A CHARACTERIZATION OF THE MAXIMUM MODULUS OF FUNCTIONS REGULAR AT THE ORIGIN

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Introduction

1. Let $f(z)$ be regular near $z=0$ and put

$$(1.1) \qquad M(\rho) = M[\rho, f] = \max_{|z|=\rho} |f(z)| .$$

It is well known that if $f(z)$ is regular for $|z| \leqq \rho$, $M(\rho)$ is an increasing function of ρ, and it was proved by Hadamard [2] [1] that log $M(\rho)$ is a convex function of log ρ. Blumenthal [1] further showed that $M(\rho)$ is itself an analytic function of ρ, except at an isolated number of points $\rho_1 < \rho_2 < \dots < \rho_n < \dots$, so that $M(\rho)$ is represented by distinct analytic functions in the intervals $\rho_n \leqq \rho \leqq \rho_{n+1}$.

The fact that $M(\rho)$ need not be given by just one analytic function makes the problem of its characterization for instance for the class of entire functions $f(z)$ very difficult. We shall solve here a simpler problem, namely the local characterization of $M(\rho)$ near $\rho = 0$. This amounts to characterizing a certain class of functions $M(\rho)$, regular at $\rho = 0$ and real for real ρ, corresponding (by 1.1) to the class of all $f(z)$ regular at $z=0$.

We shall see that for z lying on a certain analytic arc $z = \alpha(\rho)$, the maximum modulus is attained. Since $\alpha(\rho)$ is regular at $\rho = 0$, we shall be able to make ρ complex and obtain a locally $1:1$ correspondence between z and ρ. Using this approach we shall show that $M(\rho)$ is itself a regular function of ρ, which attains its maximum modulus on the positive real axis, so that $M(\rho)$ is its own maximum modulus (Theorem 1). We shall also obtain an equivalent criterion for this to happen in terms of the coefficients of the power series expansion of $M(\rho)$ near $\rho = 0$ (Theorem II).

In Part II we shall study the class of functions $M(\rho)$ having this property, with a view to obtaining all functions $f(z)$ such that (1.1) holds

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1. Numbers in brackets refer to the bibliography at the end of the paper.

for small ρ. It turns out that if

(1.2) $$M(\rho) = 1 + a_k \rho^k + \dots, \quad a_k \neq 0,$$

with $k = 1$ or 2, then $f(z)$ exists uniquely, satisfying (1.1) and $f(0) = 1$, and attaining its maximum modulus for small ρ on an assigned analytic arc through the origin (Theorems III and IV). This result fails to hold, however, whenever $k \geqq 3$ in (1.2) (Theorem V).

Part 1.

2. The following two lemmas are fundamental to our theory.

Lemma 1. *Suppose that $f(z)$ is regular on the circle $|z| = r$, and that*

$$\frac{\partial}{\partial \Theta} |f(re^{i\Theta})| = 0, \quad \text{at } \Theta = \Theta_0.$$

Then if $z_0 = re^{i\theta_0}$ and $f(z_0) \neq 0$, $z_0 \dfrac{f'(z_0)}{f(z_0)}$ is real.

We put

$$u(z) = \log |f(z)|.$$

Then $u(z)$ is harmonic at $z = z_0$, since $f(z_0) \neq 0$. Also we have at $\Theta = \Theta_0$

$$\frac{\partial}{\partial \Theta} u(re^{i\Theta}) = 0.$$

Expressing $u(z)$ in terms of x, y this gives

$$\frac{\partial}{\partial \Theta} u[r\cos\Theta, \ r\sin\Theta] = r\left[\frac{\partial u}{\partial y}\cos\Theta - \frac{\partial u}{\partial x}\sin\Theta\right] = 0,$$

i.e., at $z = z_0$,

$$-\operatorname{Im}\left\{re^{i\theta_0}\left(\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right)\right\} = 0,$$

$$\operatorname{Im}\left\{z_0 \frac{d}{dz}\log f(z)\right\} = 0, \quad z = z_0,$$

so that $z_0 \dfrac{f'(z_0)}{f(z_0)}$ is real as required.

We deduce immediately

Lemma 2. *Suppose with the hypotheses of lemma 1, that z approaches z_0 along a curve γ, $z = \alpha(\rho)$ where $\rho = |z|$, and that γ has a tangent making a positive angle with the circle $|z| = r$ at z_0. Then we have at $\rho = r$,*

$$r\frac{d}{d\rho}\log|f[\alpha(\rho)]| = z_0\frac{f'(z_0)}{f(z_0)}.$$

We write $z = \rho e^{i\theta}$,

$$u(z) = \log |f(z)| = u(\rho, \Theta).$$

Then if ρ, Θ are close to r, Θ_0, we have

$$u(\rho, \Theta) - u(r, \Theta_0) = \frac{\partial u}{\partial \rho}(\rho - r) + \frac{\partial u}{\partial \Theta}(\Theta - \Theta_0) + o\{|\rho - r| + |\Theta - \Theta_0|\}.$$

If further $re^{i\theta}$ lies on γ, so that $|\Theta - \Theta_0| = O|\rho - r|$,

and since also $\dfrac{\partial u}{\partial \Theta} = 0$, $\Theta = \Theta_0$, this becomes as $z \to z_0$ along γ

$$\frac{u(z) - u(z_0)}{\rho - r} \to \frac{\partial u}{\partial \rho}$$

i. e.

$$\frac{d}{d\rho} \log |f[\alpha(\rho)]| = \frac{\partial u}{\partial \rho} = \frac{\partial}{\partial \rho} \log |f(\rho e^{i\theta})|. \tag{2.1}$$

Also we have from the Cauchy-Riemann equations and our hypothesis at $\Theta = \Theta_0$

$$\frac{\partial}{\partial \rho} \arg f(\rho e^{i\theta}) = -\frac{1}{\rho} \frac{\partial}{\partial \Theta} \log |f(\rho e^{i\theta})| = 0.$$

Thus (2.1) may be written

$$\frac{d}{d\rho} \log |f[\alpha(\rho)]| = \frac{\partial}{\partial \rho} \log f(\rho e^{i\theta_0}) = e^{i\theta_0} \frac{f'(z_0)}{f(z_0)}$$

or

$$r \frac{d}{d\rho} \log |f[\alpha(\rho)]| = z_0 \frac{f'(z_0)}{f(z_0)},$$

which proves the lemma.

