Note on Quantum Probability.

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Summary. – When a state of a physical system dynamically changes to another state, it is important to know the correlation existing between the initial state and the final state. This correlation is described by a compound state (measure) in classical systems. In this note, we show a way how to construct such a compound state in quantum systems which is an extension of the classical compound state.

It is rather important in many physical sciences to study the dynamical change of states of a system. One of the most general description of this state change for classical systems is suggested in the communication theory of Shannon.

A state of a classical dynamical system is expressed by a probability measure on that system and its dynamical change is generally considered as follows (¹): let X, H be compact Hausdorff spaces and $\mathcal{F}_X$, $\mathcal{F}_Y$ be their Borel fields, respectively. We denote the set of all regular probability measures (states) on $(X, \mathcal{F}_X)$ by $P(X)$ and on $(Y, \mathcal{F}_Y)$ by $P(Y)$. A mapping $\lambda\colon X\times\mathcal{F}_Y\to R^+$ satisfying the following two conditions is called a channel: i) $\lambda(x,\cdot)\in P(Y)$ for each fixed $x\in X$ and ii) $\lambda(\cdot, Q)$ is a continuous measurable function on X for each fixed $Q\in\mathcal{F}_Y$. This mapping is often called a transition (or Markov) kernel and is useful to study, for instance, information transmission and stochastic processes. A channel so defined provides a mechanism of state change. Namely, a state $\varphi\in P(X)$ is transferred to a state $\psi\in P(Y)$ under a channel λ such as

$$\psi(Q)=\int_X \lambda(x,Q)\varphi(\mathrm{d}x)\,, \qquad Q\in\mathcal{F}_Y\,. \tag{1}$$

Moreover, in order to study the process of state change and the property of a channel itself, we need a compound state (joint probability) indicating the correlation existing

(¹) H. UMEGAKI: *J. Math. Anal. Appl.*, **25**, 41 (1969).

between the initial state φ and the final state ψ. The compound state Φ is given by

(2) $$\Phi(Q_1, Q_2) = \int_{Q_1} \lambda(x, Q_2)\,\varphi(\mathrm{d}x)$$

for any $Q_1 \in \mathscr{F}_X$ and $Q_2 \in \mathscr{F}_Y$.

In quantum dynamics, we take two C^*-systems $(\mathscr{A}, \mathfrak{S}(\mathscr{A}))$ and $(\mathscr{B}, \mathfrak{S}(\Lambda))$, one of which describes an initial (input) system corresponding to $(X, \mathscr{F}_X, P(X))$ above and another describes a final (output) system corresponding to $(Y, \mathscr{F}_Y, P(Y))$. Here $\mathscr{A}$ (respectively, $\mathscr{B}$) is a C^*-algebra with unity $I_{\mathscr{A}}$ (respectively, $I_{\mathscr{B}}$) and $\mathfrak{S}(\mathscr{A})$ (respectively, $\mathfrak{S}(\mathscr{B})$) is the set of all states (*i.e.* normalized positive linear functionals) on $\mathscr{A}$ (respectively, $\mathscr{B}$) (2). Then let us consider a mapping Λ^* from $\mathfrak{S}(\mathscr{A})$ to $\mathfrak{S}(\mathscr{B})$ such that its dual map Λ: $:\mathscr{B} \to \mathscr{A}$ is completely positive (2) with $\Lambda I_{\mathscr{B}} = I_{\mathscr{A}}$. This mapping Λ^* is called a channel between two quantum-dynamical systems (3). In particular, when $\mathscr{A} = C(X)$, the set of all continuous functions on X, and $\mathscr{B} = C(Y)$, the formula (1) defines a channel Λ^* from $P(X)$ to $P(Y)$ (*i.e.* $\psi = \Lambda^* \varphi$) because every probability measure φ on $(X, \mathscr{F}_X)$ can be regarded as a state on $C(X)$ by the Riesz-Markov-Kakutani theorem. We meet several channels in several fields of physics. For example, time evolution automorphism group, dynamical semi-group and conditional expectation on a certain algebra are typical channels.

Now it is well known (4) that the joint probability measure does not generally exist in quantum systems. Hence it has been difficult to define a compound state describing the correlation existing between an initial state $\varphi \in \mathfrak{S}(\mathscr{A})$ and its final state $\Lambda^*\varphi \in \mathfrak{S}(\mathscr{B})$. The aim of this note is to construct such a compound state and to show that our compound state is an extension of the classical one given by (2).

For an initial state φ and the final state $\Lambda^*\varphi$, a compound state Φ on $\mathscr{A} \otimes \mathscr{B}$ of φ and $\Lambda^*\varphi$ should satisfy the following two conditions: i) $\Phi(A \otimes I_{\mathscr{B}}) = \varphi(A)$ for any $A \in \mathscr{A}$ and ii) $\varphi(I_{\mathscr{A}} \otimes B) = \Lambda^*\varphi(B)$ for any $B \in \mathscr{B}$. There exist many states satisfying these conditions, for instance, $\Phi_0 = \varphi \otimes \Lambda^*\varphi$ is such a state. But this state does not carry any correlation between φ and $\Lambda^*\varphi$.

