THE HADAMARD PRODUCT AND LINEAR TRANSFORMATIONS OF CLASSES OF ANALYTIC FUNCTIONS[†]

By

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Introduction

Let $\mathscr{A}$ denote the space of functions analytic in $|z|<1$ given the usual topology of local uniform convergence. Toeplitz [17] showed that a continuous linear functional Λ on $\mathscr{A}$ can be represented by a function $g(\zeta)$ analytic in $|\zeta|\leqq 1$ as follows: for $f(z)=\Sigma_0^\infty a_n z^n \in \mathscr{A}$

$$\Lambda f=\sum_0^\infty c_n a_n=\left(\sum_0^\infty c_n a_n \zeta^n\right)_{\zeta=1}=(g(\zeta)*f(\zeta))_{\zeta=1}$$

$$=\lim_{\zeta\to 1}\frac{1}{2\pi}\int_0^{2\pi} f(\zeta e^{-i\theta})g(e^{i\theta})d\theta$$

where $g(\zeta)=\Sigma_0^\infty c_n\zeta^n$ and $*$ denotes the Hadamard product or convolution of two power series. In his duality method Ruscheweyh [11] has shown how Toeplitz's result can be used in a very effective way to establish the exact range of a linear functional on a class $C\subset\mathscr{A}$ in cases where C can be shown to lie in the second dual of a relatively simple family of functions. The information gained in this way is in general more precise than that obtained by constructing the extreme points of C. For example, one is able to study extremal problems concerning the ratio of two linear functionals. Furthermore in certain interesting cases the first dual family can be shown to be closed under convolution and this family then represents a structure preserving multiplier class. The ideas here have been in use for many years in connection with the theory of polynomials and can be traced back to Szego's interpretation of Grace's Theorem [7]. More recently the conjecture of Polya and Schoenberg [8] concerning the Hadamard product of two convex univalent functions was solved affirmatively essentially by showing that the class of convex functions formed a first dual space with the close-to-convex functions (more

[†] This work was completed during the academic year 1977/78 while the author was visiting at the University of Kentucky.

precisely their derivatives) lying in the second dual [13]. These results and methods have since been further extended [6, 12, 16].

In this paper we establish a new second dual theorem from which we shall deduce a general Convolution Theorem which extends the above-mentioned convolution theorems. As well as the already known results concerning functions starlike of various orders, we shall obtain new information concerning functions of bounded argument, functions of bounded boundary rotation, products of the form $\Pi_1^n(1+x_kz)^{\lambda_k}$ and various other classes of non-zero analytic functions in the disc. The methods we use will establish a link with the theory of polynomials and we shall give generalisations of Grace's Theorem to rational functions. As part of our argument we will show how Grace's Theorem is itself a very simple consequence of the Fundamental Theorem of Algebra and an ingenious Algebraic Convolution Lemma based on the properties of Moebius transformations and essentially due to Ruscheweyh [12]. In the final part of the paper we shall show how the Convolution Theorem can be used to study general linear transformations between the classes. As an application we generalise Polya and Schoenberg's original problem and give a general criterion for linear operators on $\mathscr{A}$ to preserve the property of convex univalence.

1. Preliminary definitions and statement of the main result

1.1. Suppose that $k(z)$ is analytic and $\neq 0$ in $|z|<1$ and that λ is real. We write $k \in \Pi_\lambda$ when

$$\text{Re}\,\frac{zk'(z)}{k(z)}\begin{cases} <\dfrac{\lambda}{2} & \text{if} \quad \lambda>0, \\ >\dfrac{\lambda}{2} & \text{if} \quad \lambda<0, \\ \equiv 0 & \text{if} \quad \lambda=0. \end{cases} \tag{1.1.1}$$

Finite products of the form

$$k(z)=c\prod_{k=1}^{n}(1+x_kz)^{\lambda_k} \tag{1.1.2}$$

where $|x_k|=1$, $c\neq 0$, $\Sigma_{k=1}^n \lambda_k=\lambda$, λ_k have the same sign (i.e. that of λ) are dense in Π_λ.

For $\alpha \geqq 0$, $\beta \geqq 0$ we write $f \in K(\alpha,\beta)$ if $f \in \mathscr{A}$ and can be written in the form

$$f(z)=k(z)H(z) \tag{1.1.3}$$

where $k \in \Pi_{\alpha-\beta}$ and $H \in \mathscr{A}$ is non-zero and satisfies $|\arg H(z)| \leqq \frac{1}{2}\pi \min(\alpha,\beta)$ in $|z|<1$. In particular $f \neq 0$. For example if $g \in \Pi_\alpha$, $h \in \Pi_\beta$ then $f=g/h \in K(\alpha,\beta)$.

The functions in $K(\alpha, \beta)$ form rather general classes of non-zero analytic functions and can be characterised in an "intrinsic" way by a Kaplan-type condition [5]. We give this condition as Theorem 2.2. We may ask to what extent the functions in $K(\alpha, \beta)$ are "dominated" by the relatively simple functions

$$\frac{(1+xz)^{\alpha}}{(1-yz)^{\beta}} \qquad (|x| \leqq 1, |y| \leqq 1) \tag{1.1.4}$$

for various types of extremal problems. Our basic theorem is concerned with this problem. If $\phi \in \mathcal{A}$ we write $\phi \in T(\alpha, \beta)$ if

$$\phi(z) * \frac{(1+xz)^{m}(1+uz)^{\gamma}}{(1-z)^{\beta}} \neq 0 \tag{1.1.5}$$

for $|x| = |u| = 1$, $|z| < 1$, where $m = [\alpha]$, $\gamma = \{\alpha\}$ (i.e. m is the largest integer not exceeding α and $m + \gamma = \alpha$).

1.2. The Duality Theorem. *Suppose that $\alpha \geqq 1$, $\beta \geqq 1$ and that $\phi \in T(\alpha, \beta)$. Then for $f \in K(\alpha, \beta)$,*

$$\phi(z) * f(z) \neq 0 \qquad (|z| < 1). \tag{1.2.1}$$

Assuming the functions in $K(\alpha, \beta)$ are normalised by the condition $f(0) = 1$, the theorem states in the terminology of Ruscheweyh [11] that $K(\alpha, \beta)$ lies in the second dual of the class of functions

$$\frac{(1+xz)^{m}(1+uz)^{\gamma}}{(1-yz)^{\beta}} \qquad (|x| \leqq 1, |u| \leqq 1, |y| \leqq 1) \tag{1.2.2}$$

and any continuous linear functional on $\mathcal{A}$ will have the same range on $K(\alpha, \beta)$ as on this smaller class. Among other things this implies that the class of functions (1.2.2) and $K(\alpha, \beta)$ have identical closed convex hulls.

From the Duality Theorem we will deduce the following more complete statement.

1.3. The Convolution Theorem. *Suppose that $\alpha \geqq 1$, $\beta \geqq 1$.*

(i) *If $\phi, \psi \in T(\alpha, \beta)$ then $\phi * \psi \in T(\alpha, \beta)$.*

(ii) *If $0 \leqq \varepsilon \leqq \min(\alpha, \beta)$, then for $\phi \in T(\alpha, \beta)$ and $f \in K(\alpha - \varepsilon, \beta - \varepsilon)$, $\phi * f \in K(\alpha - \varepsilon, \beta - \varepsilon)$.*

(iii) *If $\alpha \geqq \alpha' \geqq 1$, $\beta \geqq \beta' \geqq 1$, then $T(\alpha, \beta) \subset T(\alpha', \beta')$.*

(iv) *If $1 \leqq \lambda \leqq \min(\alpha, \beta)$ and $0 \leqq \delta \leqq \lambda$, then for $\phi \in T(\alpha, \beta)$, $h \in K(\alpha - \lambda, \beta - \lambda)$, $|\arg L| \leqq \delta\pi/2$,*

$$\left|\arg\left(\frac{\phi * hL}{\phi * h}\right)\right| \leqq \frac{\delta\pi}{2}. \tag{1.3.1}$$

We shall point out applications and special cases of this result in Sections 4 and 5. For now we observe that it extends very considerably the results established in proving the Polya–Schoenberg conjecture, which is the case $\alpha = 1$, $\beta = 3$ [13]. The cases $\alpha = 1$, $\beta \geqq 1$ were established by Suffridge [16], Lewis [6] and Ruscheweyh [12]. The methods of [13] and Ruscheweyh's extension of these methods [12] will play a central role in the proof. The constant interaction of a few simple techniques together with the fundamental nature of the result make it desirable to give a substantially complete proof.

2. Preparatory lemmas

In this section we develop the principle subsidiary results needed for the main proof which we give in Section 3.

2.1. Lemma. *Let $g(z_1, \cdots, z_n)$ be analytic in the polydisc $|z_k| < 1$ $(1 \leqq k \leqq n)$ and suppose that*

$$\phi(z) * g(zx_1, \cdots, zx_n) \neq 0 \tag{2.1.1}$$

for $|x_k| = 1$ $(1 \leqq k \leqq n)$ and $|z| < 1$. Then (2.1.1) *holds for $|x_k| \leqq 1$ $(1 \leqq k \leqq n)$ and $|z| < 1$.*

Proof. The case $n = 1$ is trivial. It clearly suffices to prove the case $n = 2$. This follows easily from the following two variable lemma: *if $f(z, \zeta)$ is analytic for $|z| < 1, |\zeta| < 1$ and $\neq 0$ whenever $|z| = |\zeta|$, then $f(z, \zeta) \neq 0$ for $|z| < 1, |\zeta| < 1$.* To prove this we note that $f(0,0) \neq 0$ so $f(z, \zeta) \neq 0$ for $|z| < r$, $|\zeta| < r$ and a suitable $r > 0$. Hence we can write

$$\frac{1}{f(z, \zeta)} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} c_{m,n} z^m \zeta^n \tag{2.1.2}$$

the series being absolutely convergent for $|z| < r$, $|\zeta| < r$. The conclusion will follow by analytic continuation if we show that this series is absolutely convergent for $|z| < 1$, $|\zeta| < 1$. By hypothesis we can write

$$\frac{1}{f(zx, z)} = \sum_{0}^{\infty} a_k(x) z^k \tag{2.1.3}$$

for $|x| = 1$, $|z| < 1$. Comparing this with (2.1.2) for $|z| < r$, we deduce

$$a_k(x) = \sum_{m+n=k} c_{m,n} x^m \qquad (|x| = 1)$$

for $k = 0, 1, 2, \cdots$. By Cauchy's formula

$$|a_k(x)| \leqq \frac{M(R)}{R^k} \qquad (0 < R < 1)$$

where $M(R) = \max_{|z|=|\zeta|=R} 1/|f(z,\zeta)|$. Hence

$$|c_{m,n}| \leqq \frac{M(R)}{R^k} \qquad (0 < R < 1,\ m + n = k).$$

We obtain for $0 < \rho < R < 1$ and $|z| < \rho$, $|\zeta| < \rho$

$$\begin{aligned}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} |c_{m,n}||z|^m|\zeta|^n &\leqq \sum_{k=0}^{\infty}\sum_{m+n=k} |c_{m,n}|\rho^k \\ &\leqq \sum_{k=0}^{\infty}(k+1)M(R)\left(\frac{\rho}{R}\right)^k \\ &< \infty.\end{aligned}$$

2.2. Theorem. *If f is analytic and non-zero in $|z| < 1$, then $f \in K(\alpha, \beta)$ if and only if, for $\theta_1 < \theta_2 < \theta_1 + 2\pi$ and $0 < r < 1$*

$$\begin{aligned}-\alpha\pi + \tfrac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) &\leqq \arg f(re^{i\theta_2}) - \arg f(re^{i\theta_1}) \\ &\leqq \tfrac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) + \beta\pi.\end{aligned} \tag{2.2.1}$$

The two inequalities are equivalent, i.e. each implies the other.