The idea of both these lemmas goes back to Blumenthal [1], but he did not use the Function-theoretic formulation in terms of $f(z)$, which is the main tool of our Theory.

3. Lemma 1 shows us that the points where $f(z)$ attains its maximum modulus lie among the points where $z_0 \dfrac{f'(z_0)}{f(z_0)}$ is real. We next investigate the set of points near the origin where a regular function is real. It will appear that this set consists of a finite number of regular arcs. To make this concept precise we define

Definition: *A regular arc (R-arc) is defined to be the set of points γ, given by a function $z = \alpha(t)$, $0 \leqq t \leqq \varepsilon$, where $\alpha(t)$ is regular at $t = 0$, $\alpha(0) = 0$ $\alpha'(0) \neq 0$.*

We shall suppose ε so small that γ does not cross itself. Apart from this we do not distinguish between arcs given by the same function α and different ε's. If $\alpha(t)$, $0 \leqq t \leqq \varepsilon$, $\beta(t)$, $0 \leqq t \leqq \eta$ yield the same set of points γ, we shall say that $\alpha(t)$, $\beta(t)$ are different representations of γ.

There is one representation applicable to every R-arc and for this reason particularly useful, in which $|\alpha(t)| = t$. We have

Lemma 3. *Suppose γ is an R-arc. Then for some $\varepsilon > 0$ there exists a representation of γ in the form*

$$z = \rho e^{i\Theta(\rho)}, \quad 0 \leqq \rho \leqq \varepsilon$$

where $\Theta(\rho)$ is real for real ρ, and regular at $\rho = 0$. Conversely such an expression always yields an R-arc.

The converse part is trivial. To prove the lemma let

$$z = t\,\psi(t)$$

be a representation of γ, where $\psi(0) \neq 0$. Then $\log \psi(t)$ is regular at $t = 0$, and we may write

$$\log \psi(t) = \psi_1(t) + i\,\psi_2(t)$$

where $\psi_1(t)$, $\psi_2(t)$ are regular at $t = 0$ and real for real t. Thus we may write for z on γ

$$z = te^{\psi_1(t)} \,.\, e^{i\psi_2(t)}.$$

We put

$$\rho = te^{\psi_1(t)}.$$

Then ρ is a regular function of t at $t = 0$, real for real t and

$$\frac{d\rho}{dt} = |\,\psi(0)\,| \neq 0$$

at $t = 0$. Thus we may invert and put

$$t = \lambda(\rho)$$

where $\lambda(\rho)$ is regular at $\rho = 0$, and real and positive for real and positive ρ. Hence

$$z = \rho e^{i\,\psi_2[\lambda(\rho)]}$$

gives the required representation, since $\Theta(\rho) = \psi_2[\lambda(\rho)]$ is real for real ρ and regular at $\rho = 0$.

4. We next show that the set near the origin, where a regular function is real consists of a finite number of R-arcs.

Lemma 4. *Let*

$$g(z) = a_k z^k + a_{k+1} z^{k+1} + \dots, \quad k > 0, \ a_k \neq 0,$$

be regular at $z = 0$. *Then if* ε *is a sufficiently small positive number there exist exactly 2k R-arcs,* $\gamma_1, \gamma_2, \dots, \gamma_{2k}$ *with the following properties :*

(i) *The function* $g(z)$ *is real at those points of* $|z| \leqq \varepsilon$ *which lie on the arcs* γ_ν *and only those.*

(ii) *On the arcs* $\gamma_1, \gamma_3, \dots, \gamma_{2k-1}$, $g(z)$ *is negative, while on* $\gamma_2, \gamma_4, \dots, \gamma_{2k}$ $g(z)$ *is positive.*

(iii) *The arcs* γ_ν, $\gamma_{\nu+1}$, $\nu = 1, \dots, 2k-1$ *and* γ_{2k}, γ_1 *intersect at an angle* $\frac{\pi}{k}$ *with each other at* $z = 0$ *and do not intersect elsewhere in* $|z| \leqq \varepsilon$.

The lemma is almost trivial. We include a proof merely for completeness. We put

$$g(z) = a_k w^k,$$

so that

$$w = z\left(1 + \frac{a_{k+1}}{a_k} z + \dots\right)^{\frac{1}{k}}. \tag{4.1}$$

Then $g(z)$ is real, when $a_k w^k$ is real, and this occurs on $2k$ halflines in the w-plane of the form

$$w = re^{i\Theta_\nu}, \ \nu = 1, \dots, 2k.$$

where r is real and

$$\Theta_{\nu+1} - \Theta_\nu = \frac{\pi}{k}.$$

Also adding $\frac{\pi}{k}$ to Θ changes the sign of $a_k w^k$. Thus if $\Theta = \Theta_2$ makes $f(z)$ positive, so do $\Theta_4, \Theta_6, \dots, \Theta_{2k}$ and $\Theta_1, \Theta_3, \dots, \Theta_{2k-1}$ make $f(z)$ negative.

