REMARKS ON APPROXIMATE CONTROLLABILITY

By

J. L. LIONS

Dedicated to Professor Shmuel Agmon

1. Introduction

Let us consider in a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary Γ the wave equation

$$(1.1) \qquad \frac{\partial^2 y}{\partial t^2} - \Delta y = 0, \quad x \in \Omega, \quad 0 < t < T$$

with *boundary control* v.

More precisely let Γ_0 be a subset of Γ, and let us define

$$(1.2) \qquad \Sigma_0 = \Gamma_0 \times (0, T), \quad \Sigma_0^* = \Sigma \setminus \Sigma_0, \quad \Sigma = \Gamma \times (0, T).$$

We shall assume that the control v *is applied on* Σ_0, i.e.

$$(1.3) \qquad y = \begin{cases} v & \text{on } \Sigma_0, \\ 0 & \text{on } \Sigma_0^*. \end{cases}$$

The problem of "*Exact Controllability*" can be stated *in a way that we shall have to make precise*, as follows: we are given T; we start from

$$(1.4) \qquad y(0) = 0, \quad \frac{\partial y}{\partial t}(0) = 0 \quad \text{in } \Omega$$

(where $y(0)$ denotes the function $x \to y(x, 0), \ldots$) and we are given two functions z^0 and z^1, *in function spaces that we have to make precise*; we want to drive the system from $\{0, 0\}$ to $\{z^0, z^1\}$, i.e. we want to find v (*in a function space that one has to make precise*) such that if $y(x, t; v)$ denotes the solutions of (1.1), (1.3), (1.4), then

$$(1.5) \qquad y(\cdot, T; v) = z^0, \quad \frac{\partial y}{\partial t}(\cdot, T; v) = z^1 \quad \text{in } \Omega. \qquad \square$$

The choices of the function spaces where v belongs and where z^0, z^1 belong are clearly related. In order to fix ideas[1] we choose

$$v \in L^2(\Sigma_0) \tag{1.6}$$

(all functions are assumed to be real valued).

If there is one v (at least) to verify (1.1)...(1.6) for *every* couple $\{z^0, z^1\}$, one says that there is Exact Controllability (E.C). But of course this makes sense only if we *make precise the space where* $\{z^0, z^1\}$ *belongs.* □

A first step is simple. Indeed it is known (J .L. Lions [13, 14]) that under (1.6) the unique solution of (1.1), (1.3), (1.4) satisfies

$$\begin{aligned} &y \in C([0,T]; L^2(\Omega)), \\ &\frac{\partial y}{\partial t} \in C([0,T]; H^{-1}(\Omega)). \end{aligned} \tag{1.7}$$

In (1.7), $C([0,T];X)$ denotes the space of continuous functions from $[0,T] \to X$, and

$$\begin{aligned} &H^{-1}(\Omega) = \text{dual space of } H_0^1(\Omega), \\ &H_0^1(\Omega) = \text{Sobolev space} = \left\{ \varphi | \varphi, \frac{\partial \varphi}{\partial x_1}, \cdots, \frac{\partial \varphi}{\partial x_n} \in L^2(\Omega), \varphi = 0 \text{ on } \Gamma \right\}. \end{aligned}$$

Therefore it is natural to choose

$$z^0 \in L^2(\Omega), \quad z^1 \in H^{-1}(\Omega). \tag{1.8}$$

We remark that *if* there is a solution, *then* there are *infinitely many* solutions, so that it is natural to look for *"the best control" (if it exists).* The final formulation of the problem is now:

$$\begin{aligned} &\text{given } z^0 \text{ and } z^1 \text{ satisfying (1.8), find inf } \frac{1}{2} \int_{\Sigma_0} v^2 d\Sigma \\ &\text{among all } v\text{'s such that(1.1)...(1.6) hold true.} \end{aligned} \tag{1.9}$$

This problem always make sense if we further agree that the inf $= +\infty$ is there is no v, i.e. if there is not E.C. □

The E.C. problem depends on Σ_0.

Because of the finite speed of propagation of singularities, it is obvious that the E.C. can take place *only if* T *is large enough.* □

[1] cf. J. L. Lions [13] for other choices.

One can construct *exactly* the space where $\{z^0, z^1\}$ should belong in order to have *a finite minimum* in (1.9).

