SPECTRAL ANALYSIS OF DARBOUX TRANSFORMATIONS FOR THE FOCUSING NLS HIERARCHY

By

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Abstract. We study Darboux-type transformations associated with the focusing nonlinear Schrödinger equation (NLS_-) and their effect on spectral properties of the underlying Lax operator. The latter is a formally $\mathcal{J}$-self-adjoint (but non-self-adjoint) Dirac-type differential expression of the form

$$M(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\bar{q} & -\frac{d}{dx} \end{pmatrix},$$

satisfying $\mathcal{J}M(q)\mathcal{J} = M(q)^*$, where $\mathcal{J}$ is defined by $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathcal{C}$, and $\mathcal{C}$ denotes the antilinear conjugation map in $\mathbb{C}^2$, $\mathcal{C}(a,b)^\top = (\bar{a}, \bar{b})^\top$, $a, b \in \mathbb{C}$. As one of our principal results, we prove that under the most general hypothesis $q \in L^1_{\text{loc}}(\mathbb{R})$ on q, the maximally defined operator $D(q)$ generated by $M(q)$ is actually $\mathcal{J}$-self-adjoint in $L^2(\mathbb{R})^2$. Moreover, we establish the existence of Weyl–Titchmarsh-type solutions $\Psi_+(z,\cdot) \in L^2([R,\infty))^2$ and $\Psi_-(z,\cdot) \in L^2((-\infty,R])$ for all $R \in \mathbb{R}$ of $M(q)\Psi_\pm(z) = z\Psi_\pm(z)$ for z in the resolvent set of $D(q)$.

The Darboux transformations considered in this paper are the analogue of the double commutation procedure familiar in the KdV and Schrödinger operator contexts. As in the corresponding case of Schrödinger operators, the Darboux transformations in question guarantee that the resulting potentials q are locally nonsingular. Moreover, we prove that the construction of N-soliton NLS_- potentials $q^{(N)}$ with respect to a general NLS_- background potential $q \in L^1_{\text{loc}}(\mathbb{R})$, associated with the Dirac-type operators $D(q^{(N)})$ and $D(q)$, respectively, amounts to the insertion of N complex conjugate pairs of $L^2(\mathbb{R})^2$-eigenvalues $\{z_1, \bar{z}_1, \dots, z_N, \bar{z}_N\}$ into the spectrum $\sigma(D(q))$ of $D(q)$, leaving the rest of the spectrum (especially, the essential spectrum $\sigma_e(D(q))$) invariant, that is,

$$\sigma\big(D(q^{(N)})\big) = \sigma(D(q)) \cup \{z_1, \bar{z}_1, \dots, z_N, \bar{z}_N\},$$

$$\sigma_e\big(D(q^{(N)})\big) = \sigma_e(D(q)).$$

These results are obtained by establishing the existence of bounded transformation operators which intertwine the background Dirac operator $D(q)$ and the Dirac operator $D(q^{(N)})$ obtained after N Darboux-type transformations.

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1 Introduction

Various methods of inserting eigenvalues in spectral gaps of one-dimensional Schrödinger operators $H(q)$ associated with differential expressions of the type

$$(1.1) \qquad L(q) = -\frac{d^2}{dx^2} + q$$

in $L^2(\mathbb{R})$ (or in $L^2((a, \infty))$, $a \geq -\infty$), with q real-valued and locally integrable, have attracted an enormous amount of attention. This is due to their prominent role in such diverse fields as the inverse scattering approach, supersymmetric quantum mechanics, level comparison theorems, as a tool to construct soliton solutions of the Korteweg–de Vries (KdV) hierarchy relative to (general) KdV background solutions, and in connection with Bäcklund transformations for the KdV hierarchy. The literature on this subject is too extensive for us to go into detail here, but we refer to the detailed accounts given in [15], [16], [17, App. G], [18], [19] and the references cited therein. Historically, these methods of inserting eigenvalues go back to Jacobi [26] and Darboux [10], with decisive later contributions by Crum [9], Schmincke [42], and, especially, Deift [11].

Two particular methods have turned out to be of special importance: The single commutation method, also called the Crum–Darboux method [9], [10] (actually going back at least to Jacobi [26]) and the double commutation method, to be found, for instance, in the seminal work of Gel'fand and Levitan [14]. (The latter can be obtained by a composition of two separate single commutation steps, explaining the name double commutation.)

The single commutation method, although very simply implemented, has the distinct disadvantage of relying on positivity properties of certain solutions ψ of $H(q)\psi = \lambda\psi$, which confines its applicability to the insertion of eigenvalues below the spectrum of $H(q)$ (assuming $H(q)$ to be bounded from below). A complete spectral characterization of this method has been provided by Deift [11] (see also [42]) on the basis of unitary equivalence of $A^*A|_{\ker(A)^\perp}$ and $AA^*|_{\ker(A^*)^\perp}$ for a densely defined closed linear operator A in a (complex, separable) Hilbert space.

The double commutation method, on the other hand, allows one to insert eigenvalues into *any* spectral gap of $H(q)$. Although relatively simply implemented also, a complete spectral characterization of the double commutation method for Schrödinger-type operators was more recently achieved in [15] on the basis of Weyl–Titchmarsh m-function techniques and subsequently in [19] (for general Sturm–Liouville operators on arbitrary intervals) using a functional analytic approach based on the notion of (intertwining) transformation operators.

In this paper, we concentrate on the analogue of the double commutation method

for Dirac-type operators associated with the Lax operator for the focusing nonlinear Schrödinger (NLS$_-$) hierarchy. For q locally integrable, the Dirac-type operator corresponding to the Lax differential expression in the NLS$_-$ case is associated with the 2×2 matrix-valued differential expression

$$(1.2) \qquad M(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\overline{q} & -\frac{d}{dx} \end{pmatrix}$$

for $x \in \mathbb{R}$ (cf., e.g., [13, Part I, Sect. I.2], [48], and [49]). The maximally defined Dirac-type operator associated with $M(q)$ in the (two-component) Hilbert space $L^2(\mathbb{R})^2$ is then denoted by $D(q)$. By way of contrast, the corresponding (formally self-adjoint) Lax differential expression for the defocusing NLS$_+$ case is given by

$$(1.3) \qquad i \begin{pmatrix} \frac{d}{dx} & -q \\ \overline{q} & -\frac{d}{dx} \end{pmatrix} .$$

It turns out there is no natural analogue of the single commutation method for the Dirac operators associated with the focusing and defocusing nonlinear Schrödinger hierarchies (NLS$_\pm$). However, the complexified version of the NLS$_\pm$ hierarchies, the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy, supports two natural analogues of the single commutation method. In order to describe them briefly, we recall that the Dirac-type Lax differential expression associated with the AKNS hierarchy is given by

$$(1.4) \qquad M(p,q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix}$$

(cf., e.g., [1] and [17, Ch. 3]) in terms of two locally integrable coefficients p, q on $\mathbb{R}$. The focusing (NLS$_-$) and defocusing (NLS$_+$) nonlinear Schrödinger hierarchies are then associated with the constraints

$$(1.5) \qquad \text{NLS}_\pm: \; p(x) = \pm \overline{q(x)},$$

respectively. In this paper, we concentrate on the focusing NLS$_-$ case only.

The two analogues of the single commutation method for the AKNS case, which are usually called elementary Darboux transformations, can then be described as follows. Suppose that
(1.6)
$$M(p,q)\Psi(z_1,x) = z_1\Psi(z_1,x), \quad \Psi(z_1,x) = (\psi_1(z_1,x), \psi_2(z_1,x))^\top, \; (z_1,x) \in \mathbb{C} \times \mathbb{R},$$

and
(1.7)
$$M(p,q)\widetilde{\Psi}(\tilde{z}_1,x) = \tilde{z}_1\widetilde{\Psi}(\tilde{z}_1,x), \quad \widetilde{\Psi}(\tilde{z}_1,x) = (\tilde{\psi}_1(\tilde{z}_1,x), \tilde{\psi}_2(\tilde{z}_1,x))^\top, \; (\tilde{z}_1,x) \in \mathbb{C} \times \mathbb{R}.$$

Then the two elementary Darboux transformations in the AKNS context are given by (cf. [29], [30])

$$(1.8) \qquad (p, q) \mapsto (\hat{p}_{z_1}, \hat{q}_{z_1}),$$

where

$$(1.9) \qquad \begin{aligned} \hat{p}_{z_1}(x) &= -\psi_2(z_1, x)/\psi_1(z_1, x), \\ \hat{q}_{z_1}(x) &= q'(x) - \psi_2(z_1, x)/\psi_1(z_1, x)q(x)^2 + 2iz_1 q(x), \end{aligned}$$

and

$$(1.10) \qquad (p, q) \mapsto (\check{p}_{\tilde{z}_1}, \check{q}_{\tilde{z}_1}),$$

where

$$(1.11) \qquad \begin{aligned} \check{p}_{\tilde{z}_1}(x) &= -p'(x) + \tilde{\psi}_1(\tilde{z}_1, x)/\tilde{\psi}_2(\tilde{z}_1, x)p(x)^2 + 2i\tilde{z}_1 p(x), \\ \check{q}_{\tilde{z}_1}(x) &= \tilde{\psi}_1(\tilde{z}_1, x)/\tilde{\psi}_2(\tilde{z}_1, x). \end{aligned}$$

Similar to the case of Schrödinger operators, the analogue of the double commutation method for Dirac-type operators associated with (1.4) is then obtained by an appropriate composition of the two elementary Darboux transformations (1.9) and (1.11). This two-step procedure is denoted by

$$(1.12) \qquad (p, q) \mapsto (p^{(1)}_{z_1, \tilde{z}_1}, q^{(1)}_{z_1, \tilde{z}_1})$$

and leads to (cf., e.g., [29], [30], [36, Sect. 4.2], [41])

$$(1.13) \qquad \begin{aligned} p^{(1)}_{z_1, \tilde{z}_1}(x) &= p(x) - 2i(\tilde{z}_1 - z_1)\psi_2(z_1, x)\tilde{\psi}_2(\tilde{z}_1, x)/W(\Psi(z_1, x), \tilde{\Psi}(\tilde{z}_1, x)), \\ q^{(1)}_{z_1, \tilde{z}_1}(x) &= q(x) - 2i(\tilde{z}_1 - z_1)\psi_1(z_1, x)\tilde{\psi}_1(\tilde{z}_1, x)/W(\Psi(z_1, x), \tilde{\Psi}(\tilde{z}_1, x)). \end{aligned}$$

(Here $W(F, G)$ denotes the Wronskian of $F, G \in \mathbb{C}^2$). In contrast to (1.8), (1.9) and (1.10), (1.11), the two-step procedure (1.12), (1.13) with $\tilde{z}_1 = \overline{z_1}$, is compatible with the NLS$_\pm$ cases, and one explicitly obtains the following Darboux-type transformation in the NLS$_-$ case,

$$(1.14) \qquad q(x) \mapsto q^{(1)}_{z_1}(x) = q(x) + 4\operatorname{Im}(z_1)\frac{\psi_1(z_1, x)\overline{\psi_2(z_1, x)}}{|\psi_1(z_1, x)|^2 + |\psi_2(z_1, x)|^2}.$$

The transformation (1.14) for Dirac-type operators assoiated with (1.2) in the NLS$_-$ context represents the analogue of the double commutation method for Schrödinger operators and leads to locally nonsingular NLS$_-$ potentials $q^{(1)}_{z_1}$, assuming q to be free of local singularities.

By analogy to the KdV and Schrödinger operator case, one expects the NLS$_-$ potential $q_{z_1}^{(1)}(x)$ to produce an eigenvalue at the spectral point z_1 for the associated Dirac operator $D(q_{z_1}^{(1)})$, assuming z_1 to be a point in the resolvent set of the "background" operator $D(q)$. Actually, by a simple symmetry consideration, one expects a pair of eigenvalues $(z_1, \overline{z_1})$ in the point spectrum of $D(q_{z_1}^{(1)})$. To prove this fact and to show that the remaining spectral characteristics (especially, the essential spectrum of $D(q)$) remain invariant under the Darboux-type transformation (1.14) is the principal purpose of this paper. More precisely, if we denote by $q_{z_1,\dots,z_N}^{(N)}$ the NLS$_-$ potential obtained after an N-fold iteration of the Darboux-type transformation and by $D(q_{z_1,\dots,z_N}^{(N)})$ the resulting Dirac operator, we prove that

$$(1.15) \qquad \sigma\big(D(q_{z_1,\dots,z_N}^{(N)})\big) = \sigma(D(q)) \cup \{z_1, \overline{z}_1, \dots, z_N, \overline{z}_N\},$$

$$(1.16) \qquad \sigma_{\mathrm{p}}\big(D(q_{z_1,\dots,z_N}^{(N)})\big) = \sigma_{\mathrm{p}}(D(q)) \cup \{z_1, \overline{z}_1, \dots, z_N, \overline{z}_N\},$$

$$(1.17) \qquad \sigma_{\mathrm{e}}\big(D(q_{z_1,\dots,z_N}^{(N)})\big) = \sigma_{\mathrm{e}}(D(q)).$$

(Here $\sigma(S)$, $\sigma_{\mathrm{p}}(S)$, and $\sigma_{\mathrm{e}}(S)$ denote the spectrum, point spectrum, and essential spectrum of a densely defined closed operator S in a complex separable Hilbert space $\mathcal{H}$; cf. Section 5 for more details on spectra.) Actually, we go a step beyond (1.15)–(1.17) and establish the existence of bounded transformation operators which intertwine $D(q_{z_1,\dots,z_N}^{(N)})$ and $D(q)$.

When trying to prove results of the type (1.15)–(1.17) for Dirac-type operators associated with the differential expression (1.2) in the NLS$_-$ context, one is at a distinct disadvantage compared to the case of Schrödinger operators with real-valued potentials. While L in (1.1) is formally self-adjoint for q real-valued, $M(q)$ in (1.2) is never self-adjoint (except in the trivial case $q = 0$). As a consequence, the original approach to a complete spectral characterization of the double commutation method for Schrödinger operators in [15], based on Weyl–Titchmarsh theory and hence on spectral theory, is doomed from the start, as there simply is no spectral function and general L^2-eigenfunction expansion, etc., for non-self-adjoint Dirac-type operators associated with (1.2) under our general hypothesis $q \in L_{\mathrm{loc}}^1(\mathbb{R})$. That leaves only one possible line of attack, the analogue of the transformation operator approach developed in the general Sturm–Liouville context in [19]. As it turns out in Section 6, this approach is indeed successful, although it requires more sophisticated and elaborate arguments compared to those in [19].

While the differential expression $M(q)$ in (1.2) is never formally self-adjoint (if $q \neq 0$), it is formally $\mathcal{J}$-self-adjoint, that is,

$$(1.18) \qquad \mathcal{J} M(q) \mathcal{J} = M(q)^*.$$

Here $\mathcal{J}$ is defined by

$$(1.19) \qquad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C},$$

and $\mathcal{C}$ denotes the antilinear conjugation map in $\mathbb{C}^2$,

$$(1.20) \qquad \mathcal{C}(a,b)^\top = (\bar{a}, \bar{b})^\top, \quad a, b \in \mathbb{C}.$$

As one of our principal results in this paper, we prove in Section 3 that under the most general hypothesis $q \in L^1_{\text{loc}}(\mathbb{R})$, the maximally defined Dirac operator $D(q)$ associated with $M(q)$ is in fact $\mathcal{J}$-self-adjoint,

$$(1.21) \qquad \mathcal{J} D(q) \mathcal{J} = D(q)^* \ (= D(-q)).$$

As an aside, we should mention that the corresponding maximally defined Lax operator associated with the defocusing NLS_+ differential expression (1.3) is in fact self-adjoint, assuming $q \in L^1_{\text{loc}}(\mathbb{R})$ only (this is proved in the references mentioned in Section 3). This should be contrasted with the case of one-dimensional Schrödinger differential expressions $L(q)$ in (1.1) (the Lax differential expression associated with the KdV hierarchy). If $q \in L^1_{\text{loc}}(\mathbb{R})$ in (1.1) is real-valued, then $L(q)$ is formally self-adjoint, but the maximally defined operator $H(q)$ in $L^2(\mathbb{R})$ associated with $L(q)$ may not be self-adjoint. The latter situation occurs precisely when $L(q)$ is in the limit circle case (as opposed to the limit point case) at $+\infty$ and/or $-\infty$ (cf. [8, Ch. 9]). This is in sharp contrast to the focusing (respectively, defocusing) NLS case where $D(q)$ is always $\mathcal{J}$-self-adjoint (respectively, self-adjoint).

Summarizing, we derive the following principal new results in this paper, assuming the optimal condition $q \in L^1_{\text{loc}}(\mathbb{R})$.

- $\mathcal{J}$-self-adjointness of $D(q)$.
- The existence of Weyl–Titchmarsh-type solutions of $M(q)\Psi = z\Psi$ for all $z \in \rho(D(q))$.
- The existence and boundedness of transformation operators intertwining the operators $D(q^{(N)})$ and $D(q)$.
- A spectral analysis of NLS_- Darboux transformations (cf. (1.15)–(1.17)).

Finally, we briefly describe the content of each section. Section 2 introduces our main notation and then reviews Darboux transformations for AKNS and NLS_- systems. Section 3 is devoted to a proof of the $\mathcal{J}$-self-adjointness property (1.21) of $D(q)$, assuming $q \in L^1_{\text{loc}}(\mathbb{R})$ only. Section 4 constructs eigenvalues of $D\left(q_{z_1}^{(1)}\right)$ at pairs $z_1, \bar{z}_1$, $z_1 \in \rho(D(q))$ and associated $L^2(\mathbb{R})^2$-eigenfunctions. Section 5 derives some basic spectral properties of general Dirac-type operators $D(q)$ and establishes

the existence of Weyl–Titchmarsh-type solutions associated with $M(q)$. This shows a remarkable similarity to self-adjoint systems and appears to be without precedent in this non-self-adjoint context. Our final Section 6 establishes the existence of bounded transformation operators intertwining $D\left(q_{z_1}^{(1)}\right)$ and $D(q)$ and then employs these transformation operators to prove the spectral properties (1.15)–(1.17).

All results in the principal part of this paper, Sections 3–6, are proved under the optimal condition $q \in L^1_{loc}(\mathbb{R})$. Moreover, practically all results in Sections 3–6 are new as long as one goes beyond bounded or periodic potentials q. In particular, Theorem 6.14 (characterizing transformation operators) and Theorem 6.15 (proving (1.15)–(1.17)) appear to be the first of their kind under any assumptions on q.

In this paper, we confine ourselves to a stationary (i.e., time-independent) approach only. Applications to the time-dependent focusing NLS_ equation and to nonlinear optics will be made in a subsequent paper [3].

2 Darboux-type transformations for AKNS and NLS_ systems

In this section, we take a close look at Darboux-type transformations for non-self-adjoint Dirac-type differential expressions $M(q)$ (cf. (2.3)) applicable to AKNS systems, with special emphasis on the case of the focusing nonlinear Schrödinger equation NLS_ (cf. (2.7)).

Throughout this paper, we use the following notation: $' = d/dx$; for a matrix A with complex-valued entries, $A^\top$ denotes the transposed matrix, $\overline{A}$ the matrix with complex conjugate entries, and $A^* = \overline{A}^\top = \overline{A^\top}$ the adjoint matrix. Occasionally, we use the following 2×2 matrices,

(2.1)
$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_4 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \sigma_5 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sigma_6 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $A = (a_1, a_2)^\top$ and $B = (b_1, b_2)^\top$ are 2×1 column-vectors, then $(A, B)_{\mathbb{C}^2} = B^*A = a_1\overline{b_1} + a_2\overline{b_2}$ denotes the usual scalar product in $\mathbb{C}^2$, $\|A\|_{\mathbb{C}^2} = (|a_1|^2 + |a_2|^2)^{1/2}$ the associated norm, and $A^\perp = (a_2, -a_1) = A^\top\sigma_4$ the 1×2 row-vector perpendicular to A (in the sense that $A^\perp A = 0$). We also use the notation

(2.2) $W(A, B) = B^\perp A = -A^\perp B = a_1 b_2 - a_2 b_1 = \left(A, \overline{B^\perp}\right)_{\mathbb{C}^2}$

for the Wronskian of A and B. The space of 2×2 matrices with entries in $\mathbb{C}$ is denoted by $\mathbb{C}^{2\times 2}$, and the operator norm of a 2×2 matrix A induced by the

usual norm in $\mathbb{C}^2$ is denoted by $\|A\|_{\mathbb{C}^{2\times 2}}$. In the following, $\Omega \subseteq \mathbb{R}$ denotes an open subset of $\mathbb{R}$ and $AC_{loc}(\Omega)^{2\times m}$, $m = 1, 2$, denotes the set of $2 \times m$ matrices with locally absolutely continuous entries on Ω (for 2×1 columns we use the corresponding notation $AC_{loc}(\Omega)^2$). We define[1] $L^p(\Omega)^{2\times m}$ and $L^p_{loc}(\Omega)^{2\times m}$, $m = 1, 2$, to consist of $2 \times m$ matrices with entries in $L^p(\Omega)$ and $L^p_{loc}(\Omega)$, respectively. In the special case $\Omega = \mathbb{R}$ and $F, G \in L^2(\mathbb{R})^2$, the scalar product of F and G is denoted by $(F,G)_{L^2} = \int_{\mathbb{R}} dx\, (F(x), G(x))_{\mathbb{C}^2}$ with associated norm of F given by $\|F\|_{L^2} = \left(\int_{\mathbb{R}} dx\, \|F(x)\|^2_{\mathbb{C}^2} \right)^{1/2}$. Finally, the open complex upper (respectively, lower) half-plane is denoted by $\mathbb{C}_+$ (respectively, $\mathbb{C}_-$); the domain, range, and kernel (null space) of a linear operator T are denoted by $\text{dom}(T)$, $\text{ran}(T)$, and $\ker(T)$, respectively.

Hypothesis 2.1. *Let $\Omega \subseteq \mathbb{R}$ open and assume $p, q \in L^1_{loc}(\Omega)$.*

Assuming Hypothesis 2.1 and $z \in \mathbb{C}$, we introduce the 2×2 matrix $U(z, p, q)$ and the 2×2 matrix-valued differential expression $M(p, q)$ by

$$(2.3) \qquad U(z, p, q) = \begin{pmatrix} -iz & q \\ p & iz \end{pmatrix}, \quad M(p, q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ p & -\frac{d}{dx} \end{pmatrix}.$$

The functions p and q in (2.3) are referred to as AKNS *potentials* since $M(p, q)$ is the Lax differential expression associated with the AKNS hierarchy (see, e.g., [1] and [17, Ch. 3]). The particularly important special case $p = -\bar{q}$ will be referred to as the NLS_ case (because of the obvious connection of (2.3) with the zero curvature representation and the Lax operator for the focusing nonlinear Schrödinger equation; see, e.g., [13, Part 1, Sect. I.2], [48], and [49]), and then q is called an NLS_ *potential*. For given $z \in \mathbb{C}$, $\Omega \subseteq \mathbb{R}$, and AKNS potentials (p, q), a function $\Psi(z, \cdot) \in AC_{loc}(\Omega)^2$ is called a *z-wave function associated with* (p, q) *on* Ω if $\Psi'(z, x) = U(z, p, q)\Psi(z, x)$ holds for a.e. $x \in \Omega$, that is, if $\Psi = (\psi_1, \psi_2)^\top$ satisfies the first-order system of differential equations

$$(2.4)$$
$$\psi_1'(z, x) = -iz\psi_1(z, x) + q(x)\psi_2(z, x), \quad \psi_2'(z, x) = iz\psi_2(z, x) + p(x)\psi_1(z, x)$$

for a.e. $x \in \Omega$. Equivalently, $\Psi(z) = (\psi_1(z), \psi_2(z))^\top \in AC_{loc}(\Omega)^2$ is a z-wave function associated with (p, q) on Ω if and only if $M(p, q)\Psi(z) = z\Psi(z)$ on Ω in the distributional sense. If for some $z \in \mathbb{C}$, $\Psi(z)$ and $\Phi(z)$ are z-wave functions associated with (p, q) on Ω, their Wronskian is well-known to be constant with

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[1] For brevity we write $L^p(\Omega)$ for $L^p(\Omega; dx)$ and suppress the Lebesgue measure dx whenever possible.

respect to $x \in \Omega$,

$$(2.5) \qquad \frac{d}{dx} W(\Psi(z,x), \Phi(z,x)) = 0, \quad x \in \Omega.$$

More generally, if $\Psi(z_1)$ and $\Phi(z_2)$ are z_1- and z_2-wave functions associated with (p,q) on Ω, then

$$(2.6)$$
$$\frac{d}{dx} W(\Psi(z_1,x), \Phi(z_2,x)) = i(z_2 - z_1)[\psi_1(z_1,x)\phi_2(z_2,x) + \psi_2(z_1,x)\phi_1(z_2,x)]$$
$$= i(z_2 - z_1)\Psi(z_1,x)^\top \sigma_1 \Phi(z_2,x),$$

$$\Psi(z_1) = (\psi_1(z_1), \psi_2(z_1))^\top, \quad \Phi(z_2) = (\phi_1(z_2), \phi_2(z_2))^\top, \quad z_1, z_2 \in \mathbb{C}, \quad x \in \Omega.$$

In the NLS$_-$ case $p = -\bar{q}$ we use the notation

$$(2.7) \qquad U(z,q) = \begin{pmatrix} -iz & q(x) \\ -\overline{q(x)} & iz \end{pmatrix}, \quad M(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\bar{q} & -\frac{d}{dx} \end{pmatrix}$$

instead of (2.3); we then call any distributional solution $\Psi(z)$ of $M(q)\Psi(z) = z\Psi(z)$ an NLS$_-$ z-wave function associated with q.