Since w given by (4.1) is a regular function of z near $z = 0$, $\frac{dw}{dz} = 1$, we may choose ε so small, that w is univalent in $|z| \leqq \varepsilon$, so that the correspondence is 1 : 1 and both ways regular. Let γ_ν be the set of those points in $|z| \leqq \varepsilon$, which correspond to $w = re^{i\theta_\nu}$. Then it is clear that the γ_ν have the properties (i), (ii) and (iii) of lemma 4, since the z, w correspondence is 1 : 1 and conformal.

Let $z = \varphi(w)$ be the inverse of (4.1). Then the set γ_ν consists of all points of the form $z = \varphi(re^{i\theta_\nu})$, which lie in $|z| \leqq \varepsilon$. We may choose ε so small that γ_ν consists of a single R-arc corresponding to $0 \leqq r \leqq \eta_\nu$. This proves that the γ_ν are R-arcs and completes the proof of lemma 4.

We next investigate the behaviour of $g(z)$ on the γ_ν. We have

Lemma 5. *Let γ_ν, $\nu = 1, \ldots, 2k$ be the arcs of the last lemma and let $g_\nu(\rho)$ be the value necessarily real which $g(z)$ takes at the intersection of γ_ν and $|z| = \rho$ (The intersection is unique if ρ is small enough). Then*

(i) $$g_\nu(\rho) = (-1)^\nu |a_k| \rho^k + \ldots,$$

is a regular function of ρ near $\rho = 0$, real for real ρ.

(ii) *There exists a strictly increasing function $\mu(\rho)$, $0 \leqq \rho \leqq \varepsilon$ for sufficiently small $\varepsilon > 0$, such that we have for each ν either*

$$\text{a)} \quad g_\nu(\rho) \equiv \mu(\rho)$$

$$\text{or b)} \quad g_\nu(\rho) < \mu(\rho), \quad 0 < \rho < \varepsilon.$$

Further a) *holds for at least one index ν.*

Let γ_ν be given as in lemma 3 by

$$z = \rho e^{i\theta_\nu(\rho)}.$$

Then we have

$$g_\nu(\rho) = g[\rho e^{i\theta_\nu(\rho)}],$$

which is clearly regular near $\rho = 0$. Also from the definition of the γ_ν, $g_\nu(\rho)$ is real for real positive ρ. Further

$$\left| \frac{g_\nu(\rho)}{\rho^k} \right| = \left| \frac{g[\rho e^{i\theta_\nu(\rho)}]}{\rho^k} \right| \to a_k \text{ as } \rho \to 0,$$

since $|\rho e^{i\theta_\nu(\rho)}| = |\rho|$ for real ρ. Thus since $g_\nu(\rho)$ is real for real positive ρ, negative for ν odd and positive for ν even, lemma 5 (i) follows.

Consider next

$$\lambda(\rho) = g_\nu(\rho) - g_{\nu'}(\rho)$$

for two indices ν, ν'. Then $\lambda(\rho)$ is real for small positive $\rho \leqq \varepsilon$, and if

ε is small enough $\lambda(\rho)$ has either constant sign or is identically zero. Thus, if $g_\nu(\rho)$, $g_{\nu'}(\rho)$ are not identical, one of them is larger than the other for small positive ρ. We may thus find a greatest among the $g_\nu(\rho)$, for small positive ρ, which has the properties required of $\mu(\rho)$ in lemma 5 (ii). In fact

$$\mu(\rho) = |a_k| \rho^k + \dots$$

and hence $\mu(\rho)$ is necessarily strictly increasing for small positive ρ.

This completes the proof of lemma 5.

We shall call the $g_\nu(\rho)$ *the real values of* $g(z)$ *(on the circle* $|z| = \rho$ *for small positive* ρ *being understood). We shall say that the function* $\mu(\rho)$ *of lemma* 5 (ii) *is the greatest real value of* $g(z)$. *We shall also say that* $g(z)$ *attains the value* $g_\nu(\rho)$ *on all arcs* $g_{\nu'}$, *for which* $g_{\nu'}(\rho) \equiv g_\nu(\rho)$.

5. We can now relate our problem of the maximum modulus to the preceding work. In studying the maximum modulus of $f(z)$ near $z = 0$, we may suppose without loss in generality that $f(0) = 1$. For if

$$\varphi(z) = \alpha z^\lambda + \dots$$

has maximum modulus $M(\rho)$, then $\dfrac{\varphi|z|}{\alpha z^\lambda} = f(z)$ has maximum modulus $\dfrac{M(\rho)}{|\alpha| \rho^\lambda}$. We shall always presuppose this normalization in future. We then have

Lemma 6. *Let*

$$f(z) = 1 + a_k z^k + \dots$$

be regular near $z = 0$, *and let* $M(\rho) = M[\rho, f] = \max\limits_{|z|=\rho} |f(z)|$.

Then for some $\varepsilon > 0$ *and* $|z| \leqq \varepsilon$, $\mu(\rho) = \rho \dfrac{M'(\rho)}{M(\rho)}$ *is the greatest real value of* $g(z) = z \dfrac{f'(z)}{f(z)}$. *Further the points of* $|z| \leqq \varepsilon$ *for which*

$$|z| = \rho, \quad |f(z)| = M(\rho) \tag{5.1}$$

consists of those R-arcs, where $g(z)$ *attains its greatest real value.*

We know from lemma 1, that the points on which (5.1) holds lie on the arcs γ_ν, $\nu = 1, \dots, 2k$ on which $g(z)$ is real. Let $f_\nu(\rho)$ be the value of $|f(z)|$ at the intersection of γ_ν and $|z| = \rho$. Then $M(\rho)$ is the largest of $f_\nu(\rho)$, $\nu = 1, \dots, 2k$.

