ON LINEAR ACCESSIBILITY AND THE CONFORMAL MAPPING OF CONVEX DOMAINS

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Introduction. Following Biernacki [1] a domain D in the plane will be called *linearly accessible* if its complement D^c can be written as a union of half-lines. Lewandowski [3, 4] has shown that the close-to-convex domains of Kaplan [2] are precisely those which are linearly accessible in the "strict sense" of Biernacki: the complement can be written as a union of mutually disjoint half-lines (except that the endpoint of one half-line can lie on another half-line). Kaplan's definition can be phrased as follows: a domain D_z in the z-plane is close-to-convex if there is a schlicht conformal mapping $w = f(z)$ of D_z onto a convex domain D_w with the property that $\operatorname{Re} f'(z) > 0$ for every $z \in D_z$. Although a great deal of work has been done on close-to-convex domains and their conformal mappings onto the unit disc, no results whatever (even by Biernacki) have appeared on the more general linearly accessible class. Nevertheless these domains arise in a very natural way in connection with the conformal mapping of one convex domain onto another. Moreover, as we will see, it appears likely that there is an alternative characterisation of linear accessibility in terms of conformal mappings onto convex domains very similar to the close-to-convex condition. The work of this paper centres round this connection between convexity and linear accessibility and we will establish a sufficient condition for a domain to be linearly accessible. The study of the conformal mappings of the unit disc onto linearly accessible domains will not be attempted here, although in conclusion we will mention without proof some results we have obtained concerning these.

1. We begin with two definitions. Let γ be a simple arc $z(t)$ $(a \leqq t \leqq b)$. We will say that γ is *close-to-linear* if there is a system of parallel lines

covering the plane with the property that each line in the system intersects γ in at most a single point or a line segment. It is clear from this definition that *every point z not on γ lies on a half-line which does not meet γ.* A simple arc γ satisfying this latter condition will be said to be an *arc of linear type.* We mention a few examples illustrating these definitions: a line segment, an arc consisting of two adjoining line segments, a Z shape and a semi-circle are all close-to-linear arcs; any proper sub-arc of a simple closed convex curve is an arc of linear type. In particular a circular arc which contains properly a semi-circle is an arc of linear type which is not close-to-linear.

2. It is convenient to give an alternative characterisation for each type of arc. For this purpose we assume that the arcs are regular (i.e. $z'(t)$ exists, is continuous and $\neq 0$).

Lemma 2.1. *A simple regular arc $\gamma: z = z(t)$ $(a \leqq t \leqq b)$ is close-to-linear if, and only if,*

$$|\arg z'(t_2) - \arg z'(t_1)| \leqq \pi \tag{2.1.1}$$

for any two points t_1 and t_2 in $[a, b]$.

Proof. Suppose first that γ is close-to-linear. Without loss of generality we may assume that every line parallel to the imaginary axis intersects γ in at most a line segment, and further that $\operatorname{Re} z(a) \leqq \operatorname{Re} z(b)$. It is then clear that $\operatorname{Re} z'(t) \geqq 0$ $(a \leqq t \leqq b)$ and the condition (2.1.1) follows immediately. Conversely, the condition implies that there is a real number λ such that

$$|\arg z'(t) - \lambda| \leqq \frac{\pi}{2} \quad (a \leqq t \leqq b),$$

i.e. that $\operatorname{Re} e^{-i\lambda} z'(t) \geqq 0$ $(a \leqq t \leqq b)$. Thus

$$\operatorname{Re} e^{-i\lambda}(z(t_2) - z(t_1)) = \int_{t_1}^{t_2} \operatorname{Re} e^{-i\lambda} z'(t) dt \geqq 0,$$

where equality occurs if, and only if, $\operatorname{Re} e^{-i\lambda} z'(t) = 0$ for $t_1 \leqq t \leqq t_2$. Hence every line parallel to the imaginary axis meets the arc $e^{-i\lambda} z(t)$ in at most a line segment, and the lemma follows.

Lemma 2.2. *A simple regular* arc γ: $z = z(t)$ $(a \leqq t \leqq b)$ *is of linear type if, and only if, for any three points* t_0, t_1 *and* t_2 *in* $[a, b]$ *we have*

$$\left| \arg \frac{z(t_2) - z(t_0)}{t_2 - t_0} - \arg \frac{z(t_1) - z(t_0)}{t_1 - t_0} \right| \leqq \pi . \tag{2.2.1}$$

Proof. Suppose first that γ is of linear type and let us assume that the proposition is false. We can then find three points t_0, t_1 and t_2 in $[a, b]$ such that

$$\pi < \left| \arg \frac{z(t_2) - z(t_0)}{t_2 - t_0} - \arg \frac{z(t_1) - z(t_0)}{t_1 - t_0} \right| < 2\pi . \tag{2.2.2}$$

