0, with respect to $\mathfrak{R}'_z$.

We can also determine the behaviour of $\{t, z\}$ on a deleted neighbourhood of a simple isolated boundary point z_0 of $\mathfrak{R}'_z$. For, if the parabolic transformation S corresponding to one of the associated boundary points t_0 of $\mathfrak{G}$ takes the form $t' = S(t)$, where

$$\frac{1}{t' - t_0} = \frac{1}{t - t_0} + \mu,$$

then, within a cuspidal neighbourhood of t_0 we have

$$z(t) = z_0 + \sum_{n=1}^{\infty} a_n \exp\left\{\frac{2\pi i n}{\mu}\, \frac{1}{t - t_0}\right\}.$$

From this we deduce that (3.1) holds on a deleted neighbourhood of z_0, where $\Phi(z)$ is holomorphic at z_0 and

$$\Phi(z_0) = \tfrac{1}{4}. \tag{3.3}$$

Accordingly, if all the points of $\mathfrak{B}_z^{\infty}$ are simple, the behaviour of $R(z)$ at its singularities on $\mathfrak{R}_z$ is determined.

We write $\mathfrak{R}_z^*$ for the union of $\mathfrak{R}'_z$ and the set $\mathfrak{B}_z^{\infty\prime}$ of simple isolated boundary points of $\mathfrak{R}'_z$, and put

$$\mathfrak{B}_z^* = \mathfrak{B}_z^0 \cup \mathfrak{B}_z^{\infty\prime}.$$

Let $\mathcal{M}$ denote the set of all automorphic forms $F(z)$ of dimension -4 (and with multiplier system 1) for G, such that (i) $F(z)$ is holomorphic on $\mathfrak{R}_z^* - \mathfrak{B}_z^*$, (ii) on a deleted neighbourhood of a point $z_0 \in \mathfrak{B}_z^*$

$$F(z) = \Phi(z)\,(z - z_0)^{-2},$$

where $\Phi(z)$ is holomorphic at z_0 and (3.2) holds, where $p = p(z_0)$ for $z_0 \in \mathfrak{B}_z^0$ and $1/p = 0$ for $z \in \mathfrak{B}_z^{\infty\prime}$. We also denote by $\mathcal{H}$ the set of all automorphic forms $F(z)$ of dimension -4 for G, which are holomorphic on $\mathfrak{R}_z^*$.

If $\mathfrak{R}_z^*$ is the extended z-plane, $\mathcal{H}$ is empty. On the other hand, if $\mathfrak{R}_z^*$ is the circle $|z| < 1$ and G is horocyclic with parabolic fixed points, $\mathcal{H}$

will not, in general, be empty, and will include automorphic forms which are not holomorphic at these cusps.

We note that, if $\infty \in \mathfrak{R}'_z$ and $z = 1/\zeta$, then, by (1.5),

$$\{t, z\} = \zeta^4 \{t, \zeta\},$$

so that, since ∞ is an ordinary point,

$$R(z) = \zeta^4 \Psi(\zeta), \tag{3.4}$$

where $\Psi(\zeta)$ is holomorphic at $\zeta = 0$.

We are now in a position to state

Theorem 1. *Suppose that the equation* (1.1) *is invariant under* G, *that the points of* $\mathfrak{B}^*_z$ *are all elliptic fixed points of* G, *and that* $F(z)$ *is any function of* $\mathcal{M}$. *Then*

$$R(z) = F(z) + h(z),$$

where $h(z) \in \mathcal{H}$.

Proof. If we write

$$h(z) = R(z) - F(z),$$

then $h(z)$ is an automorphic form of dimension -4 for Γ; it is holomorphic on $\mathfrak{R}^*_z$ except possibly at points $z_0 \in \mathfrak{B}^*_z$, where

$$h(z) = (z - z_0)^m H(z)$$

for some integer $m \geqq -1$, $H(z_0) \neq 0$ and $H(z)$ is holomorphic at z_0.

Now suppose that a point $z_0 \in \mathfrak{B}^*_z$ is an elliptic fixed point of G of period $k \geqq 2$ and that T is a transformation of G of period k for which $Tz_0 = z_0$. We show that this assumption implies that $m \geqq 0$ (see [2, § 48]).