The arc γ_ν is not tangential to $|z| = \rho$ for small ρ.

Thus we may use lemma 2 and obtain for $0 \leqq \rho \leqq \varepsilon$

$$\rho \frac{d}{d\rho} \log f_\nu(\rho) = g_\nu(\rho).$$

Since $f_\nu(0) = 1$, we deduce that

$$f_\nu(\rho) = \exp\left\{\int_0^\rho g_\nu(t) \frac{dt}{t}\right\}. \tag{5.2}$$

Now since $g_\nu(\rho) \leqq \mu(\rho)$ with the notation of lemma 5, with equality for one or more indices ν, we deduce from (5.2) that

$$M(\rho) = \max_{\nu=1,\ldots,2k} f_\nu(\rho) = \exp\left\{\int_0^\rho \mu(t) \frac{dt}{t}\right\} \tag{5.3}$$

and further that $M(\rho) = f_\nu(\rho)$, if and only if $g(z)$ attains the real value $\mu(\rho)$ on γ_ν. Thus the points of $|z| = \rho$ for which $|f(z)| = M(|z|)$, are just those points which lie on the γ_ν on which $g(z)$ attains the value $\mu(\rho)$. The rest of the lemma follows from (5.3) and the proof of lemma 6 is complete.

6. We can now state our first main Theorem, characterizing the function $M(\rho)$. We have

Theorem I. *Suppose that*

$$f(z) = 1 + a_k z^k + \ldots, \quad a_k \neq 0$$

is regular at $z = 0$. *Let*

$$M(\rho) = M[\rho, f] = \max_{|z|=\rho} |f(z)|.$$

Then we have

$$M(\rho) = 1 + |a_k| \rho^k + \ldots, \tag{i}$$

is a regular function of ρ *near* $\rho = 0$.

(ii) *If* $M(\rho)$ *is continued into the complex domain then for some* $\varepsilon > 0$, *and* $0 \leqq \rho \leqq \varepsilon$, $0 \leqq \Theta \leqq 2\pi$ *we have the inequality*

$$|M(\rho e^{i\Theta})| \leqq M(\rho).$$

(iii) *The points* z *in* $|z| \leqq \varepsilon$, *such that* $|z| = \rho$, $|f(z)| = M(\rho)$, *form* *at most k R-arcs, which make angles of* $\frac{2p\pi}{k}$ *with each other at* $z=0$, *where* p *is a positive integer.*

The important property is (ii), which shows that the maximum modulus functions $M(\rho)$, are just those functions, regular at $\rho=0$, and real for real ρ, which attain their maximum modulus on the positive real axis, i.e., those which are their own maximum modulus.

We know from lemma 6 that for small ρ

$$\rho \frac{M'(\rho)}{M(\rho)} = \mu(\rho), \tag{6.1}$$

where $\mu(\rho)$ is the greatest real value of $z \frac{f'(z)}{f(z)}$. By lemma 5, $\mu(\rho)$ is regular at $\rho=0$ and has there an expansion of the form

$$\mu(\rho) = k\,|a_k|\,\rho^k + \dots \tag{6.2}$$

since

$$z\frac{f'(z)}{f(z)} = k\,a_k\,z^k + \dots$$

near $z=0$. From (6.1), (6.2) we infer that $M(\rho)$ is regular near $\rho=0$, and has a power series expansion of the form

$$M(\rho) = 1 + |a_k|\,\rho^k + \dots$$

This proves (i). Again by lemma 6, $f(z)$ attains its maximum modulus on just those R-arcs on which $z\frac{f'(z)}{f(z)}$ attains its greatest real value, and these arcs have the properties required in Theorem I (iii) by lemma 4. In fact the arcs of Theorem I must be among the arcs γ_ν of lemma 4 with ν even.

7. It remains to prove Theorem I (ii) and the proof uses the whole of our preceding theory. Suppose that (ii) is false, so that for small ρ $M(\rho)$ does not attain its maximum modulus on the positive real axis. Then it follows from lemma 6, that $\mu(\rho)$ given by (6.1) does not have its greatest real value on the positive real axis. Thus for all small positive k, we can find a complex ρ such that $|\rho|=k$, and $\mu(\rho)$ is real and satisfies

$$\mu(\rho) > \mu(k).$$

For each small k choose such a value $\rho=\rho_1$ and put $\rho_2=\overline{\rho_1}$. Then since

$\mu(\rho)$ is real for real ρ, we have

$$\mu(\rho_2) = \overline{\mu(\rho_1)} = \mu(\rho_1)$$

so that

$$\mu(\rho_1) = \mu(\rho_2) > \mu(k) \tag{7.1}$$

and

$$\rho_2 = \bar{\rho}_1, \quad |\rho_1| = |\rho_2| = k. \tag{7.2}$$

Now let γ be an R-arc on which $f(z)$ attains its maximum modulus for small positive ρ. We write γ in the form

$$z = \rho e^{i\Theta(\rho)}, \tag{7.3}$$

where $\Theta(\rho)$ is real for real ρ, as we may do by lemma 3. Also if z, ρ are related as in (7.3) and ρ is real and positive, it follows from lemma 6 that we have

$$z\frac{f'(z)}{f(z)} = \rho\frac{M'(\rho)}{M(\rho)} = \mu(\rho). \tag{7.4}$$

It follows by analytic continuation that the identity (7.4) continues to hold when z, ρ are complex numbers related as in (7.3). In particular if $z = z_1, z_2, z_k$, correspond to the numbers $\rho = \rho_1, \rho_2, k$ of (7.2), we have from (7.1)

$$z_1\frac{f'(z_1)}{f(z_1)} = z_2\frac{f'(z_2)}{f(z_2)} > z_k\frac{f'(z_k)}{f(z_k)}. \tag{7.5}$$