Let us assume in the first instance that $t_0 < t_1 < t_2$. For convenience of notation and without loss of generality we will assume further that $z(t_0) = 0$, $z(t_1) = z_1 > 0$ and $z(t_2) = z_2$ satisfies $\operatorname{Im} z_2 < 0$. Since γ is simple and regular, for sufficiently small $r > 0$, the closed disc Δ of radius r and centre z_1 intersects γ in a single simple arc γ' passing through z_1. The inequality (2.2.2) implies that every ray from the origin making an angle between 0 and $\arg z_2 > \pi$ meets γ for some value of t in (t_1, t_2). Put $\alpha = \arg z_2$, so that $\pi < \alpha < 2\pi$, and choose r sufficiently small so that the angle subtended at the origin by the circle C bounding Δ is $\varepsilon < \alpha - \pi$. We now choose two points ζ_1 and ζ_2 on the complementary open circular arcs of C whose endpoints coincide with those of γ'. Neither ζ_1 nor ζ_2 lie on γ, so we can find half-lines l_1 and l_2 with respective endpoints ζ_1 and ζ_2 which fail to meet γ. Let C_1 and C_2 be the complementary circular arcs of C with endpoints ζ_1 and ζ_2, it being assumed that l_1 and l_2 do not meet the interior of Δ. The complement of Δ is divided by l_1 and l_2 into at most three open components: D_1 with boundary C_1 and a subset of $l_1 \cup l_2$, D_2 with boundary C_2 and a subset of $l_1 \cup l_2$, and possibly a sector S formed by l_1 and l_2. If S exists it is separated from C and contains no points of γ. If we suppose that $0 \in D_1$, then we can find s_1 and s_2 satisfying $t_0 < s_1 < t_1 < s_2 < t_2$ such that $z(t) \in D_1$ for $t_0 \leqq t < s_1$, $z(t) \in D_2$ for $s_2 < t \leqq t_2$ and $z(t) \in \Delta$ for $s_1 \leqq t \leqq s_2$. By hypothesis every ray $l_\theta = \{w / \arg w = \theta\}$ such that $\frac{1}{2}\varepsilon < \theta < \alpha$ meets γ for values of $t > s_2$, and therefore meets D_2. Hence every such ray intersects either l_1 or l_2. Let A_1 be the set of θ in $(\frac{1}{2}\varepsilon, \alpha)$ such that l_θ meets l_1, and define A_2 similarly. A_1 and A_2 are clearly open intervals of length $< \pi$, and since their union is the interval $(\frac{1}{2}\varepsilon, \alpha)$ which has length $> \pi$, their intersection is a non-empty open interval (x, y). We may suppose then that $A_1 = (\frac{1}{2}\varepsilon, y)$, $A_2 = (x, \alpha)$, where $x < y$. Now since ζ_1 and ζ_2 lie in the sector $|\arg w| \leqq \varepsilon$, it is clear that $x > \pi - \frac{1}{2}\varepsilon$ and $y < \pi + \frac{1}{2}\varepsilon$, and that l_1 has the same direction as l_x and l_2 as l_y. If l_x meets l_2 at w_2 and l_y meets l_1 at w_1, then w_1 and w_2 lie in the sector $\pi - \frac{1}{2}\varepsilon < \arg w < \pi + \frac{1}{2}\varepsilon$ and l_1 and l_2 meet also in this sector at w_0, which is the fourth vertex completing the parallelogram 0, w_1, w_2 along l_1 and l_2. It follows that w_0 is the vertex of the sector S and that the ray from 0 through w_0 does not meet D_2, so does not meet γ. This contradiction establishes (2.2.1) in this case. The case $t_1 < t_2 < t_0$ follows similarly. The case $t_1 < t_0 < t_2$ can be deduced from the previous cases and we omit the details.

For the converse we assume (2.2.1) satisfied and choose $z_0 \notin \gamma$. Assume that every half-line from z_0 meets γ. For simplicity of notation we can assume that $z_0 = 0$. We can find points t_1 and t_2 in $[a, b]$ such that $z(t_1) < 0$, $z(t_2) > 0$ and $(z(t_1), z(t_2))$ does not meet γ. We may suppose that $t_1 < t_2$. The inequality (2.2.1) implies that there is a line L_1 through $z(t_1)$ such that $z(t)$ lies on one side of L_1 for $t < t_1$ and on the other side for $t > t_1$ (meeting L_1 being permitted), and a similar line L_2 through $z(t_2)$. We consider three cases:

(i) L_1 parallel to L_2 but not coinciding with the real axis; in this case for $t_1 < t < t_2$, $z(t)$ lies in a half-strip bounded by L_1, L_2 and $[z(t_1), z(t_2)]$ and no point of γ lies in the opposite half-strip; hence the half-line from 0 parallel to L_1 and into the latter half-strip fails to meet γ, a contradiction;

(ii) L_1 and L_2 coincide with the real axis; we may suppose then that $\operatorname{Re} z(t) \geqq 0$ for $t > t_1$; it follows that for some $t < t_2$, $\operatorname{Re} z(t) > 0$ and therefore $\operatorname{Re} z(t) \geqq 0$ for $t < t_2$; thus $\operatorname{Re} z(t) \geqq 0$ for all t and we again obtain a contradiction;

(iii) L_1 and L_2 meet at a point z_3; let S be the open sector with vertex at z_3 and which contains 0; let S' be the opposite open sector; every ray from 0 lying entirely in S can meet γ only for t in (t_1, t_2); it is therefore clear that γ does not meet the open triangle whose vertices are $z(t_1)$, $z(t_2)$ and z_3;

also γ cannot meet S'; therefore by hypothesis $z_3 \in \gamma$, say $z_3 = z(t_3)$; if $t_3 < t_1$ then $z_3 \notin \gamma' : z = z(t)$ $(t_2 \leqq t \leqq b)$; hence there is an open disc of centre z_3 not meeting γ'; there is a ray from 0 passing through this disc and into S' which never meets the region in which $z(t)$ lies for $t < t_1$, and therefore never meets γ; similarly if $t_3 > t_2$; this final contradiction completes the proof.

3. We now present a result which contains within it some interesting structural information about the conformal mapping of one convex domain onto another.