If $\Psi(z)$ is a z-wave function associated with (p,q), and $\Psi(z,x_0) = 0$ for some $x_0 \in \Omega$, then $\Psi(z,x)$ vanishes identically for all x in an open neighborhood of x_0 in Ω by the unique solvability of the Cauchy problem for (2.4). Therefore, we always assume that $\Psi(z,x) \neq 0$ for all $x \in \Omega$.

If $q = 0$ (and analogueously if $p = 0$) a.e. on Ω, then the system (2.4) decomposes and yields

$$(2.8) \qquad \begin{aligned} \psi_1(z,x) &= C_1 \exp(-izx), \\ \psi_2(z,x) &= C_1 \left(\int_{x_0}^x dx'\, p(x') \exp(-2izx') \right) \exp(izx) + C_2 \exp(izx) \end{aligned}$$

for some constants $C_1, C_2 \in \mathbb{C}$, $x_0 \in \Omega$.

For further use, we collect some simple consequences of (2.4). First we introduce the antilinear (i.e., conjugate linear) involution $\mathcal{J}$ defined by

$$(2.9) \qquad \mathcal{J} = \sigma_1 \mathcal{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{C}, \quad \mathcal{J}^2 = I_2,$$

with $\mathcal{C}$ the antilinear conjugation map

$$(2.10) \qquad \mathcal{C}(a,b)^\top = (\bar{a}, \bar{b})^\top, \quad a, b \in \mathbb{C}.$$

Moreover, we introduce the antilinear operator $\mathcal{K}$ defined by

$$(2.11) \qquad \mathcal{K} = \sigma_4 \mathcal{C} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathcal{C}, \quad \mathcal{K}^2 = -I_2.$$

We also note that

$$(2.12) \qquad \|\mathcal{J}F\|_{\mathbb{C}^2} = \|F\|_{\mathbb{C}^2}, \quad \|\mathcal{K}F\|_{\mathbb{C}^2} = \|F\|_{\mathbb{C}^2}, \quad F \in \mathbb{C}^2.$$

Lemma 2.2. *Assume Hypothesis 2.1, $z \in \mathbb{C}$, and suppose*

$$\Psi(z) = (\psi_1(z), \psi_2(z))^\top$$

is a z-wave function associated with (p, q) on Ω. Then the following assertions hold.

(i) $\phi(z, x) = -\psi_2(z, x)/\psi_1(z, x)$ *satisfies the Riccati-type equation*

$$(2.13) \qquad -\phi'(z, x) + q(x)\phi(z, x)^2 + 2iz\phi(z, x) - p(x) = 0$$

on the set $\{x \in \Omega : \psi_1(z, x) \neq 0\}$.

(ii) $\varphi(z, x) = \psi_1(z, x)/\psi_2(z, x)$ *satisfies the Riccati-type equation*

$$(2.14) \qquad -\varphi'(z, x) - p(x)\varphi(z, x)^2 - 2iz\varphi(z, x) + q(x) = 0$$

on the set $\{x \in \Omega : \psi_2(z, x) \neq 0\}$.

Assume in addition the NLS$_-$ case $p = -\bar{q}$. Then the following assertions hold.

(iii) *If $\Psi(z) = (\psi_1(z), \psi_2(z))^\top$ is a z-wave function associated with q, then*

$$(2.15) \qquad \mathcal{K}\Psi(z) = \sigma_4 \mathcal{C}\Psi(z) = \left(\overline{\psi_2(z)}, -\overline{\psi_1(z)}\right)^\top = (\Psi(z)^\perp)^*$$

is a $\bar{z}$-wave function associated with q and

$$(2.16) \qquad M(q) = -\mathcal{K}M(q)\mathcal{K}.$$

(iv) *The following identity holds,*

$$(2.17) \qquad \left(\|\Psi(z, x)\|_{\mathbb{C}^2}^2\right)' = 2\mathrm{Im}(z)\left[|\psi_1(z, x)|^2 - |\psi_2(z, x)|^2\right].$$

(v) *$M(q)$ and $M(q)^*$ are formally unitarily equivalent in the sense that*

$$(2.18) \qquad M(q)^* = \sigma_3 M(q)\sigma_3 = M(-q).$$

In addition, $M(q)$ is formally $\mathcal{J}$-self-adjoint in the sense that

$$(2.19) \qquad \mathcal{J}M(q)\mathcal{J} = M(q)^*.$$

Consider AKNS potentials $p, q \in L^1_{loc}(\Omega)$. Fix $z, z_1, \tilde{z}_1 \in \mathbb{C}$, a z_1-wave function $\Psi(z_1)$, and a $\tilde{z}_1$-wave function $\tilde{\Psi}(\tilde{z}_1)$ associated with (p, q). Our objective is to construct new potentials $p^{(1)}, q^{(1)} \in L^1_{loc}(\Omega^{(1)})$ for some open set $\Omega^{(1)} \subseteq \Omega$ and the corresponding z-wave functions associated with $(p^{(1)}, q^{(1)})$ on $\Omega^{(1)}$. In the NLS_ case $p = -\bar{q}$, we choose $\tilde{z}_1 = \overline{z_1}$ and $\tilde{\Psi}(\overline{z_1}) = \mathcal{K}\Psi(z_1)$; see Lemma 2.2 (iii).

Remark 2.3. Let $\Gamma \in AC_{loc}(\Omega)^{2\times 2}$, $A, B \in L^1(\Omega)^{2\times 2}$ for some open subset $\Omega \subseteq \mathbb{R}$, suppose that

$$(2.20) \qquad \Gamma'(x) + \Gamma(x)A(x) - B(x)\Gamma(x) = 0 \quad \text{for a.e. } x \in \Omega,$$

and assume that $\Phi \in AC_{loc}(\Omega)^2$ satisfies the first-order system $\Phi' = A\Phi$ on Ω. Then the function $\Phi^{(1)}$, defined by $\Phi^{(1)} = \Gamma\Phi$, satisfies $\Phi^{(1)} \in AC_{loc}(\Omega)^2$ and the first-order system $(\Phi^{(1)})' = B\Phi^{(1)}$ on Ω.

Lemma 2.4. *Assume Hypothesis 2.1 and $z, z_1 \in \mathbb{C}$. In addition, suppose $\Psi(z_1) = (\psi_1(z_1), \psi_2(z_1))^\top$ is a z_1-wave function associated with (p, q) on Ω and introduce*

$$(2.21) \qquad \hat{\Omega}_{z_1} = \{x \in \Omega : \psi_1(z_1, x) \neq 0\}.$$

Define $\hat{p}_{z_1}$, $\hat{q}_{z_1}$, $\hat{\Gamma}_0(q, \hat{p}_{z_1})$, and $\hat{\Gamma}(z, q, \hat{p}_{z_1})$ on $\hat{\Omega}_{z_1}$ by

$$(2.22) \quad \hat{p}_{z_1}(x) = -\psi_2(z_1, x)/\psi_1(z_1, x), \quad \hat{q}_{z_1}(x) = q'(x) + \hat{p}_{z_1}(x)q(x)^2 + 2iz_1 q(x),$$

$$(2.23) \quad \hat{\Gamma}_0(x, q, \hat{p}_{z_1}) = -\frac{1}{2}\begin{pmatrix} q(x)\hat{p}_{z_1}(x) & q(x) \\ \hat{p}_{z_1}(x) & 1 \end{pmatrix},$$

$$(2.24) \quad \hat{\Gamma}(z, x, q, \hat{p}_{z_1}) = i(z - z_1)\sigma_5 + \hat{\Gamma}_0(x, q, \hat{p}_{z_1}).$$

Then $\Gamma = \hat{\Gamma}$ satisfies (2.20) on $\hat{\Omega}_{z_1}$, with A and B given by

$$(2.25) \qquad A(z, x) = U(z, p, q), \quad B(z, x, z_1) = U(z, \hat{p}_{z_1}, \hat{q}_{z_1}), \quad x \in \hat{\Omega}_{z_1}.$$

Proof. Using Lemma 2.2 (i) one verifies that $\Gamma = \hat{\Gamma}_0$ satisfies (2.20) with $A = U(z_1, p, q)$ and $B = U(z_1, \hat{p}_{z_1}, \hat{q}_{z_1})$. Since $U(z) = U(z_1) - i(z - z_1)\sigma_3$, the conclusion follows. $\qquad\qquad\square$

Similarly, by Lemma 2.2 (ii), one has the following analogueous result.

Lemma 2.5. *Assume Hypothesis 2.1 and let* $z, \tilde{z}_1 \in \mathbb{C}$. *In addition, suppose that* $\check{\Psi}(\tilde{z}_1) = (\check{\psi}_1(\tilde{z}_1), \check{\psi}_2(\tilde{z}_1))^\top$ *is a* $\tilde{z}_1$-*wave function associated with* (p,q) *on* Ω *and introduce*

$$(2.26) \qquad \check{\Omega}_{\tilde{z}_1} = \{x \in \Omega : \check{\psi}_2(\tilde{z}_1, x) \neq 0\}.$$

Define $\check{p}_{\tilde{z}_1}$, $\check{q}_{\tilde{z}_1}$, $\check{\Gamma}_0(p, \check{q}_{\tilde{z}_1})$, *and* $\check{\Gamma}(z, p, \check{q}_{\tilde{z}_1})$ *on* $\check{\Omega}_{\tilde{z}_1}$ *by*

$$(2.27) \quad \check{p}_{\tilde{z}_1}(x) = -p'(x) + \check{q}_{\tilde{z}_1}(x)p(x)^2 + 2i\tilde{z}_1 p(x), \quad \check{q}_{\tilde{z}_1}(x) = \check{\psi}_1(\tilde{z}_1, x)/\check{\psi}_2(\tilde{z}_1, x),$$

$$(2.28) \quad \check{\Gamma}_0(x, p, \check{q}_{\tilde{z}_1}) = \frac{1}{2}\begin{pmatrix} -1 & \check{q}_{\tilde{z}_1}(x) \\ p(x) & -p(x)\check{q}_{\tilde{z}_1}(x) \end{pmatrix},$$

$$(2.29) \quad \check{\Gamma}(z, x, p, \check{q}_{\tilde{z}_1}) = i(z - \tilde{z}_1)\sigma_6 + \check{\Gamma}_0(x, p, \check{q}_{\tilde{z}_1}).$$

Then $\Gamma = \check{\Gamma}$ *satisfies* (2.20) *on* $\check{\Omega}_{\tilde{z}_1}$, *with* A *and* B *given by*

$$(2.30) \qquad A(z, x) = U(z, p, q), \quad B(z, x, \tilde{z}_1) = U(z, \check{p}_{\tilde{z}_1}, \check{q}_{\tilde{z}_1}), \quad x \in \check{\Omega}_{\tilde{z}_1}.$$

The Darboux-type transformations characterized by (2.24) and (2.29) are also called elementary Darboux transformations. They have been discussed, for instance, in [29] and [30]. In the special context of algebro-geometric AKNS solutions, the effect of elementary Darboux transformations on the underlying compact hyperelliptic curve (in connection with the insertion and deletion of eigenvalues as well as the isospectral case) was studied in detail in [16], [17, App. G] (see also [20], [21]).

Next, we construct the transformation matrix $\Gamma(z, \Psi(z_1), \check{\Psi}(\tilde{z}_1))$ that satisfies equation (2.20) with $A(z, x) = U(z, p, q)$ and $B(z, x) = U\big(z, p^{(1)}_{z_1, \tilde{z}_1}, q^{(1)}_{z_1, \tilde{z}_1}\big)$ as the product of $\check{\Gamma}(z, \check{p}_{z_1}, \check{q}_{\tilde{z}_1})$ and $\check{\Gamma}(z, q, \hat{p}_{z_1})$. Since we choose $\tilde{z}_1 = \overline{z_1}$ in the NLS_ context, we omit the $\tilde{z}_1$-dependence in $p^{(1)}$, $q^{(1)}$, $\Omega^{(1)}$, $\Phi^{(1)}$, etc., in the NLS_ case in the following.

Theorem 2.6. *Assume Hypothesis 2.1.*

(i) *Suppose* $z, z_1, \tilde{z}_1 \in \mathbb{C}$ *and assume that* $\Psi(z_1) = (\psi_1(z_1), \psi_2(z_1))^\top$ *and* $\check{\Psi}(\tilde{z}_1) = (\check{\psi}_1(\tilde{z}_1), \check{\psi}_2(\tilde{z}_1))^\top$ *are* z_1- *and* $\tilde{z}_1$-*wave functions, respectively, associated with* (p, q) *on* Ω. *In addition, introduce*

$$(2.31) \qquad \Omega^{(1)}_{z_1, \tilde{z}_1} = \{x \in \Omega : W(\Psi(z_1, x), \check{\Psi}(\tilde{z}_1, x)) \neq 0\}.$$

Define $p^{(1)}_{z_1, \tilde{z}_1}$, $q^{(1)}_{z_1, \tilde{z}_1}$, *and* $\Gamma(z, \Psi(z_1), \check{\Psi}(\tilde{z}_1))$ *on* $\Omega^{(1)}_{z_1, \tilde{z}_1}$ *by*

(2.32)
$$p^{(1)}_{z_1, \tilde{z}_1}(x) = p(x) - 2i(\tilde{z}_1 - z_1)\psi_2(z_1, x)\check{\psi}_2(\tilde{z}_1, x)/W(\Psi(z_1, x), \check{\Psi}(\tilde{z}_1, x)),$$
(2.33)
$$q^{(1)}_{z_1, \tilde{z}_1}(x) = q(x) - 2i(\tilde{z}_1 - z_1)\psi_1(z_1, x)\check{\psi}_1(\tilde{z}_1, x)/W(\Psi(z_1, x), \check{\Psi}(\tilde{z}_1, x)),$$

$$\Gamma(z, x, \Psi(z_1), \tilde{\Psi}(\tilde{z}_1)) = -\frac{i}{2}zI_2 - \frac{i}{2}W(\Psi(z_1, x), \tilde{\Psi}(\tilde{z}_1, x))^{-1}$$

$$(2.34) \qquad \times \left(\tilde{z}_1\tilde{\Psi}(\tilde{z}_1, x)\Psi(z_1, x)^{\perp} - z_1\Psi(z_1, x)\tilde{\Psi}(\tilde{z}_1, x)^{\perp}\right).$$

Then Γ satisfies the first-order system

$$\Gamma'(z, x, \Psi(z_1), \tilde{\Psi}(\tilde{z}_1)) + \Gamma(z, x, \Psi(z_1), \tilde{\Psi}(\tilde{z}_1))U(z, p, q)$$

$$(2.35) \qquad - U\left(z, p_{z_1, \tilde{z}_1}^{(1)}, q_{z_1, \tilde{z}_1}^{(1)}\right)\Gamma(z, x, \Psi(z_1), \tilde{\Psi}(\tilde{z}_1)) = 0$$

a.e. on $\Omega_{z_1, \tilde{z}_1}^{(1)}$. Thus, if $\Upsilon(z)$ is a z-wave function associated with (p, q) on Ω, then $\Upsilon_{z_1, \tilde{z}_1}^{(1)}(z)$, defined by

$$(2.36) \qquad \Upsilon_{z_1, \tilde{z}_1}^{(1)}(z, x) = \Gamma(z, x, \Psi(z_1), \tilde{\Psi}(\tilde{z}_1))\Upsilon(z, x),$$

is a z-wave function associated with $(p_{z_1, \tilde{z}_1}^{(1)}, q_{z_1, \tilde{z}_1}^{(1)})$ on $\Omega_{z_1, \tilde{z}_1}^{(1)}$.

(ii) *Assume the* NLS_ *case $p = -\overline{q}$ and $z, z_1 \in \mathbb{C}$. Then, for $\tilde{z}_1 = \overline{z_1}$, $\tilde{\Psi}(\tilde{z}_1) = \mathcal{K}\Psi(z_1) = (\overline{\psi_2(z_1)}, -\overline{\psi_1(z_1)})^{\mathsf{T}}$, and*

$$(2.37) \qquad \Omega_{z_1}^{(1)} = \{x \in \Omega : W(\Psi(z_1, x), \mathcal{K}\Psi(z_1, x)) \neq 0\},$$

formulas (2.33)–(2.34) simplify to

$$(2.38)$$
$$q_{z_1}^{(1)}(x) = q(x) + 4\mathrm{Im}(z_1)\psi_1(z_1, x)\overline{\psi_2(z_1, x)}\|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2},$$
$$p_{z_1}^{(1)}(x) = p(x) - 4\mathrm{Im}(z_1)\psi_2(z_1, x)\overline{\psi_1(z_1, x)}\|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2}$$

$$(2.39) \qquad = -\overline{q_{z_1}^{(1)}(x)},$$

$$\Gamma(z, x, \Psi(z_1), \mathcal{K}\Psi(z_1)) = -(i/2)(z - z_1)I_2$$

$$(2.40) \qquad + \mathrm{Im}(z_1)\|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1, x)\Psi(z_1, x)^{\perp}$$

for $x \in \Omega_{z_1}^{(1)}$. In particular, if $\Upsilon(z)$ is a z-wave function associated with q on Ω, then $\Upsilon_{z_1}^{(1)}(z)$, defined by

$$(2.41) \qquad \Upsilon_{z_1}^{(1)}(z, x) = \Gamma(z, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Upsilon(z, x),$$

is a z-wave function associated with $q_{z_1}^{(1)}$ on $\Omega_{z_1}^{(1)}$ (which may vanish identically w.r.t. $x \in \Omega_{z_1}^{(1)}$; cf. Remark 2.8).

Proof. First, we use Lemma 2.4 for $z = \tilde{z}_1$ and $\Psi(z_1)$ to construct $\hat{p}_{z_1}$, $\hat{q}_{z_1}$, $\hat{\Gamma}(\tilde{z}_1, q, \hat{p}_{z_1})$ as in (2.22)–(2.24). Define $\hat{\Psi}_{z_1}(\tilde{z}_1) = \hat{\Gamma}(\tilde{z}_1, q, \hat{p}_{z_1})\Psi(\tilde{z}_1)$. By

Lemma 2.4 and Remark 2.3, we conclude that $\hat{\Psi}_{z_1}(\tilde{z}_1)$ is a $\tilde{z}_1$-wave function associated with $(\hat{p}_{z_1}, \hat{q}_{z_1})$. Moreover,

$$
\begin{aligned}
\hat{\Psi}_{z_1}(\tilde{z}_1, x) = {} & i(\tilde{z}_1 - z_1)\psi_1(\tilde{z}_1, x)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
& - \frac{1}{2}\begin{pmatrix} q(x) & 0 \\ 0 & 1 \end{pmatrix}(\hat{p}_{z_1}(x)\psi_1(\tilde{z}_1, x) + \psi_2(\tilde{z}_1, x))\begin{pmatrix} 1 \\ 1 \end{pmatrix},
\end{aligned}
\tag{2.42}
$$

where $\hat{p}_{z_1}$ is defined in (2.22). We now apply Lemma 2.5, replacing $\tilde{\Psi}(\tilde{z}_1)$ by $\hat{\Psi}_{z_1}(\tilde{z}_1)$ and (p, q) by $(\hat{p}_{z_1}, \hat{q}_{z_1})$. Then $\check{q}_{z_1, \tilde{z}_1} = \hat{\psi}_{1,z_1}(\tilde{z}_1)/\hat{\psi}_{2,z_1}(\tilde{z}_1)$, as required by (2.27), coincides with $q^{(1)}_{z_1, \tilde{z}_1}$, as defined in (2.33),

$$
\check{q}_{z_1, \tilde{z}_1}(x) = q^{(1)}_{z_1, \tilde{z}_1}(x).
\tag{2.43}
$$

By formula (2.27) for $\check{p}_{\tilde{z}_1, z_1}$ and Lemma 2.2 (i) for $\hat{p}_{z_1} = -\psi_2(z_1)/\psi_1(z_1)$, one infers

$$
\begin{aligned}
\check{p}_{\tilde{z}_1, z_1}(x) &= -\hat{p}'_{z_1}(x) + q^{(1)}_{z_1, \tilde{z}_1}(x)\hat{p}_{z_1}(x)^2 + 2i\tilde{z}_1\hat{p}_{z_1}(x) \\
&= -\big(q(x)\hat{p}_{z_1}(x)^2 + 2iz_1\hat{p}_{z_1}(x) - p(x)\big) + q^{(1)}_{z_1, \tilde{z}_1}(x)\hat{p}_{z_1}(x)^2 + 2i\tilde{z}_1\hat{p}_{z_1}(x) \\
&= p^{(1)}_{z_1, \tilde{z}_1}(x).
\end{aligned}
\tag{2.44}
$$

Using (2.29) and (2.22)–(2.24), one computes

$$\tag{2.45}$$

$$
\begin{aligned}
& \check{\Gamma}(z, \hat{p}_{z_1}, \check{q}_{z_1, \tilde{z}_1})\hat{\Gamma}(z, q, \hat{p}_{z_1}) \\
&= -\left((z - \tilde{z}_1)\sigma_6 + \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -\hat{p}_{z_1} \end{pmatrix}\begin{pmatrix} 1 & -\check{q}_{z_1, \tilde{z}_1} \\ 1 & -\check{q}_{z_1, \tilde{z}_1} \end{pmatrix}\right) \\
&\quad \times \left((z - z_1)\sigma_5 + \frac{i}{2}\begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{p}_{z_1} & 1 \\ \hat{p}_{z_1} & 1 \end{pmatrix}\right) \\
&= -(i/2)zI_2 - (i/2)W(\Psi(z_1), \tilde{\Psi}(\tilde{z}_1))^{-1} \\
&\quad \times \begin{pmatrix} \tilde{z}_1\psi_2(z_1)\tilde{\psi}_1(\tilde{z}_1) - z_1\psi_1(z_1)\tilde{\psi}_2(\tilde{z}_1) & -(\tilde{z}_1 - z_1)\psi_1(z_1)\tilde{\psi}_1(\tilde{z}_1) \\ (\tilde{z}_1 - z_1)\psi_2(z_1)\tilde{\psi}_2(\tilde{z}_1) & -\tilde{z}_1\psi_1(z_1)\tilde{\psi}_2(\tilde{z}_1) + z_1\psi_2(z_1)\tilde{\psi}_1(\tilde{z}_1) \end{pmatrix} \\
&= \Gamma(z, \Psi(z_1), \tilde{\Psi}(\tilde{z}_1)).
\end{aligned}
$$

To check (2.35), one uses Lemmas 2.4 and 2.5,

$$\Gamma'(z, \Psi(z_1), \check{\Psi}(\check{z}_1))$$
$$= \check{\Gamma}'(z, \hat{p}_{z_1}, \check{q}_{z_1, \check{z}_1})\hat{\Gamma}(z, q, \hat{p}_{z_1}) + \check{\Gamma}(z, \hat{p}_{z_1}, \check{q}_{z_1, \check{z}_1})\hat{\Gamma}'(z, q, \hat{p}_{z_1})$$
$$= [-\check{\Gamma}(z, \hat{p}_{z_1}, \check{q}_{z_1, \check{z}_1})U(z, \hat{p}_{z_1}, \hat{q}_{z_1})$$
$$\quad + U(z, \check{p}_{z_1, \check{z}_1}, \check{q}_{z_1, \check{z}_1})\check{\Gamma}(z, \hat{p}_{z_1}, \check{q}_{z_1, \check{z}_1})]\hat{\Gamma}(z, q, \hat{p}_{z_1})$$
$$\quad + \check{\Gamma}(z, \hat{p}_{z_1}, \check{q}_{z_1, \check{z}_1})[-\hat{\Gamma}(z, q, \hat{p}_{z_1})U(z, p, q) + U(z, \hat{p}_{z_1}, \hat{q}_{z_1})\hat{\Gamma}(z, q, \hat{p}_{z_1})]$$

$$(2.46) \qquad = U(z, \check{p}_{z_1, \check{z}_1}, \check{q}_{z_1, \check{z}_1})\Gamma(z, \Psi(z_1), \check{\Psi}(\check{z}_1)) - \Gamma(z, \Psi(z_1), \check{\Psi}(\check{z}_1))U(z, p, q).$$

Formulas (2.39)–(2.40) follow from (2.32)–(2.34), since by Lemma 2.2 (iii),

$$(2.47) \qquad W(\Psi(z_1), \check{\Psi}(\overline{z_1})) = W(\Psi(z_1), \mathcal{K}\Psi(z_1)) = -\|\Psi(z_1)\|_{\mathbb{C}^2}^2. \qquad \square$$

For a general treatment of Bäcklund (Darboux) and gauge transformations and their interrelations we refer, for instance, to [36, Sect. 4.1] and [41].