Since $k > 0$, the point z_k is the intersection of the arc γ of (7.3) and the circle $|z| = k$. Also ρ_1, ρ_2 are reflections of each other in the real ρ axis. Hence we have

$$\Theta(\rho_1) = \overline{\Theta(\rho_2)}$$

since the function $\Theta(\rho)$ of (7.3) is real for real ρ and so

$$|z_1 z_2| = |\rho_1\rho_2|\,|e^{i[\Theta(\rho_1) + \overline{\Theta(\rho_1)}]}| = |\rho_1\rho_2| = k^2,$$

using (7.2) and (7.3). Thus at least one of z_1, z_2, $z = z_1$ say, satisfies $|z_1| \leqq k$. Combining this with (7.5) we see that $z\frac{f'(z)}{f(z)}$ does not attain its greatest real value on the arc γ for small k, since we can find a point z_1 no further from the origin than the point on $|z| = k$ and γ, where $z\frac{f'(z)}{f(z)}$ takes a larger real value. From this it follows by lemma 6, that

$f(z)$ cannot attain its maximum modulus on γ, for small z, contrary to hypothesis. Thus our original assumption, that $M(\rho)$ is not its own maximum modulus for small ρ must have been incorrect, and Theorem I (ii) is proved. This completes the proof of Theorem I.

8. In Theorem I we have characterized those functions, which are the maximum modulus of some other function. We now prove an equivalent criterion, which depends only on the power series expansion of $M(\rho)$ near $\rho = 0$. This is

Theorem II. *Let*

$$M(\rho) = 1 + a_k \rho^k + \dots, \quad a_k \neq 0 \tag{8.1}$$

be real for real ρ. *Then* $M(\rho)$ *is the maximum modulus* $M[\rho, f(z)]$ *of some function regular for* $|z| \leqq \varepsilon$, *where* ε *is some positive number if and only if the following conditions are satisfied*:

(i) *The series* (8.1) *has a positive radius of convergence*;

(ii) $a_k > 0$;

(iii) *for every positive integer* j, *let* $\beta(j)$ *be the function defined as follows*: *let* ν *be the smallest positive integer if any such that* νj *is not a multiple of* k *and* $a_\nu \neq 0$; *then*

$$\beta(j) = a_\nu .$$

If no such integers exist, we put $\beta(j) = 0$.

Then $\beta(j) \geqq 0$ *for every positive integer* j.

The criterion of Theorem II, although we shall deduce it from Theorem I, does not use any properties of $M(\rho)$ outside its original domain of definition $0 \leqq \rho \leqq \varepsilon$.

9. To prove the Theorem we need two further lemmas.

Lemma 7. *Let*

$$M(z) = 1 + a_k z^k + \dots, \quad a_k > 0,$$

be real for real z, *and let*

$$\mu(\rho) = \max_{|z| = \rho} |M(z)|.$$

Then given $\varepsilon > 0$, *there exists* δ *such that if* $0 < \rho < \delta$ *and* Θ *is real,*

$$z = \rho e^{i\theta}, \quad |M(z)| = \mu(\rho)$$

we have

$$\left| \Theta - \frac{2\pi j}{k} \right| < \varepsilon ,$$

where j is an integer. Further if $j = 0$, $\Theta = 0$.

We know from lemma 6 that $M(z)$ attains its maximum modulus on those R-arcs on which $z\frac{M'(z)}{M(z)}$ attains its greatest real value. Now $z\frac{M'(z)}{M(z)}$ is certainly real and positive on the positive real axis for small z, hence the other R-arcs on which $z\frac{M'(z)}{M(z)}$ is real and positive must by lemma 4 make an angle $\frac{2\pi j}{k}$ with the positive real axis, where j is a non-zero integer. This proves lemma 7.

We have next

Lemma 8. *Suppose the power series (8.1) has a positive radius of convergence and let* $M(z)$ *be its sum for $\rho = z$ (possibly complex). Let $\beta(j) = \beta$ be defined as in Theorem II* (iii) *and put $\alpha = \frac{2\pi j}{k}$. Then there exists $\varepsilon > 0$, such that for $|z| < \varepsilon$, $|\arg z| < \varepsilon$ we have*

(i) $|M(z)| > |M(ze^{i\alpha})|$ *if* $\beta > 0$,

(ii) $|M(z)| < |M(ze^{i\alpha})|$ *if* $\beta < 0$,

(iii) $M(z) \equiv M(ze^{i\alpha})$ *if* $\beta = 0$.

We put

$$M(z) = M_1(z) + M_2(z) \tag{9.1}$$

where $M_1(z)$ is the sum of all those powers $a_\nu z^\nu$ for which k is a factor of νj.

$$M_1(z) = \sum_{k \mid \nu j} a_\nu z^\nu . \tag{9.2}$$

If $\beta = 0$, then $M_2(z) \equiv 0$ and in this case $M(z) \equiv M_1(z)$. Also we have clearly always

$$M_1(z) = M_1(ze^{i\alpha}) = M_1(ze^{\frac{2\pi i j}{k}}). \tag{9.3}$$

Thus (iii) follows.