Theorem 3.1. *Let $w = f(z)$ be a schlicht conformal mapping of a convex domain D_z onto a convex domain D_w. For any three points z_0, z_1 and z_2 in D_z we have*

$$\left| \arg\frac{f(z_2)-f(z_0)}{z_2 - z_0} - \arg\frac{f(z_1)-f(z_0)}{z_1 - z_0} \right| < \pi . \tag{3.1.1}$$

Remark 3.2. Roughly speaking this result is geometrically obvious if the points z_0, z_1 and z_2 are boundary points of D_z occurring say in that order as the boundary is traversed in the positive direction: for then the angle subtended by z_1 and z_2 at z_0 lies between 0 and π, and since the image points occur in the same order on the boundary of D_w, the angle subtended by $f(z_1)$ and $f(z_2)$ at $f(z_0)$ also lies between 0 and π.

Lemma 3.3. *Let $h(\zeta)$ be regular and univalent in $|\zeta| < 1$, $h(0) = 0$, and let the image Ω of $|\zeta| < 1$ be starlike with respect to 0. Let ζ_1 and ζ_2 be two points in the open unit disc. Then there is a real number α, depending on ζ_1 and ζ_2 but not on the particular function h, such that*

$$-\frac{\pi}{2} + \alpha < \tfrac{1}{2} \arg \frac{h(\zeta_2)}{\zeta_2} - \tfrac{1}{2} \arg \frac{h(\zeta_1)}{\zeta_1} < \frac{\pi}{2} + \alpha . \tag{3.3.1}$$

Proof. By the well-known Integral Representation formula we can write

$$\frac{h(\zeta)}{h'(0)} = \zeta \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \log \frac{1}{(1 - \zeta e^{-it})^2} dV(t)\right)$$

where $V(t)$ is an increasing function such that

$$\int_0^{2\pi} dV(t) = 2\pi .$$

We thus have

$$\tfrac{1}{2}\arg \frac{h(\zeta_2)}{\zeta_2} - \tfrac{1}{2}\arg\frac{h(\zeta_1)}{\zeta_1} = \frac{1}{2\pi}\int_0^{2\pi} \arg\frac{1-\zeta_1 e^{-it}}{1-\zeta_2 e^{-it}}\, dV(t). \tag{3.3.2}$$

The mapping

$$\tau = \frac{1-\zeta_1\sigma}{1-\zeta_2\sigma} \qquad (|\sigma| \leqq 1)$$

sends the closed disc $|\sigma| \leqq 1$ onto a closed disc not containing the origin. Therefore there is a real number α such that for all real t

$$-\frac{\pi}{2}+\alpha < \arg\frac{1-\zeta_1 e^{-it}}{1-\zeta_2 e^{-it}} < \frac{\pi}{2}+\alpha .$$

We deduce (3.3.1) from (3.3.2).

(3.4) Proof of 3.1. Let $w = \phi(\zeta)$ be a schlicht conformal mapping of $|\zeta| < 1$ onto D_w and let $\psi(\zeta) = f^{-1}(\phi(\zeta))$. $\phi(\zeta)$ and $\psi(\zeta)$ are then convex schlicht functions in $|\zeta| < 1$, and therefore [6] the functions

$$h(\zeta) = \zeta\left(\frac{\phi(\zeta)-\phi(\zeta_0)}{\zeta-\zeta_0}\right)^2, \quad k(\zeta) = \zeta\left(\frac{\psi(\zeta)-\psi(\zeta_0)}{\zeta-\zeta_0}\right)^2$$

are starlike in $|\zeta| < 1$ for any $\zeta_0 (|\zeta_0| < 1)$. Let ζ_1 and ζ_2 be any two points in $|\zeta| < 1$. It then follows from Lemma (3.3) that

$$\left| \tfrac{1}{2}\arg\frac{h(\zeta_2)}{k(\zeta_2)} - \tfrac{1}{2}\arg\frac{h(\zeta_1)}{k(\zeta_1)} \right| < \pi$$

so that

$$\left| \arg\frac{\phi(\zeta_2)-\phi(\zeta_0)}{\psi(\zeta_2)-\psi(\zeta_0)} - \arg\frac{\phi(\zeta_1)-\phi(\zeta_0)}{\psi(\zeta_1)-\psi(\zeta_0)} \right| < \pi .$$

Since the points ζ_0, ζ_1 and ζ_2 are arbitrary in $|\zeta| < 1$, we can write $z_0 = \psi(\zeta_0)$, $z_1 = \psi(\zeta_1)$ and $z_2 = \psi(\zeta_2)$. Then $f(z_0) = \phi(\zeta_0)$, $f(z_1) = \phi(\zeta_1)$ and $f(z_2) = \phi(\zeta_2)$ and the result follows.

It is also interesting to note that this result is characteristic of convex domains D_z. To be precise we have

(3.5) *Let D_z be a simply-connected domain with the property that every schlicht conformal mapping $w = f(z)$ of D_z onto a convex domain D_w satisfies* (3.1.1) *for any three points z_0, z_1 and z_2 in D_z. Then D_z is convex.*

Proof. Let $z = \psi(\zeta)$ be a schlicht conformal mapping of $|\zeta| < 1$ onto D_z and let $w = \phi(\zeta)$ be an arbitrary schlicht conformal mapping of $|\zeta| < 1$ onto a convex domain D_w. Then the hypothesis implies that for any three points ζ_0, ζ_1 and ζ_2 in $|\zeta| < 1$ we have

$$\left| \arg \frac{\psi(\zeta_2) - \psi(\zeta_0)}{\phi(\zeta_2) - \phi(\zeta_0)} - \arg \frac{\psi(\zeta_1) - \psi(\zeta_0)}{\phi(\zeta_1) - \phi(\zeta_0)} \right| < \pi .$$