For any weak *-compact convex subset $\mathscr{S}$ of $\mathfrak{S}(\mathscr{A})$, there exists a maximal measure μ such that φ is the barycentre of μ and μ is pseudosupported by the set $e\times\mathscr{S}$ of all extreme points of $\mathscr{S}$ in the sense that $\mu(Q) = 1$ for every Baire subset Q of $\mathscr{S}$ with $Q \supset e\times\mathscr{S}$ (5). In this case, we write

(3) $$\varphi = \int_{(e\times\mathscr{S})} \omega\,\mathrm{d}\mu\,.$$

Note that the above maximal measure μ is not always unique, and we denote the set of all such measures by $M_\varphi(\mathscr{S})$.

We now construct a « true » compound state of φ and $\Lambda^*\varphi$. For each $\mu \in M_\varphi(\mathscr{S})$, define

(4) $$\varphi_\mu = \int_{(e\times\mathscr{S})} \omega \otimes \Lambda^*\omega\,\mathrm{d}\mu\,.$$

It is easy to see that this state Φ_μ satisfies the conditions i) and ii) mentioned above. Let us now show that the compound state defined by (4) is indeed an extension of the

(2) M. TAKESAKI: *Theory of operator algebra I* (Berlin, 1981).
(3) M. OHYA: *J. Math. Anal. Appl.*, **84**, No. 2, 318 (1981).
(4) K. URBANIK: *Studia Math.*, **21**, 117 (1961).

classical one. When $\mathscr{S} = P(X)$, the extremal decomposition of a state $\varphi \in P(X)$ is unique and given by

$$\varphi = \int_X \delta_x \varphi(\mathrm{d}x)\,, \tag{5}$$

where δ_x is the Dirac measure concentrated at a point $x \in X$. Since $(\Lambda^* \delta_x)(Q_2) = \lambda(x, Q_2)$ for any $Q_2 \in \mathscr{F}_Y$, we have

$$\Phi_\varphi(Q_1 \times Q_2) = \int_X \delta_x(Q_1) \Lambda^* \delta_x(Q_2) \varphi(\mathrm{d}x) = \int_X I_{Q_1}(x) \lambda(x, Q_2) \varphi(\mathrm{d}x) = \int_{Q_1} \lambda(x, Q_2) \varphi(\mathrm{d}x)\,,$$

for any $Q_1 \in \mathscr{F}_X$ and $Q_2 \in \mathscr{F}_Y$.

We finally consider the case of $\mathscr{A} = C(\mathscr{H}_1) = C(\mathscr{H}_1) + CI$, where $C(\mathscr{H})$ is the set of all compact operators on a separable Hilbert space $\mathscr{H}$. Then $\mathfrak{S}(\mathscr{A})$ contains the set $T(\mathscr{H}_1)_{+,1}$ of all positive trace class operators on $\mathscr{H}_1$ with unit trace and so does $\mathfrak{S}(\mathscr{B})$. Moreover, a channel Λ^* is a trace-preserving completely positive map from $T(\mathscr{H}_1)_{+,1}$ to $T(\mathscr{H}_2)_{+,1}$. In this case, if an extremal decomposition of a state $\varrho \in T(\mathscr{H}_1)_{+,1}$ is given by $\varrho = \sum_n \lambda_n \varrho_n$, then our compound state is

$$\sigma = \sum_n \lambda_n \varrho_n \otimes \Lambda^* \varrho_n\,.$$

Among these compound states, the following is the most important:

$$\sigma_E = \sum_n \lambda_n E_n \otimes \Lambda^* E_n\,. \tag{6}$$

The symbols appearing in (6) mean the following: *a*) λ_n is the eigenvalue of ϱ, and the eigenvalue of multiplicity m is repeated precisely m times. *b*) $E_n = |x_n\rangle\langle x_n|$ in Dirac's notation, where x_n is an eigenvector associated with λ_n and it is unique if and only if λ_n is nondegenerate (5). *c*) E represents $\{E_n\}$. The compound state σ_E depends on E, hence on the choice of the eigenvectors $\{x_n\}$. This compound state is first introduced in (2), but it is a special case of our compound state Φ_μ as just discussed.

The compound state constructed for quantum systems plays a similar role as the joint probability measure for classical systems, so that it is useful to formulate quantum communication theory (6) and to study the dynamics of state change (7).

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(5) R. SCHATTEN: *Norm ideals of completely continuous operators* (Berlin, 1970).
(6) M. OHYA: *On compound state and mutual information in quantum information theory*, to appear in *IEEE Information Theory*.
(7) M. OHYA: *Remarks on dynamical change of states*, preprint.