Proof. If $f = kH$ where $k \in \Pi_{\alpha-\beta}$ and $|\arg H| \leqq \frac{1}{2}\pi \min(\alpha,\beta)$, then

$$\Delta_{[\theta_1,\theta_2]} \arg k \left\{\begin{array}{ll} \leqq \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) & \text{if} \quad \alpha \geqq \beta \\ \geqq \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) & \text{if} \quad \alpha < \beta \end{array}\right.$$

and so

$$\Delta_{[\theta_1,\theta_2]} \arg f \left\{\begin{array}{ll} \leqq \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) + \beta\pi & \text{if} \quad \alpha \geqq \beta, \\ \geqq -\alpha\pi + \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) & \text{if} \quad \alpha < \beta. \end{array}\right.$$

Since $\arg f(re^{i\theta})$ is periodic with period 2π we obtain on applying this to the interval $[\theta_2, \theta_1 + 2\pi]$

$$\Delta_{[\theta_1,\theta_2]} \arg f = -\Delta_{[\theta_2,\theta_1+2\pi]} \arg f \begin{cases} \geqq \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1)-\alpha\pi & \text{if } \alpha \geqq \beta, \\ \leqq \beta\pi + \frac{1}{2}(\alpha-\beta)(\theta_2-\theta_1) & \text{if } \alpha < \beta. \end{cases}$$

Conversely, assume the condition holds. If $\alpha = \beta$ the result is clear. If $\alpha > \beta$, let $\mu = 2/(\alpha - \beta)$ and set $g(z) = zf^{-\mu}(z)$. We obtain for $\theta_2 > \theta_1$

$$\Delta_{[\theta_1,\theta_2]} \arg g \geqq -\mu\beta\pi.$$

Writing $t_r(\theta) = \inf_{\theta' \geqq \theta}(\arg g(re^{i\theta'})) + \mu\beta\pi/2$, we see that $t_r(\theta)$ is increasing, $t_r(\theta + 2\pi) = t_r(\theta) + 2\pi$ and

$$|t_r(\theta) - \arg g(re^{i\theta})| \leqq \frac{\mu\beta\pi}{2}.$$

By Kaplan's method [5] we find there is a function $h(z)$ starlike in $|z| < 1$ such that

$$|\arg h - \arg g| \leqq \frac{\mu\beta\pi}{2}$$

and hence $g = h/F$ where $F \in \mathscr{A}$ satisfies $|\arg F| \leqq \mu\beta\pi/2$. Thus $f = (z/g)^{1/\mu} = F^{1/\mu}(z/h)^{1/\mu}$ which is the required form. If $\alpha < \beta$, let $\mu = 2/(\beta - \alpha)$ and $g = zf^{\mu}$. We find $\Delta_{[\theta_1,\theta_2]} \arg g \geqq -\alpha\mu\pi$ and obtain h starlike such that $g = hF$ where $|\arg F| \leqq \mu\alpha\pi/2$. This gives $f = (h/z)^{1/\mu}F^{1/\mu}$, the required form.

2.3. From this result we easily deduce the following properties of the $K(\alpha, \beta)$ classes:

(i) $\alpha' \leqq \alpha,\ \beta' \leqq \beta \Rightarrow K(\alpha', \beta') \subset K(\alpha, \beta)$,

(ii) $\Pi_\lambda = \begin{cases} K(\lambda, 0) & \text{if } \lambda \geqq 0, \\ K(0, -\lambda) & \text{if } \lambda < 0, \end{cases}$

(iii) $f \in K(\alpha, \beta) \Leftrightarrow 1/f \in K(\beta, \alpha)$,

(iv) $f \in K(\alpha, \beta),\ g \in K(\alpha', \beta') \Rightarrow fg \in K(\alpha + \alpha', \beta + \beta')$,

(v) $f \in K(\alpha, \alpha) \Leftrightarrow$ for some μ real, $|\arg(e^{i\mu}f)| \leqq \alpha\pi/2$.

The next result [10, 13] is a reflection of the fact that a starlike domain is starlike in every direction. We give a geometrical proof.

2.4. Theorem. *Let $h(z)$ be starlike in $|z| < 1$ and let n be a natural number. Then for each ζ_1 ($|\zeta_1| = 1$) we can find $\zeta_2, \cdots, \zeta_n$ satisfying $|\zeta_k| = 1$ and μ real such that*

(2.4.1) $$\left|\arg\left(e^{i\mu}\frac{h(z)}{P(z)}\right)\right| < \frac{\pi}{n} \qquad (|z|<1)$$

where

(2.4.2) $$P(z) = z\prod_{k=1}^{n}(1-\zeta_k z)^{-2/n}.$$

Proof. The class of functions of the form (2.4.2) with fixed ζ_1 is clearly compact and hence without loss of generality we may assume that $h(z)$ is analytic and starlike in $|z|<1+\varepsilon$ for suitable $\varepsilon>0$. Let D denote the image domain $\{w = h(z): |z|<1\}$. The point $\bar{\zeta}_1$ is the inverse image of a point w_1 on ∂D. With $[0, w_1]$ as the first ray construct the n rays from the origin making equal angles $2\pi/n$, so that the rays meet ∂D at points $w_1, w_2, \cdots, w_n$ respectively. Let $\bar{\zeta}_2, \cdots, \bar{\zeta}_n$ be the inverse images on $|z|=1$ of $w_2, \cdots, w_n$ and define $P(z)$ by (2.4.2). Then P maps $|z|<1$ onto the plane cut along n radial slits making equal angles of $2\pi/n$, and $\arg P(z)$ is constant on the arcs $(\bar{\zeta}_k, \bar{\zeta}_{k+1})$ on $|z|=1$ with jumps of $2\pi/n$ at these points. The value of $\arg P$ at $\bar{\zeta}_k$ is exactly half-way into the jump. Hence we see that for any θ_1, θ_2 real

$$|(\arg P(e^{i\theta_2}) - \arg P(e^{i\theta_1})) - (\arg h(e^{i\theta_2}) - \arg h(e^{i\theta_1}))| \leqq 2\pi/n,$$

i.e. the difference in swing cannot exceed the angle $2\pi/n$. Hence for some real μ

$$|\arg P(e^{i\theta}) - \arg h(e^{i\theta}) - \mu| \leqq \pi/n$$

and the conclusion follows easily using Poisson's formula.

The following simple lemma is frequently applied in our theory [13].

2.5. Lemma. *Suppose ϕ, g are analytic in $|z|<1$ and that*

(2.5.1) $$\phi(z) * \frac{1+xz}{1-yz}\, g(z) \neq 0$$

for $|x| = |y| = 1$, $|z|<1$. Then if $F\in\mathcal{A}$ and $\operatorname{Re} F > 0$,

(2.5.2) $$\operatorname{Re}\frac{\phi * gF}{\phi * g} > 0.$$

In particular

(2.5.3) $$\phi * gF \neq 0.$$

Proof. When $x = -y$ we obtain $\phi * g \neq 0$. By Lemma 2.1 and simple manipulations we obtain for $|x| \leqq 1$, $|y| = 1$

$$\frac{\phi(z) * \dfrac{1+yz}{1-yz}\, g(z)}{\phi(z) * g(z)} \neq \frac{x\bar{y}-1}{x\bar{y}+1}$$

and hence (2.5.2) holds for $F(z) = (1+yz)/(1-yz)$. If $\operatorname{Re} F > 0$ we have by Herglotz's Theorem

$$F(z) = \int_T \frac{1+yz}{1-yz}\, d\mu(y) + ic$$

where μ is a positive measure on the unit circle T and c is a real constant. Hence

$$\operatorname{Re} \frac{\phi * gF}{\phi * g} = \int_T \operatorname{Re} \frac{\phi(z) * \dfrac{1+yz}{1-yz}\, g(z)}{\phi(z) * g(z)}\, d\mu > 0.$$

2.6. Corollary. *If $F \in \mathcal{A}$ takes all its values in a convex domain D, then the inequality* (2.5.1) *implies that*

$$\frac{\phi * gF}{\phi * g}$$

takes all its values in D.

The next result plays an important algebraic role. The proof is based on an ingenious idea due to Ruscheweyh [12].

2.7. Lemma. *Suppose that for given complex numbers $\alpha_1, \cdots, \alpha_n$ the class of functions $\phi \in \mathcal{A}$ and satisfying*

$$\text{(2.7.1)} \qquad \phi(z) * \prod_{k=1}^{n} (1 + x_k z)^{\alpha_k} \neq 0 \qquad (|x_k| = 1,\ 1 \leqq k \leqq n,\ |z| < 1)$$

has the property that all functions ϕ in the class also satisfy

$$\text{(2.7.2)} \qquad \phi(z) * \prod_{j=1}^{m} (1 + x_j z)^{\beta_j} \neq 0 \qquad (|x_j| = 1,\ 1 \leqq j \leqq m,\ |z| < 1).$$

Then it is also the case that all the functions ϕ satisfy

$$\phi(z) * (1+xz)^{\lambda} \prod_{j=1}^{m} (1+x_j z)^{\beta_j} \neq 0 \tag{2.7.3}$$

for $|x| \leqq 1$, $|x_j| \leqq 1$ $(1 \leqq j \leqq m)$ *and* $|z| < 1$, *where*

$$\lambda = \sum_1^n \alpha_k - \sum_1^m \beta_j .$$

Symbolically: ((2.7.1) $\Rightarrow$ (2.7.2)) $\Rightarrow$ (2.7.3).