From this it is easily deduced that for any two points ζ_1 and ζ_2 in $|\zeta| < 1$ we have

$$\left| \arg \frac{G(\zeta_2)}{F(\zeta_2)} - \arg \frac{G(\zeta_1)}{F(\zeta_1)} \right| < 2\pi \tag{3.5.1}$$

where $G(\zeta) = \zeta\psi'(\zeta)$ and $F(\zeta) = \zeta\phi'(\zeta)$. Thus it is sufficient to show that if for a given $G(\zeta)$, (3.5.1) holds for every starlike $F(\zeta)$, then $G(\zeta)$ is starlike. To establish this result we begin by observing that $\arg G(\zeta)/\zeta$ is a bounded harmonic function and therefore

$$U(t) = \lim_{r \to 1} \arg G(re^{it})$$

exists for almost all real t. Also $V(t) = \lim_{r \to 1} \arg F(re^{it})$ exists for all t when $F(z)$ is starlike. By hypothesis

$$\left| (U(t_2) - V(t_2)) - (U(t_1) - V(t_1)) \right| \leqq 2\pi$$

for all t_1 and t_2 on the set E where $U(t)$ exists. If $t_2 > t_1$ we have therefore

$$U(t_2) - U(t_1) \geqq -2\pi + V(t_2) - V(t_1).$$

This holds for all increasing functions $V(t)$ which satisfy $V(t+2\pi) = V(t) + 2\pi$ for all t. In particular $V(t)$ may be step function with at least one jump of 2π occurring between t_1 and t_2. We deduce that $U(t_2) - U(t_1) \geqq 0$. Thus $U(t)$ is increasing on E. We may extend $U(t)$ to all real values by setting

$$U(t) = \tfrac{1}{2}(\inf\{U(\tau)/\tau \geqq t,\ \tau \in E\} + \sup\{U(\tau)/\tau \leqq t,\ \tau \in E\}).$$

The resulting function is then increasing for all t and $U(t+2\pi) = U(t) + 2\pi$. Now if $|\zeta| < R < 1$ we have by Poisson's formula

$$\frac{\zeta G'(\zeta)}{G(\zeta)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{Re^{it}+\zeta}{Re^{it}-\zeta}\operatorname{Re}\frac{Re^{it}G'(Re^{it})}{G(Re^{it})}dt$$

$$= 1 + \frac{i}{\pi}\int_0^{2\pi} \frac{\zeta Re^{it}}{(Re^{it} - \zeta)^2}\arg\frac{G(Re^{it})}{Re^{it}}dt,$$

the third expression being obtained by an integration by parts. We observe that the final integrand is bounded as $R \to 1$ and converges almost everywhere to $\zeta e^{-it}(1-\zeta e^{-it})^{-2}(U(t)-t)$. Therefore by the Lebesgue bounded convergence theorem

$$\frac{\zeta G'(\zeta)}{G(\zeta)} = 1 + \frac{i}{\pi}\int_0^{2\pi} \frac{\zeta e^{-it}}{(1-\zeta e^{-it})^2}(U(t)-t)dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{1+\zeta e^{-it}}{1-\zeta e^{-it}}dU(t).$$

Thus $\operatorname{Re}\zeta G'(\zeta)/G(\zeta) > 0$ and the result follows.

4. This section is concerned with the interpretation of the inequality (3.1.1). For this purpose we introduce the following definition: we will say a domain D is *of convex type* if there is a schlicht conformal mapping $w = f(z)$ of D onto a convex domain with the property that

$$\left|\arg\frac{f(z_2)-f(z_0)}{z_2-z_0} - \arg\frac{f(z_1)-f(z_0)}{z_1-z_0}\right| < \pi \tag{4.1}$$

for any three points z_0, z_1 and z_2 in D. We deduce immediately from (3.1):

(4.2) *A schlicht conformal mapping of a convex domain D_z onto a convex domain D_w maps every convex sub-domain of D_z onto a domain of convex type in D_w.*

(4.3) *Every close-to-convex domain is a domain of convex type.*

For if $w = f(z)$ is a schlicht conformal mapping of D_z onto a convex domain D_w, and if $\operatorname{Re} f'(z) > 0$ in D_z, then for any two points z_1 and z_2 in D_z we have

$$\operatorname{Re} \frac{z_2 - z_1}{f(z_2 - f(z_1)} = \int_0^1 \operatorname{Re} \frac{1}{f'(z(t))} dt > 0$$

where $f(z(t)) = (1 - t)f(z_1) + tf(z_2)$ $(0 \leqq t \leqq 1)$, and the conclusion is immediate.
On the other hand

Theorem 4.4. *Every domain of convex type is linearly accessible.*

Although this can be proved directly, we prefer to deduce it as a corollary of the results which follow:

(4.5) *If D is a domain of comvex type, then any two points z_1 and z_2 in D can be joined by an arc of linear type lying wholly in D.*

Proof. Let $w = f(z)$ be a schlicht conformal mapping of D onto a convex domain D_w with the inequality (4.1) satisfied. Let γ be the arc in D joining z_1 to z_2 whose image is the line segment $[f(z_1), f(z_2)]$. Thus γ has the equation $z(t) = f^{-1}((1 - t)w_1 + tw_2)$ $(0 \leqq t \leqq 1)$, where $w_1 = f(z_1)$, $w_2 = f(z_2)$. We have for any three points t_0, t_1 and t_2 in $[0,1]$,

$$\left| \arg \frac{z(t_2) - z(t_0)}{t_2 - t_0} - \arg \frac{z(t_1) - z(t_0)}{t_1 - t_0} \right|$$

$$= \left| \arg \frac{z(t_2) - z(t_0)}{f(z(t_2)) - f(z(t_0))} - \arg \frac{z(t_1) - z(t_0)}{f(z(t_1)) - f(z(t_0))} \right| < \pi .$$

Thus γ is of linear type by Lemma 2.2.