Finally, we add a few more facts valid in the NLS_ case.

Remark 2.7. Assume the NLS_ case $p = -\bar{q}$. If $\Upsilon(z) = (v_1(z), v_2(z))^\top$ is a z-wave function associated with q, then $\mathcal{K}\Upsilon(z, x)$ is a $\bar{z}$-wave function associated with q, and

$$\Upsilon_{z_1}^{(1)}(\bar{z}, x) = \Gamma(\bar{z}, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Upsilon(\bar{z}, x)$$
$$= \Gamma(\bar{z}, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\mathcal{K}\Upsilon(z, x)$$
$$= \left(\overline{v_{2, z_1}^{(1)}(z, x)}, -\overline{v_{1, z_1}^{(1)}(z, x)}\right)^\top$$

$$(2.48) \qquad = \mathcal{K}\Upsilon_{z_1}^{(1)}(z, x)$$

is a $\bar{z}$-wave function associated with $q_{z_1}^{(1)}$ (cf. Lemma 2.2(iii)).

Remark 2.8. Assume the NLS_ case $p = -\bar{q}$.

(i) Take $z = z_1$ and $\Upsilon(z_1) = \Psi(z_1)$ in Theorem 2.6. Then

$$(2.49) \qquad \Psi_{z_1}^{(1)}(z_1, x) = \Gamma(z_1, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Psi(z_1, x) = 0,$$

since $\Psi(z_1)^\perp \Psi(z_1) = 0$.

(ii) Take $z = \overline{z_1}$ and $\mathcal{K}\Upsilon(z_1) = \mathcal{K}\Psi(z_1)$ in Theorem 2.6. Then

$$(2.50) \qquad \mathcal{K}\Psi_{z_1}^{(1)}(z_1, x) = \Gamma(\overline{z_1}, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\mathcal{K}\Psi(z_1, x) = 0,$$

since $\Psi(z_1)^\perp \mathcal{K}\Psi(z_1) = \|\Psi(z_1)\|_{\mathbb{C}^2}^2$ by Lemma 2.2(iii).

In the NLS$_-$ case, Theorem 2.6 also shows that $q_{z_1}^{(1)}$ is locally nonsingular whenever q is. More precisely, one has the following result.

Corollary 2.9. *Assume the* NLS$_-$ *case* $p = -\bar{q}$ *and suppose* $z_1 \in \mathbb{C}$. *Then, if* $q \in L_{\text{loc}}^p(\mathbb{R})$ *for some* $p \in [1, \infty) \cup \{\infty\}$ *(respectively, if* $q \in C^k(\mathbb{R})$ *for some* $k \in \mathbb{N}_0$*), the* NLS$_-$ *potential* $q_{z_1}^{(1)}$ *given by* (2.38) *also satisfies* $q_{z_1}^{(1)} \in L_{\text{loc}}^p(\mathbb{R})$ *(respectively,* $q_{z_1}^{(1)} \in C^k(\mathbb{R})$*).*

Proof. Since $\left|\psi_1(z_1, x)\overline{\psi_2(z_1, x)}\right| \|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2} \leq 1/2$, one concludes from (2.38) that $(q_{z_1}^{(1)} - q) \in L^\infty(\mathbb{R})$. Again by (2.38), q and $q_{z_1}^{(1)}$ share the same L_{loc}^p and C^k properties since by (2.4) (with $p = -\bar{q}$) one has $\partial_x^m \psi_j(z_1, \cdot) \in \text{AC}_{\text{loc}}(\mathbb{R})$, $j = 1, 2$, whenever $\partial_x^m q \in L_{\text{loc}}^1(\mathbb{R})$. □

3 $\mathcal{J}$-self-adjointness of NLS$_-$ Dirac-type operators

It is known that the Dirac-type Lax differential expression in the defocusing NLS$_+$ case is always in the limit point case at $\pm\infty$. Put differently, the maximally defined Dirac-type operator corresponding to the defocusing NLS$_+$ case (cf. (1.3)) is always self-adjoint. Classical references in this context are [34, Sect. 8.6], [47], which use some additional conditions (such as real-valuedness and/or regularity) of the coefficient q. A simple proof of this fact under most general conditions on q was recently communicated to us by Hinton [25] (cf. also [6], [7] and [31] for matrix-valued extensions of this result). In this section, we show that the analogueous result holds for Dirac-type differential expressions $M(q)$ in (2.7) in the focusing NLS$_-$ case, when self-adjointness is replaced by $\mathcal{J}$-self-adjointness.

First, we recall some basic facts about $\mathcal{J}$-symmetric and $\mathcal{J}$-self-adjoint operators in a complex Hilbert space $\mathcal{H}$ (see, e.g., [12, Sect. III.5] and [22, p. 76]) with scalar product denoted by $(\cdot, \cdot)_{\mathcal{H}}$ (linear in the first and antilinear in the second place) and corresponding norm denoted by $\|\cdot\|_{\mathcal{H}}$. Let $\mathcal{J}$ be a conjugation operator in $\mathcal{H}$, that is, $\mathcal{J}$ is an antilinear involution satisfying

$$(3.1) \qquad (\mathcal{J}u, v)_{\mathcal{H}} = (\mathcal{J}v, u)_{\mathcal{H}} \quad \text{for all } u, v \in \mathcal{H}, \quad \mathcal{J}^2 = I.$$

In particular,

$$(3.2) \qquad (\mathcal{J}u, \mathcal{J}v)_{\mathcal{H}} = (v, u)_{\mathcal{H}}, \quad u, v \in \mathcal{H}.$$

A densely defined linear operator S in $\mathcal{H}$ is called $\mathcal{J}$-symmetric if

$$(3.3) \qquad S \subseteq \mathcal{J}S^*\mathcal{J} \quad \text{(equivalently, if } \mathcal{J}S\mathcal{J} \subseteq S^*).$$

Clearly, (3.3) is equivalent to

$$(3.4) \qquad (\mathcal{J}u, Sv)_{\mathcal{H}} = (\mathcal{J}Su, v)_{\mathcal{H}}, \quad u, v \in \mathrm{dom}(S).$$

Here S^* denotes the adjoint operator of S in $\mathcal{H}$. If S is $\mathcal{J}$-symmetric, so is its closure $\overline{S}$. The operator S is called $\mathcal{J}$-self-adjoint if

$$(3.5) \qquad S = \mathcal{J}S^*\mathcal{J} \quad (\text{equivalently, if } \mathcal{J}S\mathcal{J} = S^*).$$

Finally, a densely defined, closable operator T is called essentially $\mathcal{J}$-self-adjoint if its closure $\overline{T}$ is $\mathcal{J}$-self adjoint, that is, if

$$(3.6) \qquad \overline{T} = \mathcal{J}T^*\mathcal{J}.$$

Next, assuming S to be $\mathcal{J}$-symmetric, one introduces the following inner product $(\cdot, \cdot)_*$ on $\mathrm{dom}(\mathcal{J}S^*\mathcal{J}) = \mathrm{dom}(S^*\mathcal{J})$ according to [28] (see also [39]),

$$(3.7) \qquad (u, v)_* = (\mathcal{J}u, \mathcal{J}v)_{\mathcal{H}} + (S^*\mathcal{J}u, S^*\mathcal{J}v)_{\mathcal{H}}, \quad u, v \in \mathrm{dom}(\mathcal{J}S^*\mathcal{J}),$$

which renders $\mathrm{dom}(\mathcal{J}S^*\mathcal{J})$ a Hilbert space. Denoting the identity operator in $\mathcal{H}$ by I, we have the following theorem.

Theorem 3.1 (Race [39]). *Let S be a densely defined closed $\mathcal{J}$-symmetric operator. Then*

$$(3.8) \qquad \mathrm{dom}(\mathcal{J}S^*\mathcal{J}) = \mathrm{dom}(S) \oplus_* \ker((S^*\mathcal{J})^2 + I),$$

where $\oplus_*$ means the orthogonal direct sum with respect to the inner product $(\cdot, \cdot)_*$. In particular, a densely defined closed $\mathcal{J}$-symmetric operator S is $\mathcal{J}$-self-adjoint if and only if

$$(3.9) \qquad \ker((S^*\mathcal{J})^2 + I) = \{0\}.$$

We apply (3.9) to (maximally defined) Dirac-type operators associated with the differential expression $M(q)$ in (2.7) relevant to the focusing NLS_- hierarchy and prove the fundamental fact that such Dirac operators are always $\mathcal{J}$-self-adjoint under most general conditions on the coefficient q (cf. Theorem 3.5).

It is convenient to make the following NLS_- assumption throughout the remainder of this section.

Hypothesis 3.2. *Suppose $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ and assume the NLS_- case $p = -\overline{q}$.*

Assuming Hypothesis 3.2, we now introduce the following maximal and minimal Dirac-type operators in $L^2(\mathbb{R})^2$ associated with the differential expression $M(q)$,

(3.10) $\quad D_{\max}(q)F = M(q)F,$

$$F \in \text{dom}(D_{\max}(q)) = \{ G \in L^2(\mathbb{R})^2 \ : \ G \in AC_{\text{loc}}(\mathbb{R})^2; \ M(q)G \in L^2(\mathbb{R})^2 \},$$

(3.11) $\quad D_{\min}(q)F = M(q)F,$

$$F \in \text{dom}(D_{\min}(q)) = \{ G \in \text{dom}(D_{\max}(q)) \ : \ \text{supp}(G) \text{ is compact} \}.$$

It follows by standard techniques (see, e.g., [34, Ch. 8] and [47]) that under Hypothesis 3.2, $D_{\min}(q)$ is densely defined and closable in $L^2(\mathbb{R})^2$ and $D_{\max}(q)$ is a densely defined closed operator in $L^2(\mathbb{R})^2$. Moreover (cf. (2.18)), one infers (see, e.g., [34, Lemma 8.6.2] and [47] in the analogueous case of symmetric Dirac operators)

(3.12) $\qquad \overline{D_{\min}(q)} = D_{\max}(-q)^* \text{ or, equivalently, } D_{\min}(q)^* = D_{\max}(-q).$

The following result is the crucial ingredient in the proof of Theorem 3.5, the principal result of this section.

Theorem 3.3. *Assume Hypothesis 3.2. Let $N(q)$ be the (formally self-adjoint) differential expression*

(3.13) $$N(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ \overline{q} & \frac{d}{dx} \end{pmatrix},$$

and let $\widetilde{D}_{\max}(q)$ be the maximally defined Dirac-type operator in $L^2(\mathbb{R})^2$ associated with $N(q)$,

(3.14) $\quad \widetilde{D}_{\max}(q)F = N(q)F,$

$$F \in \text{dom}(\widetilde{D}_{\max}(q)) = \{ G \in L^2(\mathbb{R})^2 \ : \ G \in AC_{\text{loc}}(\mathbb{R})^2; \ N(q)G \in L^2(\mathbb{R})^2 \}.$$

Then the following hold.

(i) *We have*

(3.15) $$M(-q)M(q) = N(q)^2.$$

(ii) *Let $U_q = U_q(x)$ satisfy the initial value problem*

(3.16) $$U'_q = \begin{pmatrix} 0 & q \\ -\overline{q} & 0 \end{pmatrix} U_q, \quad U_q(0) = I_2.$$

Then $\{U_q(x)\}_{x\in\mathbb{R}}$ is a family of unitary matrices in $\mathbb{C}^2$ with entries in $AC_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfying

(3.17) $$U_q^{-1} N(q) U_q = i\frac{d}{dx} I_2.$$

(iii) *Let $\mathcal{U}_q$ denote the multiplication operator with $U_q(\cdot)$ on $L^2(\mathbb{R})^2$. Then $\widetilde{D}_{\text{max}}(q)$ is unitarily equivalent to the maximally defined operator in $L^2(\mathbb{R})^2$ associated with the differential expression $i\frac{d}{dx} I_2$,*

(3.18) $$\mathcal{U}_q^{-1} \widetilde{D}_{\text{max}}(q) \mathcal{U}_q = \left(i\frac{d}{dx} I_2\right)_{\text{max}},$$

$$\text{dom}\left(\left(i\frac{d}{dx} I_2\right)_{\text{max}}\right) = H^{1,2}(\mathbb{R})^2$$
$$= \left\{ F \in L^2(\mathbb{R})^2 \ : \ F \in AC_{\text{loc}}(\mathbb{R})^2; \ F' \in L^2(\mathbb{R})^2 \right\}.$$

Moreover,

(3.19)

$$\mathcal{U}_q^{-1} D_{\text{max}}(-q) D_{\text{max}}(q) \mathcal{U}_q = \left(-\frac{d^2}{dx^2} I_2\right)_{\text{max}},$$

$$\text{dom}\left(\left(-\frac{d^2}{dx^2} I_2\right)_{\text{max}}\right) = H^{2,2}(\mathbb{R})^2$$
$$= \left\{ F \in L^2(\mathbb{R})^2 \ : \ F, F' \in AC_{\text{loc}}(\mathbb{R})^2; \ F', F'' \in L^2(\mathbb{R})^2 \right\}.$$

Proof. That $N(q)$ is formally self-adjoint and $M(-q)M(q) = N(q)^2$ as stated in (i) is an elementary matrix calculation.

To prove (ii), we note that the initial value problem (3.16) is well-posed in the sense of Carathéodory since $q \in L^1_{\text{loc}}(\mathbb{R})$ (cf., e.g., [23, Lemma IX.2.2], [24, p. 45–46]) with a solution matrix U_q with entries in $AC_{\text{loc}}(\mathbb{R})$. Moreover, for each $x \in \mathbb{R}$, $U_q(x)$ is a unitary matrix in $\mathbb{C}^2$, since $U_q' = -B(q)U_q$, with $B(q) = \left(\begin{smallmatrix} 0 & -q \\ \overline{q} & 0 \end{smallmatrix}\right)$ skew-adjoint. Thus, the entries $U_{q,j,k}$, $1 \le j, k \le 2$ of U_q (as well as those of U_q^{-1}) actually satisfy

(3.20) $$U_{q,j,k} \in AC_{\text{loc}}(\mathbb{R}) \cap L^\infty(\mathbb{R}), \quad 1 \le j, k \le 2.$$

Next, fix $F \in AC_{\text{loc}}(\mathbb{R})^2$, such that $\mathcal{U}_q^{-1} F \in H^{1,2}(\mathbb{R})^2$. Then

$$U_q\left(i\frac{d}{dx} I_2\right) U_q^{-1} F = i\frac{d}{dx} F + iU_q\frac{d}{dx}(U_q^{-1})F$$

$$= i\frac{d}{dx} F + iU_q(U_q^{-1} B(q)^*)F$$

(3.21) $$= N(q)F,$$

where we have used the fact that $(U_q^{-1})' = U_q^{-1}B(q)^*$. Thus, (ii) follows.

Moreover, by (3.21) one concludes $\mathrm{dom}(\tilde{D}_{max}(q)) = \mathcal{U}_q H^{1,2}(\mathbb{R})^2$ by (3.20) and the fact that U_q is unitary in $\mathbb{C}^2$. This proves (3.18).

Clearly (i) and (ii) yield the relation

$$(3.22) \qquad U_q^{-1}M(-q)M(q)U_q = -\frac{d^2}{dx^2}I_2.$$

Thus, (3.19) follows once we prove the following facts:

$(3.23) \qquad$ (i) $U_q F \in L^2(\mathbb{R})^2$ if and only if $F \in L^2(\mathbb{R})^2$,

$(3.24) \qquad$ (ii) $U_q F \in AC_{loc}(\mathbb{R})^2$ if and only if $F \in AC_{loc}(\mathbb{R})^2$,

$(3.25) \qquad$ (iii) $M(q)U_q F \in L^2(\mathbb{R})^2$ if and only if $F' \in L^2(\mathbb{R})^2$,

$(3.26) \qquad$ (iv) $M(q)U_q F \in AC_{loc}(\mathbb{R})^2$ if and only if $F' \in AC_{loc}(\mathbb{R})^2$,

$(3.27) \qquad$ (v) $M(-q)M(q)U_q F \in L^2(\mathbb{R})^2$ if and only if $F'' \in L^2(\mathbb{R})^2$.

Clearly, (3.23) and (3.27) hold since U_q is unitary in $\mathbb{C}^2$. Also, (3.24) is valid, since $U_{q,j,k}, U_{q,j,k}^{-1} \in AC_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $j, k = 1, 2$. Next, an explicit computation yields

$$(3.28) \qquad M(q)U_q F = i \begin{pmatrix} U_{q,1,1}f_1' + U_{q,1,2}f_2' \\ -U_{q,2,1}f_1' - U_{q,2,2}f_2' \end{pmatrix}, \quad F = (f_1, f_2)^\top.$$

Introducing

$$(3.29) \qquad V_q = \sigma_3 U_q \sigma_3 = \begin{pmatrix} U_{q,1,1} & -U_{q,1,2} \\ -U_{q,2,1} & U_{q,2,2} \end{pmatrix},$$

one infers $V_{q,j,k}, V_{q,j,k}^{-1} \in AC_{loc}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, $j, k = 1, 2$ and

$$(3.30) \qquad V_q^{-1}M(q)U_q F = i(f_1', -f_2')^\top,$$

and hence (3.25) and (3.26) hold. This proves (3.19). $\qquad\square$

Remark 3.4. Note that by (3.18), $\tilde{D}_{max}(q)$, the maximally defined, self-adjoint operator in $L^2(\mathbb{R})^2$ associated with the 2×2 matrix-valued differential expression $N(q)$, has a purely absolutely continuous spectrum which equals $\mathbb{R}$,

$$(3.31) \qquad \sigma(\tilde{D}_{max}(q)) = \sigma_{ac}(\tilde{D}_{max}(q)) = \mathbb{R}.$$

(Refer to Section 5 for a discussion of various spectral decompositions. In the present context, $\sigma(T)$ and $\sigma_{ac}(T)$ denote the spectrum and absolutely continuous spectrum of a self-adjoint operator T in a separable complex Hilbert space $\mathcal{H}$.)

The principal result of this section then reads as follows.

Theorem 3.5. *Assume Hypothesis 3.2. Then the minimally defined Dirac-type operator* $D_{\min}(q)$ *associated with the Lax differential expression*

$$(3.32) \qquad M(q) = i \begin{pmatrix} \frac{d}{dx} & -q \\ -\bar{q} & -\frac{d}{dx} \end{pmatrix}$$

introduced in (3.11) is essentially $\mathcal{J}$*-self-adjoint in* $L^2(\mathbb{R})^2$*, that is,*

$$(3.33) \qquad \overline{D_{\min}(q)} = \mathcal{J} D_{\min}(q)^* \mathcal{J},$$

where $\mathcal{J}$ *is the conjugation defined in (2.9). Moreover,*

$$(3.34) \qquad \overline{D_{\min}(q)} = D_{\max}(q);$$

hence $D_{\max}(q)$ *is* $\mathcal{J}$*-self-adjoint.*

Proof. First recall (cf. (3.12)) that

$$(3.35) \qquad D_{\min}(q)^* = D_{\max}(-q)$$

and also note

$$(3.36) \qquad \mathcal{J} D_{\max}(-q) \mathcal{J} = D_{\max}(q).$$

Since $\overline{D_{\min}(q)}$ is closed and $\mathcal{J}$-symmetric (cf. (2.19)), its $\mathcal{J}$-self-adjointness is equivalent (cf. (3.9)) to

$$(3.37) \quad \ker\left(D_{\min}(q)^* \mathcal{J} D_{\min}(q)^* \mathcal{J} + I_2\right) = \ker(D_{\max}(-q) D_{\max}(q) + I_2) = \{0\}.$$

Since, by (3.19), $D_{\max}(-q) D_{\max}(q)$ is unitarily equivalent to $(-d^2/dx^2 I_2)_{\max} \geq 0$, one concludes $D_{\max}(-q) D_{\max}(q) \geq 0$; hence (3.37) obviously holds. Then (3.34) follows from (3.33) and (3.35), since

$$(3.38) \qquad \overline{D_{\min}(q)} = \mathcal{J} D_{\min}(q)^* \mathcal{J} = \mathcal{J} D_{\max}(-q) \mathcal{J} = D_{\max}(q). \qquad \square$$

As mentioned in the introductory paragraph to this section, Theorem 3.5 in the $\mathcal{J}$-self-adjoint context can be viewed as an analogue of the result of the corresponding (self-adjoint) Dirac operator relevant in the defocusing NLS$_+$ case of the nonlinear Schrödinger equation.

4   Constructing $L^2(\mathbb{R})^2$-wave functions for $\mathcal{J}$-self-adjoint Dirac-type operators

In this section, we discuss how to construct $L^2(\mathbb{R})^2$-wave functions for non-self-adjoint (but $\mathcal{J}$-self-adjoint) Dirac-type operators associated with the Lax differential expression for the NLS$_-$ system.

By Remark 2.8, in order to obtain a nonzero z_1-wave function associated with $q_{z_1}^{(1)}$, we have to apply the transformation matrix $\Gamma(z_1, \Psi(z_1), \mathcal{K}\Psi(z_1))$ to a z_1-wave function $\Phi(z_1)$ associated with q that is linearly independent with the original z_1-wave function $\Psi(z_1)$ associated with q. Similarly, in order to obtain a nonzero $\overline{z_1}$-wave function associated with $q_{z_1}^{(1)}$, we have to apply the transformation matrix $\Gamma(\overline{z_1}, \Psi(z_1), \mathcal{K}\Psi(z_1))$ to a $\overline{z_1}$-wave function $\mathcal{K}\Phi(z_1)$ associated with q that is linearly independent with the original $\overline{z_1}$-wave function $\mathcal{K}\Psi(z_1)$ associated with q. The function $\Phi(z_1)$ is constructed as follows.

Let $\Psi(z) = (\psi_1(z), \psi_2(z))^\top$, $z \in \mathbb{C}$, be a z-wave function associated with q on $\mathbb{R}$ and introduce

$$(4.1) \qquad \Omega_z = \{x \in \mathbb{R} : \psi_1(z, x)\psi_2(z, x) \neq 0\}.$$

Next, consider

$$(4.2) \qquad \Psi^{\#}(z, x) = (1/2)(\psi_2(z, x)^{-1}, -\psi_1(z, x)^{-1})^\top, \quad x \in \Omega_z,$$

such that

$$(4.3) \qquad W(\Psi^{\#}(z, x), \Psi(z, x)) = 1.$$

Let $x_0, x \in \Omega_z$ be such that $[x_0, x] \subseteq \Omega_z$ and define

$$(4.4) \qquad R(z, x, x_0) = -\frac{1}{2}\int_{x_0}^{x} dx' \left(\frac{\overline{q(x')}}{\psi_2(z, x')^2} + \frac{q(x')}{\psi_1(z, x')^2} \right),$$

$$(4.5) \qquad \Phi(z, x) = \Psi^{\#}(z, x) + R(z, x, x_0)\Psi(z, x), \quad [x_0, x] \subseteq \Omega_z.$$

Using (2.4), we have $(\Psi^{\#})' = U(p, q)\Psi^{\#} - R'\Psi$. Thus, $W(\Phi, \Psi) = 1$ and $\Phi(z)$ is a z-wave function associated with q, since

$$(4.6) \qquad \Phi' = (\Psi^{\#})' + R'\Psi + R\Psi' = U\Psi^{\#} - R'\Psi + R'\Psi + RU\Psi = U\Phi.$$

We note that $\Psi^{\perp}\Psi^{\#} = 1$ implies $\mathcal{K}\Psi(z)\Psi(z)^{\perp}\Phi(z) = \mathcal{K}\Psi(z)$.

Thus, if $\Gamma(z, \Psi(z_1), \mathcal{K}\Psi(z_1))$ is defined using $\Psi(z_1)$ as in (2.40), then the z_1-wave function $\Phi_{z_1}^{(1)}(z_1)$ associated with $q_{z_1}^{(1)}$, as prescribed in Theorem 2.6, is computed

as follows,

$$\Phi_{z_1}^{(1)}(z_1, x) = \Gamma(z_1, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Phi(z_1, x)$$

(4.7)
$$= \operatorname{Im}(z_1)\|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1, x).$$

Moreover, by Remark 2.7, $\mathcal{K}\Phi_{z_1}^{(1)}(z_1)$ is computed as

$$\mathcal{K}\Phi_{z_1}^{(1)}(z_1, x) = \Gamma(\overline{z_1}, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\mathcal{K}\Phi(z_1, x)$$

(4.8)
$$= -\operatorname{Im}(z_1)\|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2}\Psi(z_1, x).$$

By (4.7) and (2.40), for each $z \in \mathbb{C}$, the z-wave function $\Phi_{z_1}^{(1)}(z)$ associated with $q_{z_1}^{(1)}$ (constructed using the z-wave function $\Phi(z)$ associated with q) is computed by

$$\Phi_{z_1}^{(1)}(z, x) = \Gamma(z, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Phi(z, x)$$
$$= -(i/2)(z - z_1)\Phi(z, x) + \Phi_{z_1}^{(1)}(z_1, x)\Psi(z_1, x)^{\perp}\Phi(z, x)$$

(4.9)
$$= -(i/2)(z - z_1)\Phi(z, x) - W(\Psi(z_1, x), \Phi(z, x))\Phi_{z_1}^{(1)}(z_1, x).$$

Formulas (2.38) and (4.7) now imply

(4.10)
$$q_{z_1}^{(1)}(x) = q(x) + 4\phi_{1,z_1}^{(1)}(z_1, x)\psi_1(z_1, x),$$

where $\Phi_{z_1}^{(1)}(z, x) = (\phi_{1,z_1}^{(1)}(z, x), \phi_{2,z_1}^{(1)}(z, x))^{\top}$, $\Psi(z, x) = (\psi_1(z, x), \psi_2(z, x))^{\top}$.