This has been introduced in J. L. Lions [13, 15], using the HUM method (Hilbert Uniqueness Method). The construction is as follows (and we refer to the quoted papers to see *why* this is so): we consider the wave equation

$$\begin{aligned}&\frac{\partial^2\varphi}{\partial t^2} - \Delta\varphi = 0 \quad \text{in } \Omega \times (0,T),\\ &\varphi = 0 \quad \text{on } \Sigma,\\ &\varphi(0) = \varphi^0, \quad \frac{\partial\varphi}{\partial t}(0) = \varphi^1 \quad \text{in } \Omega\end{aligned} \tag{1.10}$$

and we define

$$\|\{\varphi^0, \varphi^1\}\|_F = \Big(\int_{\Sigma_0} \big(\frac{\partial\varphi}{\partial\nu}\big)^2 d\Sigma\Big)^{1/2}. \tag{1.11}$$

By virtue of Holmgren's uniqueness theorem (cf. L. Hörmander [11]), $\|\{\varphi^0, \varphi^1\}\|_F$ is a norm on, say, $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ ($\mathcal{D}(\Omega) = C^\infty$ functions in Ω with compact support) provided T is large enough. We define next F as the Hilbert space obtained by completion of $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$ for (1.11). Then one proves that *there is* E.C. *iff*

$$\{z^1, z^0\} \in F' = \text{dual space of} F. \tag{1.12}$$

The next step is then to make F precise. □

If Γ_0 is "large enough" and if T is large enough depending on Γ_0, then

$$F = H_0^1(\Omega) \times L^2(\Omega), \tag{1.13}$$

and in this case (1.12) amounts to (1.8): *one has* E.C. *in the "natural" function spaces.*

After a solution was found for many particular cases (Lop Fat Ho, V. Komornik, E. Zuazua; cf. Bibliography of J. L. Lions [13]), *the necessary and sufficient condition on* Σ_0 *for* (1.13) *to hold true* has been given by C. Bardos, G. Lebeau and J. Rauch [1] using microlocal analysis. In short, every light ray should meet Γ_0 in a time $\leq T$, a condition that Bardos, Lebeau and Rauch call *geometrical control.* □

Let us now consider a situation *where one does not have geometrical control.*

Let Ω be the shaded region on Fig. 1, i.e. a square minus two small squares.

Let us assume that Γ_0 is the *outside* boundary. Then there is *not* E.C. if z^0 and z^1 satisfy (1.8). Indeed a ray can go back and forth for ever between the two small

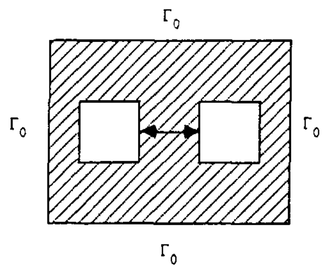

Fig. 1.

squares without ever crossing Γ_0. In such a situation the space F defined as above (since (1.11) is a norm if T is large enough) is *larger* than $H_0^1(\Omega) \times L^2(\Omega)$ (and it may contain elements which are *not* distributions in Ω), so that the dual F' is *smaller* ("much smaller"...) than $H^{-1}(\Omega) \times L^2(\Omega)$. This remark is general. Let Γ_0 be given arbitrarily "small" in Γ, and let T be given such that

(1.14) $T > 2d(\Omega, \Gamma_0), \quad d(\Omega, \Gamma_0) = \sup_{x \in \Omega} \text{distance } \{x, \Gamma_0\}, \quad$ with curves inside Ω.

Then (1.11) defines (by completion of $\mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$) a "*very large*" Hilbert space F.[2]

One has E.C. *in the sense: problem* (1.9) *admits a unique solution* iff

$$\{z^1, z^0\} \in F'. \qquad \square \tag{1.15}$$

Let us consider now a *perforated domain* (Fig. 2), i.e. a situation analogous to the one in Fig. 1 but *with a large number* of holes (a situation which arises in many applied problems) and let us assume (as in the situation of Fig. 1) that we can act *only* on the *external boundary* (a very natural assumption).

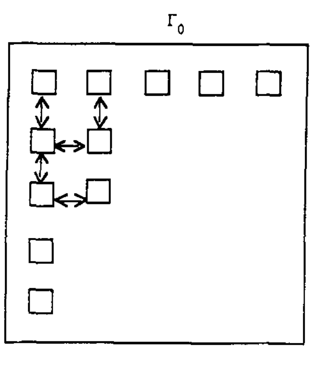

Fig. 2.