Proof. We denote the three classes by $P(\alpha)$, $P(\beta)$ and $P(\lambda, \beta)$. By Lemma 2.1 the conditions (2.7.1) and (2.7.2) will hold with the parameters in the closed unit disc. Also by Hurwitz's Theorem it will be sufficient to show that (2.7.3) holds with the parameters in the open unit disc. Choose $|x| < 1$ and set

$$a(\tau) = \frac{\tau + x}{1 + \bar{x}\tau}, \qquad b(\tau) = \frac{\tau + \bar{x}}{1 + x\tau}$$

which give automorphisms of $|\tau| < 1$. Then

$$1 + a(\tau)z = \frac{1+xz}{1+\bar{x}\tau}(1 + b(z)\tau). \tag{2.7.4}$$

Thus for $|\zeta_k| \leqq 1$, $|w| \leqq 1$ and $\alpha = \Sigma_1^n \alpha_k$ we have

$$\prod_{k=1}^{n} (1 + a(\zeta_k w)z)^{\alpha_k} = (1+xz)^{\alpha} \prod_{k=1}^{n} \left(\frac{1 + b(z)\zeta_k w}{1 + \bar{x}\zeta_k w} \right)^{\alpha_k}$$

and hence if $\phi \in P(\alpha)$

$$\phi(z) * (1+xz)^{\alpha} \prod_{k=1}^{n} (1 + b(z)\zeta_k w)^{\alpha_k} \neq 0$$

for $|z| < 1$, $|w| \leqq 1$, $|\zeta_k| \leqq 1$. We write this as

$$\prod_{k=1}^{n} (1 + \zeta_k w)^{\alpha_k} *_w \left\{ \phi(z) *_z (1+xz)^{\alpha} \frac{1}{1 - b(z)w} \right\} \neq 0$$

and deduce that for each z $(|z| < 1)$ the function

$$w \to \phi(z) *_z (1+xz)^{\alpha} \frac{1}{1-b(z)w} \qquad (|w| < 1)$$

is in $P(\alpha)$. Therefore it is also in $P(\beta)$ and we obtain

$$\prod_{j=1}^{m} (1+\zeta_j w)^{\beta_j} *_w \left\{\phi(z) *_z (1+xz)^{\alpha} \frac{1}{1-b(z)w}\right\} \neq 0$$

for $|\zeta_j| \leqq 1$, $|w|<1$, $|z|<1$. This gives

$$\phi(z) *_z (1+xz)^{\lambda} \prod_{j=1}^{m} (1+za(\zeta_j w))^{\beta_j} \neq 0$$

and the conclusion follows.

As an application of this lemma we prove a unit disc version of Grace's Theorem for polynomials. In addition to other applications of the lemma Grace's Theorem will itself form part of our proof of the convolution theorem.

2.8. Grace's Theorem. *Let $P(z)$ be a polynomial of degree at most n such that*

(2.8.1) $$P(z) * (1+z)^n \neq 0 \qquad (|z|<1).$$

Then if $Q(z)$ is a polynomial of degree at most n with no zeros in $|z|<1$,

(2.8.2) $$P(z) * Q(z) \neq 0 \qquad (|z|<1).$$

Proof. By the Fundamental Theorem of Algebra we can write

$$P(z) * (1+z)^n = c \prod_{k=1}^{n} (1+z_k z)$$

where $c \neq 0$, $|z_k| \leqq 1$ $(1 \leqq k \leqq n)$. Hence

$$P(z) * 1 + z = c\left(1 + \frac{1}{n}\sum_{1}^{n} z_k z\right) \neq 0 \qquad (|z|<1).$$

Thus $P * (1+z)^n \neq 0 \Rightarrow P * 1 + z \neq 0$. By the lemma we deduce that

$$P(z) * (1+xz)^{n-1}(1+yz) \neq 0 \qquad (|x| \leqq 1, |y| \leqq 1, |z|<1)$$

and in particular $P(z) * (1+z)^{n-1} \neq 0$ (which is the Gauss–Lucas Theorem). Repeating the argument and re-applying the lemma we obtain inductively

$$P(z) * \prod_1^n (1 + x_k z) \neq 0 \qquad (|x_k| \leqq 1, |z| < 1).$$

The conclusion follows from the Fundamental Theorem of Algebra.

3. The main proof

In this section we give the proofs of Theorems 1.2 and 1.3. The proof of Theorem 1.2 consists of a chain of arguments which builds by proving successively more and more general cases. Theorem 1.3 is then deduced from Theorem 1.2.

3.1. We consider first the class $T(1, \beta)$ where $\beta \geqq 1$. Suppose that $\phi \in \mathscr{A}$ satisfies

$$\phi(z) * \frac{1 + xz}{(1-z)^\beta} \neq 0 \qquad (|x| = 1, |z| < 1). \tag{3.1.1}$$

Our first goal will be to show that for $|x| = |y| = 1$ and $g \in \Pi_{1-\beta}$,

$$\phi(z) * \frac{1 + xz}{1 - yz} g(z) \neq 0 \qquad (|z| < 1). \tag{3.1.2}$$

The initial crucial step is to show that (3.1.1) implies

$$\phi(z) * \frac{1 + xz}{(1-z)^{\beta-1}} \neq 0 \qquad (|x| = 1, |z| < 1). \tag{3.1.3}$$

Suppose for the moment that we have established this implication for every $\beta \geqq 1$. Then Lemma 2.7 gives us the first split:

$$\phi(z) * \frac{1 + xz}{(1 - y_1 z)(1 - y_2 z)^{\beta-1}} \neq 0 \tag{3.1.4}$$

for $|x| \leqq 1, |y_1| \leqq 1, |y_2| \leqq 1, |z| < 1$. Applying this lemma inductively we obtain

$$\phi(z) * \frac{1 + xz}{\left(\prod_1^n (1 - y_k z)\right)(1 - yz)^\gamma} \neq 0 \tag{3.1.5}$$

where $n = [\beta]$, $\gamma = \{\beta\}$, so that $n \geqq 1$ and $0 \leqq \gamma < 1$. Applying Lemma 2.5 we deduce that for $\operatorname{Re} F > 0$

$$\phi(z) * \frac{F(z)}{\left(\prod_1^{n-1} (1 - y_k z)\right)(1 - yz)^\gamma} \neq 0. \tag{3.1.6}$$

Thus to prove (3.1.2) it is sufficient to show that given $|x| = |y| = 1$ and $g \in \Pi_{1-\beta}$, we can find y_k, ε and c on the unit circle and $\operatorname{Re} F > 0$ such that

$$\frac{1+xz}{1-yz}\, g(z) = \frac{cF(z)}{\left(\prod_1^{n-1} (1-y_k z)\right)(1-\varepsilon z)^{\gamma}}\,.$$

Choose $\varepsilon = y$ and note that $h(z) = g(z)(1-yz)^{\gamma-1} \in \Pi_{-n}$. Thus we wish to show that given $|x| = 1$ and $h \in \Pi_{-n}$, we can find y_k and c such that

$$\operatorname{Re}\left\{\bar{c}(1+xz)h(z)\prod_1^{n-1}(1-y_k z)\right\} > 0.$$

This follows immediately from Theorem 2.4. From (3.1.2) and Lemma 2.5 we obtain for $g \in \Pi_{1-\beta}$ and $\operatorname{Re} F > 0$,

$$\phi(z) * g(z)F(z) \neq 0 \qquad (|z| < 1) \tag{3.1.7}$$

and this gives Theorem 1.2 in the case $\alpha = 1$.

3.2. To complete this part of the chain of reasoning it remains to establish the first link (3.1.1) $\Rightarrow$ (3.1.3). The case $\beta = 1$ is clear. Assume $\beta > 1$. Then the implication is equivalent to

$$\left|\frac{\phi(z) * z(1-z)^{-\beta}}{\phi(z) * (1-z)^{-\beta}}\right| < 1 \qquad (|z| < 1)$$

$$\Rightarrow \left|\frac{\phi(z) * z(1-z)^{1-\beta}}{\phi(z) * (1-z)^{1-\beta}}\right| < 1 \qquad (|z| < 1).$$

Let

$$\omega(z) = \frac{\phi(z) * z(1-z)^{1-\beta}}{\phi(z) * (1-z)^{1-\beta}}\,.$$

Then ω is meromorphic in $|z| < 1$ and $\omega(0) = 0$. We wish to show that $|\omega(z)| < 1$ ($|z| < 1$). If this is not the case, then by the Clunie–Jack Lemma [4] we can find ζ satisfying $|\zeta| < 1$ such that

$$|\omega(\zeta)| = 1, \qquad \zeta\omega'(\zeta) = k\omega(\zeta)$$

where $k \geqq 1$. Simple algebraic manipulations, which we shall omit, give the relation

$$\frac{\phi(\zeta)*\zeta(1-\zeta)^{-\beta}}{\phi(\zeta)*(1-\zeta)^{-\beta}}=\frac{(k+\beta-2)\omega(\zeta)}{\beta-1+(k-1)\omega(\zeta)}$$

provided $\beta-1+(k-1)\omega(\zeta)\neq 0$. Hence by hypothesis

$$|(k-1)+(\beta-1)|<|\beta-1+(k-1)\omega(\zeta)|$$
$$\leqq|\beta-1|+|k-1|.$$

Since $k\geqq 1$, $\beta>1$, this is clearly false. If $\beta-1+(k-1)\omega(\zeta)=0$, then also $k+\beta-2=0$, which implies $\beta\leqq 1$, a contradiction.

3.3. We now consider the class $T(m,\beta)$ where m is a natural number and $\beta\geqq 1$. Suppose that ϕ satisfies

(3.3.1) $$\phi(z)*\frac{(1+xz)^m}{(1-z)^\beta}\neq 0 \qquad (|x|=1, |z|<1).$$

By Lemma 2.1 this holds for $|x|\leqq 1$. It can be re-written

(3.3.2) $$(1+x)^m *_x \left(\phi(z) *_z \frac{1+xz+\cdots+x^m z^m}{(1-z)^\beta}\right)\neq 0$$

for $|x|\leqq 1$, $|z|<1$. Hence by Grace's Theorem

(3.3.3) $$P(x) *_x \left(\phi(z) *_z \frac{1+xz+\cdots+x^m z^m}{(1-z)^\beta}\right)\neq 0$$

for any polynomial $P\neq 0$ in $|x|<1$ and of degree $\leqq m$, the inequality certainly being valid for $|x|<1$, $|z|<1$. We obtain

(3.3.4) $$\phi(z)*\frac{P(xz)}{(1-z)^\beta}\neq 0$$

and by Hurwitz's Theorem this is valid for $|x|\leqq 1$, $|z|<1$. In particular we deduce that

(3.3.5) $$\phi(z)*\frac{1+xz}{(1-z)^\beta}\neq 0 \qquad (|x|\leqq 1, |z|<1),$$

and so $\phi\in T(1,\beta)$. Thus if $\{\beta_k\}_1^N$ are positive numbers whose sum is $\beta-1$ we obtain from (3.1.2)

$$\phi(z) * \frac{1+xz}{1-yz} \prod_1^N (1-y_k z)^{-\beta_k} \neq 0 \qquad (|x| \leqq 1, |y| \leqq 1, |y_k| \leqq 1, |z|<1). \tag{3.3.6}$$

Applying Lemma 2.7 to the implication (3.3.1) $\Rightarrow$ (3.3.6) we obtain

$$\phi(z) * \frac{(1+x_1 z)(1+x_2 z)^{m-1}}{(1-yz)\prod_1^N (1-y_k z)^{\beta_k}} \neq 0 \tag{3.3.7}$$

and since we may choose $x_1 = x_2 = x$ we may again apply Grace's Theorem to obtain

$$\phi(z) * \frac{1+xz}{1-yz} \frac{\prod_{j=1}^{m-1}(1+x_j z)}{\prod_{k=1}^N (1-y_k z)^{\beta_k}} \neq 0. \tag{3.3.8}$$

Applying Hurwitz's Theorem and Lemma 2.5 we deduce that for $g \in \Pi_{1-\beta}$ and $\operatorname{Re} F > 0$,

$$\phi(z) * g(z) F(z) \prod_1^{m-1} (1+x_j z) \neq 0. \tag{3.3.9}$$

We will deduce from this that for $g \in \Pi_{1-\beta}$, $h \in \Pi_{m-1}$, $\operatorname{Re} F > 0$,

$$\phi(z) * g(z) h(z) F(z) \neq 0 \qquad (|z|<1). \tag{3.3.10}$$

This will follow from Lemma 2.5 and (3.3.9) if we can show that for $|x| = |y| = 1$,

$$\frac{1+xz}{1-yz} gh = gG \prod_1^{m-1} (1+x_j z)$$

for suitable x_j on the unit circle and G satisfying for a real μ, $\operatorname{Re}(e^{i\mu} G) > 0$. We may assume $m > 1$, so we require

$$\left(\frac{1}{h}\right)^{2/(m-1)} = \left(\frac{1}{G}\right)^{2/(m-1)} \left(\frac{1+xz}{(1-yz)\prod_1^{m-1}(1+x_k z)}\right)^{2/(m-1)}.$$

Choose $x_1 = x$. Since $z(1/h)^{2/(m-1)}$ is starlike, the conclusion follows from Theorem 2.4.