Suppose next $\beta \neq 0$. In this case we have for some $\nu > k$,

$$M_2(z) = \beta z^\nu + \ldots .$$

We put

$$z = \rho (\cos \Theta + i \sin \Theta),$$

so that

$$z^{\nu} = \rho^{\nu} (\cos \nu \Theta + i \sin \nu\Theta)$$

and

(9.4) $$M_2 (z) = \beta\rho^{\nu} [\cos \nu\Theta + i \sin \nu \Theta + o (1)],$$

where $o (1)$ denotes a function of z, which tends to zero uniformly as $z \to 0$. We may write using (9.3)

(9.5) $$M_1 (z) = M_1 (ze^{i\alpha}) = u (z) + iv (z).$$

We have from (9.1), (9.4), (9.5)

$$|M (z)|^2 = \{u (z) + \beta\rho^{\nu} [\cos \nu \Theta + o (1)]\}^2 + \{v (z) + \beta\rho^{\nu} [\sin \nu \Theta + o (1)]\}^2 .$$

This gives

(9.6) $$|M (z)|^2 - |M (ze^{i\alpha})|^2 = 2\beta\rho^{\nu} u (z) [\cos \nu \Theta - \cos \nu (\Theta + \alpha) + o (1)],$$

since from (9.2) $u (z) \to 1$, $v (z) \to 0$ as $z \to 0$. Now by hypothesis $\nu\alpha$ is not a multiple of 2π, so that $\cos \nu\alpha < 1$. Hence if ε is sufficiently small and $|\Theta| < \varepsilon$, $0 < \rho < \varepsilon$ then

$$2\rho^{\nu} u (z) [\cos \nu \Theta - \cos \nu (\Theta + \alpha) + o (1]$$

is positive, so that from (9.6)

$$| M (z)| \gtrless | M (ze^{i\alpha}) |$$

according as

$$\beta \gtrless 0 .$$

This completes the proof of lemma 8.

10. We can now prove Theorem II. We know from Theorem I, that $M (\rho)$ is the maximum modulus of some function $f (z)$ if and only if $M (z)$ attains its own maximum modulus on the positive real axis. Let $M (z)$ be the function of Theorem II and suppose that for $|z| < \varepsilon$, $|\arg z| < \varepsilon$ and every $\alpha = \frac{2\pi ij}{k}$ for $j = 1, 2, \ldots, k - 1$, the results of lemma 8 hold. Next choose $\delta < \varepsilon$ so small that the points $z = \rho e^{i\theta}$, $0 < \rho < \delta$ such that

$$| M (z) | = \mu (\rho)$$

in the notation of lemma 7 all satisfy

$$|\Theta - \alpha_j| < \varepsilon \tag{10.1}$$

where $\alpha_j = \dfrac{2\pi j}{k}$ for some $j = 0, 1, 2, \ldots, k-1$.

Then if $\beta(j) \geqq 0$ for every j, we have from lemma 8, if $z = \rho e^{i\theta}$ and (10.1) is satisfied with

$$|M(ze^{-i\alpha_j})| \geqq |M(z)|, \tag{10.2}$$

so that the maximum modulus is certainly also attained for some Θ with $|\Theta| < \varepsilon$, and hence $\Theta = 0$ by lemma 7. Thus if the conditions (i), (ii) and (iii) of Theorem II are satisfied $M(z)$ attains its maximum modulus for small z on the positive real axis, so that $M(\rho)$ is the maximum modulus of the function $M(z)$.

Conversely (i) and (ii) are also necessary for this, by Theorem I. Suppose next that (iii) is not satisfied, so that $\beta(j) < 0$ for some j. Then we have from lemma 8 for small positive ρ

$$\left| M\left(\rho e^{\frac{2\pi i j}{k}}\right) \right| > M(\rho),$$

so that $M(z)$ does not attain its maximum modulus on the positive real axis for small z, and hence by Theorem I, $M(\rho)$ cannot be the maximum modulus of any function $f(z)$. This shows that the conditions (iii) of Theorem II are also necessary in order that $M(\rho)$ should be the maximum modulus of any function $f(z)$. This completes the proof of Theorem II.

Remark: *We note that* M (z) *attains its maximum modulus on the positive real axis and nowhere else if and only if* $\beta(j) > 0$, *for* $j = 1, 2, \ldots, k-1$. For in this case we cannot have $\beta(j) = 0$ since otherwise, by lemma 8, we have

$$M(\rho) = M(\rho e^{2\pi i j/k}).$$

Conversely if $\beta(j) > 0$ for $j = 1, 2, \ldots, k-1$, and

$$|M(\rho e^{i\theta})| = M(\rho)$$

for some small ρ and $0 \leqq \Theta \leqq 2\pi$ we have from lemma 7, for some j,

$$\left| \Theta - \frac{2\pi j}{k} \right| < \varepsilon$$

and hence if j is not a multiple of k we deduce from lemma 8

$$|M[\rho e^{i(\theta-2\pi j/k)}]| > M(\rho)$$

which gives a contradiction. Thus we must have $j=0$, $|\Theta|<\varepsilon$ and hence from lemma 7, $\Theta=0$. This proves the remark.

Part II.

11. We have now solved our first problem, namely the characterization of the functions

$$M(\rho)=\max_{|z|=\rho}|f(z)|$$

which are for small positive ρ the maximum modulus of some function $f(z)$, regular at $z=0$. The conditions are that $M(\rho)$ is for small positive ρ the sum of a convergent power series in ρ, some of whose coefficients (depending on the index of the first non-vanishing coefficient) must be positive.