[2] The characterization of F is in general an open question. Particular cases have been solved by A. Haraux [10].

Then of course there are an even "larger" number of rays which are trapped without ever reaching Γ_0.

It means that F is getting "larger and larger", and that the precise construction will become more and more complicated. On the other hand, from a practical viewpoint, we apply the control only on Γ_0, the external boundary.

In order to avoid (at least for some time in the arguments) the difficulty of the complicated nature of the space F, *we propose now a "slight" change in the formulation of the problem.* This will be the goal of Section 2 below, the solution of this modified problem being presented in Sections 3 and 4. □

Before proceeding further, a few Remarks are in order.

Remark 1.1 The first systematic report on E.C. for distributed systems has been given by D. L. Russell [23]. Among many other things, the connections between E.C. and stabilization were shown in this paper. (For the question of stabilization, not addressed here, we refer also to J. Lagnese [12], J. L. Lions [15] and E. Zuazua [25].)

The question of "how well" one can *stabilize* by acting *only* on a "tiny" part Γ_0 of Γ seems to be an interesting *open problem.*

Remark 1.2 What has been said above is *not* restricted to second-order hyperbolic operators. For many other situations of *reversible* operators (and also with *other boundary conditions*) we refer to J. L. Lions [13, 16]. All we are going to say extends to these situations, but it is not presented here. □

Remark 1.3 The situation is quite different for *irreversible systems*, such as the heat equation. But what we are going to say applies to parabolic systems. cf. J. L. Lions [17]. □

Remark 1.4 The questions of E.C. (or of Approximate Controllability as presented in Section 2 below) lead to many very interesting problems in the case of *perforated domains*. How is it, in particular, possible to pass to the limit when the number of holes increases? For interesting results, we refer to D. Cioranescu, P. Donato and E. Zuazua [4]. □

Remark 1.5 One can deduce from HUM an *optimality system* which leads to *numerical algorithms*. We refer to R. Glowinski, C. H. Li and J. L. Lions [9], R. G. Glowinski, W. Kinton and M. Wheeler [7] and R. Glowinski and C. H. Li [8]. □

2. Approximate Controllability

Let Γ_0 be an *arbitrary* open subset of Γ and let $y(x, t; v) = y(v)$ be defined by (1.1), (1.3), (1.4). It follows from Holmgren's theorem that if

$$T > T_0 = 2d(\Omega, \Gamma_0) \quad (\text{cf.}(1.14)) \tag{2.1}$$

then

$$\left\{ y(T; v), \frac{\partial y}{\partial t}(T; v) \right\} \text{ spans a } \textit{dense} \text{ subset of } L^2(\Omega) \times H^{-1}(\Omega), \text{ when } v \text{ spans } L^2(\Sigma_0). \tag{2.2}$$

This is the *approximate controllability.*

Suppose then we are given

$$\{z^0, z^1\} \in L^2(\Omega) \times H^{-1}(\Omega) \tag{2.3}$$

and let us introduce:

$$B_0 \text{ (resp. } B_1) = \text{unit ball of } L^2(\Omega) \text{ (resp. } H^{-1}(\Omega)). \tag{2.4}$$

Then, for every α_0, α_1 given > 0, it follows from (2.2) that there exists $v \in L^2(\Sigma_0)$ such that

$$y(T; v) \in z^0 + \alpha_0 B_0, \quad \frac{\partial y}{\partial t}(T; v) \in z^1 + \alpha_1 B_1. \tag{2.5}$$

Actually there exist infinitely many v's satisfying (2.5).

It becomes now natural to replace problem (1.9) by

(2.6) given $z^0, z^1, \alpha_0 > 0, \alpha_1 > 0$ and given T satisfying (2.1),
find $\inf \frac{1}{2} \int_{\Sigma_0} v^2 d\Sigma$
among all v's such that (2.5) holds true. □

Remark 2.1 Problem (2.6) is identical to (1.9) if $\alpha_0 = \alpha_1 = 0$.

We know that in that case it admits a unique solution for every couple $\{z^0, z^1\}$ satisfying (2.3) if Γ_0 and T are large enough, and for couples such that $\{z^1, z^0\} \in F'$ otherwise. When $\alpha_0, \alpha_1 > 0$, *problem* (2.6) *admits a unique solution for every couple satisfying* (2.3) (assuming (2.1)).

In what follows we are going to characterize (by an optimality system) this unique solution.