3.4. We now complete the proof of Theorem 1.2 in the case $\alpha = m$. Consider first the case $\beta \geqq m$. Then by (3.3.10) we have for $g \in \Pi_{m-\beta}$, (3.4.1) $$\phi(z)*g(z)\prod_{k=1}^{m}\frac{1+x_kz}{1-y_kz}\neq 0.$$

applying Lemma 2.5 m times we obtain for $|\arg H|\leqq m\pi/2$,

(3.4.2) $$\phi * gH\neq 0,$$

the desired result. Next suppose that $\beta < m$. Let $n=[\beta]$, $\gamma=\{\beta\}$. Then by (3.3.10) we have for $u\in\Pi_{-\gamma}$, $v\in\Pi_{m-n}$,

(3.4.3) $$\phi(z)*u(z)v(z)\prod_{1}^{n}\frac{1+x_kz}{1-y_kz}\neq 0.$$

Applying Lemma 2.5 n times we obtain for $|\arg L|\leqq n\pi/2$,

(3.4.4) $$\phi * Luv\neq 0.$$

Let $g\in\Pi_{m-\beta}$ and $|\arg G|\leqq(\beta-1)\pi/2$. Then

$$\frac{1+xz}{1-yz}Gg=\left(\frac{1+xz}{1-yz}\right)^{1-\gamma}G(1-yz)^{-\gamma}(1+xz)^{\gamma}g=Luv$$

where

$$L=\left(\frac{1+xz}{1-yz}\right)^{1-\gamma}G,$$

so $|\arg(e^{i\mu}L)|\leqq n\pi/2$, $u=(1-yz)^{-\gamma}\in\Pi_{-\gamma}$, $v=(1+xz)^{\gamma}g\in\Pi_{m-n}$. Thus by (3.4.4)

(3.4.5) $$\phi*\frac{1+xz}{1-yz}Gg\neq 0.$$

Hence by Lemma 2.5 we have for $|\arg H|\leqq\beta\pi/2$, $g\in\Pi_{m-\beta}$

(3.4.6) $$\phi * gH\neq 0,$$

the desired result.

3.5. We now complete the proof of Theorem 1.2. Suppose that $\alpha\geqq 1$, $\beta\geqq 1$ and let $m=[\alpha]$, $\gamma=\{\alpha\}$. If $\phi\in T(\alpha,\beta)$, then

(3.5.1) $$\phi(z)*\frac{(1+xz)^m(1+uz)^{\gamma}}{(1-z)^{\beta}}\neq 0$$

for $|x| \leqq 1$, $|u| \leqq 1$, $|z| < 1$. Hence for $|w| < 1$, $|x| \leqq 1$, $|y| \leqq 1$, $|u| \leqq 1$ and $|z| < 1$

$$\frac{(1+xw)^m}{(1-yw)^\beta} *_w \left\{\phi(z) *_z \frac{(1+uz)^\gamma}{1-wz}\right\} = \phi(z) *_z \frac{(1+uz)^\gamma(1+xwz)^m}{(1-ywz)^\beta} \neq 0,$$

and so the function

$$w \to \phi(z) *_z \frac{(1+uz)^\gamma}{1-wz} \in T(m,\beta)$$

for each fixed z ($|z| < 1$). Hence for $f \in K(m,\beta)$

(3.5.2) $$\phi(z) * (1+uz)^\gamma f(wz) \neq 0$$

for $|u| \leqq 1$, $|w| \leqq 1$, $|z| < 1$ by Hurwitz's Theorem. Let $g \in \Pi_\gamma$ and consider

(3.5.3) $$\phi(z) * \frac{(1+xz)^m}{(1-yz)^\beta} g(z).$$

This will be non-zero by (3.5.2) provided that we can find $|u| \leqq 1$ and $f \in K(m,\beta)$ such that

$$\frac{(1+xz)^m}{(1-yz)^\beta} g(z) = (1+uz)^\gamma f(z).$$

Choose $u = x$. Then $(1+xz)^{m-\gamma}(1-yz)^{-\beta} g(z) \in K(m,\beta)$ by 2.3. Thus (3.5.3) is non-zero and in the same way that we proved the implication (3.5.1) $\Rightarrow$ (3.5.2) we obtain for $g \in \Pi_\gamma$ and $f \in K(m,\beta)$,

(3.5.4) $$\phi * gf \neq 0.$$

Thus to complete the proof of Theorem 1.2 it is sufficient to show that if $k \in \Pi_{\alpha-\beta}$, $|\arg G| \leqq \frac{1}{2}\pi(\min(\alpha,\beta)-1)$ and $|x| = |y| = 1$, we can write

(3.5.5) $$\frac{1+xz}{1-yz} kG = ghL$$

where $g \in \Pi_\gamma$, $h \in \Pi_{m-\beta}$, $|\arg L| \leqq \frac{1}{2}\pi \min(m,\beta)$, for the conclusion will then follow from (3.5.4) and Lemma 2.5. We consider three cases:

(i) $\beta \leqq m$. Choose

$$g = k^{\gamma/(\alpha-\beta)}, \quad h = k^{(m-\beta)/(\alpha-\beta)}, \quad L = \frac{1+xz}{1-yz} G.$$

(ii) $m < \beta < m + \gamma$. Then $|\arg G| \leqq (\beta - 1)\pi/2$. Choose

$$g = (1 + xz)^{\beta - m}k, \quad h = (1 - yz)^{m - \beta}, \quad L = \left(\frac{1 + xz}{1 - yz}\right)^{m - \beta + 1} G.$$

(iii) $m + \gamma \leqq \beta$. Then $|\arg G| \leqq (m + \gamma - 1)\pi/2$. Choose

$$g = (1 + xz)^{\gamma}, \quad h = (1 - yz)^{-\gamma}k, \quad L = \left(\frac{1 + xz}{1 - yz}\right)^{1 - \gamma} G.$$

The proof is complete.

3.6. Proof of Theorem 1.3. Let $\rho = \min(\alpha, \beta)$. If $|x| = |y| = 1$, $g \in \Pi_{\alpha - \beta}$ and $|\arg G| \leqq (\rho - 1)\pi/2$, then

$$\frac{1 + xz}{1 - yz}\, gG \in K(\alpha, \beta),$$

and hence by Theorem 1.2 and Lemma 2.5 we have for $\phi \in T(\alpha, \beta)$ and $\operatorname{Re} F > 0$,

$$\operatorname{Re} \frac{\phi * gGF}{\phi * gG} > 0. \tag{3.6.1}$$

Suppose that $h \in K(\alpha - \lambda, \beta - \lambda)$, $|\arg L| \leqq \delta\pi/2$ where $1 \leqq \lambda \leqq \rho$ and $0 \leqq \delta \leqq \lambda$. Then $h = gM$ where $g \in \Pi_{\alpha - \beta}$ and $|\arg M| \leqq (\rho - \lambda)\pi/2$. Let $k = [\delta]$, $\sigma = \{\delta\}$. If $k = 0$, then $0 \leqq \delta < 1$ and $|\arg M| \leqq (\rho - \lambda)\pi/2 \leqq (\rho - 1)\pi/2$, so by (3.6.1) and Corollary 2.6

$$\left| \arg \left(\frac{\phi * gML}{\phi * gM}\right) \right| \leqq \frac{\delta\pi}{2}$$

since $|\arg w| \leqq \delta\pi/2$ is a convex region. This gives (1.3.1) in the case $k = 0$. If $k \geqq 1$, we write $L = L_1 \cdots L_k R$ where $\operatorname{Re} L_i > 0$ $(1 \leqq i \leqq k)$ and $|\arg R| \leqq \sigma\pi/2$. Then iterating (3.6.1) we obtain

$$\phi * gML_1 \cdots L_k R = (\phi * gMR)\tilde{L}_1 \cdots \tilde{L}_k$$

where $\operatorname{Re} \tilde{L}_i > 0$ $(1 \leqq i \leqq k)$. Also by the case $k = 0$, $\phi * gMR = (\phi * gM)\tilde{R}$ where $|\arg \tilde{R}| \leqq \sigma\pi/2$. Thus $\phi * hL = (\phi * h)\tilde{L}$ where $|\arg \tilde{L}| \leqq \delta\pi/2$, which proves (1.3.1). In the case $\lambda = \rho$ we deduce that for $g \in \Pi_{\alpha - \beta}$ and $|\arg H| \leqq \varepsilon\pi/2$, where $0 \leqq \varepsilon \leqq \rho$, (3.6.2) $$\left|\arg\left(\frac{\phi * gH}{\phi * g}\right)\right| \leqq \frac{\varepsilon\pi}{2}.$$

In particular putting $\varepsilon = 1$ we have for $\operatorname{Re} F > 0$

(3.6.3) $$\operatorname{Re}\frac{\phi * gF}{\phi * g} > 0.$$

Applying Corollary 2.6 we see that

$$\operatorname{Re}\frac{\phi * zg'}{\phi * g} = \operatorname{Re}\frac{\phi * g\frac{zg'}{g}}{\phi * g}\begin{cases} <\frac{1}{2}(\alpha-\beta) & \text{if} \quad \alpha > \beta \\ >\frac{1}{2}(\alpha-\beta) & \text{if} \quad \alpha < \beta \\ \equiv 0 & \text{if} \quad \alpha = \beta \end{cases}$$

and so $\phi * g \in \Pi_{\alpha-\beta}$. Hence if $0 \leqq \varepsilon \leqq \rho$ and $f \in K(\alpha-\varepsilon, \beta-\varepsilon)$ we can write $f = gH$ with $g \in \Pi_{\alpha-\beta}$ and $|\arg H| \leqq (\rho-\varepsilon)\pi/2$, and so by (3.6.2),

$$\phi * f = (\phi * g)\tilde{H} = \tilde{g}\tilde{H}$$

where $\tilde{g} \in \Pi_{\alpha-\beta}$ and $|\arg \tilde{H}| \leqq (\rho-\varepsilon)\pi/2$. Thus $\phi * f \in K(\alpha-\varepsilon, \beta-\varepsilon)$ which proves (ii). To prove (i) we note that if $|x| = |u| = 1$, the function $(1+xz)^m(1+uz)^\gamma(1-z)^{-\beta} \in K(\alpha, \beta)$, and hence by (ii) for $\psi \in T(\alpha, \beta)$,

(3.6.4) $$\psi(z) * \frac{(1+xz)^m(1+uz)^\gamma}{(1-z)^\beta} \in K(\alpha, \beta).$$

Hence for $\phi \in T(\alpha, \beta)$ the convolution of ϕ with (3.6.4) is non-zero by Theorem 1.2, and thus $\phi * \psi \in T(\alpha, \beta)$. Finally to prove (iii) we note that by (2.3)(i), $K(\alpha', \beta') \subset K(\alpha, \beta)$. Also if $|x| = |u| = 1$

$$\frac{(1+xz)^{[\alpha']}(1+uz)^{\{\alpha'\}}}{(1-z)^{\beta'}} \in K(\alpha', \beta')$$

and so is in $K(\alpha, \beta)$. Hence if $\phi \in T(\alpha, \beta)$ its convolution with this function is non-zero by Theorem 1.2, and hence $\phi \in T(\alpha', \beta')$.