The transformation $\zeta = Tz$ may be taken in the form

$$\frac{\zeta - z_0}{\zeta - z_1} = \rho \frac{z - z_0}{z - z_1}, \tag{3.5}$$

where $\rho = e^{2\pi i/k}$ and z_1 is the other elliptic fixed point and is finite. We have, by (1.5) and (3.5),

$$(cz+d)^{-2} = \frac{d\zeta}{dz} = \frac{(\zeta - z_0)(\zeta - z_1)}{(z - z_0)(z - z_1)} = \rho \left(\frac{\zeta - z_1}{z - z_1}\right)^2.$$

In particular,

$$(cz_0 + d)^{-2} = \rho. \tag{3.6}$$

Now

$$h(Tz) = (Tz - Tz_0)^m H(Tz) = \frac{(z-z_0)^m}{(cz+d)^m (cz_0+d)^m} H(Tz),$$

and so

$$h(Tz) = (cz+d)^4 h(z) = (cz+d)^4 (z-z_0)^m H(z).$$

On division we obtain

$$1 = (cz+d)^{m+4} (cz_0+d)^m H(z)/H(Tz).$$

On letting $z \to z_0$, we deduce from this and (3.6) that

$$1 = (cz+d)^{2m+4} = \rho^{-(m+2)}.$$

It follows that $m \equiv -2 \pmod{k}$, and, since $m \geqq -1$,

$$m \geqq k - 2 \geqq 0.$$

Thus $h(z)$ is holomorphic at z_0 and so $h(z) \in \mathcal{H}$.

In many cases functions $F(z) \in \mathcal{M}$ can be constructed as sums of Poincaré series such as

$$(3.7) \qquad F(z; z_0) = \frac{1}{4k}\left(1 - \frac{1}{p^2}\right) \sum_{T \in G} \frac{1}{(cz+d)^4 (Tz - z_0)^2},$$

and

$$(3.8) \qquad F^*(z; z_0) = \frac{1}{8k}\left(1 - \frac{1}{p^2}\right) \sum_{T \in G} \frac{(z_0 - Pz_0)^2}{(cz+d)^4 (Tz-z_0)^2 (Tz - Pz_0)^2}.$$

Here k is the period of the elliptic fixed point z_0. In (3.7) we suppose that $\infty \notin \mathfrak{R}_s^*$; in (3.8), $\infty \in \mathfrak{R}_s^*$ is allowed and P is a transformation of G for which z_0 is not a fixed point. It is easily checked that both $F(z; z_0)$ and $F^*(z; z_0)$ are automorphic functions of dimension -4 for G and that they behave in the required way in the neighbourhood of points congruent to z_0. By taking finite or infinite sums of series of the forms (3.7) and (3.8) it may be possible to construct a function $F(z)$ belonging to $\mathcal{M}$. Thus the problem of determining the Schwarzian derivative is reduced to that of finding the holomorphic function $h(z)$.

Theorem 2. *Suppose that the equation* (1.1) *is invariant for a group G for which z_0 is a simple parabolic fixed point. Then $R(z)$ vanishes at z_0.*

Proof. We have already seen in § 2 that z_0 lies over a simple isolated boundary point of $\mathfrak{R}'_B$ at $w = 0$. We then have, as shown at the beginning of § 3, that

(3.9) $$\tfrac{1}{2}\{t\,,\,w\} = \Phi(w)\,w^{-2},$$

where $\Phi(w)$ is holomorphic at 0 and $\Phi(0) = \frac{1}{4}$.

Put, in the notation of (1.9), (2.1), (2.2) and (2.3),

$$R_T(\zeta) = (cz+d)^4\,R(z).$$

Then $R_T(\zeta)$ is an automorphic form of dimension -4 for the conjugate group TGT^{-1} and has a parabolic fixed point at ∞ in the upper half-plane. The statement that $R(z)$ vanishes at z_0 means that $R_T(\zeta) \to 0$ as $\operatorname{Im}\zeta \to +\infty$, i.e. as $w \to 0$. Now we have, by (1.5) and (3.9),

$$\begin{aligned} R_T(\zeta) &= \tfrac{1}{2}(cz+d)^4\{t\,,\,z\} = \tfrac{1}{2}\{t\,,\,\zeta\} \\ &= \tfrac{1}{2}\{t\,,\,w\}\left(\frac{dw}{d\zeta}\right)^2 + \tfrac{1}{2}\{w\,,\,\zeta\} \\ &= (2\pi i)^2\,\{\Phi(w) - \tfrac{1}{4}\}, \end{aligned}$$

from which the desired result follows.