By the same argument we have

(4.6) *A schlicht conformal mapping of one convex domain onto another sends line segments onto arcs of linear type.*

Theorem 4.7. *Let D be a simply–connected domain with the property that any two points in D can be joined by an arc of linear type which lies wholly in D. Then D is linearly accessible.*

Proof. Let $z_0 \in D^c$, the complement of D, and assume that every half-line with endpoint z_0 meets D. Without loss of generality we may assume that $z_0 = 0$. There are then points $z_1 < 0$ and $z_2 > 0$ which lie in D, and there is an arc γ_1 in D of linear type joining z_1 to z_2. The change in $\arg z$ along γ_1 has absolute value an odd multiple of π which cannot exceed 2π since by the definition of an arc of linear type, there is a ray from 0 not meeting γ_1. Thus $\Delta \arg z$ along γ_1 is $+\pi$ or $-\pi$. In the first case we choose $z_3 \in D$ on the positive imaginary axis, in the second case on the negative imaginary axis. There are arcs of linear type in D joining z_3 to z_1 and z_3 to z_2. At least one of these arcs, say γ_2, is such that $|\Delta \arg z|$ along γ_2 is $3\pi/2$: for otherwise there is a closed curve in D which has non-zero winding number about 0, and since D is simply-connected we then have $0 \notin D^c$. Inductively, for each $n = 1, 2, \cdots$ we can find an arc γ_n in D of linear type such that $|\Delta \arg z|$ along γ_n is $2\pi - \pi/2^{n-1}$, and such that if a_n and b_n are the arguments of the end-points of γ_n, $a_n \to \alpha$ and $b_n \to \alpha + 2\pi$. Let σ be the ray $\arg z = \alpha$ and let $\zeta \in D \cap \sigma$. Let Δ be an open disc in D with centre ζ. For sufficiently large n, the rays from 0 through the endpoints ζ_n, ζ_n' of γ_n meet Δ. By the same reasoning as before there is an arc Γ in D of linear type which joins ζ to either ζ_n or ζ_n' (say ζ_n) such that $|\Delta \arg z|$ along Γ is $> 2\pi - \pi/2^{n-1}$. But then by suitable choice of $\zeta' \in \Delta$, we can construct an arc Γ' in D of linear type joining ζ' to ζ_n such that $|\Delta \arg z|$ for $z \in \Gamma'$ is $> 2\pi$, and this, as we have already observed, is impossible.

(4.8) (4.4) is an immediate consequence of (4.5) and (4.7). As a particular consequence of these results we have that *a convex function in the unit disc maps every convex sub-domain of the disc onto a linearly accessible domain,* *and conversely the inverse image of a convex domain is a linearly accessible sub-domain of the unit disc.* Another particular case is that of a *bilinear mapping*

$$w = f(z) = \frac{az + b}{cz + d} \qquad (ad - bc \neq 0)$$

defined in a convex domain D_z not containing the point $z = -d/c$. The image D_w is then a domain of convex type: for D_z being convex is contained in a half-plane H having as boundary a line L through the point $-d/c$; $f(z)$ maps H onto another half-plane and the conclusion follows from (4.2).

(4.9) With the help of this example we can exhibit *a domain of convex type which is not close-to-convex*: indeed if C_1 and C_2 are circles meeting at just one point ζ, and if C_1 lies inside C_2, then the crescent-shaped region between the circles is a domain of convex type (and is clearly not close-to-convex), for the mapping $w = 1/(z - \zeta)$ maps the region onto a strip; since a strip is convex the conclusion is immediate.

5. It will come as no surprise that the notion of "close-to-linear" arc is the analogue of arc "of linear type" which arises in connection with close-to-convex domains. In fact we have

Theorem 5.1. *A simply-connected domain D is close-to-convex if, and only if, any two points in D can be joined by a close-to-linear arc lying wholly in D.*

To prove the necessity: let $w = f(z)$ be a schlicht conformal mapping of D onto a convex domain D_w such that $\mathrm{Re} f'(z) > 0$ for all $z \in D$. Let z_1 and z_2 lie in D, let $w_1 = f(z_1)$ and $w_2 = f(z_2)$ and let γ be the arc $z(t) = f^{-1}((1 - t)w_1 + tw_2)$ $(0 \leqq t \leqq 1)$. Then for any two points t_1 and t_2 in $[0,1]$ we have

$$\left|\arg z'(t_2) - \arg z'(t_1)\right| = \left|\arg f'(z(t_1)) - \arg f'(z(t_2))\right| < \pi$$

and therefore γ is close-to-linear by Lemma 2.1.

The sufficiency is rather harder to establish and we need first to weaken considerably the conditions on the complement of a domain which are sufficient for it to be linearly accessible in the strict sense.