4. The classes $K(\alpha, \beta)$

The convolution theorem provides us with information which appears to be unavailable on the basis of extreme point theory.

4.1. Theorem. *Let Λ_1, Λ_2 be continuous linear functionals on $\mathcal{A}$ and suppose that*

$$\Lambda_2\left(\frac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\right)\neq 0$$

for $|x|\leqq 1$, $|u|\leqq 1$, $|y|\leqq 1$, *where* m *is a natural number,* $0\leqq\gamma<1$, $\beta\geqq 1$. *Let* $f(z)=1+\sum_1^\infty a_nz^n\in K(m+\gamma,\beta)$. *Then for suitable* $|x|\leqq 1$, $|u|\leqq 1$, $|y|\leqq 1$,

$$\frac{\Lambda_1 f}{\Lambda_2 f}=\frac{\Lambda_1\left(\dfrac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\right)}{\Lambda_2\left(\dfrac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\right)}. \tag{4.1.1}$$

In particular for suitable x, u, y

$$\Lambda_1 f=\Lambda_1\left(\frac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\right). \tag{4.1.2}$$

This is immediate from Theorem 1.2 and Ruscheweyh's Duality Theorem [11]. Another easy consequence of the Duality Principle concerns linear operators on $\mathscr{A}$. Let M be a compact space of analytic functions $g\in\mathscr{A}$ and let M^* denote the "dual" of M:

$$M^*=\{f\in\mathscr{A}: f(0)=1,\ g*f\neq 0,\ \text{for } g\in M\}.$$

4.2. Theorem. *Let* Λ *be a continuous linear operator on* $\mathscr{A}$ *and suppose that*

$$\Lambda\left(\frac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\right)\in M^*$$

for $|x|\leqq 1$, $|u|\leqq 1$, $|y|\leqq 1$, *where* m *is a natural number,* $0\leqq\gamma<1$, $\beta\geqq 1$. *Then if* $f(0)=1$ *and* $f\in K(m+\gamma,\beta)$, $\Lambda f\in M^*$.

Proof. Choose $h\in M$ and $|z|<1$. For $f\in\mathscr{A}$ let

$$\lambda f=(h*\Lambda f)(z),$$

so λ is a continuous linear functional on $\mathscr{A}$. If $f\in K(m+\gamma,\beta)$ then for suitable x, u, y

$$(h*\Lambda f)(z)=\left(h*\Lambda\frac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta}\right)\neq 0.$$

Also $\Lambda f(0)$ is a continuous linear functional on $\mathscr{A}$ so for $f\in K(m+\gamma,\beta)$ with $f(0)=1$, $\Lambda f(0)=1$. Thus $\Lambda f\in M^*$.

Deeper questions concerning linear mappings will be discussed in Section 5. However reducing such questions to consideration of relatively simple members of the given classes will remain a constant theme. We apply Theorem 4.1 to the coefficient problem for $K(\alpha,\beta)$. We make use of the now standard notation $f \ll g$ [2].

4.3. Theorem. *Let* $f(z) = 1 + \Sigma_1^\infty a_n z^n \in K(\alpha,\beta)$ *where* $\alpha \geqq 1$, $\beta \geqq 1$. *If* $\beta \geqq \min(1+\{\alpha\}, 2-\{\alpha\})$ *then*

$$f(z) \ll \frac{(1+z)^\alpha}{(1-z)^\beta}. \tag{4.3.1}$$

In particular this holds for $\alpha \geqq 1$, $\beta \geqq \frac{3}{2}$.

Proof. Write $\alpha = m + \gamma$ where m is a natural number and $0 \leqq \gamma < 1$. By Theorem 4.1 the extremal function for any coefficient must lie among the functions

$$\frac{(1+xz)^m(1+uz)^\gamma}{(1-yz)^\beta} \qquad (|x| \leqq 1, |u| \leqq 1, |y| \leqq 1). \tag{4.3.2}$$

If $\beta \geqq 1 + \gamma$ we write this as

$$(1+xz)^{m-1}\frac{(1+xz)(1+uz)^\gamma}{(1-yz)^{1+\gamma}}\frac{1}{(1-yz)^{\beta-1-\gamma}}.$$

Clearly $(1+xz)^{m-1} \ll (1+z)^{m-1}$, $(1-yz)^{-\beta+1+\gamma} \ll (1-z)^{-\beta+1+\gamma}$. Also for a suitable real μ

$$\left|\arg\left(e^{i\mu}\frac{(1+xz)(1+uz)^\gamma}{(1-yz)^{1+\gamma}}\right)\right| \leqq (1+\gamma)\frac{\pi}{2}$$

and hence as shown by Brannan [1]

$$\frac{(1+xz)(1+uz)^\gamma}{(1-yz)^{1+\gamma}} \ll \left(\frac{1+z}{1-z}\right)^{1+\gamma}$$

and the conclusion is clear. If $\beta \geqq 2-\gamma$, we note that by Brannan's theorem [1]

$$\frac{(1+uz)^\gamma}{(1-yz)^\beta} \ll \frac{(1+z)^\gamma}{(1-z)^\beta}$$

and again the conclusion is clear.

4.4. Remark. It is natural to conjecture that the conclusion holds for every $\alpha \geqq 1$, $\beta \geqq 1$. The crucial case is clearly $m = 1$, $\beta = 1$. A more general question concerns the extreme points of $K(\alpha, \beta)$. Are these the functions $(1+xz)^{\alpha}(1-yz)^{-\beta}$ where $|x| = |y| = 1$, $x \neq -y$? If true, Brannan's result would give the coefficient conjecture.

4.5. Various well-known classes are contained in the $K(\alpha, \beta)$ classes. A function f is close-to-convex of order β [2] if, and only if, $f' \in K(\beta, \beta+2)$. These classes contain the functions of bounded boundary rotation at most $2\pi(\beta+1)$ [2].

For $\lambda < 0$ the class $\Pi_\lambda = K(0, -\lambda)$ is directly related to the class $S^*_{1+\frac{1}{2}\lambda}$ of functions starlike of order $1+\frac{1}{2}\lambda$. $k \in \Pi_\lambda$ if, and only if, $zk \in S^*_{1+\frac{1}{2}\lambda}$. Note that the functions in $T(1, 1-\lambda)$ preserve Π_λ under convolution, a result first proved for $\lambda = -1, -2$ in [13] and for general $\lambda < 0$ in [16]. Apart from the extensive literature on polynomials I am not aware of any detailed study of Π_λ for $\lambda > 0$. Note that for $\lambda > 0$ the functions in $T(\lambda+1, 1)$ preserve the structure of Π_λ under convolution.

It seems well worth giving a separate statement of the Convolution Theorem for the case $\alpha = \beta$.

4.6. Theorem. *Let $\alpha \geqq 1$ and suppose that $\phi \in T(\alpha, \alpha)$ and $\phi(0) = 1$. Let h and $L \in \mathcal{A}$ and satisfy*

$$|\arg h| \leqq (\alpha - \lambda)\frac{\pi}{2}, \qquad |\arg L| \leqq \frac{\delta\pi}{2}, \tag{4.6.1}$$

where $1 \leqq \lambda \leqq \alpha$, $0 \leqq \delta \leqq \lambda$. Then

$$\left|\arg \frac{\phi * hL}{\phi * h}\right| \leqq \frac{\delta\pi}{2}. \tag{4.6.2}$$

In particular if $H \in \mathcal{A}$ satisfies $|\arg H| \leqq \alpha\pi/2$, then

$$|\arg(\phi * H)| \leqq \alpha\pi/2. \tag{4.6.3}$$

The case $\alpha = 1$ is very well-known. The case $\alpha = 2$ has a geometrical interpretation. A domain D in the plane is called *linearly accessible* if its complement D^c can be written as a union of half-lines (equivalently ∞ is visible from every boundary point of D). Every close-to-convex domain is linearly accessible. We refer to [14, 15] for a detailed discussion of the concept.

4.7. Theorem. *Suppose $f \in \mathcal{A}$ takes all its values in a linearly accessible domain D. If $\phi \in \mathcal{A}$ satisfies $\phi(0) = 1$ and*

(4.7.1) $$\phi(z) * \left(\frac{1+xz}{1-z}\right)^2 \neq 0 \qquad (|x|=1, |z|<1),$$

then $\phi * f(z)$ takes all its values in D.

Proof. Suppose that $w_0 \notin D$, so there is a half-line l with end-point w_0 such that $l \subset D^c$. Since f takes no values on l, for some real μ,

$$|\arg(e^{i\mu}(f(z)-w_0))| < \pi \qquad (|z|<1).$$

Hence by (4.6.3)

$$\phi(z) * e^{i\mu}(f(z)-w_0) \neq 0$$

and so $\phi(z) * f(z) \neq w_0$.

4.8. Remark. Since the functions $(1+xz)^2(1-z)^{-2}$ ($|x|=1$, $x \neq -1$) are mappings of the disc onto the plane cut along a ray from the origin, the condition (4.7.1) is both necessary and sufficient for this "linearly accessible preserving" property under convolution. The class of such functions is closed under convolution. Writing the condition as a quadratic non-zero condition in the variable x whose coefficients are functions of z, the condition on ϕ can be transformed into a direct inequality relating ϕ and $z\phi'$: we make use of the fact that $ax^2+bx+c \neq 0$ in $|x| \leqq 1$ if, and only if,

(4.8.1) $$|a\bar{b} - b\bar{c}| < |c|^2 - |a|^2.$$

Here we have

$$a = \phi * \frac{z^2}{(1-z)^2} = z\phi' - \phi + 1,$$

$$b = \phi * \frac{2z}{(1-z)^2} = 2z\phi',$$

$$c = \phi * \frac{1}{(1-z)^2} = \phi + z\phi'.$$

After some algebra we obtain the inequality

(4.8.2) $$4(\operatorname{Im} z\phi')^2 < \operatorname{Re}(2\phi - 1)\operatorname{Re}\{(1+4z\phi')(2\bar{\phi}-1)\}.$$

As further applications of the Convolution Theorem we establish two generalisations of Grace's Theorem to rational functions.