4. Applications.

In the applications of Theorems 1 and 2 which we make we shall sometimes relax the restrictions made at the beginning of § 2 regarding the point $z = \infty$; for it may be convenient to take G in a form such that ∞ is a fixed point. We can still apply the Theorems provided that we ensure that the functions behave in the correct way at ∞. For, if $\varphi(z)$ is an automorphic function for G of dimension -4, then, in the notation of (1.9),

(4.1) $$\varphi_T(z) = (cz+d)^{-4}\,\varphi(Tz)$$

is an automorphic form of dimension -4 for the conjugate group $G_T = T^{-1}GT$. By choosing T suitably we can make the restrictions on $z = \infty$ hold for G_T and then derive the information we require from Theorems 1 and 2 by making use of (4.1). In particular, if ∞ is a branch-point of order $p-1$ for $\widehat{\mathfrak{R}}$ over $\mathfrak{R}'_z$, we find from (3.1) and (3.2) that

$$\tfrac{1}{2}\{t\,,\,z\} = \tfrac{1}{2}\zeta^4\{t\,,\,\zeta\} = \zeta^2\,\Phi(\zeta),$$

where $\Phi(\zeta)$ is holomorphic at $\zeta = 1/z = 0$, and

$$\Phi(0) = \frac{1}{4}\left(1 - \frac{1}{p^2}\right).$$

We begin by considering equations of the form

(4.2) $$u^m = g(z),$$

where m is an integer greater than unity and $g(z)$ is an automorphic form of dimension $-k$, and with multiplier system v, for a properly discontinuous of group G. Then

$$g(Tz) = v(T)(cz+d)^k g(z), \tag{4.3}$$

for all $z \in \mathfrak{R}_z$ and all $T \in G$. The non-zero number $v(T)$ depends only on T and k; $\mathfrak{R}_z$ is the domain on which $g(z)$ is defined. Normally we require k to be an integer divisible by m, but in certain cases this is inessential, for example if $\mathfrak{R}_z$ is simply connected and none of the points $T^{-1}\infty = -d/c$ are interior points. The equation (4.2) is then invariant under the group G.

If we take $\mathfrak{R}'_z$ to be $\mathfrak{R}_z$, the Riemann surface $\mathfrak{R}_f$ is regular over $\mathfrak{R}_z$ and we can take $\widehat{\mathfrak{R}}$ to be the universal covering surface of $\mathfrak{R}_f = \mathfrak{R}'_f$. However, it is not necessary to do this. We now consider some particular cases.

Let $s(\nu_1, \nu_2, \nu_3; z)$ be the Riemann-Schwarz triangle-function which has elliptic fixed points of periods ν_1, ν_2 and ν_3 at the three vertices A_1, A_2 and A_3 of one of the pairs of adjacent triangles which form a fundamental region for the associated group $G(\nu_1, \nu_2, \nu_3)$; the values of $s(\nu_1, \nu_2, \nu_3; z)$ at these three vertices are 1, 0 and ∞, respectively. We take $\mathfrak{R}_z$ to be (i) the extended z-plane, (ii) the finite z-plane, and (iii) the circle $|z| < 1$, according as the number

$$\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3}$$

exceeds, equals or is less than unity. Here ν_1, ν_2 and ν_3 are integers greater than unity. We also permit them to take the value ∞, in which case the associated vertex A_i becomes a parabolic fixed point of $G(\nu_1, \nu_2, \nu_3)$. This can only arise in case (iii), when the group is horocyclic, or in case (ii), when ν_1, ν_2 and ν_3 are $2, 2, \infty$, in some order, and the parabolic fixed point is the point at infinity. In case (i) the groups are the dihedral, tetrahedral, octahedral and icosahedral groups, and equations invariant under these groups were considered in [10].