(5.2) *Let D be a domain and suppose that given any two boundary points of D we can find a half-line or two disjoint half-lines containing the points and not meeting D. Then D is close-to-convex.*

This hypothesis can be weakened further, and it is enough to assume that the points to which the hypothesis applies form a dense subset of the boundary (e.g. accessible boundary points). We begin the proof of (5.2) by first establishing:

(5.3) *Under the hypotheses of* (5.2) *given any n points $z_1, z_2, \cdots, z_n$, on the boundary of D we can find a finite number of mutually disjoint half-lines containing the points and not meeting D.*

Proof. For any point z on the boundary of D there is a closed sector $S(z)$ in D^c, the complement of D, with vertex z and of angle $\gamma(z)$ $(0 \leqq \gamma(z) < 2\pi$ such that every half-line with endpoint z not in $S(z)$ meets D. We denote by $\mathscr{S}(z)$ the family of half-lines with endpoint z which lie in $S(z)$. We proceed by induction assuming that $n > 2$ and the result proved up to $n-1$. Assume first that $\gamma(z_n) \geqq \pi$. There is then a line L through z_n such that D lies entirely on one side of L. Without loss of generality we may assume that L is the real axis, that $z_n = 0$ and that $\operatorname{Re} z < 0$ for $z \in D$. We then have $\operatorname{Re} z_i \leqq 0$ $(i = 1, \cdots, n-1)$. By the induction hypothesis we can find a finite number of mutually disjoint half-lines in D^c containing the z_i $(1 \leqq i \leqq n-1)$. Since we require a half-line from 0 in D^c not meeting any of these, we may suppose that k of the half-lines, say $l_1, l_2, \cdots, l_k$, meet L at points $a_1 < a_2 < \cdots < a_k$ where, for some i, $a_i < 0 < a_{i+1}$, and that the remaining half-lines lie entirely in $\operatorname{Re} z < 0$. If α_j is the angle l_j makes at a_j with the positive axis, we see that $\alpha_1 \geqq \alpha_2 \geqq \cdots \geqq \alpha_k$. Choose α satisfying $\alpha_{i+1} \leqq \alpha \leqq \alpha_i$. The half-line from 0 making angle α with the positive axis lies in D^c and is disjoint from the other half-lines, the required conclusion.

We may therefore assume that each $\gamma(z_i)$ is $< \pi$ $(1 \leqq i \leqq n)$. We suppose next that $z_n \in S(z_1)$. Without loss of generality we may then assume that $z_1 = 0$, $z_n > 0$, the positive axis being one of the arms of $S(z_1)$ (since z_n is a boundary point of D, it cannot lie in the interior of $S(z_1)$). There are disjoint half-lines in D^c containing $z_1, z_2, \cdots, z_{n-1}$ and for definiteness we may assume that the one containing z_1 makes an angle γ with the positive axis, where $0 < \gamma < \pi$. The z_i $(2 \leqq i \leqq n-1)$ all lie outside the sector $S = \{0 < \arg z < \gamma\}$ and $S \subset D^c$. We may assume that exactly k of the disjoint half-lines cut the positive axis at points $a_1 < a_2 < \cdots < a_k$, where for some i, $a_i < z_n < a_{i+1}$. The angles α_j which they make decrease with j and are all $\leqq \gamma$. Choose α satisfying $\alpha_{i+1} \leqq \alpha \leqq \alpha_i$. The half-line in this direction from z_n gives the required conclusion.

Thus we may further assume that $z_i \notin S(z_j)$ for $i \neq j$. We base our proof now on the following geometrical lemma which enables us to eliminate "impossible" half-lines from the $\mathscr{S}(z_i)$:

Lemma 5.4. *Let $S(z_1)$, $S(z_2)$ and $S(z_3)$ be three closed sectors of angles $\gamma(z_i)$ $(0 \leqq \gamma(z_i) < \pi)$ such that $z_i \notin S(z_j)$ for $j \neq i$ $(i = 1,2,3,\ j = 1,2,3)$ and assume that for any two distinct points z_i and z_j there are disjoint half-lines in $\mathscr{S}(z_i)$ and $\mathscr{S}(z_j)$. Let $\mathscr{S}'(z_1)$ be $\mathscr{S}(z_1)$ less all half-lines in $\mathscr{S}(z_1)$ which meet every half-line in $\mathscr{S}(z_2)$. Then the half-lines in $\mathscr{S}'(z_1)$ form a closed sector $S'(z_1)$; further, there are disjoint half-lines in $\mathscr{S}'(z_1)$ and $\mathscr{S}(z_3)$ and every half-line in $\mathscr{S}'(z_1)$ is disjoint from at least one half-line in $\mathscr{S}(z_2)$.*

Proof. Clearly $\mathscr{S}'(z_1)$ is non-empty and we may assume that $\mathscr{S}'(z_1) \neq \mathscr{S}(z_1)$ as otherwise the lemma is trivial. Without loss of generality we may assume that z_1 and z_2 are real and that $z_1 < z_2$. For definiteness we may assume further that at least one half-line in $\mathscr{S}(z_2)$ lies (apart from z_2) in the upper half-plane. Then the whole of $S(z_2)$ (except z_2) lies in the upper half-plane, the arms making angles α_2 and β_2 with the positive axis $(0 < \alpha_2 \leqq \beta_2 < \pi)$. The corresponding angles α_1 and β_1 of $S(z_1)$ will then satisfy $0 < \alpha_1 < \alpha_2 \leqq \beta_1 < \pi$. $\mathscr{S}'(z_1)$ clearly consists of those half-lines from z_1 making an angle θ with the positive axis, where $\alpha_2 \leqq \theta \leqq \beta_1$. For any point z we denote by $l(\theta, z)$ the half-line with endpoint z making an angle θ with the positive axis. To complete the proof we need only to show that there are disjoint half-lines in $\mathscr{S}'(z_1)$ and $\mathscr{S}(z_3)$. Assume on the contrary that every half-line in $\mathscr{S}(z_3)$ meets every half-line in $\mathscr{S}'(z_1)$. Let $L(\theta)$ denote the line containing the half-line $l(\theta, z_1)$. If z_3 lies on the same side of $L(\beta_1)$