4.9. Theorem. *Let $R(z)=P(z)/Q(z)$ be a rational function with no zeros or poles in $|z|<1$, and suppose that*

$$\deg P \leqq m, \qquad \deg Q \leqq n$$

where $m \geqq 1$ and $n \geqq 0$. Let Λ_1, Λ_2 be continuous linear functionals on $\mathcal{A}$ such that

$$\Lambda_2\left(\frac{(1+xz)^m}{(1-yz)^n}\right) \neq 0 \qquad (|x|\leqq 1,\ |y|\leqq 1). \tag{4.9.1}$$

Then for suitable $|x|\leqq 1$, $|y|\leqq 1$,

$$\frac{\Lambda_1 R}{\Lambda_2 R} = \frac{\Lambda_1\left(\dfrac{(1+xz)^m}{(1-yz)^n}\right)}{\Lambda_2\left(\dfrac{(1+xz)^m}{(1-yz)^n}\right)}. \tag{4.9.2}$$

In particular, if $R(0)=1$, for suitable $|x|\leqq 1$, $|y|\leqq 1$,

$$\Lambda_1 R = \Lambda_1\left(\frac{(1+xz)^m}{(1-yz)^n}\right). \tag{4.9.3}$$

Proof. By the Fundamental Theorem of Algebra $R \in K(m,n)$ and the result is immediate from Theorem 4.1 for $m \geqq 1$, $n \geqq 1$. If $n=0$, the conclusion is a consequence of Grace's Theorem.

4.10. Remark. The conclusion is false in the case $m=0$. For example take $m=0$, $n=2$ and let $R(z)=(1-z)^{-1}$. If the theorem were true in this case, then

$$\phi * \frac{1}{1-z} = 0 \Rightarrow \phi * \frac{1}{(1-xz)^2} = 0$$

for some $|x|\leqq 1$. Equivalently

$$\phi * \frac{1}{(1-z)^2} \neq 0 \quad \text{in} \quad |z|<1$$

$$\Rightarrow \phi * \frac{1}{1-z} \neq 0 \qquad \text{in} \quad |z|<1.$$

But by Lemma 2.7 this implication gives

$$\phi * \frac{1}{(1-xz)(1-yz)} \neq 0 \qquad (|x| \leqq 1, |y| \leqq 1, |z| < 1).$$

Writing $\psi = z\phi$ the assertion is equivalent to

$$\psi'(z) \neq 0 \quad \text{in} \quad |z| < 1 \Rightarrow \psi(z) \text{ schlicht in } |z| < 1,$$

which is of course false.

4.11. Theorem. *Let $R(z) = P(z)/Q(z)$ be a rational function with no poles in $|z| < 1$ and let*

$$m = \deg P, \qquad n = \deg Q, \qquad N = \max(m, n).$$

Suppose that $\phi \in T(N, n)$ with $\phi(0) = 1$. Then

(4.11.1) $$(\phi * R)(|z| < 1) \subset R(|z| < 1).$$

Proof. Suppose that $R(z) \neq w$ in $|z| < 1$. Then

$$\frac{P - wQ}{Q} \neq 0$$

and by the Fundamental Theorem of Algebra,

$$\frac{P - wQ}{Q} \in K(N, n).$$

Hence by the Convolution Theorem for $n \geqq 1$ and Grace's Theorem for $n = 0$, if $\phi \in T(N, n)$ and $\phi(0) = 1$,

$$\phi(z) * \frac{P - wQ}{Q} \neq 0$$

and so $\phi * R \neq w$. The conclusion follows.

4.12. Remark. In particular every rational function without poles in $|z| < 1$ has non-trivial image preserving convolution multipliers depending only on the degrees of the numerator and denominator.

5. Linear transformations on the classes $T(\alpha, \beta)$

5.1. The Convolution Theorem tells us that $T(\alpha, \beta)$ is closed under convolution for $\alpha \geqq 1$, $\beta \geqq 1$ and that $\alpha \geqq \alpha' \geqq 1$, $\beta \geqq \beta' \geqq 1$ imply $T(\alpha, \beta) \subset T(\alpha', \beta')$. Thus if $\phi \in T(\alpha, \beta)$ the linear operator $\phi\, *$ gives a mapping of $T(\alpha, \beta) \to T(\alpha', \beta')$. In this section we shall discuss a general criterion for linear mappings between the classes.

Let Λ be a continuous linear operator on $\mathscr{A}$. With ζ acting as a parameter, let

$$H(z, \zeta) = \Lambda \left(\frac{1}{1 - \zeta z}\right).$$

Then $H(z, \zeta)$ is called the *kernel* of the operator Λ and for $f \in \mathscr{A}$

$$\begin{aligned} \Lambda f(z) &= (H(z, \zeta) *_\zeta f(\zeta))_{\zeta = 1} \\ &= \lim_{R \to 1} \frac{1}{2\pi} \int_0^{2\pi} f(Re^{-i\theta}) H(z, e^{i\theta})\, d\theta \end{aligned} \tag{5.1.1}$$

the limit existing and converging locally uniformly in $|z| < 1$. Kernel functions $H(z, \zeta)$ are characterised by the property of being analytic as a function of two variables in $|z| < r < 1$, $|\zeta| < 1 + \delta(r)$ for each $0 < r < 1$, where $\delta(r) > 0$. Note that the convolution operator $\phi\, *$ has kernel $\phi(z\zeta)$.

5.2. Suppose that $\alpha \geqq 1$, $\beta \geqq 1$, $\alpha' \geqq 1$, $\beta' \geqq 1$ and that we wish to show that the linear operator Λ with kernel $H(z, \zeta)$ maps $T(\alpha, \beta)$ into $T(\alpha', \beta')$. Since $T(\alpha, \beta)$ is closed under rotations and contractions of the variable, we see from (5.1.1) that the necessary and sufficient condition is: for each $\phi \in T(\alpha, \beta)$

$$\phi(\zeta) *_\zeta H(z, \zeta) *_z \frac{(1 + xz)^{m'}(1 + uz)^{\gamma'}}{(1 - z)^{\beta'}} \neq 0 \tag{5.2.1}$$

for $|\zeta| \leqq 1$, $|z| < 1$, $|x| = |u| = 1$, where $m' = [\alpha']$, $\gamma' = \{\alpha'\}$. By the Convolution Theorem this will hold for $|\zeta| < 1$ if the function

$$\zeta \to H(z, \zeta) *_z \frac{(1 + xz)^{m'}(1 + uz)^{\gamma'}}{(1 - z)^{\beta'}} \tag{5.2.2}$$

is in $K(\alpha, \beta)$ for each fixed z, x, u in the given ranges. Thus in this case $z \to H(z, \zeta) *_\zeta \phi(\zeta)$ maps $T(\alpha, \beta) \to T(\alpha', \beta')$ for each $|\zeta| < 1$. Since a local uniform limit of functions in $T(\alpha', \beta')$ is either $\equiv 0$ or in $T(\alpha', \beta')$, we see that if (5.2.2) holds for $|\zeta| < 1$ and if $\phi \in T(\alpha, \beta)$, then either $\Lambda\phi \equiv 0$ or $\Lambda\phi \in T(\alpha', \beta')$.

Note that, though complicated, this condition is a condition purely on the kernel. The particular cases which we shall consider reduce to this general criterion.

5.3. Theorem. *Suppose that* $1 \leqq \delta = \min(\alpha', \beta') \leqq \min(\alpha, \beta)$. *Let* Λ *be a continuous linear operator on* $\mathcal{A}$ *with kernel* $H(z, \zeta)$ *satisfying the following two properties*:

(i) *for each* $|\zeta| = 1$ *the function*

$$z \to H(z, \zeta) \in T(\alpha', \beta'); \tag{5.3.1}$$

(ii) *for each* $|z| < 1$ *the function*

$$\zeta \to H(z, \zeta) *_z (1+z)^{\alpha'-\beta'} \in K(\alpha - \delta, \beta - \delta). \tag{5.3.2}$$

Then for each $\phi \in T(\alpha, \beta)$, *either* $\Lambda\phi \equiv 0$ *or* $\Lambda\phi \in T(\alpha', \beta')$.

5.4. Remark. Since the functions $(1 - \zeta z)^{-1}$ are in all the $T(\alpha, \beta)$ classes, the condition (i) is clearly a *necessary* condition for the mapping property to hold. In certain interesting cases it also turns out to be sufficient, though this is not the case in general.

In order to prove the theorem we require the following lemma.

5.5. Lemma. *Let* $f(z, \zeta)$ *be a kernel function such that* $f(z, \zeta) \neq 0$ *for* $|z| < 1$, $|\zeta| = 1$, *and* $f(0, \zeta) \neq 0$ *for* $|\zeta| \leqq 1$. *Then* $f(z, \zeta) \neq 0$ *for* $|z| < 1$, $|\zeta| \leqq 1$.

The proof proceeds by considering the Laurent expansion of $1/f(z, \zeta)$ in a region $|z| \leqq r < 1$, $1 - \delta(r) \leqq |\zeta| \leqq 1 + \delta(r)$, and showing by simple power series arguments and analytic continuation that the resulting series is in fact a Taylor series.

5.6. Proof of Theorem 5.3. We consider first the case $\alpha' \leqq \beta'$, so $\delta = \alpha'$. Let

$$g(z) = \frac{1}{(1-z)^{\beta'-\alpha'}}, \qquad L(x, u, z) = \frac{(1+xz)^{m'}(1+uz)^{\gamma'}}{(1-z)^{\alpha'}}.$$

Then $g \in \Pi_{\alpha'-\beta'}$ and since $m' + \gamma' = \alpha'$, for suitable $\rho(x, u)$ real

$$|\arg(e^{i\rho(x,u)} L(x, u, z))| \leqq \frac{\alpha' \pi}{2}.$$

Thus by the Convolution Theorem and (i)

$$(5.6.1)\qquad \left|\arg\left(e^{i\rho(x,u)}\frac{H(z,\zeta)*_z\dfrac{(1+xz)^{m'}(1+uz)^{\gamma'}}{(1-z)^{\beta'}}}{H(z,\zeta)*_z(1-z)^{\alpha'-\beta'}}\right)\right|\leqq\frac{\alpha'\pi}{2}$$

for $|\zeta|=1$, $|z|<1$. Also by (ii) the denominator in this expression is non-zero for $|\zeta|<1$. If we denote the numerator by $f(z,\zeta)$, then $f(z,\zeta)\neq 0$ for $|z|<1$, $|\zeta|=1$. Also $f(0,\zeta)=H(0,\zeta)\neq 0$ for $|\zeta|<1$ by (ii). Hence by Lemma 5.5 $f(z,\zeta)\neq 0$ for $|z|<1$, $|\zeta|\leqq 1$. Thus we can apply the maximum principle to deduce that (5.6.1) holds for $|z|<1$, $|\zeta|\leqq 1$. Then applying (ii) we see that for each $|z|<1$, $|x|=|u|=1$, the function

$$\zeta\to H(z,\zeta)*_z\frac{(1+xz)^{m'}(1+uz)^{\gamma'}}{(1-z)^{\beta'}}$$

is the product of a function in $K(\alpha-\alpha',\beta-\alpha')$ and a function in $K(\alpha',\alpha')$, so is in $K(\alpha,\beta)$. The criterion (5.2.2) is thus satisfied.