Now take $g(z)$ in (4.2) to be an automorphic form for $G(\nu_1, \nu_2, \nu_3)$ of dimension $-k$ with multiplier system v, where, for simplicity, m divides k. We take $\mathfrak{B}_z$ to be the set (assumed not empty) of branch-points of the equation and suppose that no branch-points are ordinary points for $G(\nu_1, \nu_2, \nu_3)$; i.e. they are elliptic fixed points. If ν_i is finite we suppose that there is a branch-point at A_i of order $q_i - 1$, where $q_i \geq 1$. Then q_i divides m, and

$q_i = 1$ implies that there is no branch-point at A_i. If ν_i is infinite we take q_i to be infinite. We also choose positive integers r_1, r_2, r_3, possibly infinite, and write $p_i = q_i r_i$ $(i = 1, 2, 3)$. Then $\mathfrak{B}_z^0$ is taken to be the set of points congruent to vertices A_i for which p_i is finite, while $\mathfrak{B}_z^\infty$ is the set of points congruent to vertices A_i for which ν_i is finite but p_i is infinite. We take $\widehat{\mathfrak{R}}$ as a branched simply-connected covering manifold of $\mathfrak{R}'_z$ with branch-points of order $p_i - 1$ over the points of $\mathfrak{B}_z^0$; $\widehat{\mathfrak{R}}$ is regular over $\mathfrak{R}'_f$ and $\mathfrak{R}'_z$. We note that the equation (4.2) may not be irreducible, but when this happens we confine our attention to one of the associated irreducible equations.

We confine our attention to forms $g(z)$ and groups $G(\nu_1, \nu_2, \nu_3)$ for which $\mathfrak{G}$ is hyperbolic, as in the other two cases the uniformizing functions are usually rational, trigonometric or elliptic functions. If $\mathfrak{B}_z^\infty$ is not null, its points are simple isolated boundary points of $\mathfrak{R}'_z$. This is obvious in the case (iii), just mentioned, by the condition given at the end of § 2. It is also true in the parabolic case (ii), where G can be taken to be the group generated by the transformations $z' = z + 1$ and $z' = -z$. To show this, however, we cannot take $\mathfrak{Z}$ to be the extended z-plane, but may proceed as follows.

The automorphic form $g(z)$ can be written as $g(z) = g_1(z) g_2(z)$, where $g_1(z)$ is a rational function of $e^{2\pi i z}$ having zeros or poles at $z = 0$ or $z = \frac{1}{2}$, and $g_2(z)$ is holomorphic and non-zero for all finite z. If we put $w = e^{2\pi i/zn}$, where n is a positive integer, $g_1(z) = g_3(w)$, where g_3 is a rational function. Hence, if $v = u\{g_2(z)\}^{-1/m}$, the equation (4.2) become the algebraic equation

$$v^m = g_3(w),$$

and we can project $\mathfrak{R}'_f$ into the compact Riemann surface $\mathfrak{Z}$ of this equation. If n is sufficiently large, the genus of $\mathfrak{Z}$ will exceed unity, so that the points of $\mathfrak{B}_z^\infty$ are simple isolated boundary points of $\mathfrak{R}'_z$. Thus $\mathfrak{R}_z^* = \mathfrak{R}_z$ and $\mathfrak{B}_z^* = \mathfrak{B}_z$.

We now apply Theorem 1 taking

$$(4.4) \qquad F(z) = \frac{(s')^2}{4s(s-1)} \left[\frac{1 - p_1^{-2}}{\nu_1^2 (s-1)} - \frac{1 - p_2^{-2}}{\nu_2^2 s} + \frac{1 - p_3^{-2}}{\nu_3^2} \right],$$

with obvious conventions when any of p_i, ν_i $(i = 1, 2, 3)$ are infinite; here $s = s(\nu_1, \nu_2, \nu_3; z)$. It is easily checked that $F(z) \in \mathcal{M}$, so that, by the theorem $R(z) = F(z) + h(z)$, where $h(z) \in \mathcal{H}$. Also, by the same arguments as used above, parabolic fixed points for $G(\nu_1, \nu_2, \nu_3)$, when such exist, are simple parabolic fixed points, so that, by Theorem 2, $R(z)$ vanishes at such points. So does $F(z)$ and hence $h(z)$ is a holomophic automorphic form of dimension -4 which vanishes at parabolic fixed points of $G(\nu_1, \nu_2, \nu_3)$. No such forms can exist; for if $h(z)$ does not vanish identically it has zeros at A_i of multiplicity at least $1-\nu_i^{-1}$ $(i = 1, 2, 3)$ [2, p. 112] and the total number of zeros in a fundamental region is only $2-\Sigma\nu_i^{-1}$. Hence

$$R(z) = \tfrac{1}{2}\{t, z\} = F(z), \tag{4.5}$$

where $F(z)$ is given by (4.4). Thus $\{t, z\}$ is explicitly evaluated.