Remark 4.1. We emphasize that while $R(z, x, x_0)$ in (4.4), and hence $\Phi(z, x)$ in (4.5), will in general have singularities on $\mathbb{R}$, formulas (4.7)–(4.10) are well-defined for all $x \in \mathbb{R}$.

The next hypothesis is crucial in our attempt to construct z_1- and $\overline{z}_1$-wave functions in $L^2(\mathbb{R})^2$ associated with the Dirac-type differential expression $M(q_{z_1}^{(1)})$.

Hypothesis 4.2. *Suppose* $q \in L^1_{\text{loc}}(\mathbb{R})$, *assume the* NLS_ *case* $p = -\overline{q}$, *and let* $z_0 \in \mathbb{C}$. *Suppose* $\Psi(z_0)$ *to be a* z_0-*wave function associated with* q *which satisfies the condition* $\|\Psi(z_0, \cdot)\|_{\mathbb{C}^2}^{-1} \in L^2(\mathbb{R})$, *that is,*

(4.11)
$$\int_{-\infty}^{\infty} dx \, \|\Psi(z_0, x)\|_{\mathbb{C}^2}^{-2} < \infty.$$

If a z_0-wave function $\Psi(z_0)$ associated with q satisfies condition (4.11), we say that $\Psi(z_0)$ satisfies Hypothesis 4.2 at z_0.

Remark 4.3. Assume Hypothesis 3.2 and suppose that $\Psi(z)$ satisfies Hypothesis 4.2 at z. Then

(i) by Hölder's inequality, $\Psi(z) \notin L^2((-\infty, R])^2 \cup L^2([R, \infty))^2$ for all $R \in \mathbb{R}$;

(ii) $\mathcal{K}\Psi(z)$ satisfies Hypothesis 4.2 at $\bar{z}$ by Lemma 2.2 (iii).

Remark 4.4. Assume Hypothesis 3.2 and let $\lambda \in \mathbb{R}$. Then all λ-wave functions $\Psi(\lambda)$ associated with q satisfy $\|\Psi(\lambda, x)\|_{\mathbb{C}^2} = c(\lambda)$ (independently of $x \in \mathbb{R}$). In particular, $\|\Psi(\lambda, \cdot)\|_{\mathbb{C}^2}, \|\Psi(\lambda, \cdot)\|_{\mathbb{C}^2}^{-1} \notin L^2([R, \pm\infty))$, $R \in \mathbb{R}$, and hence there exists no λ-wave function associated with q that satisfies Hypothesis 4.2 at $\lambda \in \mathbb{R}$.

The principal result of this section then reads as follows.

Theorem 4.5. *Assume Hypothesis 3.2. Let $z_1 \in \mathbb{C}\backslash\mathbb{R}$ and suppose that the z_1-wave function $\Psi(z_1)$ associated with q satisfies Hypothesis 4.2 at z_1. Let $q_{z_1}^{(1)}$ be given by (2.38). Then z_1 and $\overline{z_1}$ are eigenvalues of the maximal Dirac-type operator $D_{\max}(q_{z_1}^{(1)})$ associated with $M(q_{z_1}^{(1)})$ of geometric multiplicity equal to one. The corresponding eigenfunctions $\Phi_{z_1}^{(1)}(z_1)$ and $\mathcal{K}\Phi_{z_1}^{(1)}(z_1)$ are given by (4.7) and (4.8), respectively, that is,*

$$(4.12) \qquad \Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1) \in \operatorname{dom}(D_{\max}(q_{z_1}^{(1)})),$$

$$(4.13) \qquad D_{\max}(q_{z_1}^{(1)})\Phi_{z_1}^{(1)}(z_1) = z_1 \Phi_{z_1}^{(1)}(z_1),$$

$$(4.14) \qquad D_{\max}(q_{z_1}^{(1)})\mathcal{K}\Phi_{z_1}^{(1)}(z_1) = \overline{z_1}\mathcal{K}\Phi_{z_1}^{(1)}(z_1).$$

Proof. Indeed, using (4.11) at z_1 one obtains

$$(4.15) \qquad 0 < \left\|\Phi_{z_1}^{(1)}(z_1)\right\|_{L^2}^2 = |\operatorname{Im}(z_1)|^2 \int_{-\infty}^{\infty} dx\, \|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-4} \|\Psi(z_1, x)\|_{\mathbb{C}^2}^2 < \infty.$$

In order to show that z_1 has geometric multiplicity equal to one as an eigenvalue of $D_{\max}(q_{z_1}^{(1)})$, assume that $\Phi_{z_1}^{(1)}(z_1, \cdot) \in L^2(\mathbb{R})^2$ and $\tilde{\Phi}_{z_1}^{(1)}(z_1, \cdot) \in L^2(\mathbb{R})^2$ are linearly independent z_1-wave functions associated with $q_{z_1}^{(1)}$. Then clearly $\Phi_{z_1}^{(1)}(z_1, \cdot), \tilde{\Phi}_{z_1}^{(1)}(z_1, \cdot) \in \operatorname{dom}(D_{\max}(q_{z_1}^{(1)}))$ and

$$(4.16) \qquad W(\Phi_{z_1}^{(1)}(z_1, \cdot), \tilde{\Phi}_{z_1}^{(1)}(z_1, \cdot)) \in L^1(\mathbb{R}).$$

However, $W(\Phi_{z_1}^{(1)}(z_1, x), \tilde{\Phi}_{z_1}^{(1)}(z_1, x))$ is constant with respect to $x \in \mathbb{R}$ by (2.5); thus (4.16) represents a contradiction and hence z_1 has geometric multiplicity equal to one. Analogueous arguments apply to $\overline{z_1}$. $\square$

The argument that $D_{\max}(q_{z_1}^{(1)})$ has only eigenvalues of geometric multiplicity equal to one applies of course in complete generality to any $D(q)$ with $q \in L_{\text{loc}}^1(\mathbb{R})$.

Next we show that condition (4.11) is preserved under iterations, a fact of great relevance in connection with the multi-soliton solutions relative to arbitrary backgrounds discussed at the end of Section 6.

Lemma 4.6. *Assume Hypothesis 3.2 and let $z_1, z_2 \in \mathbb{C}$. Fix a z_1-wave function $\Psi(z_1)$ and a z_2-wave function $\Phi(z_2)$ associated with q. Using $\Psi(z_1)$, construct the NLS$_-$ potential $q_{z_1}^{(1)}$ by formula (2.38) and consider the transformation matrix $\Gamma(z_2, \Psi(z_1), \mathcal{K}\Psi(z_1))$ given by formula (2.40) for $z = z_2$. Let $\Phi_{z_1}^{(1)}(z_2)$ be the z_2-wave function associated with $q_{z_1}^{(1)}$ as in (4.9). If $\Phi(z_2)$ satisfies Hypothesis 4.2 at z_2, then $\Phi_{z_1}^{(1)}(z_2)$ satisfies Hypothesis 4.2 at z_2.*

Proof. Without loss of generality, we may assume $\text{Im}(z_1)\text{Im}(z_2) \geq 0$. Formulas (2.12), (4.7), and (4.9) imply

$$\left\| \Phi_{z_1}^{(1)}(z_2) \right\|_{\mathbb{C}^2}^2$$

$$= \frac{1}{4}|z_2 - z_1|^2 \|\Phi(z_2)\|_{\mathbb{C}^2}^2 + \frac{1}{4}|\overline{z_1} - z_1|^2 \|\Psi(z_1)\|_{\mathbb{C}^2}^{-4}\|\mathcal{K}\Psi(z_1))\Psi(z_1)^\perp\Phi(z_2)\|_{\mathbb{C}^2}^2$$

$$\quad - \frac{1}{2}\text{Re}\left(((z_2 - z_1)\Phi(z_2), (\overline{z_1} - z_1)\|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1)\Psi(z_1)^\perp\Phi(z_2))_{\mathbb{C}^2}\right)$$

$$= \frac{1}{4}|z_2 - z_1|^2 \|\Phi(z_2)\|_{\mathbb{C}^2}^2 + \frac{1}{4}|\overline{z_1} - z_1|^2 \|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}|\Psi(z_1)^\perp\Phi(z_2)|^2$$

$$\quad + \frac{1}{2}\|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}|\Psi(z_1)^\perp\Phi(z_2)|^2\text{Re}((z_2 - z_1)(\overline{z_1} - z_1))$$

$$= \frac{1}{4}|z_2 - z_1|^2 \|\Phi(z_2)\|_{\mathbb{C}^2}^2 + \|\Psi(z_1)\|_{\mathbb{C}^2}^{-2}|\Psi(z_1)^\perp\Phi(z_2)|^2\text{Im}(z_1)\text{Im}(z_2)$$

$$(4.17) \qquad \geq \frac{1}{4}|z_2 - z_1|^2 \|\Phi(z_2)\|_{\mathbb{C}^2}^2.$$

Thus, if $\int_{-\infty}^{\infty} dx \, \|\Phi(z_2, x)\|_{\mathbb{C}^2}^{-2} < \infty$, then $\int_{-\infty}^{\infty} dx \, \|\Phi_{z_1}^{(1)}(z_2, x)\|_{\mathbb{C}^2}^{-2} < \infty$. $\qquad \square$

It is shown in Remark 6.1 that for any $z \in \rho\big(D_{\max}(q_{z_1}^{(1)})\big)$, all but two z-wave functions of $D_{\max}(q_{z_1}^{(1)})$ satisfy Hypothesis 4.2 at z.

5 Some spectral properties and the existence of Weyl–Titchmarsh-type solutions for $\mathcal{J}$-self-adjoint Dirac-type operators

The principal purpose of this section is to establish the existence of Weyl–Titchmarsh-type solutions for formally $\mathcal{J}$-self-adjoint Dirac differential expressions $M(q)$ associated with the focusing NLS$_-$ case. The latter are well-known to exist in the case of self-adjoint Dirac operators (in particular, in the context of the defocusing NLS$_+$ equation) and are a fundamental ingredient in the spectral analysis in the self-adjoint context (cf., for instance, [33, Chs. 3, 4]). As far as we know, no such result is known in the general $\mathcal{J}$-self-adjoint case studied in this section. Along the way, we also collect some results concerning spectral properties of $D_{\max}(q)$.

Thus, assuming the NLS$_-$ case and hence the basic Hypothesis 3.2 throughout this section, we adopt the simplified notation (cf. Section 3)

$$(5.1) \qquad D(q) = \overline{D_{\min}(q)} = D_{\max}(q), \quad q \in L^1_{loc}(\mathbb{R}).$$

We also denote by I_2 the identity operator in $L^2(\mathbb{R})^2$ (as well as in $\mathbb{C}^2$). Moreover, we find it convenient to introduce the following notations,

(5.2)

$$L^2_{loc}([-\infty, \infty)) = \{f \colon \mathbb{R} \to \mathbb{C} \text{ is measurable} : f \in L^2((-\infty, R]) \text{ for all } R \in \mathbb{R}\},$$

$$L^2_{loc}((-\infty, \infty]) = \{f \colon \mathbb{R} \to \mathbb{C} \text{ is measurable} : f \in L^2([R, \infty)) \text{ for all } R \in \mathbb{R}\}.$$

We start with the following auxiliary result (the variation of parameters formula).

Lemma 5.1. *Assume Hypothesis 3.2 and let $(z, x_0) \in \mathbb{C} \times \mathbb{R}$. Let $\Psi_1(z)$ and $\Psi_2(z)$ be linearly independent z-wave functions for $M(q)$ defined on $[x_0, \infty)$, and denote by $\Xi(z, x) = [\Psi_1(z, x), \Psi_2(z, x)]$ the 2×2 fundamental matrix solution of $M(q)\Xi(z) = z\Xi(z)$. Assume the Wronskian of Ψ_1 and Ψ_2 satisfies $W(\Psi_1(z, x), \Psi_2(z, x)) = \det(\Xi(z, x)) = 1$ for some (and hence for all) $x \in [x_0, \infty)$. Moreover, suppose $B \in L^1_{loc}([x_0, \infty))^{2 \times 2}$. Then $\Phi(z, \cdot) \in AC_{loc}(\mathbb{R})^2$ satisfies $M(q)\Phi(z) = z\Phi(z) + B\Phi(z)$ if and only if*

$$(5.3) \quad \Phi(z, x) = \Xi(z, x)C - i \int_{x_0}^x dx' \, \Xi(z, x)\Xi(z, x')^{-1}\sigma_3 B(x')\Phi(z, x'), \quad x \geq x_0$$

for some $C = (c_1, c_2)^\top \in \mathbb{C}^2$ independent of x. Moreover, $\Phi(z, x) = 0$ for all $x \geq x_0$ if and only if $C = 0$.

Proof. The computation

$$M(q)\left(\Phi(x) + i \int_{x_0}^x dx' \, \Xi(x)\Xi(x')^{-1}\sigma_3 B(x')\Phi(x')\right)$$

$$= z\Phi(x) + B(x)\Phi(x) + iM(q)\left(\Xi(x)\int_{x_0}^x dx' \, \Xi(x')^{-1}\sigma_3 B(x')\Phi(x')\right)$$

$$(5.4) \qquad = z\left(\Phi(x) + i \int_{x_0}^x dx' \, \Xi(x)\Xi(x')^{-1}\sigma_3 B(x')\Phi(x')\right)$$

shows that if Φ satisfies $M(q)\Phi(z) = z\Phi(z) + B\Phi(z)$, then Φ satisfies (5.3), since Ξ is a fundamental matrix for the first-order linear differential system $M(q)\Psi(z) = z\Psi(z)$. Conversely, if Φ satisfies (5.3), then one readily verifies that Φ satisfies $M(q)\Phi(z) = z\Phi(z) + B\Phi(z)$. That $\Phi(z, x) = 0$ for all $x \geq x_0$ if and only if $C = 0$ follows upon iterating the Volterra-type integral equation (5.3) in a standard manner. $\qquad \square$

Next, we find it convenient to recall a number of basic definitions and well-known facts in connection with the spectral theory of non-self-adjoint operators (we refer to [12, Chs. I, III, IX], [22, Sects. 1, 21–23], and [40, pp. 178–179] for more details). Let S be a densely defined closed operator in a complex Hilbert space $\mathcal{H}$. Denote by $\mathcal{B}(\mathcal{H})$ the Banach space of all bounded linear operators on $\mathcal{H}$. The spectrum $\sigma(S)$, point spectrum (the set of eigenvalues) $\sigma_p(S)$, continuous spectrum $\sigma_c(S)$, residual spectrum $\sigma_r(S)$, approximate point spectrum $\sigma_a(S)$, essential spectrum $\sigma_e(S)$, field of regularity $\pi(S)$, resolvent set $\rho(S)$, and $\Delta(S)$ are defined by

(5.5) $\sigma(S) = \{\lambda \in \mathbb{C} : (S - \lambda I)^{-1} \notin \mathcal{B}(\mathcal{H})\}$,

(5.6) $\sigma_p(S) = \{\lambda \in \mathbb{C} : \ker(S - \lambda I) \neq \{0\}\}$,

(5.7) $\sigma_c(S) = \{\lambda \in \mathbb{C} : \ker(S - \lambda I) = \{0\}$ and $\operatorname{ran}(S - \lambda I)$ is dense in $\mathcal{H}$

but not equal to $\mathcal{H}\}$,

(5.8) $\sigma_r(S) = \{\lambda \in \mathbb{C} : \ker(S - \lambda I) = \{0\}$ and $\operatorname{ran}(S - \lambda I)$ is not dense in $\mathcal{H}\}$,

(5.9) $\sigma_a(S) = \{\lambda \in \mathbb{C} :$ there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \in \operatorname{dom}(S)$ with

$\|f_n\|_{\mathcal{H}} = 1, n \in \mathbb{N}$ and $\lim_{n \to \infty} \|(S - \lambda I)f_n\|_{\mathcal{H}} = 0\}$,

(5.10)

$\sigma_e(S) = \{\lambda \in \mathbb{C} :$ there exists a sequence $\{f_n\}_{n \in \mathbb{N}} \in \operatorname{dom}(S)$ s.t. $\{f_n\}_{n \in \mathbb{N}}$

contains no convergent subsequence, $\|f_n\|_{\mathcal{H}} = 1, n \in \mathbb{N}$,

and $\lim_{n \to \infty} \|(S - \lambda I)f_n\|_{\mathcal{H}} = 0\}$,

(5.11) $\pi(S) = \{z \in \mathbb{C} :$ there exists $k_z > 0$ s.t. $\|(S - zI)u\|_{\mathcal{H}} \geq k_z \|u\|_{\mathcal{H}}$

for all $u \in \operatorname{dom}(S)\}$,

(5.12) $\rho(S) = \mathbb{C} \backslash \sigma(S)$,

(5.13)

$\Delta(S) = \{z \in \mathbb{C} : \dim(\ker(S - zI)) < \infty$ and $\operatorname{ran}(S - zI)$ is closed$\}$,

respectively. One then has

(5.14) $\sigma(S) = \sigma_p(S) \cup \sigma_c(S) \cup \sigma_r(S)$ (disjoint union)

(5.15) $= \sigma_p(S) \cup \sigma_e(S) \cup \sigma_r(S)$,

(5.16) $\sigma_c(S) \subseteq \sigma_e(S) \backslash (\sigma_p(S) \cup \sigma_r(S))$,

(5.17) $\sigma_r(S) = \sigma_p(S^*)^* \backslash \sigma_p(S)$,

$$(5.18) \qquad \sigma_{\mathrm{a}}(S) = \{\lambda \in \mathbb{C} \; : \; \text{for all } \varepsilon > 0, \text{ there exists } 0 \neq f_\varepsilon \in \mathrm{dom}(S)$$
$$\text{s.t. } \|(S - \lambda I)f_\varepsilon\|_{\mathcal{H}} \le \varepsilon \|f_\varepsilon\|_{\mathcal{H}}\}$$

$$(5.19) \qquad\qquad = \mathbb{C} \backslash \pi(S),$$

$$(5.20) \qquad \sigma(S) \backslash \sigma_{\mathrm{a}}(S) \subseteq \sigma_{\mathrm{r}}(S), \quad \sigma(S) \backslash \sigma_{\mathrm{a}}(S) \text{ is open,}$$

$$(5.21) \qquad \sigma_{\mathrm{e}}(S) = \{\lambda \in \mathbb{C} \; : \; \text{there exists a sequence } \{g_n\}_{n \in \mathbb{N}} \in \mathrm{dom}(S) \text{ s.t.}$$
$$\text{w-}\lim_{n \to \infty} g_n = 0, \|g_n\|_{\mathcal{H}} = 1, n \in \mathbb{N},$$
$$\text{and } \lim_{n \to \infty} \|(S - \lambda I)g_n\|_{\mathcal{H}} = 0\}$$

$$(5.22) \qquad\qquad = \mathbb{C} \backslash \Delta(S),$$

$$(5.23) \qquad \sigma_{\mathrm{e}}(S) \subseteq \sigma_{\mathrm{a}}(S) \subseteq \sigma(S) \text{ (all three sets are closed),}$$

$$(5.24) \qquad \rho(S) \subseteq \pi(S) \subseteq \Delta(S) \text{ (all three sets are open).}$$

Here ω^* in the context of (5.17) denotes the complex conjugate of the set $\omega \subseteq \mathbb{C}$, that is,

$$(5.25) \qquad\qquad \omega^* = \{\overline{\lambda} \in \mathbb{C} \; : \; \lambda \in \omega\}.$$

For future reference, we note that the sequence $\{f_n\}_{n \in \mathbb{N}}$ in (5.10) (and the sequence $\{g_n\}_{n \in \mathbb{N}}$ in (5.21)) is called a *singular* (or *Weyl*) sequence of S corresponding to λ. We also note that there are several versions of the concept of the essential spectrum in the non-self-adjoint context (cf. [12, Ch. IX]), but we only use the one in (5.10) (respectively, (5.21)) in this paper.

In the special case where S is $\mathcal{J}$-self-adjoint, one obtains the following simplifications (cf. [12, p. 118], [22, p. 76]):

$$(5.26) \qquad\qquad \sigma(S) = \sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{c}}(S)$$

$$(5.27) \qquad\qquad = \sigma_{\mathrm{p}}(S) \cup \sigma_{\mathrm{e}}(S),$$

$$(5.28) \qquad \sigma_{\mathrm{r}}(S) = \emptyset,$$

$$(5.29) \qquad \sigma_{\mathrm{p}}(S) = \sigma_{\mathrm{p}}(S^*)^*,$$

$$(5.30) \qquad \sigma_{\mathrm{a}}(S) = \sigma(S),$$

$$(5.31) \qquad \pi(S) = \rho(S),$$

whenever S is $\mathcal{J}$-self-adjoint. Note that $\pi(S) = \emptyset$ may occur for $\mathcal{J}$-symmetric operators (see [39]) in sharp contrast to the case of densely defined, closed, symmetric operators T, where any nonreal number is in the field of regularity $\pi(T)$.

Returning to the NLS$_-$ case, we next recall an elementary but useful consequence of (2.16) and (2.18).

Lemma 5.2. *Assume Hypothesis 3.2. Then*

(5.32) $$\mathcal{K}D(q)\mathcal{K} = -D(q),$$

(5.33) $$\sigma_3 D(q)\sigma_3 = D(q)^* = D(-q).$$

Consequently,

(5.34) $$\sigma(D(q))^* = \sigma(D(q)) = \sigma(D(-q)).$$

Moreover,

$$if\ z_0 \in \sigma_p(D(q))\ and\ D(q)F = z_0 F\ for\ some\ F \in \mathrm{dom}(D(q)),$$

(5.35) $$then\ D(q)\mathcal{K}F = \overline{z_0}\mathcal{K}F.$$

Proof. Relations (5.32) and (5.33) are clear from (2.16), (2.18), (3.10), (3.12), and (5.1). Since $(D(q)^* - zI_2)^{-1} = [(D(q) - \overline{z}I_2)^{-1}]^*$, $\sigma(D(q)^*) = \sigma(D(q))^*$ and hence (5.34) follows from (5.33). Finally, (5.35) is clear from (5.32) and $\mathcal{K}^2 = -I_2$. $\qquad\square$

Next we introduce the basic hypothesis to be assumed for the remainder of this section.

Hypothesis 5.3. *Suppose* $q \in L^1_{\mathrm{loc}}(\mathbb{R})$, *assume the* NLS_ *case* $p = -\overline{q}$, *and suppose that the ($\mathcal{J}$-self-adjoint) operator $D(q)$ has nonempty resolvent set,* $\rho(D(q)) \neq \emptyset$.

Recalling the standard notation

(5.36) $$\mathrm{nul}(T) = \mathrm{dim}(\mathrm{ker}(T)),$$

(5.37) $$\mathrm{def}(T) = \mathrm{dim}(\mathrm{ran}(T)^\perp) = \mathrm{dim}(\mathrm{ker}(T^*)) = \mathrm{nul}(T^*),$$

where T denotes a densely defined closed operator in $\mathcal{H}$, we can state the following fundamental result, establishing the existence of Weyl–Titchmarsh-type solutions for $\mathcal{J}$-self-adjoint Dirac-type operators relevant to the NLS_ case.

Theorem 5.4. *Assume Hypothesis 5.3 and pick* $z \in \rho(D(q))$. *Then there exist two unique (up to constant multiples) linearly independent z-wave functions* $\Psi_-(z, \cdot)$ *and* $\Psi_+(z, \cdot)$ *associated with* q *satisfying*

(5.38) $$\Psi_-(z, \cdot) \in L^2_{\mathrm{loc}}([-\infty, \infty))^2, \quad \Psi_+(z, \cdot) \in L^2_{\mathrm{loc}}((-\infty, \infty])^2,$$

(5.39) $$\|\Psi_-(z, \cdot)\|_{\mathbb{C}^2}^{-1} \in L^2_{\mathrm{loc}}((-\infty, \infty]), \quad \|\Psi_+(z, \cdot)\|_{\mathbb{C}^2}^{-1} \in L^2_{\mathrm{loc}}([-\infty, \infty)),$$

(5.40) $$\lim_{x \to \pm\infty} \|\Psi_\pm(z, x)\|_{\mathbb{C}^2} = \lim_{x \to \mp\infty} \|\Psi_\pm(z, x)\|_{\mathbb{C}^2}^{-1} = 0,$$

(5.41) $$\sup_{r \in \mathbb{R}} \left[\left(\int_{-\infty}^r dx\, \|\Psi_-(z, x)\|_{\mathbb{C}^2}^2 \right) \left(\int_r^\infty dx\, \|\Psi_+(z, x)\|_{\mathbb{C}^2}^2 \right) \right] < \infty.$$

The corresponding $\bar{z}$-wave functions $\mathcal{K}\Psi_\pm(z,x)$ associated with q satisfy (5.38)–(5.41) with z replaced by $\bar{z}$.