as z_2 then the assumption implies that every half-line in $\mathscr{S}(z_3)$ meets every half-line in $\mathscr{S}(z_1)$, contradicting the hypothesis. Hence z_3 lies on the opposite side to z_2 of both $L(\alpha_2)$ and $L(\beta_1)$. If $S(z_3)$ meets the segment $[z_1, z_2]$, then every half-line in $\mathscr{S}(z_3)$ meets this segment and so also meets, by the assumption, every half-line in $\mathscr{S}(z_1)$, a contradiction. Finally if $S(z_3)$ does not meet this segment, the assumption implies that every half-line in $\mathscr{S}(z_3)$ meets every half-line in $\mathscr{S}(z_2)$, a contradiction.

(5.5) We now complete the proof of (5.3). Delete from $\mathscr{S}(z_1)$ all half-lines which meet every half-line in $\mathscr{S}(z_i)$ $(i = 2, \cdots, n)$. By successive applications of (5.4) we obtain a closed sector $S^*(z_1)$ such that every half-line in $\mathscr{S}^*(z_1)$ is disjoint from at least one half-line in $\mathscr{S}(z_i)$ $(i = 2, \cdots, n)$. Beginning again with $\mathscr{S}^*(z_1), \mathscr{S}(z_2), \cdots, \mathscr{S}(z_n)$ we repeat the process on $\mathscr{S}(z_2)$ and so on. We obtain finally n closed sectors $S^*(z_1), \cdots, S^*(z_n)$ with the property that every half-line in $\mathscr{S}^*(z_i)$ is disjoint from at least one half-line in $\mathscr{S}^*(z_j)$ $(i \neq j,$ $1 \leqq i \leqq n,\ 1 \leqq j \leqq n)$. We now show by induction that there are n disjoint half-lines l_i with $l_i \in \mathscr{S}^*(z_i)$ $(1 \leqq i \leqq n)$, which will complete the proof. The result is clear if $n = 2$, so assume proved up to $n-1$. Assume $z_n = 0$ and that $S^*(z_n)$ has one arm along the positive axis, the other arm L making an angle γ with the axis, where $0 < \gamma < \pi$. We may further suppose that there are k points $a_1 < a_2 < \cdots < a_k$ on the positive axis and j points $b_1, b_2, \cdots, b_j$ on $L(e^{-i\gamma}b_1 < e^{-i\gamma}b_2 < \cdots < e^{-i\gamma}b_j)$ at which the half-lines l_i first meet $S^*(z_n)$, the remaining half-lines, if any, being disjoint from $S^*(z_n)$. No half-line meets both arms of $S^*(z_n)$. If the half-line through a_1 makes angle α with the positive axis, then clearly $\alpha \leqq \beta$. Moreover the half-line through a_i makes angle $\leqq \alpha$, and the one through b_i makes angle $\geqq \beta$. Thus l_n may be chosen to be any half-line from 0 making an angle between α and β.

(5.6) We now prove (5.2). Since the boundary of D is an infinite closed set and since the plane is separable, we can find a sequence $\{z_n\}$ of distinct points such that the closure of the set $\{z_1, z_2, \cdots\}$ is the boundary of D. For each n we can find a finite number of half-lines which are mutually disjoint, lie in D^c and contain $z_1, z_2, \cdots, z_n$. Let D_n be the plane cut along these half-lines. Then D_n is a close-to-convex domain containing D. If z_0 is a fixed point of D, then it is easily seen that the density of the sequence $\{z_n\}$ on the boundary of D implies that the kernel relative to z_0 of the sequence $\{D_n\}$ is precisely D. In view of the compactness property of the family of close-to-convex functions, we deduce immediately that D is close-to-convex.

(5.7) We can now establish the sufficiency of Theorem 5.1. Assume the condition satisfied and that D is not close-to-convex. By (4.7) D is linearly accessible and therefore by (5.2) we can find two boundary points z_1 and z_2 of D such that (with the notation of (5.3)) every half-line in $\mathscr{S}(z_1)$ meets every half-line in $\mathscr{S}(z_2)$ at points other than z_1 or z_2. We may assume that z_1 and z_2 lie on the real axis, $z_1 < z_2$ and that the arms of $S(z_i)$ make angles α_i and β_i $(i = 1,2)$ with the positive axis, where $0 < \alpha_1 \leqq \beta_1 < \alpha_2 \leqq \beta_2 < \pi$. Choose ε satisfying $0 < \varepsilon < \frac{1}{2}(\alpha_2 - \beta_1)$, and consider the half-lines l_1 and l_2 from z_1 and z_2 respectively, making angles $\beta_1 + \varepsilon$ and $\alpha_2 - \varepsilon$ with the positive axis. There is a point $\zeta_1 \in l_1 \cap D$ and a point $\zeta_2 \in l_2 \cap D$, and moreover we must have $\arg(\zeta_1 - z_2) > \beta_2$ and $\arg(\zeta_2 - z_1) < \alpha_1$. By hypothesis there is a close-to-linear arc Γ lying in D and joining ζ_1 to ζ_2. We observe that the half-lines $l(\theta, z_2)$ meet Γ for $-\pi \leqq \theta \leqq \alpha_2 - \varepsilon$, and the half-lines $l(\theta, z_1)$ meet Γ for $\beta_1 + \varepsilon \leqq \theta \leqq 2\pi$. Thus any line $L(\theta)$ through z_1 containing $l(\theta, z_1)$ for $\beta_1 + \varepsilon \leqq \theta \leqq \pi$ meets Γ at two points separated along L by z_1, and any $L(\theta)$ through z_2 containing $l(\theta, z_2)$ for $0 \leqq \theta \leqq \alpha_2 - \varepsilon$ meets Γ at two points separated along L by z_2. Since $\alpha_2 - \varepsilon > \beta_1 + \varepsilon$, Γ cannot be close-to-linear. This contradiction completes the proof.