Further, since

$$2\{z, s\} = \frac{1-\nu_1^{-2}}{(s-1)^2} + \frac{1-\nu_2^{-2}}{s^2} - \frac{1-\nu_1^{-2}-\nu_2^{-2}+\nu_3^{-2}}{s(s-1)}, \tag{4.6}$$

we conclude from (1.5), (4.4), (4.5) and (4.6), that

$$\begin{aligned} 2\{t, s\} &= 2\{z, s\} + 2\{t, z\}(s')^{-2} \\ &= \left(1-\frac{1}{p_1^2\nu_1^2}\right)\frac{1}{(s-1)^2} + \left(1-\frac{1}{p_2^2\nu_2^2}\right)\frac{1}{s^2} \\ &\quad -\left(1-\frac{1}{p_1^2\nu_1^2}-\frac{1}{p_2^2\nu_2^2}+\frac{1}{p_3^2\nu_3^2}\right)\frac{1}{s(s-1)} \\ &= 2\{t, J\}, \end{aligned} \tag{4.7}$$

where

$$J = s(p_1\nu_1, p_2\nu_2, p_3\nu_3; t). \tag{4.8}$$

By subjecting t to a suitable bilinear transformation we may therefore take

$$j(z) = s(\nu_1, \nu_2, \nu_3; z) = s(p_1\nu_1, p_2\nu_2, p_3\nu_3; t) = J(t). \tag{4.9}$$

If $S \in G(p_1\nu_1, p_2\nu_2, p_3\nu_3)$, $j\{z(St)\} = j\{z(t)\}$, so that $z(St) = T_S\{z(t)\}$ where $T_S \in G(\nu_1, \nu_2, \nu_3)$. From this we easily deduce that Γ_z is a normal subgroup of $G(p_1\nu_1, p_2\nu_2, p_3\nu_3)$ and that

$$G(\nu_1, \nu_2, \nu_3) \cong G(p_1\nu_1, p_2\nu_2, p_3\nu_3)/\Gamma_z. \tag{4.10}$$

This therefore determines Γ_z.

Let H be the largest subgroup of G for which

$$g(Tz) = g(z) \text{ for all } T \in H.$$

Then H determines a subgroup Γ^* of $G(p_1\nu_1, p_2\nu_2, p_3\nu_3)$ such that

$$\Gamma^*/\Gamma_z \cong H.$$

Γ^* need not be normal in $G(p_1\nu_1, p_2\nu_2, p_3\nu_3)$. Then $g\{z(t)\}$ is an automorphic function for Γ^*, and, when the equation (4.2) is irreducible, $u = u(t)$

is an automorphic function for a normal subgroup Γ^*_m of Γ^*, where Γ^*/Γ^*_m is isomorphic to the cyclic group C_m of order m. The equation is therefore uniformized by functions $z(t)$ and $u(t)$ belonging to the group

$$\Gamma_f = \Gamma^*_m \cap \Gamma_z, \tag{4.11}$$

and $\Gamma_f/\Gamma_z \cong C_m$.

When $g(z)$ is an automorphic function for $G(\nu_1, \nu_2, \nu_3)$, with multiplier system 1, the group structure is simpler. For $g(z)$ is then a rational function of $s(\nu_1, \nu_2, \nu_3; t)$ and we have $H = G(\nu_1, \nu_2, \nu_3)$ and

$$\Gamma^* = G(p_1\nu_2, p_2\nu_2, p_3\nu_3).$$

Certain equations of the form

$$g^*(u) = g(z) \tag{4.12}$$

can be treated similarly. For example, take

$$g^*(u) = s(\mu_1, \mu_2, \mu_3; u), \quad g(z) = s(\nu_1, \nu_2, \nu_3; z), \tag{4.13}$$

where we suppose that the numbers μ_i and ν_i are so interrelated that the equation cannot be solved rationally or by elliptic functions. Let $\lambda_1, \lambda_2, \lambda_3$ be any three positive integers, possibly infinite, such that λ_i is divisible by both μ_i and ν_i $(i = 1, 2, 3)$. We can then find $R(z)$ and show similarly that we may take