Proof. We prove the existence of the two z-wave functions following the lines of [12, Sect 10.4]. To this end, we introduce the operators

$$D_{\max}(q; -\infty)F = M(q)F,$$
$$(5.42) \quad F \in \mathrm{dom}(D_{\max}(q; -\infty)) = \{G \in L^2((-\infty,0])^2 : G \in AC_{\mathrm{loc}}((-\infty,0])^2,$$
$$M(q)G \in L^2((-\infty,0])^2\},$$

$$D_{\min}(q; -\infty)F = M(q)F,$$
$$(5.43) \quad F \in \mathrm{dom}(D_{\min}(q; -\infty)) = \{G \in \mathrm{dom}(D_{\max}(q; -\infty)) : G(0) = 0;$$
$$\mathrm{supp}(G) \subset (-\infty,0] \text{ is compact}\},$$

$$D_{\max}(q; \infty)F = M(q)F,$$
$$(5.44) \quad F \in \mathrm{dom}(D_{\max}(q; \infty)) = \{G \in L^2([0,\infty))^2 : G \in AC_{\mathrm{loc}}([0,\infty))^2,$$
$$M(q)G \in L^2([0,\infty))^2\},$$

$$D_{\min}(q; \infty)F = M(q)F,$$
$$(5.45) \quad \mathrm{dom}(D_{\min}(q; \infty)) = \{G \in \mathrm{dom}(D_{\max}(q; \infty)) : G(0) = 0;$$
$$\mathrm{supp}(G) \subset [0,\infty) \text{ is compact}\}.$$

In close analogy to [12, Theorem III.10.20], one can prove that for all $z \in \rho(D(q))$,

$$(5.46) \quad \mathrm{def}\big(\overline{D_{\min}(q)} - zI_2\big) = \mathrm{def}\big(\overline{D_{\min}(q; -\infty)} - zI_2\big) + \mathrm{def}\big(\overline{D_{\min}(q; \infty)} - zI_2\big) - 2.$$

Since $D(q) = \overline{D_{\min}(q)}$ is $\mathcal{J}$-self-adjoint and $z \in \rho(D(q))$, we necessarily have (see, e.g., [12, Theorem III.5.5]) that

$$(5.47) \quad \mathrm{def}\big(\overline{D_{\min}(q)} - zI_2\big) = \mathrm{def}(D(q) - zI_2) = 0.$$

Thus,

$$(5.48) \quad \mathrm{def}\big(\overline{D_{\min}(q; -\infty)} - zI_2\big) + \mathrm{def}\big(\overline{D_{\min}(q; \infty)} - zI_2\big) = 2.$$

We claim that the only possibility is

$$(5.49) \quad \mathrm{def}\big(\overline{D_{\min}(q; -\infty)} - zI_2\big) = \mathrm{def}\big(\overline{D_{\min}(q; \infty)} - zI_2\big) = 1.$$

Indeed, arguing by contradiction, we assume, for instance, that

$$(5.50) \quad \mathrm{def}\big(\overline{D_{\min}(q; -\infty)} - z_0I_2\big) = 0.$$

This implies

$$(5.51) \qquad\qquad \text{nul}\big(D_{\text{max}}(-q;\infty) - \overline{z_0}I_2\big) = 2,$$

since $(D_{\text{min}}(q);\infty)^* = D_{\text{max}}(-q;\infty)$. Thus all $\overline{z_0}$-wave functions for the NLS$_-$ potential $-q$ are in $L^2_{\text{loc}}((-\infty,\infty])^2$. According to Remark 4.4, this is clearly impossible for $\overline{z_0} \in \mathbb{R} \cap \rho(D(-q))$. Next we show that this is impossible also for $\overline{z_0} \in (\mathbb{C}\backslash\mathbb{R}) \cap \rho(D(-q))$. Using Lemma 2.2 (iii), we can simplify notation and replace $\overline{z_0}$ by z_0 without loss of generality. Moreover, to avoid confusion with the change $q \to -q$ and the corresponding change for $q^{(1)}_{z_0}$, we simply use q instead of $-q$ in the proof of (5.38) below.

To this end, fix $z_0 \in \rho(D(q))$ with $\text{Im}(z_0) \neq 0$ and $\Psi_1(z_0), \Psi_2(z_0)$ two linearly independent z_0-wave functions associated with the background potential q. The latter are in $L^2([0,\infty))^2$ by hypothesis (5.51). Then for any z_0-wave function $\Psi(z_0) = (\psi_1(z_0), \psi_2(z_0))^\top$ associated with the NLS$_-$ potential q, one infers (cf. (4.7) and (4.10)) that $\Phi^{(1)}_{z_0}(z_0) = \text{Im}(z_0)\mathcal{K}\Psi(z_0)\|\Psi(z_0)\|^{-2}_{\mathbb{C}^2}$ is a z_0-wave function associated with the NLS$_-$ potential

$$(5.52) \qquad\qquad q^{(1)}_{z_0} = q + 4\text{Im}(z_0)\psi_1(z_0)\overline{\psi_2(z_0)}\|\Psi(z_0)\|^{-2}_{\mathbb{C}^2}.$$

Thus $\Phi^{(1)}_{z_0}(z_0)$ satisfies

$$(5.53) \qquad z_0\Phi^{(1)}_{z_0}(z_0) = M(q^{(1)})\Phi^{(1)}_{z_0}(z_0) = M(q)\Phi^{(1)}_{z_0}(z_0) - B(z_0)\Phi^{(1)}_{z_0}(z_0),$$

where

$$(5.54) \quad B(z_0,x) = 4i\text{Im}(z_0)\|\Psi(z_0,x)\|^{-2}_{\mathbb{C}^2} \begin{pmatrix} 0 & \psi_1(z_0,x)\overline{\psi_2(z_0,x)} \\ \overline{\psi_1(z_0,x)}\psi_2(z_0,x) & 0 \end{pmatrix}$$

belongs to $L^\infty(\mathbb{R})^{2\times 2}$. In particular,

$$(5.55) \qquad\qquad \text{ess sup}_{x\in\mathbb{R}}\|B(z_0,x)\|_{\mathbb{C}^{2\times 2}} \leq 2|\text{Im}(z_0)|.$$

Since no confusion can arise, we occasionally suppress the explicit z_0-dependence in the calculations below. The variation of parameters formula (5.3) then yields the following for the fundamental system of solutions $\Xi = [\Psi_1, \Psi_2]$, $W(\Psi_1, \Psi_2) = 1$, associated with q and z_0:

$$(5.56)$$

$$\Phi^{(1)}_{z_0}(x) = \Xi(x)C - i \int_a^x dx'\, \Xi(x)\Xi(x')^{-1}\sigma_3 B(x')\Phi^{(1)}_{z_0}(x')$$

$$= c_1\Psi_1(x) + c_2\Psi_2(x) - i\Xi(x)\int_a^x dx'\, \Xi(x')^{-1}\sigma_3 B(x')\Phi^{(1)}_{z_0}(x'), \quad x \geq a > 0,$$

where $C = (c_1, c_2)^\top \neq 0$ depends on a and we assume $W(\Psi_1, \Psi_2) = 1$ according to Lemma 5.1. Hence one obtains

$$(5.57) \qquad \|\Xi(x)\|_{\mathbb{C}^{2\times 2}} \leq 2^{1/2} \max\left(\|\Psi_1(x)\|_{\mathbb{C}^2}, \|\Psi_2(x)\|_{\mathbb{C}^2}\right).$$

Moreover, writing $\Psi_j = (\psi_{1,j}, \psi_{2,j})^\top$, $j = 1, 2$, one infers

$$(5.58) \qquad \|\psi_{k,j}(z_0, \cdot)\|_{L^2([a,\infty))} \leq K(a), \quad j, k = 1, 2$$

for some constant $K(a) > 0$ with

$$(5.59) \qquad \lim_{a\uparrow\infty} K(a) = 0.$$

Hence,

$$\int_a^x dx' \|\Xi(x')^{-1}\|_{\mathbb{C}^{2\times 2}}^2 \leq 2\int_a^x dx' \max\left(\left\|\begin{pmatrix}\psi_{2,2}(x') \\ -\psi_{2,1}(x')\end{pmatrix}\right\|_{\mathbb{C}^2}^2, \left\|\begin{pmatrix}-\psi_{1,2}(x') \\ \psi_{1,1}(x')\end{pmatrix}\right\|_{\mathbb{C}^2}^2\right)$$

$$(5.60) \qquad\qquad \leq 4K(a)^2.$$

Thus, (5.55)–(5.60) yield for $x \geq a$,

(5.61)

$$\left\|\Phi_{z_0}^{(1)}(x)\right\|_{\mathbb{C}^2} \leq |c_1|\|\Psi_1(x)\|_{\mathbb{C}^2} + |c_2|\|\Psi_2(x)\|_{\mathbb{C}^2}$$

$$+ 2^{1/2}4|\mathrm{Im}(z_0)|K(a)\max\left(\|\Psi_1(x)\|_{\mathbb{C}^2}, \|\Psi_2(x)\|_{\mathbb{C}^2}\right)\left(\int_a^x dx' \left\|\Phi_{z_0}^{(1)}(x')\right\|_{\mathbb{C}^2}^2\right)^{1/2}.$$

Squaring (5.61) and integrating the result from a to x, one estimates

$$(5.62) \qquad\qquad \int_a^x dx' \left\|\Phi_{z_0}^{(1)}(x')\right\|_{\mathbb{C}^2}^2$$

$$\leq 3|c_1|^2 \int_a^x dx' \|\Psi_1(x')\|_{\mathbb{C}^2}^2 + 3|c_2|^2 \int_a^x dx' \|\Psi_2(x')\|_{\mathbb{C}^2}^2$$

$$+ 96|\mathrm{Im}(z_0)|^2 K(a)^2 \int_a^x dx' \int_a^{x'} dx'' \max\left(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\right)\left\|\Phi_{z_0}^{(1)}(x'')\right\|_{\mathbb{C}^2}^2$$

$$\leq 6K(a)^2\left(|c_1|^2 + |c_2|^2\right)$$

$$+ 96|\mathrm{Im}(z_0)|^2 K(a)^2 \int_a^x dx'' \int_{x''}^x dx' \max\left(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\right)\left\|\Phi_{z_0}^{(1)}(x'')\right\|_{\mathbb{C}^2}^2$$

$$\leq 6K(a)^2\left(|c_1|^2 + |c_2|^2\right)$$

$$+ 96|\mathrm{Im}(z_0)|^2 K(a)^2 \int_a^x dx'' \left\|\Phi_{z_0}^{(1)}(x'')\right\|_{\mathbb{C}^2}^2 \int_a^x dx' \max\left(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\right)$$

$$\leq 6K(a)^2\left(|c_1|^2 + |c_2|^2\right) + 192|\mathrm{Im}(z_0)|^2 K(a)^4 \int_a^x dx'' \left\|\Phi_{z_0}^{(1)}(x'')\right\|_{\mathbb{C}^2}^2.$$

Here we have applied the Fubini–Tonelli theorem to the integrand

$$(5.63) \qquad \max\left(\|\Psi_1(x')\|_{\mathbb{C}^2}, \|\Psi_2(x')\|_{\mathbb{C}^2}\right)\left\|\Phi_{z_0}^{(1)}(x'')\right\|_{\mathbb{C}^2}^2 \chi_{[a,x']}(x'') \geq 0$$

(χ_Λ the characteristic function of the set $\Lambda \subset \mathbb{R}$) to prove equality of the iterated integrals $\int_a^x dx' \int_a^{x'} dx'' \cdots$ and $\int_a^x dx'' \int_{x''}^x dx' \cdots$ in (5.62). Hence, if one chooses a large enough (such that $192|\mathrm{Im}(z_0)|^2 K(a)^4 < 1$), then

$$(5.64) \quad \left[1 - 192|\mathrm{Im}(z_0)|^2 K(a)^4\right] \int_a^x dx' \left\|\Phi_{z_0}^{(1)}(z_0, x')\right\|_{\mathbb{C}^2}^2 \leq 6K(a)^2\left(|c_1|^2 + |c_2|^2\right),$$

and thus, $\Phi_{z_0}^{(1)}(z_0, \cdot) \in L^2([a,\infty))^2$. Since $\left\|\Phi_{z_0}^{(1)}(z_0, \cdot)\right\|_{\mathbb{C}^2} = \|\Psi(z_0, \cdot)\|_{\mathbb{C}^2}^{-1}$ this contradicts the assumption $\Psi(z_0, \cdot) \in L^2([0,\infty))^2$. This proves (5.49).

Finally, if $\Psi_-(z)$ and $\Psi_+(z)$ satisfying (5.38) were linearly dependent then it would follow that $z \in \sigma_p(D(q))$ by (5.38), contradicting the initial assumption $z \in \rho(D(q))$. To sum up, (5.49) implies existence and uniqueness (up to constant multiples) of $\Psi_\pm(z)$ satisfying (5.38).

To prove (5.39), we assume without loss of generality that

$$(5.65) \qquad W(\Psi_-(z,x), \Psi_+(z,x)) = 1, \quad x \in \mathbb{R}.$$

Then one computes

$$(5.66) \qquad 1 + |(\Psi_-(z,x), \Psi_+(z,x))_{\mathbb{C}^2}|^2 = \|\Psi_+(z,x)\|_{\mathbb{C}^2}^2 \|\Psi_-(z,x)\|_{\mathbb{C}^2}^2 \geq 1$$

and hence,

$$(5.67) \qquad \|\Psi_\mp(z,x)\|_{\mathbb{C}^2}^{-1} \leq \|\Psi_\pm(z,x)\|_{\mathbb{C}^2}, \quad x \in \mathbb{R}.$$

Thus, $\|\Psi_-(z,\cdot)\|_{\mathbb{C}^2}^{-1} \in L^2_{\mathrm{loc}}((-\infty,\infty])$. The fact that $\|\Psi_+(z,\cdot)\|_{\mathbb{C}^2}^{-1} \in L^2_{\mathrm{loc}}([-\infty,\infty))$ in (5.39) is proved analogueously.

From (5.38) and (5.39), one infers

$$(5.68) \qquad \liminf_{x\to\pm\infty} \|\Psi_\pm(z,x)\|_{\mathbb{C}^2} = \liminf_{x\to\pm\infty} \|\Psi_\mp(z,x)\|_{\mathbb{C}^2}^{-1} = 0.$$

To prove (5.40), one first integrates (2.17) to obtain
$$(5.69)$$
$$\|\Psi_\pm(z,x_2)\|_{\mathbb{C}^2}^2 - \|\Psi_\pm(z,x_1)\|_{\mathbb{C}^2}^2 = 2\mathrm{Im}(z)\int_{x_1}^{x_2} dx\left[|\psi_{1,\pm}(z,x)|^2 - |\psi_{2,\pm}(z,x)|^2\right].$$

Thus, $\lim_{x\to\pm\infty} \|\Psi_\pm(z,x)\|_{\mathbb{C}^2}$ exist and hence equal 0 by (5.68). By (5.67), one then has

$$(5.70) \qquad \lim_{x\to\pm\infty} \|\Psi_\mp(z,x)\|_{\mathbb{C}^2}^{-1} = 0.$$

What remains to be proved is (5.41). To this end, consider the Green's function for the Dirac-type operator $D(q)$. It can be written in terms of the z-wave functions
(5.71)
$$\Psi_-(z,x) = (\psi_{1,-}(z,x), \psi_{2,-}(z,x))^\top, \quad \Psi_+(z,x) = (\psi_{1,+}(z,x), \psi_{2,+}(z,x))^\top,$$

whose existence is guaranteed by (5.38) and whose Wronskian is still assumed to satisfy (5.65). More precisely, define the 2×2 matrix-valued function on $\mathbb{R}^2$

$$(5.72) \qquad G(z,x,x') = i \begin{cases} \Psi_+(z,x)\Psi_-(z,x')^\top \sigma_1, & x > x', \\ \Psi_-(z,x)\Psi_+(z,x')^\top \sigma_1, & x < x', \end{cases} \quad z \in \rho(D(q)).$$

Explicitly,
(5.73)

$$G(z,x,x') = i \begin{cases} \begin{pmatrix} \psi_{1,+}(z,x)\psi_{2,-}(z,x') & \psi_{1,+}(z,x)\psi_{1,-}(z,x') \\ \psi_{2,+}(z,x)\psi_{2,-}(z,x') & \psi_{2,+}(z,x)\psi_{1,-}(z,x') \end{pmatrix}, & x > x', \\[2em] \begin{pmatrix} \psi_{1,-}(z,x)\psi_{2,+}(z,x') & \psi_{1,-}(z,x)\psi_{1,+}(z,x') \\ \psi_{2,-}(z,x)\psi_{2,+}(z,x') & \psi_{2,-}(z,x)\psi_{1,+}(z,x') \end{pmatrix}, & x < x'. \end{cases}$$

To prove that (5.72) does indeed represent the Green's function associated with the $\mathcal{J}$-self-adjoint operator $D(q)$, one can argue as follows. One introduces a densely defined operator $R(z)$ in $L^2(\mathbb{R})^2$ by

$$(5.74) \qquad (R(z)F)(x) = \int_\mathbb{R} dx' \, G(z,x,x')F(x'), \quad z \in \rho(D(q)), \, x \in \mathbb{R},$$

$$(5.75) \qquad F \in \mathrm{dom}(R(z)) = \{G \in L^2(\mathbb{R}))^2 : \mathrm{supp}(G) \text{ is compact}\}.$$

By inspection, one sees that

$$(5.76) \qquad R(z)F \in AC_{\mathrm{loc}}(\mathbb{R})^2 \cap L^2(\mathbb{R})^2 \quad \text{and} \quad M(q)R(z)F \in L^2(\mathbb{R})^2.$$

Thus, $R(z)$ maps $L^2(\mathbb{R})^2$-elements of compact support into the domain of $D(q)$. Moreover, an explicit computation shows that

$$(M(q) - zI)[R(z)F - (D(q) - zI)^{-1}F]$$
$$(5.77) \qquad = (D(q) - zI)[R(z)F - (D(q) - zI)^{-1}F] = 0, \quad F \in \mathrm{dom}(R(z)).$$

Since by hypothesis, $z \in \rho(D(q))$, (5.77) implies

$$(5.78) \qquad R(z)F = (D(q) - zI)^{-1}F, \quad F \in \mathrm{dom}(R(z)).$$

Hence $R(z)$ extends boundedly to all of $L^2(\mathbb{R})^2$, and its closure coincides with the resolvent $(D(q) - zI)^{-1}$ of $D(q)$. Being the closure of the Carleman operator

$R(z)$, $(D(q) - zI)^{-1}$ is a Carleman operator and hence has a Carleman integral kernel. Moreover, the integral kernels of $R(z)$ and $(D(q) - zI)^{-1}$ are easily seen to coincide when considering restriction of both operators to $L^2((a,b))^2$ for arbitrary $a, b \in \mathbb{R}$, $a < b$. This proves that (5.72) is the integral kernel of $(D(q) - zI)^{-1}$. Thus, $z \in \rho(D(q))$ is equivalent to the boundedness of the operator

$$(5.79) \qquad L^2(\mathbb{R})^2 \ni F(x) \longmapsto \int_{\mathbb{R}} G(z,x,y)F(y)\,dy$$

$$= i\Psi_+(z,x) \int_{-\infty}^x \Psi_-(z,y)^\top \sigma_1 F(y)dy$$

$$+ i\Psi_-(z,x) \int_x^\infty \Psi_+(z,y)^\top \sigma_1 F(y)dy$$

in $L^2(\mathbb{R})^2$. Taking $z \in \rho(D(q))$ and $F = (f,0)^\top$ and $F = (0,f)^\top$, one sees that the operators

$$L^2(\mathbb{R}) \ni f(x) \mapsto \psi_{j,+}(z,x) \int_{-\infty}^x \psi_{k,-}(z,y)f(y)dy$$

$$(5.80) \qquad\qquad + \psi_{j,-}(z,x) \int_x^\infty \psi_{k,+}(z,y)f(y)dy, \quad j,k = 1,2$$

are bounded in $L^2(\mathbb{R})$. The last statement implies the relations (in fact, it is equivalent to them, cf. [37] and Lemma 6.2)

$$(5.81) \quad \sup_{r \in \mathbb{R}} \left[\left(\int_{-\infty}^r |\psi_{k,-}(x)|^2\,dx \right) \left(\int_r^\infty |\psi_{j,+}(x)|^2\,dx \right) \right] < +\infty, \quad j,k = 1,2.$$

For simplicity, we consider the case $j = k = 1$. (The proof for the remaining combinations of indices j, k proceeds analogously; cf. also [4].) Assume (5.80) for $j = k = 1$; then there exists a constant $C > 0$ such that

$$\int_{\mathbb{R}} \left| \psi_{1,+}(x) \int_{-\infty}^x \psi_{1,-}(y)f(y)\,dy + \psi_{1,-}(x) \int_x^\infty \psi_{1,+}(y)f(y)dy \right|^2 dx$$

$$(5.82) \qquad \leq C \int_{\mathbb{R}} |f(x)|^2\,dx.$$

For fixed $r \in \mathbb{R}$ and $f \in L^2(\mathbb{R})$ satisfying $f(x) = 0$ for all $x > r$, the inequality (5.82) implies (restricting the interval of integration)

$$(5.83) \qquad \left(\int_r^\infty |\psi_{1,+}(x)|^2\,dx \right) \left| \int_{-\infty}^r \psi_{1,-}(y)f(y)\,dy \right|^2 \leq C \int_{-\infty}^r |f(x)|^2\,dx.$$

Thus, choosing $f(x) = \overline{\psi_{1,-}(x)}$, for $x \leq r$ and $f(x) = 0$ otherwise (then clearly $f \in L^2(\mathbb{R})$), one obtains (5.81) with $j = k = 1$. Since $\|\Psi_\pm(x)\|_{\mathbb{C}^2}^2 = |\psi_{1,\pm}(x)|^2 + |\psi_{2,\pm}(x)|^2$, (5.81) yields (5.41).

Finally, since $\|\mathcal{K}\Psi(x)\|_{\mathbb{C}^2} = \|\Psi(x)\|_{\mathbb{C}^2}$, the statement about the $\bar{z}$-wave functions associated with q holds. $\square$

The solutions $\Psi_{\pm}(z,x)$ in (5.38) are analogues of the familiar Weyl–Titchmarsh solutions in the context of self-adjoint Dirac-type operators (cf. [33, Ch. 3]).

The following is a consequence of (5.38) and Remark 4.4.

Corollary 5.5. *Assume $q \in L^1_{loc}(\mathbb{R})$ and suppose the* NLS$_-$ *case* $p = -\bar{q}$. *Then*

$$(5.84) \qquad\qquad \sigma_c(D(q)) \supseteq \mathbb{R},$$

$$(5.85) \qquad\qquad \sigma_e(D(q)) \supseteq \mathbb{R},$$

$$(5.86) \qquad\qquad \sigma_p(D(q)) \cap \mathbb{R} = \emptyset.$$

Proof. Relation (5.86) holds by Remark 4.4, which excludes the existence of an $L^2(\mathbb{R})^2$ solution F of $M(q)F = \lambda F$ near $\pm\infty$ for all $\lambda \in \mathbb{R}$. To prove (5.84), we can restrict ourselves to the case in which $\rho(D(q)) \neq \emptyset$. Pick $\lambda_0 \in \mathbb{R}$. Then Remark 4.4, (5.38), and (5.86) imply that $\lambda_0 \notin \rho(D(q)) \cup \sigma_p(D(q))$. Since $\sigma_r(D(q)) = \emptyset$ by (5.28), it follows by (5.14) that $\lambda_0 \in \sigma_c(D(q))$. Relation (5.85) is then obvious from (5.16). $\square$

As a consequence of (5.84), our frequent assumption $z \in \rho(D(q))$ (especially in Section 6) automatically implies $z \in \mathbb{C}\backslash\mathbb{R}$.

Interesting restrictions on the permissible location of eigenvalues of $\mathcal{J}$-self-adjoint Dirac-type operators $D(q)$ under strong additional constraints on q were recently derived in [27].