3. Duality Method

We introduce the operator

$$L : v \in L^2(\Sigma_0) \to \left\{ y(T;v), \frac{\partial y}{\partial t}(T;v) \right\} \in L^2(\Omega) \times H^{-1}(\Omega) \tag{3.1}$$

which is linear and continuous from $L^2(\Sigma_0) \to L^2(\Omega) \times H^{-1}(\Omega)$.

We shall need its adjoint:

$$L^* : L^2(\Omega) \times H_0^1(\Omega) \to L^2(\Sigma_0). \tag{3.2}$$

The operator L^* is defined as follows. Let $\{a^0, a^1\}$ be given in $L^2(\Omega) \times H_0^1(\Omega)$. We define φ as the solution of

$$\begin{aligned} &\frac{\partial^2 \varphi}{\partial t^2} - \Delta \varphi = 0 \quad \text{in } \Omega \times (0,T), \\ &\varphi(T) = -a^1, \quad \frac{\partial \varphi}{\partial t}(T) = a^0 \quad \text{in } \Omega, \\ &\varphi = 0 \quad \text{on } \Sigma. \end{aligned} \tag{3.3}$$

Then

$$L^* \{a^0, a^1\} = \frac{\partial \varphi}{\partial \nu}\Big|_{\Sigma_0}. \tag{3.4}$$

We remark that if $\{a^0, a^1\} \in L^2(\Omega) \times H_0^1(\Omega)$ then it is known that $\partial\varphi/\partial\nu \in L^2(\Sigma)$ (cf. J. L. Lions [14]), so that (3.4) defines an operator which is linear and continuous from $L^2(\Omega) \times H_0^1(\Omega) \to L^2(\Sigma_0)$. If now we multiply (3.3) by $y(v)$ and if we apply Green's formula (which is valid by definition of weak solutions, cf. J.L. Lions and E. Magenes [19]), we obtain:

$$\left(\frac{\partial \varphi}{\partial t}(T), y(T;v) \right) - \left(\varphi(T), \frac{\partial y}{\partial t}(T;v) \right) - \int_{\Sigma_0} \frac{\partial \varphi}{\partial \nu} y(v) d\Sigma = 0$$

i.e.

$$\int_{\Sigma_0} \frac{\partial \varphi}{\partial \nu} v d\Sigma = \langle Lv, \{a^0, a^1\} \rangle \tag{3.5}$$

hence (3.4) follows. □

We introduce now two functionals:

$$F(v) = \frac{1}{2} \int_{\Sigma_0} v^2 d\Sigma \quad \text{on } L^2(\Sigma_0), \tag{3.6}$$

$$G(\{f^0, f^1\}) = \begin{cases} 0, \quad \text{if} & f^0 \in z^0 + \alpha_0 B_0, \\ & f^1 \in z^1 + \alpha_1 B_1, \\ +\infty \quad \text{otherwise.} \end{cases} \tag{3.7}$$

Using these notations, problem (2.6) can be reformulated as:

$$\inf_v F(v) + G(Lv), \quad v \in L^2(\Sigma_0). \tag{3.8}$$

We now apply duality theory in the sense of Fenchel. Using a general result of R. T. Rockafellar [22] (cf. also I. Ekeland and R. Temam [6]), we have

$$\begin{aligned} \inf F(v) + G(Lv) = -\inf F^*(L^*\{a^0, a^1\}) + G^*(-\{a^0, a^1\}), \\ \{a^0, a^1\} \in L^2(\Omega) \times H_0^1(\Omega) \end{aligned} \tag{3.9}$$

where

$$F^*(v) = \sup < v, \hat{v} > -F(\hat{v}) \tag{3.10}$$

and an analogous definition for G^*. One has

$$F^*(v) = F(v) \tag{3.11}$$

and

$$\begin{aligned} G^*(\{a^0, a^1\}) &= \sup[(a^0, f^0) + (a^1, f^1) - G(f^0, f^1)] \\ &= \sup_{B_0 \times B_1} (a^0, z^0 + \alpha_0 B_0) + (a^1, z^1 + \alpha_1 B_1), \end{aligned}$$

i.e.

$$G^*(\{a^0, a^1\}) = (a^0, z^0) + \alpha_0 ||a_0||_{L^2(\Omega)} + (a^1, z^1) + \alpha_1 ||a_1||_{H_0^1(\Omega)}. \tag{3.12}$$