We now attack the problem of characterizing all functions with a given maximum modulus of this type. Let

(11.1) $$M(\rho)=1+a_k\rho^k+\dots,\quad a_k\neq 0$$

satisfy the conditions of Theorem II. Then we shall see (Theorem IV) that it is always uniquely possible to find a normalized function $f(z)$ having maximum modulus $M(\rho)$ and attaining it on an assigned R-arc γ for small positive ρ, when $k=1$ or 2. However if (11.1) holds with $k\geqq 3$, this result is false (Theorem V).

We first prove the following result:

Theorem III. *Let* M(ρ), *regular near* $\rho=0$ *and real for real* ρ, *be given by* (11.1). *Let* γ *be an R-arc given by*

(11.2) $$z=\rho e^{i\Theta(\rho)}$$

where $\Theta(\rho)$ *is regular near* $\rho=0$ *and real for real* ρ. *Then there exists a unique function* $f(z)$ *regular near* $z=0$ *and having for some* $\varepsilon>0$ *the following properties*:

(i) $|f[\rho e^{i\Theta(\rho)}]|=M(\rho),\quad 0<\rho<\varepsilon,$

(ii) $\dfrac{\partial}{\partial\Theta}|f(\rho e^{i\Theta})|=0,\quad \Theta=\Theta(\rho),\quad 0<\rho<\varepsilon,$

(iii) $f(0)=1$.

If there exists $f(z)$ having maximum modulus (or minimum modulus!) $M(\rho)$ and attaining the maximum (or minimum) modulus on γ, it must certainly satisfy (i) and (ii). By further multiplying by a constant $e^{i\lambda}$ we may assume (iii) satisfied also. Thus the conditions (i) to (iii) of Theorem III are certainly necessary for the result we are seeking. In order that the function $f(z)$ of that Theorem should in effect have its maximum modulus $M(\rho)$, it is further necessary that $M(\rho)$ should satisfy the conditions (ii) and (iii) of Theorem II. We shall see that when $k = 1$ or 2 this further condition is also sufficient, but not when $k \geqq 3$. There is thus a sharp distinction between these two cases.

12. We proceed to prove Theorem III. Let z be related to ρ by (11.2), where ρ, z are small complex numbers and let $\rho = \rho(z)$ be the inverse function of (11.2) clearly regular at $z = 0$. We put

(12.1) $$g(z) = g[\rho(z)] = \rho \frac{M'(\rho)}{M(\rho)}$$

and

(12.2) $$f(z) = \exp \int_0^z g(z) \frac{dz}{z}.$$

Then $f(z)$ is the required function in Theorem III.

In fact $g(z)$ is clearly regular and vanishes at $z = 0$, and so $f(z)$ is regular at $z = 0$ and satisfies (iii).

Again we have from (12.1), (12.2)

(12.3) $$z \frac{f'(z)}{f(z)} = g(z) = \rho \frac{M'(\rho)}{M(\rho)},$$

if z, ρ are related as in (11.2) and so in particular $z \frac{f'(z)}{f(z)}$ is real for z on γ, and the argument of lemma 1 shows that this is equivalent to (ii). Lastly it now follows from lemma 2, that if

$$\mu(\rho) = |f[\rho e^{i\theta(\rho)}]|$$

then we have for small positive ρ, $z = \rho e^{i\theta(\rho)}$,

$$\rho \frac{\mu'(\rho)}{\mu(\rho)} = z \frac{f'(z)}{f(z)} = \rho \frac{M'(\rho)}{M(\rho)},$$

using (12.3), and hence

$$M(\rho) = k\mu(\rho)$$

where k is a constant. Since $M(0) = f(0) = 1$, we deduce $k = 1$, which proves (i). This completes the existence part of Theorem III. To prove uniqueness, note that by lemmas 1 and 2 any function $f(z)$ satisfying (i) and (ii) must satisfy (12.3) when (11.2) holds, and this condition together with (iii) implies (12.1), (12.2) and determines $f(z)$ uniquely.

13. Lastly we investigate under what circumstances the function $f(z)$ of Theorem III actually has the maximum modulus $M(\rho)$. We have in this connection first

Theorem IV. *Suppose that $M(\rho)$ given by* (11.1) *satisfies the conditions of Theorem II and $k = 1$ or 2. Then if γ is an R-arc, there exists a unique function $f(z)$ such that $f(0) = 1$ and*

$$|f(z)| \leqq M(|z|)$$

with equality for z on γ.

Let γ be given by (11.2). Then by Theorem III, if $f(z)$ exists having the properties required in Theorem IV, then $f(z)$ is unique. We know also that $f(z)$ must be given by (12.1) and (12.2). Let us investigate this function $f(z)$. The R-arcs where $f(z)$ attains its maximum modulus must lie among those where $z\dfrac{f'(z)}{f(z)}$ is real and positive, by lemma 6. The arc γ certainly has this property by (12.3), (11.2) and Theorem II (ii). If $k = 1$ it is by lemma 4 the only arc with this property so that in this case Theorem IV follows.

If $k = 2$ and (11.2) holds, we have from (12.3)

$$z\frac{f'(z)}{f(z)} = \rho\frac{M'(\rho)}{M(\rho)} = a_2\rho^2 + \dots . \tag{13.1}$$

By condition (ii) of Theorem II, $a_2 > 0$, so that the right hand side of (13.1) is positive for small real and positive or negative ρ. It follows from lemma 4 that $\rho\dfrac{M'(\rho)}{M(\rho)}$ is real and positive on only two R-arcs near $\rho = 0$, and since the positive and negative real axis give two such R-arcs, there can be no others. Thus the only points on $|z| = \rho$, where $z\dfrac{f'(z)}{f(z)}$ can attain its greatest positive value are given by

$$z = \rho e^{i\theta(\rho)}, \quad z = -\rho e^{i\theta(-\rho)},$$

with real positive ρ. At these points we have respectively

$$|f(z)| = M(\rho), \quad |f(z)| = M(-\rho)$$

and since $M(z)$ is its own maximum modulus for small complex z we have

$$M(\rho) \geq M(-\rho).$$

Hence the maximum of $|f(z)|$ on $|z| = \rho$ occurs at $z = \rho e^{i\Theta(\rho)}$, which, together with (i) and (iii) of Theorem III, proves Theorem IV.