Remark 5.6. Given normalized Weyl–Titchmarsh-type solutions $\Psi_{\pm}(z,x,x_0)$ of $M(q)\Psi(z,x) = z\Psi(z,x)$ for $z \in \rho(D(q))$ satisfying

$$(5.87) \qquad\qquad \psi_{1,\pm}(z,x_0,x_0) = 1, \quad z \in \rho(D(q))$$

for some $x_0 \in \mathbb{R}$, one can formally introduce associated Weyl–Titchmarsh m-functions as follows. Denote by $\Xi(z,x,x_0)$ a normalized 2×2 fundamental system of solutions of

$$(5.88) \qquad\qquad M(q)\Psi(z,x) = z\Psi(z,x), \quad z \in \mathbb{C}$$

at some $x_0 \in \mathbb{R}$, that is, $\Xi(z,x,x_0)$ satisfies (5.88) for a.e. $x \in \mathbb{R}$ and

$$(5.89) \qquad\qquad \Xi(z,x_0,x_0) = I_2, \quad z \in \mathbb{C}.$$

One then partitions $\Xi(z,x,x_0)$ as

$$(5.90) \qquad \Xi(z,x,x_0) = (\Theta(z,x,x_0)\ \Phi(z,x,x_0)) = \begin{pmatrix} \theta_1(z,x,x_0) & \phi_1(z,x,x_0) \\ \theta_2(z,x,x_0) & \phi_2(z,x,x_0) \end{pmatrix},$$

where $\theta_j(z, x, x_0)$ and $\phi_j(z, x, x_0)$, $j = 1, 2$, are entire with respect to $z \in \mathbb{C}$ and normalized according to (5.89). Then the normalized Weyl–Titchmarsh solutions $\Psi_\pm(z, x, x_0)$ can be expressed in terms of the basis $(\Theta(z, x, x_0)\ \Phi(z, x, x_0))$ as

$$(5.91) \qquad \Psi_\pm(z, x, x_0) = \Theta(z, x, x_0) + m_\pm(z, x_0)\Phi(z, x, x_0), \quad z \in \rho(D(q))$$

for some coefficients $m_\pm(z, x_0)$. Clearly, $m_\pm(z, x_0)$ are analytic on

$$\rho(D(q)) \setminus \{z \in \mathbb{C} : \Phi(z, \cdot, x_0) \in L^2((x_0, \pm\infty))^2\};$$

they are the obvious analogues of the half-line Weyl–Titchmarsh coefficients, familiar from (second-order scalar and first-order 2×2) self-adjoint differential and difference operators (cf., e.g., [2, § VII.1], [33, Chs. 2, 3]). It is tempting to conjecture that appropriate boundary values of $m_\pm(z, x_0)$ as z approaches $\sigma(D(q))$ encode the spectral information on $D(q)$, but this is left for future investigations.

6   Transformation operators for $\mathcal{J}$-self-adjoint Dirac-type operators

The aim of this section is to construct transformation operators in $L^2(\mathbb{R})^2$ that intertwine the $\mathcal{J}$-self-adjoint operators $D(q)$ and $D(q_{s,1}^{(1)})$ corresponding to the Lax differential expressions $M(q)$ and $M(q_{s,1}^{(1)})$ in the focusing NLS$_-$-case and to use these transformation operators to relate the spectra of $D(q)$ and $D(q_{s,1}^{(1)})$, the principal goal of this paper.

In the following, we always assume Hypothesis 5.3 and freely use the notation established in Section 5 for $D(q)$ as the maximally defined $\mathcal{J}$-self-adjoint Dirac operator in the focusing NLS$_-$ case and the Weyl–Titchmarsh-type solutions $\Psi_\pm(z, x)$, $z \in \rho(D(q))$, established in Theorem 5.4.

We start with an elementary but important observation.

Remark 6.1. Since the two z-wave functions $\Psi_-(z)$ and $\Psi_+(z)$ of $D(q)$ are linearly independent,

$$(6.1) \qquad W(\Psi_-(z, x), \Psi_+(z, x)) = c(z) \neq 0,$$

all other (nontrivial) z-wave functions Ψ satisfy

$$(6.2) \qquad \Psi(z) = \alpha\Psi_-(z) + \beta\Psi_+(z) \quad \text{for some } \alpha, \beta \in \mathbb{C}\setminus\{0\}.$$

In addition, as we prove next,

$$(6.3) \qquad \|\Psi(z, \cdot)\|_{\mathbb{C}^2}^{-1} \in L^2(\mathbb{R}).$$

Indeed, $\Psi(z) = \alpha\Psi_-(z) + \beta\Psi_+(z)$; hence

$$(6.4) \qquad \|\Psi(z)\|_{\mathbb{C}^2}^{-1} \le \big| |\alpha| \|\Psi_+(z)\|_{\mathbb{C}^2} - |\beta| \|\Psi_-(z)\|_{\mathbb{C}^2} \big|^{-1},$$

(5.40), and the fact that $\Psi_\pm(z,\cdot) \in AC_{\mathrm{loc}}(\mathbb{R})^2$, yield the existence of constants $C_\pm > 0$ such that

$$\|\Psi(z,x)\|_{\mathbb{C}^2}^{-1} = \big(\|\Psi_\pm(z,x)\|_{\mathbb{C}^2}\|\Psi(z,x)\|_{\mathbb{C}^2}^{-1}\big)\|\Psi_\pm(z,x)\|_{\mathbb{C}^2}^{-1}$$
$$(6.5) \qquad\qquad \le C_\pm\|\Psi_\pm(z,x)\|_{\mathbb{C}^2}^{-1}, \quad x \in \mathbb{R}.$$

By (5.39), this implies that all z-wave functions $\Psi(z)$ associated with q, except $\Psi_\pm(z)$, satisfy (4.11). Hence, Hypothesis 5.3 guarantees the existence of z-wave functions $\Psi(z)$ satisfying Hypothesis 4.2 for all $z \in \rho(D(q))$. In particular, all but two z-wave functions of $D(q)$ (viz., $\Psi_\pm(z)$) satisfy Hypothesis 4.2 at $z \in \rho(D(q))$.

Without loss of generality we restrict our considerations to the special case $\alpha = \beta = 1$ in (6.2) for the remainder of this section up to (6.126), that is, we choose

$$(6.6) \qquad \Psi(z) = \Psi_-(z) + \Psi_+(z)$$

in the following.

Next, we pick some fixed $z_1 \in \rho(D(q))$. We take $\Psi(z_1)$ as in (6.6), $\Psi(z_1) = \Psi_-(z_1) + \Psi_+(z_1)$, where $\Psi_\pm(z_1)$ satisfy (5.38)–(5.41) with z replaced by z_1, and let $\mathcal{K}\Psi(z_1)$ be the corresponding $\overline{z_1}$-wave function associated with q. By Theorem 5.4, $\mathcal{K}\Psi_\pm(z_1,\cdot)$ satisfy (5.38)–(5.41) with z replaced by $\overline{z_1}$. Moreover,

$$(6.7) \qquad \mathcal{K}\Psi(z_1) = \mathcal{K}\Psi_-(z_1) + \mathcal{K}\Psi_+(z_1).$$

Define the NLS potential $q_{z_1}^{(1)}$ as in (2.38) and consider the z_1-wave function $\Phi_{z_1}^{(1)}(z_1) = (\phi_1^{(1)}(z_1), \phi_2^{(1)}(z_1))^\top$ associated with $q_{z_1}^{(1)}$ as defined in (4.7),

$$(6.8) \qquad \Phi_{z_1}^{(1)}(z_1,x) = \mathrm{Im}(z_1)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1,x),$$

and the $\overline{z_1}$-wave function $\mathcal{K}\Phi_{z_1}^{(1)}(z_1)$ associated with $q_{z_1}^{(1)}$ as defined in (4.8),

$$(6.9) \qquad \mathcal{K}\Phi_{z_1}^{(1)}(z_1,x) = -\mathrm{Im}(z_1)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}\Psi(z_1,x).$$

Recall that according to (4.10) the new NLS$_-$ potential is then given by

$$(6.10) \qquad q_{z_1}^{(1)}(x) = q(x) + 4\phi_1^{(1)}(z_1,x)\psi_1(z_1,x).$$

Of course, both $D(q)$ and $D\big(q_{z_1}^{(1)}\big)$ (associated with the differential expressions $M(q)$ and $M\big(q_{z_1}^{(1)}\big)$, respectively) are $\mathcal{J}$-self-adjoint by Theorem 3.5 and Corollary 2.9.

In order to motivate the introduction of transformation operators, one can argue as follows. Since

$$(6.11) \qquad \Gamma(z, \Psi(z_1), \mathcal{K}\Psi(z_1)) = -(i/2)(z - z_1)I_2 + \Phi_{z_1}^{(1)}(z_1)\Psi(z_1)^{\perp},$$

one computes, for every z-wave function $\Upsilon(z, \cdot)$ associated with q,

$$(6.12) \qquad \Upsilon_{z_1}^{(1)}(z, x) = \Gamma(z, x, \Psi(z_1), \mathcal{K}\Psi(z_1))\Upsilon(z, x)$$

$$= -(i/2)(z - z_1)\Upsilon(z, x) + \Phi_{z_1}^{(1)}(z_1, x)\Psi(z_1, x)^{\perp}\Upsilon(z, x)$$

$$= -(i/2)(z - z_1)\Upsilon(z, x) - \Phi_{z_1}^{(1)}(z_1, x)W(\Psi(z_1, x), \Upsilon(z, x))$$

$$= -(i/2)(z - z_1)\Upsilon(z, x) - \Phi_{z_1}^{(1)}(z_1, x)W(\Psi_-(z_1, x) + \Psi_+(z_1, x), \Upsilon(z, x))$$

$$= -(i/2)(z - z_1)\left\{ \Upsilon(z, x) + 2\Phi_{z_1}^{(1)}(z_1, x) \right.$$

$$\left. \times \left[\int_{-a}^{x} dx' \, \Psi_-(z_1, x')^{\top}\sigma_1\Upsilon(z, x') - \int_{x}^{a} dx' \, \Psi_+(z_1, x')^{\top}\sigma_1\Upsilon(z, x') \right] \right\}$$

$$- \Phi_{z_1}^{(1)}(z_1, x)[W(\Psi_-(z_1, -a), \Upsilon(z, -a)) + W(\Psi_+(z_1, a), \Upsilon(z, a))], \quad a > 0.$$

To arrive at (6.12), we have used the integrated form of (2.6). Replacing $\Upsilon(z)$ by $F \in L^2(\mathbb{R})^2$ in (6.12), noting that

$$(6.13) \qquad \liminf_{a\uparrow\infty} |W(\Psi_\pm(z_1, \pm a), F(\pm a))| = 0, \quad F \in L^2(\mathbb{R})^2,$$

and repeating the same argument with $\Upsilon(z)$ replaced by $\mathcal{K}\Upsilon(z)$ then leads to the introduction of the following transformation operators T_{z_1} and $\tilde{T}_{\overline{z_1}}$ in $L^2(\mathbb{R})^2$,

$$(6.14) \quad L^2(\mathbb{R})^2 \ni F(x) \mapsto (T_{z_1}F)(x) = F(x) + 2\Phi_{z_1}^{(1)}(z_1, x)$$

$$\times \left[\int_{-\infty}^{x} dx' \, \Psi_-(z_1, x')^{\top}\sigma_1 F(x') - \int_{x}^{\infty} dx' \, \Psi_+(z_1, x')^{\top}\sigma_1 F(x') \right],$$

$$(6.15) \quad L^2(\mathbb{R})^2 \ni F(x) \mapsto (\tilde{T}_{\overline{z_1}}F)(x) = F(x) + 2\mathcal{K}\Phi_{z_1}^{(1)}(z_1, x)$$

$$\times \left[\int_{-\infty}^{x} dx' \, (\mathcal{K}\Psi_-(z_1, x'))^{\top}\sigma_1 F(x') - \int_{x}^{\infty} dx' \, (\mathcal{K}\Psi_+(z_1, x'))^{\top}\sigma_1 F(x') \right],$$

$$z_1 \in \rho(D(q)).$$

That T_{z_1} and $\tilde{T}_{\overline{z_1}}$ are in fact bounded operators in $L^2(\mathbb{R})^2$ follows from Lemmas 6.2 and 6.3 below.

Lemma 6.2 (Talenti [44], Tomaselli [46] (see also Chisholm and Everitt [4], Chisholm, Everitt and Littlejohn [5], and Muckenhoupt [37]). *Let* $f \in L^2(\mathbb{R})$, $U \in L^2((-\infty, R])$, $V \in L^2([R, \infty))$ *for all* $R \in \mathbb{R}$. *Then the following assertions* (i)–(iii) *are equivalent.*

(i) *There exists a finite constant $C > 0$ such that*

$$(6.16) \qquad \int_{\mathbb{R}} dx \left| U(x) \int_x^\infty dx' \, V(x') f(x') \right|^2 \le C \int_{\mathbb{R}} dx \, |f(x)|^2.$$

(ii) *There exists a finite constant $D > 0$ such that*

$$(6.17) \qquad \int_{\mathbb{R}} dx \left| V(x) \int_{-\infty}^x dx' \, U(x') f(x') \right|^2 \le D \int_{\mathbb{R}} dx \, |f(x)|^2.$$

(iii)

$$(6.18) \qquad \sup_{r \in \mathbb{R}} \left[\left(\int_{-\infty}^r dx \, |U(x)|^2 \right) \left(\int_r^\infty dx \, |V(x)|^2 \right) \right] < \infty.$$

Lemma 6.3. *Assume Hypothesis 5.3 and $z_1 \in \rho(D(q))$. Then the operators*

$$(6.19) \qquad L^2(\mathbb{R}) \ni f(x) \mapsto \frac{1}{\|\Psi_\pm(z_1,x)\|_{\mathbb{C}^2}} \int_{\pm\infty}^x dx' \, \|\Psi_\pm(z_1,x')\|_{\mathbb{C}^2} \, f(x'),$$

$$(6.20) \qquad L^2(\mathbb{R}) \ni f(x) \mapsto \|\Psi_\pm(z_1,x)\|_{\mathbb{C}^2} \int_{\mp\infty}^x dx' \, \frac{1}{\|\Psi_\pm(z_1,x')\|_{\mathbb{C}^2}} f(x')$$

are bounded in $L^2(\mathbb{R})$.

Proof. Using (5.38), (5.39), (5.41), and (5.67), we obtain

$$(6.21) \qquad \sup_{r \in \mathbb{R}} \left(\int_{-\infty}^r dx \, \|\Psi_-(z_1,x)\|_{\mathbb{C}^2}^2 \right) \left(\int_r^\infty dx \, \|\Psi_-(z_1,x)\|_{\mathbb{C}^2}^{-2} \right) < \infty$$

and

$$(6.22) \qquad \sup_{r \in \mathbb{R}} \left(\int_{-\infty}^r dx \, \|\Psi_+(z_1,x)\|_{\mathbb{C}^2}^{-2} \right) \left(\int_r^\infty dx \, \|\Psi_+(z_1,x)\|_{\mathbb{C}^2}^2 \right) < \infty.$$

By Lemma 6.2, (6.21) is a necessary and sufficient condition for the operators associated with $\Psi_-(z_1, \cdot)$ in (6.19) and (6.20) to be bounded in $L^2(\mathbb{R})$. Similarly, (6.22) is a necessary and sufficient condition for the operators associated with $\Psi_+(z_1, \cdot)$ in (6.19) and (6.20) to be bounded in $L^2(\mathbb{R})$. $\square$

Thus, (6.19), (4.7), and (4.8) imply the following result.

Corollary 6.4. *Assume Hypothesis 5.3 and $z_1 \in \rho(D(q))$. Then the operators T_{z_1} and $\widetilde{T}_{\overline{z_1}}$ defined in (6.14) and (6.15) are bounded operators in $L^2(\mathbb{R})^2$.*

Proof. By (6.8), we have $\left\|\Phi_{z_1}^{(1)}(z_1,x)\right\|_{\mathbb{C}^2} = |\text{Im}(z_1)| \|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1}$. Applying (6.5) then yields the existence of constants $C_\pm > 0$ such that

$$
\begin{aligned}
\left\|\Phi_{z_1}^{(1)}(z_1,x)\right\|_{\mathbb{C}^2} &= |\text{Im}(z_1)| \|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1} \\
(6.23) \qquad &\leq |\text{Im}(z_1)| C_\pm \|\Psi_\pm(z_1,x)\|_{\mathbb{C}^2}^{-1}, \quad x \in \mathbb{R}.
\end{aligned}
$$

At this point, an application of Lemma 6.3 proves the boundedness of T_{z_1} in $L^2(\mathbb{R})^2$. Since $\|\mathcal{K}G\|_{\mathbb{C}^2} = \|G\|_{\mathbb{C}^2}$ for all $G \in \mathbb{C}^2$, the same arguments prove boundedness of $\tilde{T}_{\overline{z_1}}$ in $L^2(\mathbb{R})^2$. $\qquad\square$

In order to motivate the introduction of the inverse transformation operators, we invert the matrix $\Gamma(z, \Psi(z_1), \mathcal{K}\Psi(z_1))$ to obtain

$$
(6.24) \quad \Gamma(z, \Psi(z_1), \mathcal{K}\Psi(z_1))^{-1} = c(z,z_1)\left[-(i/2)(z - \overline{z_1})I_2 + \mathcal{K}\Psi(z_1)\mathcal{K}\Phi_{z_1}^{(1)}(z_1)^\perp \right],
$$

where

$$
(6.25) \qquad c(z,z_1) = -4(z - z_1)^{-1}(z - \overline{z_1})^{-1}.
$$

Thus, for z-wave functions $\Upsilon_{z_1}^{(1)}(z,x)$ associated with $q_{z_1}^{(1)}$,

$$
\begin{aligned}
(6.26) \qquad \Upsilon(z,x) &= \Gamma(z, x, \Psi(z_1), \mathcal{K}\Psi(z_1))^{-1}\Upsilon_{z_1}^{(1)}(z,x) \\
&= c(z,z_1)\left[-(i/2)(z - \overline{z_1})\Upsilon_{z_1}^{(1)}(z,x) + \mathcal{K}\Psi(z_1,x)\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x)^\perp \Upsilon_{z_1}^{(1)}(z,x) \right] \\
&= c(z,z_1)\left[-(i/2)(z - \overline{z_1})\Upsilon_{z_1}^{(1)}(z,x) - \mathcal{K}\Psi(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x), \Upsilon_{z_1}^{(1)}(z,x)\big) \right] \\
&= c(z,z_1)\Big\{ -(i/2)(z - \overline{z_1})\Upsilon_{z_1}^{(1)}(z,x) \\
&\qquad\quad - [\mathcal{K}\Psi_-(z_1,x) + \mathcal{K}\Psi_+(z_1,x)]W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x), \Upsilon_{z_1}^{(1)}(z,x)\big) \Big\} \\
&= 2i(z - z_1)^{-1}\bigg\{ \Upsilon_{z_1}^{(1)}(z,x) + 2\mathcal{K}\Psi_+(z_1,x)\int_{-a}^x dx'\, \big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top \sigma_1 \Upsilon_{z_1}^{(1)}(z,x') \\
&\qquad\qquad - 2\mathcal{K}\Psi_-(z_1,x)\int_x^a dx'\, \big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top \sigma_1 \Upsilon_{z_1}^{(1)}(z,x') \bigg\} \\
&\quad - c(z,z_1)\Big[\mathcal{K}\Psi_-(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,-a), \Upsilon_{z_1}^{(1)}(z,-a)\big) \\
&\qquad\qquad + \mathcal{K}\Psi_+(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,a), \Upsilon_{z_1}^{(1)}(z,a)\big) \Big], \quad a > 0.
\end{aligned}
$$

Replacing $\Upsilon_{z_1}^{(1)}(z)$ by $F \in L^2(\mathbb{R})^2$ in (6.26) and repeating the same argument for $\mathcal{K}\Upsilon_{z_1}^{(1)}(z)$ instead of $\Upsilon_{z_1}^{(1)}(z)$ then leads to the introduction of the following (inverse)

transformation operators $\widehat{S}_{\overline{z_1}}$ and $\widetilde{S}_{z_1}$ in $L^2(\mathbb{R})^2$,

$$L^2(\mathbb{R})^2 \ni F(x) \mapsto (\widehat{S}_{\overline{z_1}}F)(x) = F(x)$$

$$+ 2\mathcal{K}\Psi_+(z_1, x) \int_{-\infty}^x dx' \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1, x')\right)^\top \sigma_1 F(x')$$

(6.27) $$- 2\mathcal{K}\Psi_-(z_1, x) \int_x^\infty dx' \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1, x')\right)^\top \sigma_1 F(x'),$$

$$L^2(\mathbb{R})^2 \ni F(x) \mapsto (\widetilde{S}_{z_1}F)(x) = F(x)$$

(6.28) $$+ 2\Psi_+(z_1, x) \int_{-\infty}^x dx' \, \Phi_{z_1}^{(1)}(z_1, x')^\top \sigma_1 F(x')$$

$$- 2\Psi_-(z_1, x) \int_x^\infty dx' \, \Phi_{z_1}^{(1)}(z_1, x')^\top \sigma_1 F(x'), \quad z_1 \in \rho(D(q)).$$

Equations (6.20), (4.7), and (4.8) then imply the following result.

Corollary 6.5. *Assume Hypothesis 5.3 and $z_1 \in \rho(D(q))$. Then the operators $\widehat{S}_{\overline{z_1}}$ and $\widetilde{S}_{z_1}$ defined in (6.27) and (6.28) are bounded operators in $L^2(\mathbb{R})^2$.*

Proof. Again using the estimate (6.23), one can follow the arguments in the proof of Corollary 6.4. $\square$

We define two vectors $F, G \in L^2(\mathbb{R})^2$ to be $\mathcal{J}$-orthogonal if

(6.29) $$(F, \mathcal{J}G)_{L^2} = 0,$$

and we then write

(6.30) $$F \perp_{\mathcal{J}} G.$$

Since

(6.31) $$(F(x), \mathcal{J}G(x))_{\mathbb{C}^2} = (\mathcal{J}G(x))^* F(x) = G(x)^\top \sigma_1 F(x),$$

$F \perp_{\mathcal{J}} G$ is equivalent to

$$(F, \mathcal{J}G)_{L^2} = \int_{\mathbb{R}} dx \, G(x)^\top \sigma_1 F(x) = 0, \quad F, G \in L^2(\mathbb{R})^2.$$

We also introduce the following notation of the $\mathcal{J}$-orthogonal complement $\mathcal{V}^{\perp_{\mathcal{J}}}$ to a subset $\mathcal{V} \subset L^2(\mathbb{R})^2$,

(6.32) $$\mathcal{V}^{\perp_{\mathcal{J}}} = \{F \in L^2(\mathbb{R})^2 : F \perp_{\mathcal{J}} G \text{ for all } G \in \mathcal{V}\}.$$

In analogy to the orthogonality property of eigenvectors corresponding to different (necessarily real) eigenvalues of a symmetric operator in some complex Hilbert

space $\mathcal{H}$, one infers the $\mathcal{J}$-orthogonality of eigenvectors corresponding to different eigenvalues of a $\mathcal{J}$-symmetric operator S in $\mathcal{H}$. Indeed, $Sf_j = z_j f_j$, $j = 1, 2$, with $z_1 \neq z_2$ implies

(6.33)
$$\overline{z_2}(\mathcal{J} f_1, f_2)_{\mathcal{H}} = (\mathcal{J} f_1, Sf_2)_{\mathcal{H}} = (\mathcal{J} Sf_1, f_2)_{\mathcal{H}} = (\mathcal{J} z_1 f_1, f_2)_{\mathcal{H}} = \overline{z_1}(\mathcal{J} f_1, f_2)_{\mathcal{H}}$$

and hence

(6.34) $\qquad (z_1 - z_2)(f_1, \mathcal{J} f_2)_{\mathcal{H}} = 0, \quad \text{implying} (f_1, \mathcal{J} f_2)_{\mathcal{H}} = 0.$

Next, let σ_0 be an isolated subset of $\sigma\big(D\big(q_{z_1}^{(1)}\big)\big)$ in the sense that σ_0 can be surrounded by a positively oriented rectifiable simple closed countour $\gamma_{\sigma_0} \subset \rho\big(D\big(q_{z_1}^{(1)}\big)\big)$ separating σ_0 from the remaining spectrum $\sigma\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\sigma_0$. Then the Riesz projection onto the spectral subspace corresponding to σ_0 is given by

(6.35) $\qquad P_{z_1}^{(1)}(\sigma_0) = -\dfrac{1}{2\pi i} \displaystyle\int_{\gamma_{\sigma_0}} dz \left(D\big(q_{z_1}^{(1)}\big) - zI_2\right)^{-1}.$

The spectral subspace $\Sigma_{z_1}^{(1)}(\sigma_0)$ corresponding to σ_0 is then the range of the Riesz projection, $\Sigma_{z_1}^{(1)}(\sigma_0) = P_{z_1}^{(1)}(\sigma_0)L^2(\mathbb{R})^2$.

We record the following result.