$$g^*(u) = g(z) = s(\lambda_1, \lambda_2, \lambda_3; t), \tag{4.14}$$

so that Γ_z and Γ_u (the maximal group to which $u(t)$ belongs) are normal subgroups of $G(\lambda_1, \lambda_2, \lambda_3)$ and

$$G(\mu_1, \mu_2, \mu_3) \cong G(\lambda_1, \lambda_2, \lambda_3)/\Gamma_u, \quad G(\nu_1, \nu_2, \nu_3) \cong G(\lambda_1, \lambda_2, \lambda_3)/\Gamma_z. \tag{4.15}$$

We have $\Gamma_f = \Gamma_u \cap \Gamma_z$, and

$$\Gamma_z/\Gamma_f \cong G(\mu_1, \mu_2, \mu_3).$$

The associated surface $\hat{\mathfrak{R}}$ will be branched over $\mathfrak{R}'_f$ except when λ_i is taken to be the least common multiple of μ_i and ν_i $(i = 1, 2, 3)$.

These results extend and generalize some of the results obtained in [10]. The equations $e^u + z^p = 1$ and $e^u + e^z = 1$ considered by Dalzell [1], can be treated in the same way. Similar arguments may be used when triangle-functions are replaced by other automorphic functions $g(z)$, such as polygon functions, although formulae for $\{z, g\}$ analogous to (4.6) may only be known in special cases.

In all the examples considered above, $\hat{\mathfrak{R}}$ could have been taken to be the unbranched universal covering surface of $\mathfrak{R}'_f$. We conclude by giving an example where $\widehat{\mathfrak{R}}$ must be a branched covering surface of $\mathfrak{R}'_f$. Suppose that the equation (1.1) is of the form (4.12) where g and g^* are rational functions. At branch-points of $\mathfrak{R}_f$ over $\mathfrak{R}_z$ we suppose that $g(z) = 1$, 0 or ∞; similarly, at branch-points over the u-plane we suppose that $g^*(u) = 1$, 0 or ∞. Then positive integers λ_1, λ_2 and λ_3 can be chosen so that the equations (4.14) determine $u(t)$ and $z(t)$ as single-valued automorphic functions for certain subgroups Γ_u and Γ_z of $\Gamma = G(\lambda_1, \lambda_2, \lambda_3)$. Let m be the number of sheets of $\mathfrak{R}_f$ over the z-plane. The groups Γ_u and Γ_z will not, in general, be normal subgroups of Γ. However, if Γ_u^* is the subgroup of Γ which leaves invariant each of the m solutions u of the equation, then Γ_u^* is normal in Γ and the factor group is isomorphic to the Galois group of the equation $g^*(u) = s$ over the field of its coefficients. A normal subgroup Γ_z^* can be defined in a similar way.

As an example take

$$(4.15) \qquad g^*(u) = \psi_{m,n}(u) = \frac{(m+n-1)!}{(m-1)!\,(n-1)!} \int_0^u v^{m-1}(1-v)^{n-1}\,dv,$$

where m and n are positive integers. When $g^{*\prime}(u) = 0$, $g^*(u) = 1$ or 0, so that the branch-points u yield $g^*(u) = 1$, 0 or ∞. The function $g(z)$ may be chosen similarly, or may be taken to be a triangle-function $s(\gamma_1, \gamma_2, \gamma_3; z)$, not necessarily rational. The branch-points need not necessarily be elliptic fixed points for some group G.

For example, the equation $\psi_{3,2}(u) = \psi_{2,2}(z)$, i.e.

$$4u^3 - 3u^4 = 3z^2 - 2z^3 = s,$$

can be uniformized by taking

$$s = s(2, 6, 12; t).$$

The uniformizing functions $u(t)$ and $z(t)$ belong to a subgroup of index 12 in the group $G(2, 6, 12)$. The surface $\widehat{\mathfrak{R}}$ is branched over $\mathfrak{R}_f$ at two points over $z = -\frac{1}{2}$, at one point over $z = 0$ and at two points over $z = \frac{3}{2}$. The orders of the branch-points are 1 (twice), 2, and 1,5, respectively.

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(Received April 4, 1958)