We note incidentally that in Theorem IV strict inequality holds except when z is on γ for small z, except that when $k = 2$ and $M(\rho) \equiv M(-\rho)$, equality also holds at the point

$$z = -\rho e^{i\Theta(-\rho)}.$$

where $\Theta(\rho)$ is the function of (11.2).

14. If $k \geqq 3$, the result of Theorem IV breaks down, as we shall show in Theorem V below. In this case the problem of classifying all functions whose maximum modulus is $M(\rho)$ appears much more difficult.

Theorem V. *Let* $M(\rho)$ *be given by* (11.1) *with* $k \geqq 3$. *Then there exists an R-arc,* γ *such that no function* $f(z)$ *exists satisfying for any* $\varepsilon > 0$, *and all* $|z| \leqq \varepsilon$ *the inequality*

$$|f(z)| \leqq M(|z|)$$

with equality for z *on* γ.

If contrary to this Theorem there is such a function $f(z)$ we may without loss in generality assume that $f(0) = 1$, so that $f(z)$ satisfies the conditions (i) to (iii) of Theorem III. We take for γ the arc given by

$$z = \rho e^{i\alpha\rho} \tag{14.1}$$

where α is a suitable large positive constant depending on $M(\rho)$. Then if $f(z)$ exists having maximum modulus $M(\rho)$ and attaining it on γ, it follows from Theorem III, that we must have

$$z\frac{f'(z)}{f(z)} = \rho\frac{M'(\rho)}{M(\rho)} = (k\, a_k\rho^k + b\rho^{k+1} + \dots) \tag{14.2}$$

where $b = (k+1)\, a_{k+1}$. We deduce from (14.1) that

$$z = \rho + i\alpha\,\rho^2 + \dots$$

so that

(14.3) $$\rho = z - i\alpha z^2 + \dots$$

and substituting from (14.3) in (14.2) we obtain

$$z\frac{f'(z)}{f(z)} = ka_k z^k(1 - ki\alpha z + \dots) + bz^{k+1} + \dots$$
$$= ka_k z^k + (b - k^2 i\alpha a_k) z^{k+1} + \dots$$

whence

$$\log f(z) = a_k z^k + \left(\frac{b - k^2 i\alpha a_k}{k+1}\right) z^{k+1} + \dots .$$

We put $z = \rho e^{i\theta}$ in the above and take real parts. This gives

(14.4) $$\log |f(\rho e^{i\theta})| =$$
$$a_k\rho^k \cos k\Theta + \frac{\rho^{k+1}}{k+1}[b\cos(k+1)\Theta + k^2\alpha a_k \sin(k+1)\Theta] + \dots .$$

We now put $\Theta = \mp\frac{2\pi}{k}$. Then since $k \geqq 3$, we have $\sin(k+1)\Theta = \sin\Theta \neq 0$.

We may suppose $\alpha a_k \sin(k+1)\Theta > 0$.

Hence $\alpha a_k \sin(k+1)\Theta$ can be made as large as we please by choosing α sufficiently large. We obtain in (14.4) an expansion of the form

(14.5) $$\log|f(\rho e^{i\theta})| = a_k\rho^k + \beta\rho^{k+1} + \dots$$

where the constant β can be made as large as we please by a suitable choice of α. On the other hand when $|z| = \rho$ and z lies on γ, we have

$$\log|f(z)| = \log M(\rho) = a_k\rho^k + \frac{b}{k+1}\rho^{k+1} + \dots,$$

where b is the constant in (14.2). Hence if α is so large that $\beta > b/(k+1)$ we have for all small positive ρ, and one of $\Theta = \mp\frac{2\pi}{k}$

$$\log|f(\rho e^{i\theta})| > \log|f(\rho e^{i\alpha\rho})| .$$

so that $f(z)$ does not attain its maximum modulus on γ. This contradicts our original assumption that Theorem V is false and completes the proof of that Theorem.

15. In conclusion it may be worthwhile to point out some open questions. Blumenthal [1] raised the problem of characterizing the maximum modulus of entire functions. A more modest aim would be to obtain conditions for a function $M(\rho)$, regular at $\rho = 0$ and real for real ρ to be the maximum modulus of an entire function for sufficiently small ρ, and to decide in what way such a function $M(\rho)$ can locally be the maximum modulus of several entire functions. In this way it might be possible to settle the problem of whether two entire functions of the form $w = f(z)$, which have the same maximum modulus for all values of $\rho = |z|$ are necessarily obtained from each other by reflections and rotations in the z- and w-planes [2].

A characterization of the maximum modulus of the smaller class of polynomials might also be of interest.

In connection with the preceding work it may be possible to generalize the Theorem I that a maximum modulus is always locally near the origin its own maximum modulus to a Theorem in the large.

Further Theorem V opens the question of how to characterize all functions having a given maximum modulus

$$M(\rho) = 1 + a_k \rho^k + \dots$$

in the neighbourhood of the origin, when $k \geqq 3$.

2. See also B. Anderson [3], who used essentially lemmas 1 and 2 in an investigation of functions equivalent under rotation and reflection.

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