We can now write down explicitly the *dual problem* of the right-hand side of (3.9). We (hopefully) simplify notations by setting $\rho^0 = -a^1, \rho^1 = a^0$. We define ρ as the solution of

$$\begin{aligned} &\frac{\partial^2 \rho}{\partial t^2} - \Delta\rho = 0 \quad \text{in } \Omega \times (0, T), \\ &\rho(T) = \rho^0, \quad \frac{\partial \rho}{\partial t}(T) = \rho^1, \quad \rho = 0 \text{ on } \Sigma, \\ &\rho^0 \in H_0^1(\Omega), \quad \rho^1 \in L^2(\Omega). \end{aligned} \tag{3.13}$$

Then (3.9) becomes

$$\inf F(v) + G(Lv) = -\inf \mathcal{J}(\rho^0, \rho^1) \tag{3.14}$$

where

$$\mathcal{J}(\rho^0, \rho^1) = \frac{1}{2}\int_{\Sigma_0}\left(\frac{\partial \rho}{\partial \nu}\right)^2 d\Sigma + (\rho^0, z^1) - (\rho^1, z^0) + \alpha_1 \|\rho^0\|_{H_0^1} + \alpha_0 \|\rho^1\|_{L^2(\Omega)}. \tag{3.15}$$

Remark 3.1 It is not obvious a *priori* that the problem

$$\inf \mathcal{J}(\rho^0, \rho^1), \quad \rho^0 \in H_0^1(\Omega), \quad \rho^1 \in L^2(\Omega) \tag{3.16}$$

admits a unique solution. It is true because of the Rockafellar duality theorem and the use of Holmgren's uniqueness theorem, i.e. by *reversing* the argument, we transform (3.16) back into (3.8). □

Remark 3.2 If Σ_0 is "large enough" then

$$\int_{\Sigma_0}\left(\frac{\partial \rho}{\partial \nu}\right)^2 d\Sigma \geq c[\|\rho^0\|^2_{H_0^1(\Omega)} + \|\rho^1\|^2_{L^2(\Omega)}] \tag{3.17}$$

(cf. J. L. Lions [13] and the Bibliography therein). In this case the fact that (3.16) admits a unique solution is obvious *and remains true even if* $\alpha_0 = \alpha_1 = 0$. □

Remark 3.3 If $\{\rho^0, \rho^1\}$ is the unique solution of (3.16), then the optimal control v is given by

$$v = \frac{\partial \rho}{\partial \nu} \quad \text{on } \Sigma_0, \tag{3.18}$$

where ρ is given by (3.13). □

Remark 3.4 The dual formulation has been used in numerical computations (cf. Glowinski, Li and Lions [9]) for $\alpha_0 = \alpha_1 = 0$ and Σ_0 "large enough". □

Remark 3.5 If

$$\alpha_0 \geq \|z^0\|_{L^2(\Omega)}, \quad \alpha_1 \geq \|z^1\|_{H^1(\Omega)} \tag{3.19}$$

then the optimal solutions are $\rho^0 = 0, \rho^1 = 0$ and $v = 0$. □

Remark 3.6 Let us denote by $v_\alpha = v_{\alpha_0, \alpha_1}$ the optimal control given by formula (3.18).

If (3.17) *holds true* then one can show that

$$v_\alpha \to v \quad \text{in } L^2(\Sigma_0)$$

when $\alpha \to 0$, where v denotes the solution of (1.9) under conditions (1.5).

If (3.17) *does not hold true*, then the behaviour of

$$\inf(F(v) + G(Lv)) = F(v_\alpha) + G(Lv_\alpha) = \mu(\alpha)$$

as a function of α when $\alpha \to 0$, *depends on the properties of* z^0 and z^1.

If $\{z^1, z^0\} \in F'$ (with the notations introduced in Section 2) then one has again the *same* convergence result as above.