Secondly suppose that $\beta'<\alpha'$ so $\delta=\beta'$. Let $g(z)=(1+xz)^{\alpha'-\beta'}\in\Pi_{\alpha'-\beta'}$ and

$$L(x,u,z)=\frac{(1+xz)^{m'+\beta'-\alpha'}(1+uz)^{\gamma'}}{(1-z)^{\beta'}}.$$

Then $m'+\beta'-\alpha'=\beta'-\gamma'>0$ and hence for suitable $\rho(x,u)$ real

$$|\arg(e^{i\rho(x,u)}L(x,u,z))|\leqq\frac{\beta'\pi}{2}.$$

Similarly to before we deduce that the function (5.2.2) is the product of a function in $K(\alpha-\beta',\beta-\beta')$ and a function in $K(\beta',\beta')$, so is in $K(\alpha,\beta)$.

5.7. Corollary. *Suppose that $\alpha\geqq 1$, $\beta\geqq 1$ and that Λ is a continuous linear operator on $\mathscr{A}$ with kernel $H(z,\zeta)$ satisfying*

(i) *for each $|\zeta|=1$ the function*

$$z\to H(z,\zeta)\in T(\alpha,\beta);$$

(ii) *for each $|z|<1$ the function*

$$\zeta\to H(z,\zeta)*_z(1+z)^{\alpha-\beta}\in\prod_{\alpha-\beta}.$$

Then Λ maps $T(\alpha,\beta)$ into $T(\alpha,\beta)\cup\{0\}$.

5.8. Definition. A continuous linear operator on $\mathscr{A}$ is said to be *origin preserving* if $\Lambda f(0) = f(0)$ for every $f \in \mathscr{A}$. It is immediately verified that Λ is origin preserving if, and only if, the kernel $H(z, \zeta)$ satisfies $H(0, \zeta) = 1$.

5.9. Theorem. *If $\alpha \geqq 1$ and if Λ is a continuous linear operator origin preserving on $\mathscr{A}$, then Λ maps $T(\alpha, \alpha)$ into $T(\alpha, \alpha)$ if, and only if, for each $|\zeta| = 1$,*

$$z \to \Lambda\left(\frac{1}{1 - \zeta z}\right) \in T(\alpha, \alpha).$$

As an application we prove a structure preserving theorem for the $K(\alpha, \beta)$ classes.

5.10. Theorem. *Suppose that $1 \leqq \alpha \leqq \beta$ and let Λ be a continuous linear operator on $\mathscr{A}$ with kernel $H(z, \zeta)$ satisfying the following two properties:*

(i) *for each $|\zeta| = 1$ the function*

(5.10.1) $$z \to \Lambda((1 + \zeta z)^{\alpha - \beta}) \in \prod\nolimits_{\alpha - \beta};$$

(ii) *for each $|z| < 1$ the function*

(5.10.2) $$\zeta \to H(z, \zeta) \in T(\alpha, \beta).$$

Then for each $0 \leqq \varepsilon \leqq \alpha$, Λ maps $K(\alpha - \varepsilon, \beta - \varepsilon)$ into $K(\alpha - \varepsilon, \beta - \varepsilon) \cup \{0\}$ and (in the case $\varepsilon = \alpha$) Λ maps $\Pi_{\alpha - \beta}$ into $\Pi_{\alpha - \beta}$.

Proof. Consider first the case $\alpha = 1$, $\beta > 1$ and assume that

(a) $z \to \Lambda((1 + \zeta z)^{1-\beta}) \in \Pi_{1-\beta}$ for each $|\zeta| = 1$;

(b) $\zeta \to H(z, \zeta) \in T(1, \beta)$ for each $|z| < 1$.

We note that $g \in \Pi_{1-\beta}$ if, and only if, $g(z) = \phi(z) * (1 - z)^{1-\beta}$ where $\phi \in T(1, \beta)$. For $f \in \mathscr{A}$ define

$$\Gamma f(z) = ((1 - z)^{1-\beta})_i * \Lambda(f(z) * (1 - z)^{1-\beta})$$

where h_i denotes the convolution inverse of h ($h * h_i = (1 - z)^{-1}$). Γ is a continuous linear opeator on $\mathscr{A}$ with kernel $J(z, \zeta)$ satisfying

(5.10.3) $$(1 - z)^{1-\beta} *_z J(z, \zeta) = (1 - \zeta)^{1-\beta} *_\zeta H(z, \zeta).$$

From (a) and (b) we obtain

(c) $z \to J(z,\zeta) \in T(1,\beta)$ for each $|\zeta|=1$;

(d) $\zeta \to (1-z)^{1-\beta} *_z J(z,\zeta) \in \Pi_{1-\beta}$ for each $|z|<1$.

Hence by Corollary 5.7 Γ maps $T(1,\beta)$ into $T(1,\beta)\cup\{0\}$. We show that if $\phi(z)=\Sigma_0^\infty \phi_n z^n \in T(1,\beta)$, then $\Gamma\phi \not\equiv 0$. Let $J(z,\zeta)=\Sigma_0^\infty \tau_n(z)\zeta^n$, so by (d) $\Sigma_0^\infty \tau_n(0)\zeta^n \in \Pi_{1-\beta}$. Hence $\tau_0(0)\neq 0$ and by the Convolution Theorem $\Sigma_0^\infty \phi_n\tau_n(0)\zeta^n \in \Pi_{1-\beta}$. Hence clearly

$$\left|\sum_0^\infty \phi_n\tau_n(0)\zeta^n\right| > 2^{1-\beta}|\phi_0\tau_0(0)| \qquad (|\zeta|<1)$$

and so $\Gamma\phi(0)=\Sigma_0^\infty \phi_n\tau_n(0)\neq 0$. Thus Λ maps $\Pi_{1-\beta}$ to $\Pi_{1-\beta}$.

Secondly, consider the case $\alpha=1$, $\beta=1$. Then (i) implies that $\Lambda 1$ is a non-zero constant and so Λ maps Π_0 into Π_0. Finally consider the general case $1\leqq\alpha\leqq\beta$ and put $\alpha-\beta=1-\gamma$, so $\gamma\geqq 1$. We obtain

(iii) $z \to \Lambda((1+\zeta z)^{1-\gamma}) \in \Pi_{1-\gamma}$;

(iv) $\zeta \to H(z,\zeta) \in T(\alpha,\beta) \subset T(1,\gamma)$,

since $\alpha\geqq 1$, $\beta\geqq\gamma$. Hence the previous cases show that Λ maps $\Pi_{\alpha-\beta}$ to $\Pi_{\alpha-\beta}$. Let $f\in K(\alpha-\varepsilon,\beta-\varepsilon)$ where $0\leqq\varepsilon\leqq\alpha$. Then $f=gL$ where $g\in\Pi_{\alpha-\beta}$, $|\arg L|\leqq(\alpha-\varepsilon)\pi/2$. Assume $\Lambda f\not\equiv 0$. By (ii) and the Convolution Theorem

$$\left|\arg\frac{H(z,\zeta)*_\zeta g(\zeta)L(\zeta)}{H(z,\zeta)*_\zeta g(\zeta)}\right| \leqq (\alpha-\varepsilon)\frac{\pi}{2}$$

for $|\zeta|<1$, $|z|<1$. Letting $\zeta\to 1$ we deduce that

$$\left|\arg\frac{\Lambda f(z)}{\Lambda g(z)}\right| \leqq (\alpha-\varepsilon)\frac{\pi}{2},$$

and since $\Lambda g\in\Pi_{\alpha-\beta}$, $\Lambda f\in K(\alpha-\varepsilon,\beta-\varepsilon)$.

As a final application we establish general criteria for continuous linear operators on $\mathcal{A}$ to preserve convex, starlike and close-to-convex univalent mappings of the disc. These results generalise the Convolution Theorems established in proving the Polya–Schoenberg conjecture [13].

5.11. Theorem. *Let Λ be a continuous linear operator on $\mathcal{A}$ with kernel $H(z,\zeta)$. Suppose that*

(i) *for each $|\zeta|=1$ the function*

$$z \to \Lambda\left(\frac{z}{(1-\zeta z)^2}\right) \tag{5.11.1}$$

is starlike in $|z|<1$;

(ii) *for each* $|z|<1$ *the function*

(5.11.2) $$\zeta \to \frac{H(z,\zeta)-H(z,0)}{z}$$

is convex in $|\zeta|<1$.

Then if g *is starlike,* Λg *is starlike.*

5.12. Theorem. *Let* Γ *be a continuous linear operator on* $\mathscr{A}$ *with kernel* $J(z,\zeta)$. *Suppose that*

(a) *for each* $|\zeta|=1$ *the function*

(5.12.1) $$z \to J(z,\zeta)$$

is convex in $|z|<1$;

(b) *for each* $|z|<1$ *the function*

(5.12.2) $$\zeta \to \frac{\partial}{\partial z} J(z,\zeta)$$

is starlike in $|\zeta|<1$.

Then for each convex ϕ, $\Gamma\phi$ *is convex, and for each* f *close-to-convex,* Γf *is either close-to-convex or constant.*

Proof. Let $H(z,\zeta)=\Sigma_0^\infty \sigma_n(z)\zeta^n$ so that

$$\Lambda\left(\frac{z}{(1-\zeta z)^2}\right)=\sum_1^\infty n\sigma_n(z)\zeta^{n-1}$$

is starlike in z for each $|\zeta|=1$, and so $=0$ at $z=0$. Hence $\sigma_n(0)=0$ ($n \geqq 1$). Thus for $f\in\mathscr{A}$, $\Lambda(zf(z))_{z=0}=0$. Define for $f\in\mathscr{A}$

$$\Lambda^*f(z)=\frac{1}{z}\Lambda(zf(z)),$$

so Λ^* is a continuous linear operator on $\mathscr{A}$ with kernel

$$H^*(z,\zeta)=\frac{1}{z\zeta}(H(z,\zeta)-H(z,0)).$$

The conditions (i) and (ii) imply

$$z \to \Lambda^*(1-\zeta z)^{-2} \in \prod_{-2} \text{ for each } |\zeta| = 1;$$

$$\zeta \to H^*(z,\zeta) \in T(1,3) \quad \text{for each } |z| < 1.$$

Hence by Theorem 5.10 Λ^* maps Π_{-2} to Π_{-2}, so Λ maps S_0^* to S_0^*. Furthermore Λ^* maps $K(1,3)$ to $K(1,3) \cup \{0\}$. Since f is close-to-convex if, and only if, $f' \in K(1,3)$, we see that if f is close-to-convex, $\Lambda z f'$ is either identically zero or has the form zh' where h is close-to-convex.

To prove 5.12 define for $f \in \mathscr{A}$

$$\Lambda f(z) = \frac{z}{(1-z)^2} * \Gamma\left(\log \frac{1}{1-z} * f(z)\right).$$

Then (a) and (b) imply (i) and (ii) and hence if ϕ is convex and f close-to-convex, $\Gamma(\phi(z) - \phi(0))$ is convex and $\Gamma(f(z) - f(0))$ is either close-to-convex or identically zero. Finally writing $J(z,\zeta) = \Sigma_0^\infty \tau_n(z)\zeta^n$, the condition (b) implies that $\tau_0'(z) = 0$, so $\Gamma 1 = \tau_0(z)$ is constant. Hence $\Gamma\phi$ is convex and Γf close-to-convex or constant.