(5.8) We remark finally that the usefulness of the conditions (4.7) and (5.1) for linear accessibility and close-to-convexity is somewhat limited unless one can reduce the pairs of points to be considered to boundary points of the domain. Without going into details we can assert that with suitable modifications of the proofs one can establish that *a sufficient condition for a simply-connected domain D to be linearly accessible (close–to–convex) is that given any pair of accessible boundary points of D, one can find a cross–cut in the domain joining the points which is of linear type (close-to-linear).*

6. Given a linearly accessible domain D the question arises as to whether the condition (4.7) is necessary. In view of (4.5) this question would be affirmatively answered if one could establish the following *conjecture*:

Every linearly accessible domain is of convex type.

We conclude by mentioning briefly some results relevant to this problem which we have obtained concerning the class of *linearly accessible functions* (i.e. schlicht conformal mappings of the unit disc onto linearly accessible domains).

(6.1) *$f(z)$ is linearly accessible in $|z| < 1$, if, and only if, for each r, $0 < r < 1$, and for any point z_0 ($|z_0| < 1$), we have*

$$(6.1.1) \quad \theta_2 + \arg\left(\frac{f(re^{i\theta_2}) - f(z_0)}{re^{i\theta_2} - z_0}\right)^2 - \theta_1 - \arg\left(\frac{f(re^{i\theta_1}) - f(z_0)}{re^{i\theta_1} - z_0}\right)^2 > -2\pi$$

whenever $\theta_2 > \theta_1$.

This "intrinsic" characterisation can be geometrically interpreted in the case $|z_0| = r$ as stating that the chordal angle on $f(|z| = r)$ as measured from $f(z_0)$ going once round the curve in the positive direction does not turn back from a previous direction by more than an angle π, nor turn forwards by more than 2π.

(6.2) *$f(z)$ is linearly accessible in $|z| < 1$ if, and only if, for each fixed z_0 ($|z_0| < 1$) we can find a function $g(z)$ starlike of order $\frac{1}{2}$, so satisfying*

$$(6.2.1) \qquad \operatorname{Re}\frac{zg'(z)}{g(z)} > \frac{1}{2} \qquad (|z| < 1),$$

such that

$$(6.2.2) \qquad \operatorname{Re}\left\{\frac{f(z) - f(z_0)}{z - z_0} \bigg/ \frac{g(z)}{z}\right\} > 0 \qquad (|z| < 1).$$

What is lacking in this statement is any indication as to how the functions $g(z)$ may be related as z_0 varies. If it can be shown that $g(z)$ can be chosen to have the form

$$(6.2.3) \qquad g(z) = cz\,\frac{\phi(z) - \phi(z_0)}{z - z_0}$$

where $\phi(z)$ is convex (and the same $\phi(z)$ for all z_0) and c depends only on z_0, then we have the condition for $f(z)$ to be of convex type.

(6.3) *$f(z)$ is linearly accessible in $|z|<1$ if, and only if, for any convex function $\phi(z)$ and for any three points z_0, z_1 and z_2 in $|z|<1$, we have*

$$\left| \arg\frac{f(z_2)-f(z_0)}{\phi(z_2)-\phi(z_0)} - \arg\frac{f(z_1)-f(z_0)}{\phi(z_1)-\phi(z_0)}\right| < 2\pi. \tag{6.3.1}$$

(6.4) *Let $f(z) = \sum_{1}^{\infty} a_n z^n$ be linearly accessible in $|z|<1$. Then*

$$\left| \arg\frac{zf'(z)}{f(z)} \right| < \frac{3\pi}{2}. \tag{6.4.1}$$

For each $z_0(|z_0|<1)$ the function

$$h(z) = \int_0^z \frac{f(\zeta)-f(z_0)}{\zeta - z_0}d\zeta \tag{6.4.2}$$

is close-to-convex in $|z|<1$.

If $P_n(z) = \sum_{k=1}^{n} a_k z^k$ is the n^{th} partial sum, then

$$\left| 1 - \frac{P_n(z)}{f(z)} \right| \leqq (2n+1)|z|^n \tag{6.4.3}$$

for $n = 1, 2, \cdots$ and $|z|<1$. This gives in particular

$$|a_n| \leqq (2n+1)|f(z)| + (2n-1)\left|\frac{f(z)}{z}\right| \tag{6.4.4}$$

from which we deduce

$$|a_n| \leqq (2n-1)|a_1|; \qquad |a_n| \leqq 4dn \tag{6.4.5}$$

where d is the distance from 0 of the complement of the image of $|z|<1$. The results (6.4.1) and (6.4.2) are known for close-to-convex functions [**5**]. The second inequality in (6.4.5) can be deduced easily by subordination from the original definition of linear accessibility. The inequality (6.4.3) is certainly not sharp in the interior of $|z| < 1$, and we would expect the best estimate for the second member to be something like $(n+1)|z|^n + n|z|^{n+1}$ (the estimate for $z(1-z)^{-2}$).

Proofs of the results of this section and a detailed discussion of the conjecture will be presented in the near future.

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