Lemma 6.6. *Assume Hypothesis 5.3.*

(i) *Let* $z_0, \tilde{z}_0 \in \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)$, $z_0 \neq \tilde{z}_0$. *Then the* $L^2(\mathbb{R})^2$-*eigenfunctions* $\Phi_{z_1}^{(1)}(z_0)$, $\Phi_{z_1}^{(1)}(\tilde{z}_0)$ *corresponding to* z_0 *and* $\tilde{z}_0$, *respectively, are* $\mathcal{J}$-*orthogonal,*

(6.36) $\qquad\qquad\qquad \Phi_{z_1}^{(1)}(z_0) \perp_{\mathcal{J}} \Phi_{z_1}^{(1)}(\tilde{z}_0).$

(ii) *Let* $z_0 \in \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)$ *and* σ_0 *be an isolated subset of* $\sigma\big(D\big(q_{z_1}^{(1)}\big)\big)$ *which does not contain* z_0. *Then the* $L^2(\mathbb{R})^2$-*eigenfunction* $\Phi_{z_1}^{(1)}(z_0)$ *corresponding to* z_0 *is* $\mathcal{J}$-*orthogonal to the spectral subspace* $\Sigma_{z_1}^{(1)}(\sigma_0)$ *corresponding to* σ_0,

(6.37) $\qquad\qquad\qquad \Phi_{z_1}^{(1)}(z_0) \perp_{\mathcal{J}} \Sigma_{z_1}^{(1)}(\sigma_0).$

Proof. Assertion (i) is clear from (6.33) and (6.34). To prove (ii), we choose a positively oriented rectifiable simple closed contour γ_{σ_0} which separates σ_0 and $\{z_0\}$. For $F \in L^2(\mathbb{R})^2$, we compute

$$\left(P_{z_1}^{(1)}(\sigma_0)F, \mathcal{J}\Phi_{z_1}^{(1)}(z_0)\right)_{L^2} = -\dfrac{1}{2\pi i} \int_{\gamma_{\sigma_0}} dz \left(D\big(q_{z_1}^{(1)}\big) - zI_2\right)^{-1} F, \mathcal{J}\Phi_{z_1}^{(1)}(z_0)\right)_{L^2}$$

$$= -\dfrac{1}{2\pi i} \int_{\gamma_{\sigma_0}} dz \left(F, \mathcal{J}\big(D\big(q_{z_1}^{(1)}\big) - zI_2\big)^{-1}\Phi_{z_1}^{(1)}(z_0)\right)_{L^2}$$

$$= -\dfrac{1}{2\pi i} \int_{\gamma_{\sigma_0}} dz \, (z_0 - z)^{-1}\left(F, \mathcal{J}\Phi_{z_1}^{(1)}(z_0)\right)_{L^2} = 0.$$

Here we have used the fact that since $D\big(q_{z_1}^{(1)}\big)$ is $\mathcal{J}$-self-adjoint, so is $\big(D\big(q_{z_1}^{(1)}\big) - z I_2\big)^{-1}$. $\square$

Remark 6.7. From Lemma 6.6 and Theorem 4.5, we conclude that

$$(6.38) \qquad \Phi_{z_1}^{(1)}(z_1) \perp_{\mathcal{J}} \mathcal{K}\Phi_{z_1}^{(1)}(z_1),$$

since z_1 is a nonreal eigenvalue. Thus,

$$(6.39) \qquad \Phi_{z_1}^{(1)}(z_1) \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}.$$

Remark 6.8. By Corollary 6.5, the operator $\widehat{S}_{\overline{z_1}}$, $z_1 \in \rho(D(q))$, is well-defined and bounded in $L^2(\mathbb{R})^2$. However, for future considerations, it is more appropriate to restrict it to the closed subspace $\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$. Hence, from now on, we denote by $S_{\overline{z_1}}$ the restriction of $\widehat{S}_{\overline{z_1}}$ to $\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$,

$$(6.40) \qquad S_{\overline{z_1}} = \widehat{S}_{\overline{z_1}}\Big|_{\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}}, \quad z_1 \in \rho(D(q)).$$

An elementary computation (based on (6.31)) then reveals that

$$(S_{\overline{z_1}}G)(x) = (\widehat{S}_{\overline{z_1}}G)(x) = G(x) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\, \big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1 G(x'),$$

$$(6.41)$$

$$G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}.$$

Next, we prove several results leading up to the principal theorems of this section.

Lemma 6.9. *Assume Hypothesis 5.3 and $z_1 \in \rho(D(q))$. Then*

$$(6.42) \quad \operatorname{ran}(T_{z_1}) \subseteq \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}, \quad \Phi_{z_1}^{(1)}(z_1) \in \ker(S_{\overline{z_1}}),$$

$$(6.43) \quad \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} = \operatorname{Im}(z_1)(2W(\Psi_+(z_1),\Psi_-(z_1)))^{-1} \neq 0,$$

$$(6.44) \quad T_{z_1}S_{\overline{z_1}}G = G - \frac{\big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}{\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}\Phi_{z_1}^{(1)}(z_1), \quad G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}},$$

$$(6.45) \quad S_{\overline{z_1}}T_{z_1}F = F, \quad F \in L^2(\mathbb{R})^2.$$

Proof. For brevity, we introduce

$$(6.46) \quad \xi(x;F) = \int_{-\infty}^{x} dx'\, \Psi_-(z_1,x')^{\top}\sigma_1 F(x') - \int_{x}^{\infty} dx'\, \Psi_+(z_1,x')^{\top}\sigma_1 F(x'),$$

$$F \in L^2(\mathbb{R})^2,\ x \in \mathbb{R},$$

such that

$$(6.47) \qquad (T_{z_1} F)(x) = F(x) + 2\Phi_{z_1}^{(1)}(z_1, x)\xi(x; F), \quad F \in L^2(\mathbb{R})^2.$$

Using

$$(6.48) \qquad \xi_x(x; F) = \Psi(z_1, x)^\top \sigma_1 F(x) \quad \text{for a.e. } x \in \mathbb{R},$$

we have

(6.49)

$$\left(T_{z_1} F, \mathcal{JK}\Phi_{z_1}^{(1)}(z_1)\right)_{L^2} = \left(F, \mathcal{JK}\Phi_{z_1}^{(1)}(z_1)\right)_{L^2} + 2\left(\Phi_{z_1}^{(1)}(z_1)\xi(\cdot; F), \mathcal{JK}\Phi_{z_1}^{(1)}(z_1)\right)_{L^2}$$

$$= \left(F, \mathcal{JK}\Phi_{z_1}^{(1)}(z_1)\right)_{L^2} - (\mathrm{Im}(z_1))^{-1} \int_{\mathbb{R}} dx \left(\left\|\Phi_{z_1}^{(1)}(z_1, x)\right\|_{\mathbb{C}^2}^2\right)_x \xi(x; F)$$

$$= \left(F, \mathcal{JK}\Phi_{z_1}^{(1)}(z_1)\right)_{L^2} + (\mathrm{Im} z_1)^{-1} \int_{\mathbb{R}} dx \left\|\Phi_{z_1}^{(1)}(z_1, x)\right\|_{\mathbb{C}^2}^2 \Psi(z_1, x)^\top \sigma_1 F(x)$$

$$= \left(F, \mathcal{JK}\Phi_{z_1}^{(1)}(z_1)\right)_{L^2} + \mathrm{Im}(z_1) \int_{\mathbb{R}} dx \left\|\Psi(z_1, x)\right\|_{\mathbb{C}}^{-2} \Psi(z_1, x)^\top \sigma_1 F(x)$$

$$= 0, \quad F \in L^2(\mathbb{R})^2.$$

To justify the integration by parts step in (6.49), one can argue as follows. By (6.23), the estimate

$$\left\|\Phi_{z_1}^{(1)}(z_1, x)\right\|_{\mathbb{C}^2}^2 |\xi(x; F)|$$

$$\leq |\mathrm{Im}(z_1)|^2 \|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-2} \left[\int_{-\infty}^x dx' \|\Psi_-(z_1, x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2} \right.$$

$$\left. + \int_x^\infty dx' \|\Psi_+(z_1, x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2} \right]$$

$$\leq |\mathrm{Im}(z_1)|^2 \|\Psi(z_1, x)\|_{\mathbb{C}^2}^{-1} \left[C_- \|\Psi_-(z_1, x)\|_{\mathbb{C}^2}^{-1} \int_{-\infty}^x dx' \|\Psi_-(z_1, x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2} \right.$$

$$(6.50) \qquad \left. + C_+ \|\Psi_+(z_1, x)\|_{\mathbb{C}^2}^{-1} \int_x^\infty dx' \|\Psi_+(z_1, x')\|_{\mathbb{C}^2} \|F(x')\|_{\mathbb{C}^2} \right]$$

yields

$$(6.51) \qquad \left\|\Phi_{z_1}^{(1)}(z_1, \cdot)\right\|_{\mathbb{C}^2}^2 |\xi(\cdot; F)| \in L^1(\mathbb{R}),$$

by (6.3) and (6.19). Thus,

$$(6.52) \qquad \liminf_{x \to \pm\infty} \left\|\Phi_{z_1}^{(1)}(z_1, x)\right\|_{\mathbb{C}^2}^2 |\xi(x; F)| = 0$$

(actually, $\lim_{x \to \pm\infty} |\cdots| = 0$ in (6.52) since all Lebesgue integrals involved in (6.49) are finite), which was to be proven. Hence, $\mathcal{K}\Phi_{z_1}^{(1)}(z_1) \perp_{\mathcal{J}} \mathrm{ran}(T_{z_1})$.

Next, using (2.6), Lemma 2.2 (iii), (6.8), (6.39), and (6.41), one computes

(6.53)

$$\left(S_{\overline{z_1}}\Phi_{z_1}^{(1)}(z_1)\right)(x) = \Phi_{z_1}^{(1)}(z_1,x) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\, \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\right)^{\top}\sigma_1\Phi^{(1)}(z_1,x')$$

$$= \Phi_{z_1}^{(1)}(z_1,x) - 2\mathcal{K}\Psi(z_1,x)(2\mathrm{Im}(z_1))^{-1}W\left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x),\Phi_{z_1}^{(1)}(z_1,x)\right)$$

$$= \Phi_{z_1}^{(1)}(z_1,x) - \mathcal{K}\Psi(z_1,x)(\mathrm{Im}(z_1))^{-1}\left\|\Phi_{z_1}^{(1)}(z_1,x)\right\|_{\mathbb{C}^2}^{2}$$

$$= \Phi_{z_1}^{(1)}(z_1,x) - \mathrm{Im}(z_1)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}\mathcal{K}\Psi(z_1,x)$$

$$= 0,$$

and hence $\Phi_{z_1}^{(1)}(z_1) \in \ker(S_{\overline{z_1}})$.

For the proof of (6.43), we next assume $G \in \left\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\right\}^{\perp_{\mathcal{J}}}$. Using

(6.54) $2\mathrm{Im}(z)\Psi_{\mp}(z_1,x)^{\top}\sigma_1\mathcal{K}\Psi(z_1,x) = W(\Psi_{\mp}(z_1,x),\mathcal{K}\Psi(z,x))_x$

(cf. (2.6)), we then compute

(6.55) $(T_{z_1}S_{\overline{z_1}}G)(x) = (S_{\overline{z_1}}G)(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;S_{\overline{z_1}}G)$

$$= G(x) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^{x} dx'\, \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\right)^{\top}\sigma_1 G(x') + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;G)$$

$$+ 2(\mathrm{Im}(z_1))^{-1}\Phi_{z_1}^{(1)}(z_1,x)$$

$$\times \left[\int_{-\infty}^{x} dx'\, W(\Psi_-(z_1,x'),\mathcal{K}\Psi(z_1,x'))_{x'}\int_{-\infty}^{x'} dx''\, \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x'')\right)^{\top}\sigma_1 G(x'')\right.$$

$$\left. - \int_{x}^{\infty} dx'\, W(\Psi_+(z_1,x'),\mathcal{K}\Psi(z_1,x'))_{x'}\int_{-\infty}^{x'} dx''\, \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x'')\right)^{\top}\sigma_1 G(x'')\right].$$

By (6.23), one estimates

$$\left|W(\Psi_{\mp}(z_1,x),\mathcal{K}\Psi(z_1,x))\int_{\mp\infty}^{x} dx'\, \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\right)^{\top}\sigma_1 G(x')\right|$$

$$\leq \|\Psi_{\mp}(z_1,x)\|_{\mathbb{C}^2}\|\Psi(z_1,x)\|_{\mathbb{C}^2}\int_{\mp\infty}^{x} dx'\, \left\|\Phi_{z_1}^{(1)}(z_1,x')\right\|_{\mathbb{C}^2}\|G(x')\|_{\mathbb{C}^2}$$

$$\leq \|\Psi_{\mp}(z_1,x)\|_{\mathbb{C}^2}\|\Psi(z_1,x)\|_{\mathbb{C}^2}|\mathrm{Im}(z_1)|\int_{\mp\infty}^{x} dx'\, \left\|\Psi(z_1,x')\right\|_{\mathbb{C}^2}^{-1}\|G(x')\|_{\mathbb{C}^2}$$

$$\leq |\mathrm{Im}(z_1)|C_{\pm}\|\Psi_{\mp}(z_1,x)\|_{\mathbb{C}^2}[\|\Psi_-(z_1,x)\|_{\mathbb{C}^2} + \|\Psi_+(z_1,x)\|_{\mathbb{C}^2}]$$

(6.56) $\times \int_{\mp\infty}^{x} dx'\, \left\|\Psi_{\pm}(z_1,x')\right\|_{\mathbb{C}^2}^{-1}\|G(x')\|_{\mathbb{C}^2}$;

hence the left-hand side of (6.56) is in $L^1((\mp\infty,R])$ for all $R \in \mathbb{R}$, by Lemma 6.3. Thus,

(6.57) $\displaystyle\liminf_{x\to\mp\infty}\left|W(\Psi_{\mp}(z_1,x),\mathcal{K}\Psi(z_1,x))\int_{\mp\infty}^{x} dx'\, \left(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\right)^{\top}\sigma_1 G(x')\right| = 0.$

An integration by parts, using (6.57) and

$$(6.58) \qquad (\mathrm{Im}(z_1))^{-1}\Phi_{z_1}^{(1)}(z_1,x)W(\Psi(z_1,x),\mathcal{K}\Psi(z_1,x)) = -\mathcal{K}\Psi(z_1,x),$$

then yields

$$(T_{z_1}S_{\overline{z_1}}G)(x) = G(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;G) - 2(\mathrm{Im}(z_1))^{-1}\Phi_{z_1}^{(1)}(z_1,x)$$

$$\times \left[\int_{-\infty}^{x} dx' \, W(\Psi_-(z_1,x'),\mathcal{K}\Psi(z_1,x'))\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1 G(x') \right.$$

$$\left. - \int_{x}^{\infty} dx' \, W(\Psi_+(z_1,x'),\mathcal{K}\Psi(z_1,x'))\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^{\top}\sigma_1 G(x') \right]$$

$$= G(x) + 2\Phi_{z_1}^{(1)}(z_1,x)$$

$$(6.59) \qquad \times \left[\int_{-\infty}^{x} dx' \, \Theta_-(z_1,x')^{\top}\sigma_1 G(x') - \int_{x}^{\infty} dx' \, \Theta_+(z_1,x')^{\top}\sigma_1 G(x') \right].$$

(Again, $\lim_{x\to\pm\infty} |\cdots| = 0$ in (6.57) since all Lebesgue integrals involved are finite.) Here we have introduced the abbreviation

$$(6.60) \quad \Theta_\pm(z_1,x) = \Psi_\pm(z_1,x) + \Psi(z_1,x)\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-2}W(\Psi_\pm(z_1,x),\mathcal{K}\Psi(z_1,x)).$$

Actually, using the Jacobi identity

$$(6.61) \qquad AW(B,C) + BW(C,A) + CW(A,B) = 0, \quad A,B,C \in \mathbb{C}^2,$$

we obtain
(6.62)
$$\Psi_\pm(z_1)W(\mathcal{K}\Psi(z_1),\Psi(z_1)) + \Psi(z_1)W(\Psi_\pm(z_1),\mathcal{K}\Psi(z_1)) = \mathcal{K}\Psi(z_1)W(\Psi_\pm(z_1),\Psi(z_1)),$$

which in turn implies

$$(6.63) \qquad \Theta_\pm(z_1,x) = (\mathrm{Im}(z_1))^{-1}W(\Psi_\pm(z_1,x),\Psi_\mp(z_1,x))\Phi_{z_1}^{(1)}(z_1,x).$$

Combining (6.59) and (6.63) results in

$$(T_{z_1}S_{\overline{z_1}}G)(x) = G(x) + 2(\mathrm{Im}(z_1))^{-1}W(\Psi_-(z_1,x),\Psi_+(z_1,x))\Phi_{z_1}^{(1)}(z_1,x)$$

$$(6.64) \qquad \times \int_{\mathbb{R}} dx' \, \Phi_{z_1}^{(1)}(z_1,x')^{\top}\sigma_1 G(x').$$

Thus,

$$(T_{z_1}S_{\overline{z_1}}G)(x) = G(x)$$

$$(6.65) \qquad + 2(\mathrm{Im}(z_1))^{-1}W(\Psi_-(z_1,x),\Psi_+(z_1,x))\big(G,\mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}\Phi_{z_1}^{(1)}(z_1,x).$$

Applying (6.65) to $G = \Phi_{z_1}^{(1)}(z_1) \in \{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp_{\mathcal{J}}}$, we obtain the relation

(6.66) $0 = 1 + 2(\text{Im}(z_1))^{-1}W(\Psi_-(z_1,x),\Psi_+(z_1,x))\big(\Phi_{z_1}^{(1)}(z_1),\mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}$,

which proves (6.43). Insertion of (6.66) into (6.65) proves (6.44).

Finally, let $F \in L^2(\mathbb{R})^2$. Then $T_{z_1}F \in \{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp_{\mathcal{J}}}$ by (6.42), and hence (6.41) yields

(6.67) $(S_{\overline{z_1}}T_{z_1}F)(x) = F(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;F)$

$$+ 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^x dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top \sigma_1\big[F(x') + 2\Phi^{(1)}(z_1,x')\xi(x';F)\big]$$

$$= F(x) + 2\Phi_{z_1}^{(1)}(z_1,x)\xi(x;F) + 2\mathcal{K}\Psi(z_1,x)\int_{-\infty}^x dx'\,\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x')\big)^\top \sigma_1 F(x')$$

$$- 2(\text{Im}(z_1))^{-1}\mathcal{K}\Psi(z_1,x)\int_{-\infty}^x dx'\,\big(W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x'),\Phi^{(1)}(z_1,x')\big)\big)_{x'}\xi(x',F),$$

using

(6.68) $-2\text{Im}(z_1)\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x)\big)^\top \sigma\Phi_{z_1}^{(1)}(z_1,x) = W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x),\Phi_{z_1}^{(1)}(z_1,x)\big)_x.$

The estimate

$$\big\|\mathcal{K}\Psi(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x),\Phi_{z_1}^{(1)}(z_1,x)\big)\xi(x;F)\big\|_{\mathbb{C}^2}$$

$$\leq \|\Psi(z_1,x)\|_{\mathbb{C}^2}\,2|\text{Im}(z_1)|\big\|\Phi_{z_1}^{(1)}(z_1,x))\big\|_{\mathbb{C}^2}^2|\xi(x;F)|$$

$$\leq 2|\text{Im}(z_1)|^3\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1}|\xi(x;F)|$$

$$= 2|\text{Im}(z_1)|^3\|\Psi(z_1,x)\|_{\mathbb{C}^2}^{-1}\left[\int_{-\infty}^x dx'\,\|\Psi_-(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2}\right.$$

$$\left. + \int_x^\infty dx'\,\|\Psi_+(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2}\right]$$

$$\leq 2|\text{Im}(z_1)|^3\left[C_-\|\Psi_-(z_1,x)\|_{\mathbb{C}^2}^{-1}\int_{-\infty}^x dx'\,\|\Psi_-(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2}\right.$$

(6.69) $$\left. + C_+\|\Psi_+(z_1,x)\|_{\mathbb{C}^2}^{-1}\int_x^\infty dx'\,\|\Psi_+(z_1,x')\|_{\mathbb{C}^2}\|F(x')\|_{\mathbb{C}^2}\right]$$

then proves

(6.70) $\big\|\mathcal{K}\Psi(z_1)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1),\Phi_{z_1}^{(1)}(z_1)\big)\xi(\cdot;F)\big\|_{\mathbb{C}^2} \in L^2(\mathbb{R})$,

by (6.19). An integration by parts in the last term of (6.67), using (6.8), (6.9), and

$$\liminf_{x\downarrow-\infty}\big\|\mathcal{K}\Psi(z_1,x)W\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1,x),\Phi_{z_1}^{(1)}(z_1,x)\big)\xi(x;F)\big\|_{\mathbb{C}^2}$$

(6.71) $$\leq |2\text{Im}(z_1)|\liminf_{x\downarrow-\infty}\|\Psi(z_1,x)\|_{\mathbb{C}^2}\big\|\Phi_{z_1}^{(1)}(z_1,x)\big\|_{\mathbb{C}^2}^2|\xi(x;F)| = 0$$

by (6.70) (in fact, $\lim_{x\downarrow-\infty}|\cdots| = 0$ in (6.71) since all Lebesgue integrals involved are finite), then proves $S_{\overline{z_1}}T_{z_1}F = F$ and hence (6.45). $\qquad\square$

Next, we further restrict $S_{\overline{z_1}}$ and define the operator $S_{z_1,\overline{z_1}}$ by

$$(6.72) \qquad S_{z_1,\overline{z_1}} = S_{\overline{z_1}}\big|_{\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp\mathcal{J}}} = \widehat{S}_{\overline{z_1}}\big|_{\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp\mathcal{J}}} .$$

Lemma 6.10. *Assume Hypothesis 5.3 and $z_1 \in \rho(D(q))$. Then*

$$(6.73) \qquad \ker(T_{z_1}) = \{0\}, \quad \operatorname{ran}(T_{z_1}) = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}},$$

$$(6.74) \qquad \ker(S_{\overline{z_1}}) = \operatorname{span}\{\Phi_{z_1}^{(1)}(z_1)\}, \quad \operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2.$$

Moreover, $S_{z_1,\overline{z_1}}$ is the inverse of T_{z_1}, that is,

$$(6.75) \qquad \ker(S_{z_1,\overline{z_1}}) = \{0\}, \quad \operatorname{ran}(S_{z_1,\overline{z_1}}) = L^2(\mathbb{R})^2,$$

$$(6.76) \qquad S_{z_1,\overline{z_1}}T_{z_1} = I_2 \quad on \; L^2(\mathbb{R})^2,$$

$$(6.77) \qquad T_{z_1}S_{z_1,\overline{z_1}} = I_2 \quad on \; \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}}.$$

Proof. Suppose $T_{z_1}F = 0$ for some $F \in L^2(\mathbb{R})^2$. Then (6.45) yields $0 = S_{\overline{z_1}}T_{z_1}F = F$, and hence $\ker(T_{z_1}) = \{0\}$. The assertion $\operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2$ in (6.74) is also clear from (6.45). Next, assume $S_{\overline{z_1}}G = 0$ for some $G \in \operatorname{dom}(S_{\overline{z_1}}) = \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}}$ (cf. (6.40)). Then (6.44) implies

$$(6.78) \qquad 0 = T_{z_1}S_{\overline{z_1}}G$$

$$= G - \frac{\big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}{\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}} \Phi_{z_1}^{(1)}(z_1), \quad G \in \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}},$$

and hence $G = c\Phi_{z_1}^{(1)}(z_1)$ for some $c \in \mathbb{C}$. This proves (6.75).

Next, suppose $G \in \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}}$. Then $\big(G, \mathcal{J}\Phi_{z_1}^{(1)}\big)_{L^2} = 0$ and (6.44) imply $T_{z_1}S_{\overline{z_1}}G = G$, and hence

$$(6.79) \qquad \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}} \subseteq \operatorname{ran}(T_{z_1}) \subseteq \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}},$$

taking $\operatorname{ran}(T_{z_1}) \subseteq \big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}}$ in (6.42) into account. The computation

$$(6.80) \qquad \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}T_{z_1}S_{\overline{z_1}}G\big)_{L^2} = \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}G\big)_{L^2} - \big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} = 0$$

then proves $\Phi_{z_1}^{(1)}(z_1)\perp_{\mathcal{J}}\operatorname{ran}(T_{z_1})$ and

$$(6.81) \qquad \operatorname{ran}(T_{z_1}) = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp\mathcal{J}},$$

since $\operatorname{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2$.

Finally, (6.76) and (6.77) are clear from (6.44), (6.45), (6.73), and (6.74). $\quad\square$

Next we state an auxiliary result.