5.13. Example. Suppose that we wish to know which subdomains of $|z| < 1$ are mapped by every convex univalent function onto a convex region. This was a problem first solved by Pommerenke [9] and later studied by Heins [3]. Such a domain is clearly convex. Let $w = \sigma(z)$ be univalent and map $|z| < 1$ onto such a domain. Then the linear operator

$$\Lambda\phi(z) = \phi(\sigma(z))$$

preserves convex univalence. The kernel of the operator is given by

$$H(z,\zeta) = \frac{1}{1 - \zeta\sigma(z)}.$$

Theorem 5.12 gives the following conditions as sufficient for the desired property:

(5.13.1) $$z \to \frac{1}{1-\zeta\sigma(z)} \qquad \text{convex for each } |\zeta| = 1;$$

(5.13.2) $$\zeta \to \frac{\zeta\sigma'(z)}{(1-\zeta\sigma(z))^2} \qquad \text{starlike for each } |z| < 1.$$

Clearly (5.13.1) implies that $\sigma(z)$ is univalent so $\sigma'(z) \neq 0$. Hence since $|\sigma(z)| < 1$ the condition (5.13.2) is automatically satisfied. Thus (5.13.1) is both a necessary and a sufficient condition. Writing

$$T = \operatorname{Re}\left(1 + \frac{z\sigma''(z)}{\sigma'(z)}\right)$$

this condition becomes

$$T + \operatorname{Re}\frac{2z\zeta\sigma'(z)}{1-\zeta\sigma(z)} > 0 \qquad (|\zeta| = 1, |z| < 1). \tag{5.13.3}$$

This gives

$$\left|\sigma - \frac{z\sigma'}{T}\right| < \frac{1}{2}(1 + |\sigma|^2) - \operatorname{Re}\bar{\sigma}\frac{z\sigma'}{T}. \tag{5.13.4}$$

If $c = \sigma(z) - z\sigma'(z)/T$, $\rho = |z\sigma'(z)|/T$, then ρ is the radius of curvature of the curve $\Gamma_r = \{w = \sigma(re^{i\theta})\}$ at $w = \sigma(z)$ ($|z| = r$) and c is the centre of curvature. Clearly $|c| < 1 + \rho$. Therefore

$$2|c| < 1 + |\sigma|^2 - 2\operatorname{Re}\bar{\sigma}\frac{z\sigma'}{T} = 1 + |\sigma|^2 + |c|^2 - |\sigma|^2 - \rho^2$$

and hence $|c|^2 - 2|c| + 1 > \rho^2$, which gives $|c| + \rho < 1$. Conversely, this condition implies (5.13.4). Thus the necessary and sufficient condition is that the circle of curvature at each point of Γ_r lies entirely in $|w| < 1$.

6. Concluding remarks and an open question

6.1. Apart from the case $\alpha = 1$ the definition of the classes $T(\alpha, \beta)$ is not an easy one to work with. As it implies a variety of "positivity" conditions it would be interesting to know whether any of these represent sufficient conditions. The case α not an integer is particularly interesting. Is it true that $\phi \in T(\alpha, \beta)$ if

$$\phi(z) * \frac{(1+xz)^\alpha}{(1-z)^\beta} \neq 0 \qquad (|x| = 1, |z| < 1) \tag{6.1.1}$$

where $\alpha \geqq 1$, $\beta \geqq 1$? By Lemma 2.7 it would be enough to establish either of the implications

$$(6.1.1) \Rightarrow \phi(z) * \frac{(1+xz)^{\alpha-1}}{(1-z)^\beta} \neq 0 \qquad (1|x| = 1, |z| < 1),$$

$$(6.1.1) \Rightarrow \phi(z) * \frac{1+xz}{(1-z)^\beta} \neq 0 \qquad (|x| = 1, |z| < 1).$$

This conjecture is made very tentatively as I have not even managed to show that the condition (6.1.1) represents a normal family of functions. The truth of the conjecture would settle affirmatively the coefficient conjecture for $K(\alpha, \beta)$ ($\alpha \geqq 1$, $\beta \geqq 1$).

6.2. The further study of non-zero convolution conditions seems well worth pursuing and could throw new light on old problems. For example $f(z) = z + a_2 z^2 + \cdots$ is schlicht in $|z| < 1$ if, and only if,

$$\frac{f(z)}{z} * \frac{1}{(1-xz)(1-yz)} \neq 0 \qquad (|x| = |y| = 1, |z| < 1). \tag{6.2.1}$$

The second dual problem is thus of great interest.

7. Some additional remarks

7.1. The ranges of the parameters x, y and u in Theorems 4.1, 4.2 and 4.9 can be reduced to two cases: (i) $|x| = |u| = |y| = 1$ or (ii) $x = u = -y$ and $|x| < 1$. This follows from the following sharpened version of Ruscheweyh's duality theorem:

7.2. Theorem. *Suppose that* $g(z_1, z_2, \cdots, z_n)$ *is analytic for* $|z_k| < 1$ $(1 \leqq k \leqq n)$ *and that* $g(0, \cdots, 0) = 1$. *Denote by* $T(g)$ *the class of functions* $\phi \in A$ *satisfying*

$$\phi(z) * g(x_1 z, \cdots, x_n z) \neq 0 \tag{7.2.1}$$

for $|x_k| = 1$ $(1 \leqq k \leqq n)$ *and* $|z| < 1$. *Let* $f(z) = 1 + \Sigma_1^\infty a_n z^n \in A$ *and suppose that*

$$f(z) * \phi(z) \neq 0 \qquad (|z| < 1) \tag{7.2.2}$$

for each $\phi \in T(g)$. *Then there is a probability measure* μ *on the torus* $T^n = \{|x_k| = 1, 1 \leqq k \leqq n\}$ *such that*

$$f(z) = \int_{T^n} g(x_1 z, \cdots, x_n z) d\mu. \tag{7.2.3}$$

Furthermore if Λ_1 *and* Λ_2 *are continuous linear functionals on* A *such that*

$$\Lambda_2 g(x_1 z, \cdots, x_n z) \neq 0 \tag{7.2.4}$$

for $|x_k| = 1$ $(1 \leqq k \leqq n)$ *and for* $x_1 = x_2 = \cdots = x_n$ *and* $|x_1| < 1$, *then for some values of the parameters* x_k *in this range*

(7.2.5) $$\frac{\Lambda_1 f(z)}{\Lambda_2 f(z)} = \frac{\Lambda_1 g(x_1 z, \cdots, x_n z)}{\Lambda_2 g(x_1 z, \cdots, x_n z)}.$$

7,3. The proof of (7.2.5) follows Ruscheweyh's original argument [11] with one additional ingredient to reduce the range of the parameters x_k. This is the following known lemma brought to my attention by D. Aharonov: if $h(z_1, \cdots, z_n)$ is analytic in the closed polydisc $|z_k| \leqq 1$ $(1 \leqq k \leqq n)$ and if $h(z_1, \cdots, z_n) \neq 0$ when $|z_k| = 1$ and also when $z_1 = z_2 = \cdots = z_n = z$ and $|z| < 1$, then $h(z_1, \cdots, z_n) \neq 0$ for $|z_k| \leqq 1$.

To prove the representation (7.2.3) we observe that $f(z)$ lies in the closed convex hull of the functions $g(y_1 z, \cdots, y_n z)$ with $|y_k| \leqq 1$. But for any analytic function g and parameters y_k satisfying $|y_k| \leqq 1$ we can find a probability measure μ on T^n such that

(7.3.1) $$g(y_1 z, \cdots, y_n z) = \int_{T^n} g(x_1 z, \cdots, x_n z) d\mu.$$

This is seen from the fact that the function

$$\prod_1^n \frac{1}{1 - y_k z_k} \qquad (|z_k| < 1)$$

has a representation in the form

$$\prod_1^n \frac{1}{1 - y_k z_k} = \int_{T^n} \prod_1^n \frac{1}{1 - x_k z_k} d\mu$$

(see e.g. T. Sheil-Small, *On the convolution of analytic functions*, J. Reine Angew. Math. **258** (1973), 137–152). Thus

$$g(y_1 z_1, \cdots, y_n z_n) = g(z_1, \cdots, z_n) * \prod_1^n \frac{1}{1 - y_k z_k}$$

$$= \int_{T^n} g(z_1 x_1, \cdots, z_n x_n) d\mu$$

and (7.3.1) follows. Thus f must lie in the closed convex hull of the functions $g(x_1 z, \cdots, x_n z)$ $(|x_k| = 1)$ and (7.2.3) follows from Choquet's theorem.

7.4. Theorem. *Let* $f(z) = 1 + \Sigma_1^\infty a_n z^n \in K(\alpha, \beta)$, *where* $\alpha \geqq 1$ *and* $\beta \geqq 1$. *Then there is a probability measure* μ *on the torus* T^2 *such that*

(7.4.1)
$$f(z)=\int_T \frac{(1+xz)^{\alpha}}{(1-yz)^{\beta}}\,d\mu.$$

Furthermore

(7.4.2)
$$f(z)\ll\frac{(1+z)^{\alpha}}{(1-z)^{\beta}}.$$

Proof. The following extreme point argument is due to J. Clunie.

Let $h(z)$ be an extreme point of the closed convex hull of $K(\alpha,\beta)$. By Theorem 1.2 and the preceding remarks $h(z)$ has the form

(7.4.3)
$$h(z)=\frac{(1+xz)^{m}(1+uz)^{\gamma}}{(1-yz)^{\beta}}$$

where $|x|=|u|=|y|=1$ and $m=[\alpha]$, $\gamma=\{\alpha\}$. We may assume $0<\gamma<1$. To prove the result it will be sufficient to show that $x=u$. Now

$$h(z)=\frac{(1+xz)^{m-1+\gamma}}{(1-yz)^{\beta-1}}\,\frac{(1+xz)^{1-\gamma}(1+uz)^{\gamma}}{1-yz}$$

$$=\int_T \frac{(1+xz)^{\alpha-1}}{(1-yz)^{\beta-1}}\,\frac{1+\zeta vz}{1-vz}\,d\mu(v)$$

where μ is a probability measure on T and $|\zeta|=1$. We have thus represented h as a convex combination of elements in $K(\alpha,\beta)$ and, since h is an extreme point, the measure μ must be a point mass. Hence for some $|\zeta|=|v|=1$

$$h(z)=\frac{(1+xz)^{\alpha-1}}{(1-yz)^{\beta-1}}\,\frac{1+\zeta vz}{1-vz}.$$

The relation $x=u$ follows easily.

7.5. In a forthcoming paper St. Ruscheweyh [*Some convexity and convolution theorems for analytic functions*, Math. Ann.] also obtains Theorem 7.4. In addition Ruscheweyh establishes the following interesting consequence of the convolution theorem. For $\alpha\geqq 0$, $\beta\geqq 0$ denote by $S(\alpha,\beta)$ the functions of the form

$$f(z)=\frac{g(z)}{h(z)}$$

where $g \in \Pi_\alpha$, $h \in \Pi_\beta$. If $\phi \in T(\alpha+1, \beta+1)$, then $\phi * f \in S(\alpha, \beta)$. In particular this shows that if $p(z) = z + \cdots$ belongs to the class V_k of functions of bounded boundary rotation at most $k\pi$, then so does $z\phi * p$ for each $\phi \in T(\frac{1}{2}k, \frac{1}{2}k+2)$.

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