If $\{z^1, z^0\} \notin F'$ then $\mu(\alpha) \to +\infty$ as $\alpha \to 0$. The behaviour of $\mu(\alpha)$ near $\alpha = 0$ could be an indication of "how far" $\{z^1, z^0\}$ stands from F', but this is at this stage purely formal. □

4. Optimality System

The function $\mathcal{J}$ *is not differentiable*. Therefore the solution of (3.16) is characterized by a *variational inequality* instead of an Euler equation:

$$\begin{aligned}
&\int_{\Sigma_0} \frac{\partial \rho}{\partial \nu} \frac{\partial(\hat\rho - \rho)}{\partial \nu} d\Sigma + (z^1, \hat\rho^0 - \rho^0) - (z^0, \hat\rho^1 - \rho^1) \\
&+ \alpha_1 [\|\hat\rho^0\|_{H_0^1(\Omega)} - \|\rho^0\|_{H_0^1(\Omega)}] + \alpha_0(\|\hat\rho^1\|_{L^2(\Omega)} - \|\rho^1\|_{L^2(\Omega)}) \geq 0 \\
&\forall \{\hat\rho^0, \hat\rho^1\} \in H_0^1(\Omega) \times L^2(\Omega).
\end{aligned} \tag{4.1}$$

In (4.1) $\hat\rho$ denotes the solution of (3.13) where ρ^i is replaced by $\hat\rho^i, i = 0, 1$. □

We can now transform (4.1) in a more "classical looking" Variational Inequality. Let ρ be defined by (3.13), and let y be defined by

$$\begin{aligned}
&\frac{\partial^2 y}{\partial t^2} - \Delta y = 0 \quad \text{in } \Omega \times (0, T), \\
&y(0) = 0, \quad \frac{\partial y}{\partial t}(0) = 0 \quad \text{in } \Omega, \\
&y = \partial\rho/\partial\nu \quad \text{on } \Sigma_0, \\
&\quad 0 \qquad \text{on } \Sigma_0^*.
\end{aligned} \tag{4.2}$$

We define $\Lambda : H_0^1(\Omega) \times L^2(\Omega) \to H^{-1}(\Omega) \times L^2(\Omega)$ by

$$\Lambda\{\rho^0, \rho^1\} = \left\{ -\frac{\partial y}{\partial t}(T), y(T) \right\}. \tag{4.3}$$

Multiplying (4.2) by $\hat{\rho} - \rho$ and integrating by parts, one obtains

$$\int_{\Sigma_0} \frac{\partial \rho}{\partial \nu} \frac{\partial(\hat{\rho} - \rho)}{\partial \nu} d\Sigma = \langle \Lambda\{\rho^0, \rho^1\}, \{\hat{\rho}^0, \hat{\rho}^1\} - \{\rho^0, \rho^1\}\rangle. \tag{4.4}$$

Observe that (4.4) *implies that* Λ is ≥ 0, and that

$$\Lambda^* = \Lambda.$$

We now use (4.4) in (4.1). We introduce

$$j(\rho^0, \rho^1) = \alpha_1 \|\rho^0\|_{H_0^1(\Omega)} + \alpha_0 \|\rho^1\|_{L^2(\Omega)}. \tag{4.5}$$

We obtain

$$\begin{aligned} &< \Lambda\{\rho^0, \rho^1\}, \{\hat{\rho}^0, \hat{\rho}^1\} - \{\rho^0, \rho^1\} > + j(\hat{\rho}^0, \hat{\rho}^1) - j(\rho^0, \rho^1) \\ &\qquad \geq < \{-z^1, z^0\}, \{\hat{\rho}^0, \hat{\rho}^1\} - \{\rho^0, \rho^1\} > . \end{aligned} \tag{4.6}$$

The optimality system is now as follows: *one solves* (3.13), *one solves* (4.2) *next and one defines* Λ *as in* (4.3); *the variational inequality* (4.6) *admits a unique solution* $\{\rho^0, \rho^1\}$. *Then the optimal solution* v *is given by* $\partial\rho/\partial\nu$ *on* Σ_0.

Remark 4.1 The variational inequality (4.6) has a standard form (cf. H. Brezis [2], F. Browder [3], G. Stampacchia [24], J. L. Lions and G. Stampacchia [20]) but the fact that it admits a unique solution is not obvious *a priori*, for the same reasons as in Remark 3.1. □

Remark 4.2 If $\alpha_0 = \alpha_1 = 0$, (4.6) reduces to an *equation* (the non-differentiability disappears):

$$\Lambda\{\rho^0, \rho^1\} = \{-z^1, z^0\}. \tag{4.7}$$

This is HUM (J. L. Lions, *loc. cit.*). Equation (4.7) admits a unique solution in F for $\{-z^1, z^0\} \in F'$, and F is $H_0^1(\Omega) \times L^2(\Omega)$ iff Σ_0 is "large enough", as we have said before. □

Remark 4.3 What has been made here is *constructive*. Numerical computations are therefore possible along these lines. They have not yet been made. □