Lemma 6.11. *Assume Hypothesis 3.2 and let* $\xi \in AC_{\mathrm{loc}}(\mathbb{R})$, $\Phi = (\phi_1, \phi_2)^\top \in AC_{\mathrm{loc}}(\mathbb{R})^2$, $F = (f_1, f_2)^\top \in \mathrm{dom}(D(q))$, *and*

$$(6.82) \qquad A_- \in \left\{ G \in AC_{\mathrm{loc}}(\mathbb{R})^2 \; : \; G, M(q)G \in L^2_{\mathrm{loc}}([-\infty, \infty))^2 \right\},$$

$$(6.83) \qquad A_+ \in \left\{ G \in AC_{\mathrm{loc}}(\mathbb{R})^2 \; : \; G, M(q)G \in L^2_{\mathrm{loc}}((-\infty, \infty])^2 \right\}.$$

Then

$$(6.84) \quad (M(q)\xi\Phi)(x) = i\xi'(x)(\phi_1(x), -\phi_2(x))^\top + \xi(x)(M(q)\Phi)(x) \; for \; a.e. \; x \in \mathbb{R}$$

and

$$\int_{-\infty}^{x} dx' \, A_-(x')^\top \sigma_1 (M(q)F)(x') = \int_{-\infty}^{x} dx' \, (M(q)A_-)(x')^\top \sigma_1 F(x')$$

$$(6.85) \qquad\qquad\qquad\qquad + iA_-(x)^\perp F(x),$$

$$\int_{x}^{\infty} dx' \, A_+(x')^\top \sigma_1 (M(q)F)(x') = \int_{x}^{\infty} dx' \, (M(q)A_+)(x')^\top \sigma_1 F(x')$$

$$(6.86) \qquad\qquad\qquad\qquad - iA_+(x)^\perp F(x)$$

for all $x \in \mathbb{R}$.

Proof. Assertion (6.84) follows directly from the definition of $M(q)$. To prove assertion (6.85) one integrates by parts and obtains

$$\int_{-\infty}^{x} dx' \, A_-(x')^\top \sigma_1 (M(q)F)(x') = i[a_{2,-}(x)f_1(x) - a_{1,-}(x)f_2(x)]$$

$$+ i \int_{-\infty}^{x} dx' \, (-a_{1,-}\bar{q}f_1 - a_{2,-}qf_{2,-} + a'_{1,-}f_2 - a'_{2,-}f_1)(x')$$

$$(6.87) \qquad = \int_{-\infty}^{x} dx' \, ((M(q)A_-)(x'))^\top \sigma_1 F(x') + iA_-(x)^\perp F(x),$$

using $\liminf_{x\downarrow-\infty} |A_-(x)^\perp F(x)| = 0$ (actually, $\lim_{x\downarrow-\infty} |\cdots| = 0$). Relation (6.86) is proved analogueously. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Box$

The following result shows that T_{z_1} and $S_{\overline{z_1}}$ intertwine $D\big(q_{z_1}^{(1)}\big)$ and $D(q)$.

Lemma 6.12. *Assume Hypothesis 5.3 and* $z_1 \in \rho(D(q))$. *Then*

$$(6.88) \qquad\qquad\qquad D\big(q_{z_1}^{(1)}\big)T_{z_1} = T_{z_1}D(q),$$

$$(6.89) \qquad\qquad\qquad S_{\overline{z_1}}D\big(q_{z_1}^{(1)}\big) = D(q)S_{\overline{z_1}}.$$

Proof. Using formula (4.10), we have

$$(6.90) \qquad M\big(q_{z_1}^{(1)}\big) - M(q) = 4i \begin{pmatrix} 0 & -\phi_{1,z_1}^{(1)}(z_1)\psi_1(z_1) \\ \phi_{2,z_1}^{(1)}(z_1)\psi_2(z_1) & 0 \end{pmatrix}.$$

Relation (6.84) applied to $M(q^{(1)})$ with $\Phi = \Phi_{z_1}^{(1)}(z_1)$ and (cf. (6.46))

$$\xi(x) = \xi(x;F) = \int_{-\infty}^{x} dx'\, \Psi_-(z_1,x')^\top \sigma_1 F(x') - \int_x^\infty dx'\, \Psi_+(z_1,x')^\top \sigma_1 F(x'),$$

$$(6.91) \qquad\qquad\qquad\qquad\qquad\qquad F \in \mathrm{dom}(D(q))$$

then yields

$$\big(M\big(q_{z_1}^{(1)}\big)\big(\xi(\cdot;F)\,\Phi_{z_1}^{(1)}(z_1)\big)\big)(x) = i\Psi(z_1,x)^\top \sigma_1 F(x)(\phi_{1,z_1}^{(1)}(z_1,x), -\phi_{2,z_1}^{(1)}(z_1,x))^\top$$

$$(6.92) \qquad\qquad\qquad + z_1\Phi_{z_1}^{(1)}(z_1,x)\xi(x;F),$$

since $M\big(q_{z_1}^{(1)}\big)\Phi_{z_1}^{(1)}(z_1) = z_1\Phi_{z_1}^{(1)}(z_1)$. By (6.85), with $A_- = \Psi_-(z_1)$ and $A_+ = \Psi_+(z_1)$, we obtain

$$\int_{\pm\infty}^{x} dx'\, \Psi_\pm(z_1,x')^\top \sigma_1 (M(q)F)(x') = z_1 \int_{\pm\infty}^{x} dx'\, \Psi_\pm(z_1,x')^\top \sigma_1 F(x')$$

$$(6.93) \qquad\qquad\qquad + i\Psi_\pm(z_1,x)^\perp F(x), \quad F \in \mathrm{dom}(D(q))$$

since $(M(q)\Psi_\pm(z_1))^\top = z_1\Psi_\pm(z_1)^\top$. Thus,

$$\big(M\big(q_{z_1}^{(1)}\big)T_{z_1} - T_{z_1}M(q)\big)F = \big(M\big(q_{z_1}^{(1)}\big) - M(q)\big)F$$

$$+ 2M\big(q_{z_1}^{(1)}\big)\big(\Phi^{(1)}(z_1)\xi(\cdot;F)\big) - 2\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;M(q)F)$$

$$= 4i\big(-\phi_{1,z_1}^{(1)}(z_1)\psi_1(z_1)f_2, \phi_{2,z_1}^{(1)}(z_1)\psi_2(z_1)f_1\big)^\top$$

$$+ 2i\big[\Psi(z_1)^\top \sigma_1 F\big]\big(\phi_1^{(1)}(z_1), -\phi_2^{(1)}(z_1)\big)^\top + 2z_1\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;F)$$

$$- 2z_1\Phi_{z_1}^{(1)}(z_1)\xi(\cdot;F) - 2i\big[\Psi(z_1)^\perp F\big]\big(\phi_{1,z_1}^{(1)}(z_1), \phi_{2,z_1}^{(1)}(z_1)\big)^\top$$

$$(6.94) \qquad = 0, \quad F \in \mathrm{dom}(D(q)).$$

This computation also proves that

$$(6.95) \qquad T_{z_1}\mathrm{dom}(D(q)) \subseteq \mathrm{dom}\big(D(q_{z_1}^{(1)})\big) \quad \text{and hence} \quad T_{z_1}D(q) \subseteq D\big(q_{z_1}^{(1)}\big)T_{z_1}.$$

Next, choosing $G \in \mathrm{dom}(D(q)S_{\overline{z_1}})$ (i.e., $G \in L^2(\mathbb{R})^2$ such that $S_{\overline{z_1}}G \in \mathrm{dom}(D(q))$), (6.95) implies

$$S_{\overline{z_1}}D(q_{z_1}^{(1)})T_{z_1}S_{\overline{z_1}}G = S_{\overline{z_1}}D(q_{z_1}^{(1)})\big[G - \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}^{-1}\big(G, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}\big]$$

$$(6.96) \quad = S_{\overline{z_1}}D(q_{z_1}^{(1)})G = S_{\overline{z_1}}T_{z_1}D(q)S_{\overline{z_1}}G = D(q)S_{\overline{z_1}}G, \quad G \in \mathrm{dom}(D(q)S_{\overline{z_1}}).$$

Here we have used successively (6.44), $S_{\overline{z_1}}\Phi_{z_1}^{(1)}(z_1) = 0$ (cf. (6.42)), and (6.45). Thus, we conclude

(6.97) $\text{dom}\big(D(q_{z_1}^{(1)})\big) \subseteq \text{dom}(D(q)S_{\overline{z_1}})$ and hence $S_{\overline{z_1}}D\big(q_{z_1}^{(1)}\big) \subseteq D(q)S_{\overline{z_1}}$.

Hence,

(6.98) $\text{dom}\big(D(q_{z_1}^{(1)})T_{z_1}\big) = \big\{F \in L^2(\mathbb{R})^2 \ : \ T_{z_1}F \in \text{dom}\big(D(q_{z_1}^{(1)})\big)\big\} \subseteq \text{dom}(D(q))$,

since $S_{\overline{z_1}}(T_{z_1}F) = F \in \text{dom}(D(q))$ for $F \in \text{dom}\big(D\big(q_{z_1}^{(1)}\big)T_{z_1}\big)$ by (6.97). Combining (6.95) and (6.98) then proves (6.88). Equations (6.88), (6.44), and (6.45) in turn imply

$$S_{\overline{z_1}}D(q_{z_1}^{(1)})T_{z_1}S_{\overline{z_1}} = S_{\overline{z_1}}D(q_{z_1}^{(1)})\big[I - \big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}\big)_{L^2}^{-1}\big(\cdot, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}\big]$$
(6.99) $= S_{\overline{z_1}}D(q_{z_1}^{(1)}) = S_{\overline{z_1}}T_{z_1}D(q)S_{\overline{z_1}} = D(q)S_{\overline{z_1}}$,

and hence (6.89). $\square$

Remark 6.13. One can prove, similarly to Lemma 6.12, that

$$D\big(q_{z_1}^{(1)}\big)\widetilde{T}_{\overline{z_1}} = \widetilde{T}_{\overline{z_1}}D(q) \quad \text{and} \quad \ker(\widetilde{T}_{\overline{z_1}}) = \{0\},$$
(6.100) $\widetilde{S}_{z_1}D\big(q_{z_1}^{(1)}\big) = D(q)\widetilde{S}_{z_1} \quad \text{and} \quad \ker(\widetilde{S}_{z_1}) = \text{span}\big\{\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}$

and that $\widetilde{T}_{\overline{z_1}}$ and $\widetilde{S}_{z_1}\big|_{\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\}^{\perp_{\mathcal{J}}}}$ are transformation operators (inverse to each other).

Summarizing the results of Corollaries 6.4 and 6.5 and Lemmas 6.9–6.12 obtained thus far, we are now in position to state one of the principal results of this section.

Theorem 6.14. *Assume Hypothesis 5.3 and let* $z_1 \in \rho(D(q))$. *Then* T_{z_1} *and* $S_{\overline{z_1}}$ *are bounded linear operators in* $L^2(\mathbb{R})^2$ *which intertwine the operators* $D(q)$ *and* $D\big(q_{z_1}^{(1)}\big)$,

(6.101) $D\big(q_{z_1}^{(1)}\big)T_{z_1} = T_{z_1}D(q)$,

(6.102) $S_{\overline{z_1}}D\big(q_{z_1}^{(1)}\big) = D(q)S_{\overline{z_1}}$.

Moreover,

(6.103) $\ker(T_{z_1}) = \{0\}, \quad \text{ran}(T_{z_1}) = \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$,

(6.104) $\ker(S_{\overline{z_1}}) = \text{span}\big\{\Phi_{z_1}^{(1)}(z_1)\big\}, \quad \text{ran}(S_{\overline{z_1}}) = L^2(\mathbb{R})^2$,

and $S_{z_1,\overline{z_1}}$ *(cf. (6.72)) is the inverse of* T_{z_1}, *that is,*

(6.105) $S_{z_1,\overline{z_1}}T_{z_1} = I_2 \quad \text{on } L^2(\mathbb{R})^2$,

(6.106) $T_{z_1}S_{z_1,\overline{z_1}} = I_2 \quad \text{on } \big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_{\mathcal{J}}}$.

An analogueous statement can be formulated when z_1 is replaced by $\overline{z_1}$, that is, for the operators $T_{\overline{z_1}}$ and $\widetilde{S}_{z_1}$; we choose not to dwell on it here due to the symmetry of the arguments.

We conclude with the principal spectral theoretic result of this paper.

Theorem 6.15. *Assume Hypothesis 5.3 and* [2] $z_1 \in \rho(D(q))$. *Then*

(6.107) $$\sigma\big(D\big(q_{z_1}^{(1)}\big)\big) = \sigma(D(q)) \cup \{z_1, \overline{z_1}\},$$

(6.108) $$\sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big) = \sigma_{\mathrm{p}}(D(q)) \cup \{z_1, \overline{z_1}\},$$

(6.109) $$\sigma_{\mathrm{e}}\big(D\big(q_{z_1}^{(1)}\big)\big) = \sigma_{\mathrm{e}}(D(q)).$$

In other words, constructing the new NLS_ *potential* $q_{z_1}^{(1)}$ *amounts to inserting a pair of complex conjugate (nonreal)* $L^2(\mathbb{R})^2$-*eigenvalues,* z_1 *and* $\overline{z_1}$, *into the spectrum of the background operator* $D(q)$, *leaving the rest of its spectrum invariant.*

Proof. We prove (6.108) and (6.109) from which (6.107) follows by (5.27). By Theorem 4.5, one has $z_1, \overline{z_1} \in \sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big)$. Thus, (6.108) is equivalent to

(6.110) $$\sigma_{\mathrm{p}}\big(D\big(q_{z_1}^{(1)}\big)\big) \backslash \{z_1, \overline{z_1}\} = \sigma_{\mathrm{p}}(D(q)).$$

Let

(6.111) $$X^{(1)} = \big\{ \Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1) \big\}^{\perp_J}.$$

Since $D(q_{z_1}^{(1)})$ is $\mathcal{J}$-self-adjoint, $X^{(1)}$ is a closed, $D\big(q_{z_1}^{(1)}\big)$-invariant subspace. Denote by $D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}$ the part of $D\big(q_{z_1}^{(1)}\big)$ in $X^{(1)}$ with

(6.112) $$\mathrm{dom}\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big) = X^{(1)} \cap \mathrm{dom}\big(D\big(q_{z_1}^{(1)}\big)\big).$$

From (6.101) and Theorem 6.14, it follows that

(6.113) $$D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}} = TD(q)T^{-1} \quad \text{on } X^{(1)}$$

and

(6.114) $$D(q) = T^{-1}D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}T \quad \text{on } L^2(\mathbb{R})^2,$$

with $T = T_{z_1}$ and $T^{-1} = \widehat{S}_{\overline{z_1}}\big|_{\big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}^{\perp_J}}$ (cf. Lemma 6.10). Assume that $\mu \in \sigma_{\mathrm{p}}(D(q))$ and

(6.115) $$D(q)F(\mu) = \mu F(\mu), \quad 0 \neq F(\mu) \in \mathrm{dom}(D(q));$$

---

[2]Recall that $z \in \rho(D(q))$ implies $z \in \mathbb{C}\backslash\mathbb{R}$ by Corollary 5.5.

then (6.113) implies

$$(6.116) \qquad D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}} TF(\mu) = \mu TF(\mu).$$

Since $\ker(T) = \{0\}$ (cf. (6.103)), this implies

$$(6.117) \quad \sigma_p(D(q)) \subseteq \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big) \text{ and hence } \sigma_p(D(q)) \subseteq \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\}.$$

Conversely, assuming $\nu \in \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\}$ and

$$(6.118) \qquad D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}} F^{(1)}(\nu) = \nu F^{(1)}(\nu), \quad 0 \neq F^{(1)}(\nu) \in \mathrm{dom}\big(D\big(q_{z_1}^{(1)}\big)\big),$$

we have from (6.114) that

$$(6.119) \qquad D(q)T^{-1}F^{(1)}(\nu) = \nu T^{-1}F^{(1)}(\nu).$$

Since $\ker(T^{-1}) = \{0\}$ (cf. (6.75)), one concludes $T^{-1}F^{(1)}(\nu) \neq 0$ and hence

$$(6.120) \qquad \sigma_p(D(q)) \supseteq \sigma_p\big(D\big(q_{z_1}^{(1)}\big)\big)\backslash\{z_1, \overline{z_1}\},$$

which implies (6.108).

Next, observe that (6.114) also implies

$$(6.121) \qquad \sigma_e(D(q)) = \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big).$$

To prove (6.109), we note that (6.121) implies

$$(6.122) \qquad \sigma_e(D(q)) = \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big) \subseteq \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big).$$

Conversely, let $\lambda \in \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big)$ and let $\{G_n\}_{n\in\mathbb{N}}$ be a singular sequence of $D\big(q_{z_1}^{(1)}\big)$ corresponding to λ, that is, a bounded sequence in $\mathrm{dom}\big(D\big(q_{z_1}^{(1)}\big)\big)$ without any convergent subsequence such that $\lim_{n\to\infty}\big(D\big(q_{z_1}^{(1)}\big) - \lambda I\big)G_n = 0$. Since $L^2(\mathbb{R})^2$ admits the direct sum decomposition (not to be confused with an orthogonal direct sum decomposition)

$$(6.123) \qquad L^2(\mathbb{R})^2 = X^{(1)} \dot{+} \mathrm{span}\big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}$$

(here one uses the fact that $\Phi_{z_1}^{(1)}(z_1) \perp_{\mathcal{J}} \mathcal{K}\Phi_{z_1}^{(1)}(z_1)$, cf. (6.38)), one can write

$$(6.124)$$
$$G_n = F_n + \frac{\big(G_n, \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}{\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}\Phi_{z_1}^{(1)}(z_1) + \frac{\big(G_n, \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}{\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2}}\mathcal{K}\Phi_{z_1}^{(1)}(z_1),$$

where $F_n \in X^{(1)} \cap \mathrm{dom}\big(D\big(q_{z_1}^{(1)}\big)\big)$ is the projection of G_n onto the space $X^{(1)}$, the projection being parallel to the subspace $\mathrm{span}\big\{\Phi_{z_1}^{(1)}(z_1), \mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big\}$. Since the

coefficients of $\Phi_{z_1}^{(1)}(z_1)$ and $\mathcal{K}\Phi_{z_1}^{(1)}(z_1)$ in (6.124) are bounded with respect to $n \in \mathbb{N}$, one can assume that they are convergent by restricting to a subsequence. Thus, for some real numbers c_1 and c_2, one has

$$(6.125) \qquad \left(D\big(q_{z_1}^{(1)}\big) - \lambda I\right) F_n \underset{n \to \infty}{\to} c_1 \Phi_{z_1}^{(1)}(z_1) + c_2 \mathcal{K}\Phi_{z_1}^{(1)}(z_1).$$

Since $X^{(1)}$ is $D\big(q_{z_1}^{(1)}\big)$-invariant, $F_n \in X^{(1)}$ implies $\big(D\big(q_{z_1}^{(1)}\big) - \lambda I\big)F_n \in X^{(1)}$; and since $X^{(1)}$ is closed, the limit $c_1\Phi_{z_1}^{(1)}(z_1) + c_2\mathcal{K}\Phi_{z_1}^{(1)}(z_1)$ also belongs to $X^{(1)}$. But this implies $c_1 = c_2 = 0$, because $\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} = 0$ (cf. (6.38)) and $\big(\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} \neq 0$, $\big(\mathcal{K}\Phi_{z_1}^{(1)}(z_1), \mathcal{J}\mathcal{K}\Phi_{z_1}^{(1)}(z_1)\big)_{L^2} \neq 0$ (cf. (6.43)). Thus, one concludes that $\{F_n\}_{n\in\mathbb{N}} \subset X^{(1)}$ is a singular sequence of $D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}$ corresponding to λ, since it is bounded and has no convergent subsequence. (Otherwise, by (6.124), $\{G_n\}_{n\in\mathbb{N}}$ would have a convergent subsequence, contradicting the assumption that it is a singular sequence.) It follows that $\lambda \in \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big) = \sigma_e(D(q))$, and hence $\sigma_e(D(q)) = \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big|_{X^{(1)}}\big) \supseteq \sigma_e\big(D\big(q_{z_1}^{(1)}\big)\big)$; this proves (6.109). $\qquad\square$

Theorems 6.14 and 6.15 are new. We note that they have been proved under the optimal assumption $q \in L^1_{\mathrm{loc}}(\mathbb{R})$ (but they seem to be new under virtually any assumptions on q).

In the special periodic case, where the machinery of Floquet theory can be applied, the issue of isospectral Darboux transformations is briefly mentioned in [35, Theorem 3]. This excludes the insertion of eigenvalues as in Theorem 6.15. Inserting eigenvalues into the spectrum of a self-adjoint one-dimensional Dirac operator (not applicable in the present NLS_ context) was investigated by means of transformation operators (along the lines of [19]) in [45].

We conclude this section with a few facts on N-soliton NLS_ potentials. As shown in Lemma 4.6, the insertion of pairs of complex conjugate eigenvalues into the spectrum of the background operator $D(q)$ can be iterated. To fix the proper notation, we now slightly extend our approach of Section 6 and consider a more general linear combination $\Psi_{\gamma_k}(z, x)$ of $\Psi_{\pm}(z, x)$. Assuming $z_k \in \mathbb{C}\backslash\mathbb{R}$, $k = 1, \ldots, N$, we define

$$(6.126) \qquad \Psi_{\gamma_k}(z_k, x) = \Psi_{-}(z_k, x) + \gamma_k \Psi_{+}(z_k, x), \quad \gamma_k \in \mathbb{C}\backslash\{0\}, \ k = 1 \ldots, N$$

(as opposed to our choice $\gamma_k = 1$ in (6.6)). In obvious notation, we then denote the corresponding Nth iteration of the construction of $q_{z_1}^{(1)}(x)$ presented in Sections 2 and 4 (cf. (2.38), (4.10)), identifying $\Psi(z_1, x)$ and $\Psi_{\gamma_1}(z_1, x)$, by $q_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N}^{(N)}(x)$.

In order to describe a well-known explicit formula for $q_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N}^{(N)}(x)$ (cf., e.g., [36, Sect. 4.2], [38], [43]) we introduce the quantities

$$(6.127) \qquad \varphi_k(x) = \frac{\psi_{2,-}(z_k, x) + \gamma_k \psi_{2,+}(z_k, x)}{\psi_{1,-}(z_k, x) + \gamma_k \psi_{1,+}(z_k, x)},$$

and the $2N \times 2N$ Vandermonde-type matrices
(6.128)

$$V_{2N}(x) = \begin{pmatrix} 1 & z_1 & \cdots & z_1^{N-1} & \varphi_1 & z_1\varphi_1 & \cdots & z_1^{N-1}\varphi_1 \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_N & \cdots & z_N^{N-1} & \varphi_N & z_N\varphi_N & \cdots & z_N^{N-1}\varphi_N \\ 1 & z_{N+1} & \cdots & z_{N+1}^{N-1} & \varphi_{N+1} & z_{N+1}\varphi_{N+1} & \cdots & z_{N+1}^{N-1}\varphi_{N+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{2N} & \cdots & z_{2N}^{N-1} & \varphi_{2N} & z_{2N}\varphi_{2N} & \cdots & z_{2N}^{N-1}\varphi_{2N} \end{pmatrix}$$

and

$$(6.129) \quad \widetilde{V}_{2N}(x) = \begin{pmatrix} 1 & z_1 & \cdots & z_1^{N-1} & \varphi_1 & z_1\varphi_1 & \cdots & z_1^{N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_N & \cdots & z_N^{N-1} & \varphi_N & z_N\varphi_N & \cdots & z_N^{N} \\ 1 & z_{N+1} & \cdots & z_{N+1}^{N-1} & \varphi_{N+1} & z_{N+1}\varphi_{N+1} & \cdots & z_{N+1}^{N} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 1 & z_{2N} & \cdots & z_{2N}^{N-1} & \varphi_{2N} & z_{2N}\varphi_{2N} & \cdots & z_{2N}^{N} \end{pmatrix}.$$

The Nth iteration $q^{(N)}_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N}(x)$ is then explicitly given by

$$(6.130) \qquad q^{(N)}_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N}(x) = q(x) - 2i\frac{\det(\widetilde{V}_{2N}(x))}{\det(V_{2N}(x))}.$$

Denote by $D(q^{(N)}_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N})$ the associated $\mathcal{J}$-self-adjoint operator in $L^2(\mathbb{R})^2$; repeated application of Theorem 6.15 then yields

$$(6.131) \qquad \sigma(D(q^{(N)}_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N})) = \sigma(D(q)) \cup \{z_1,\overline{z_1},\ldots,z_N,\overline{z_N}\},$$

$$(6.132) \qquad \sigma_{\mathrm{p}}(D(q^{(N)}_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N})) = \sigma_{\mathrm{p}}(D(q)) \cup \{z_1,\overline{z_1},\ldots,z_N,\overline{z_N}\},$$

$$(6.133) \qquad \sigma_{\mathrm{e}}(D(q^{(N)}_{z_1,\ldots z_N,\gamma_1,\ldots,\gamma_N})) = \sigma_{\mathrm{e}}(D(q)).$$

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