Remark 4.4 As is usual in these matters, one can derive the optimality system directly, by a *penalty method*. One starts with

$$\inf \frac{1}{2} \int_{\Sigma_0} v^2 d\Sigma + \frac{1}{2\varepsilon} \left\| \frac{\partial^2 y}{\partial t^2} - \Delta y \right\|^2_{L^2(\Omega \times (0,T))} \tag{4.8}$$

where

$$y = \begin{cases} v & \text{on } \Sigma_0 \\ 0 & \text{on } \Sigma_0^*, \end{cases} \qquad y(0) = \frac{\partial y}{\partial t}(0) = 0$$

and

$$y(T) \in z^0 + \alpha_0 B_0, \quad \frac{\partial y}{\partial t}(T) \in z^1 + \alpha_1 B_1.$$

One writes the Euler equation for (4.8) and one next lets $\varepsilon \to 0$. □

5. Final Remarks

We conclude by raising a certain number of questions related to the above Remarks.

Remark 5.1 The question of Exact Controllability for *non-linear* distributed systems leads to a large number of open questions. Interesting results have been obtained by E. Zuazua [27]. It would be interesting to see how the above considerations extend to non-linear wave equations. □

Remark 5.2 It would be of some interest to look for similar problems for the equation

(5.1)
$$\varepsilon \frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t} - \Delta y = 0$$

and to see what happens when $\varepsilon \to 0$ (and $\alpha_1 \to 0$). □

Remark 5.3 Let us consider a perforated domain as in Fig. 2. More precisely let Ω_ε be a domain Ω minus the set of holes $\mathcal{O}_\varepsilon$ arranged in a periodic manner and with period ε in all directions. We assume that these holes never meet $\partial\Omega = \Gamma_0$ and we denote by $\mathcal{S}_\varepsilon$ the union of the boundaries of all the holes inside Ω. We consider then the state $y = y_\varepsilon(x, t; v)$ given by

(5.2)
$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} - \Delta y = 0 \quad \text{in } \Omega_\varepsilon \times (0, T), \\ &y = v \quad \text{on } \Gamma_0 \times (0, T), \\ &\frac{\partial y}{\partial \nu} = 0 \quad \text{on } \mathcal{S}_\varepsilon \times (0, T), \\ &y(x, 0) = \frac{\partial y}{\partial t}(x, 0) = 0 \quad \text{in } \Omega_\varepsilon. \end{aligned}$$

We consider again the same problem as above, namely

(5.3)
$$\inf \frac{1}{2} \int_{\Sigma_0} v^2 d\Sigma$$

for all v's subject to

$$y(T;v) \in z^0 + \alpha_0 B_0, \quad \frac{\partial y}{\partial t}(T;v) \in z^1 + \alpha_1 B_1 \tag{5.4}$$

[where B_0 and B_1 depend in fact on ε, since they are here the unit balls in $L^2(\Omega_\varepsilon)$ and in $H^{-1}(\Omega_\varepsilon)$ and of course realizing that in the present situation, because of the Neumann condition on $\mathcal{S}_\varepsilon, H^{-1}(\Omega_\varepsilon)$ is not necessarily the most appropriate space here].

Let v_ε be the solution of (5.3). Do we have convergence of v_ε towards v, where v would solve the following problem; let $\mathcal{A}$ be the *homogenized* operator in Ω (cf. D. Cioranescu and J. Saint Jean Paulin [5], J. L. Lions [18], O. A. Oleinik, A. S. Shamaev and G. A. Yosifian [21] and the Bibliography therein). Let y be given by

$$\begin{aligned} &\frac{\partial^2 y}{\partial t^2} + \mathcal{A}y = 0 \quad \text{in } \Omega \times (0,T), \\ &y = v \quad \text{on } \Gamma_0 \times (0,T), \\ &y(x,0) = \frac{\partial y}{\partial t}(x,0) = 0 \quad \text{in } \Omega. \end{aligned} \tag{5.5}$$

Then is v (if it exists as the limit of v_ε) the solution of (5.3), where v is subject to (5.4) with y now the solution of (5.5), and where B_0 (resp. B_1) is now the unit ball of $L^2(\Omega)$ (resp. $H^{-1}(\Omega)$)? Questions of this type—but technically very different—are solved in D. Cioranescu, P. Donato and E. Zuazua [4].

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COLLÈGE DE FRANCE
3, RUE D'ULM
75005 PARIS, FRANCE

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